Periodicity and periodicity on average in
coupled map lattices. Rigorous results.

M.Blank

Observatoire de Nice BP 229, 06304 Nice Cedex 4, France, e-mail: blank@obs-nice.fr
Russian Academy of Sci. Inst. Gen. Genetics, Gubkin st. 3, 117809, Moscow, Russia

Abstract. - We prove the existence and genericity in a sense of periodic on average behavior for systems of coupled map lattices, when the behavior of the system is chaotic, and the existence of periodic behavior for such systems, when the uncoupled one is chaotic. The stability of both of these phenomena with respect to small random perturbations is also proved.

Key words: coupled map lattice, periodic behavior, Sinai-Bowen-Ruelle (SBR) invariant measure.

H.Chate and P.Manneville [1] showed numerically the possibility for a system with a large number of components to have a chaotic behavior at a local scale (for individual subsystems), but on average periodic at a global scale (for space averages). Recently Y.Pomeau [2] proposed a beautiful heuristic explanation of this phenomenon for the case of coupled map lattices. His explanation bases on two following assumptions. At first he considered the uncoupled system (consisted of quadratic maps with the period-three window chaotic regime) and showed that for a special choice of initial conditions (in one window for all subsystems) this situation may take place by the central limit theorem. Then supposing that sufficiently weak interaction preserves this phenomenon he deduced that this result is valid for a special choice of initial conditions. The second step of his explanation is to consider arbitrary initial conditions. He supposed that when interaction is small, but not too small, there would be something like a rough synchronization, that is after a time the system hits into a state suitable to imply the first assumption. We prove the first assumption rigorously for a sufficiently broad class of chaotic maps, so called piecewise expanding maps, and show that the second assumption is unnecessary. Moreover we construct an example, where there is periodicity on average for all initial conditions, but there is no rough synchronization in the sense above. The latter is very important, because it shows that the system may be on average periodic for all initial conditions, but with different averages. On another point of view under weak interactions such a phenomena may take place not only on average, but for the system itself, that is instead of chaotic behavior in the uncoupled system we may have a periodic behavior of the coupled one. Both of these phenomena are stable with respect to small random perturbations. We consider also some new numerical results about the bifurcation structure of coupled map lattices with respect to the value of interactions.

Let us consider a finite coupled map lattice, that is a mapping $F_\epsilon^{(d)}: X^d \to X^d$, where X^d is a unit d-dimensional cube, i.e. a direct product of d unit intervals $X_i = X$, and the i-th coordinate of the mapping may be written in the form:

$$(F_\epsilon^{(d)}(\bar{x}))_i = (1-\epsilon) f_i(x_i) + \epsilon/3 \sum_{|i-j|\le 1} f_j(x_j)$$

Here $f_i : X \to X$ for all i is a one-dimensional mapping, $\bar{x} = (x_1, \ldots, x_d) \in X^d$, ϵ is a value of the interaction between neighboring mappings, and we suppose periodic in space boundary conditions. For the sake of simplicity we suppose that $f_i = f$ and does not depend on i. We suppose additionally, that the mapping f is piecewise expanding. This means that X may be decomposed into a finite number of intervals I_i, such that the restriction of f to each of them is a twice continuously differentiable one-to-one map, whose modulus of derivative (expanding constant) is strictly larger than one. Then such a mapping possesses a stochastic attractor (see for example refs. 3 and 4), consisting of a finite number of intervals transposing and mixing (within the intervals) by the mapping f. As an example we may take in mind the following map:

$$f(x) = \begin{cases} (\{12x\}+3)/4, & \text{if } x \in [0, 1/2); \\ \{12x\}/4, & \text{if } x \in [1/2, 1], \end{cases}$$

or the tent map $f_{\text{tent}}(x) = \min(ax, -ax+a)$, $1 < a \le 2$. Here $\{x\}$ is a fractional part of the number x. The map f is a piecewise expanding map (i.e. its derivative, where it exists, is equal to $3 > 1$) and this map transposes intervals $I_1 = [0, 1/4]$ and $I_2 = [3/4, 1]$ and mixes points within them. Therefore we shall call this map transposing. All points from the interval $(1/4, 1/2)$ hit after one iteration onto I_2 and points from $[1/2, 3/4)$ to I_1. This is important, because this property guarantees that small perturbations of this map have the same property of transpositions.

We shall say that the coupled map lattice system $(F_\epsilon^{(d)}, X^d)$ has on average periodic behavior if the space-average function

$$S_\epsilon^{(d)}(t) = 1/d \sum_i^d x_i(t)$$

tends to a periodic on t function as $d \to \infty$.

If initial points for the uncoupled system (when $\epsilon = 0$) lie in the one of these intervals, then the space-average function may be considered as a sum of d identically distributed independent random variables, divided by d. Therefore $S_0^{(d)}$ converges in distribution to Gaussian process with the dispersion d^{-1} when $d \to \infty$. The idea of Y.Pomeau is that when we consider sufficiently small interaction ($\epsilon \ll 1$) the last statement remains valid. On another hand he proposed to consider not too small interactions, supposing that in that case even initial conditions being in different intervals after a time will hit into one interval. The example above shows that this may not be true, because for $\epsilon < 1/4$ the system preserves the rough arrangement of points in intervals I_1 and I_2. However, considering the space average $S_\epsilon^{(d)}$ for uncoupled system we may divide it in two parts: points, belonging to the first interval I_1 and the second I_2. Then the same argument as before gives us on average periodic behavior (with period 2) for the uncoupled system, but values of this periodic function depend on the distribution of initial points between intervals $[0, 1/2)$ and $[1/2, 1]$. The dynamics of such system is shown on Fig.1. The same argument evidently remains true for any number of transposing intervals.

Theorem 1. Let f be a piecewise expanding mapping with the expanding constant strictly greater than 2. Then for sufficiently small ϵ the coupled map lattice system $(F_\epsilon^{(d)}, X^d)$ is on average periodic.

Proof. It was proved in ref. 5 (see also ref. 6) that even for more general assumptions the system $(F_\epsilon^{(d)}, X^d)$ has Sinai-Bowen-Ruelle (SBR) invariant measure [3] $\mu_\epsilon^{(d)}$, converging in the L_1-norm as $\epsilon \to \infty$ to the direct product of equal SBR measures of the initial mapping $\mu^{(d)}$. If we suppose that f has only one SBR measure, then it follows that ergodic averages along almost all trajectories of $F_\epsilon^{(d)}$ have the same values. Now from this strong convergence to the invariant measure of the uncoupled map lattice system we have the statement of the theorem, because $S_\epsilon^{(d)} \to S_0^{(d)}$ in distribution as $\epsilon \to 0$ for any fixed dimension d.

On average periodic behavior of a chaotic system may be considered as a kind of localization: trajectories which should normally be dense in a large domains remain confined to smaller sets. It turns out that under weak interaction such a phenomena may take place not only on average, but for the system itself, that is instead of chaotic behavior in the uncoupled system we may have a periodic behavior in the coupled one. It is interesting to remark, that the condition that the expanding constant has to be larger than 2 is sufficient here. It is well known that in the one-dimensional case considering instead of the initial mapping some its power f^n we may increase the expanding constant from r to r^n. This trick gives a possibility to investigate not only piecewise expanding mappings, but also mappings that become piecewise expanding only after going to some its power. As weak interactions in coupled maps that we have considered in Theorem 1 vary derivatives only slightly it means natural that this trick works also in this situation. However this is not true.

Theorem 2. There exists a piecewise expanding mapping with the SBR measure μ with the positive density on the whole interval $[0, 1]$ such that for any fixed dimension d and all sufficiently small $\epsilon > 0$ the coupled map lattice system has a finite number of SBR measures concentrated in a finite number of cubes transposing by $F_\epsilon^{(d)}$, whose volumes tend to zero as $\epsilon \to 0$.

To prove this statement we consider the following one-dimensional mapping:

$$f(x) = \begin{cases} 1 - x/(c - c/r), & \text{if } 0 \le x \le c - c/r; \\ r(x - c) + c, & \text{if } c - c/r \le x \le c; \\ -r(x - c) + c, & \text{if } c \le x \le c + c/r; \\ (x - c - c/r)/(c - c/r), & \text{if } c + c/r \le x \le 1/2; \\ 1 - f(1 - x), & \text{if } 1/2 \le x \le 1. \end{cases}$$

Here parameters are chosen such that $1 < r < 2$ and $c = 1/(3 + 1/r)$. This is a piecewise expanding ergodic one-dimensional map, having a smooth invariant measure with a positive density on the interval $[0, 1]$. It is sufficient to prove the two-dimensional case ($d = 2$). In that case for all sufficiently small $\varepsilon > 0$ the map $F_\epsilon^{(2)}$ has two smooth invariant measures, converging as $\varepsilon \to 0$ to delta-measures, concentrated in points $(c, 1 - c)$ and $(1 - c, c)$. Indeed for $r < 2$ there exist two squares around these points, mapping into themselves under the action of $F_\epsilon^{(2)}$. The vertexes of the first of these squares are $(c - \epsilon u, 1 - c + \epsilon u)$, $(c + \epsilon v, 1 - c + \epsilon u)$, $(c + \epsilon v, 1 - c - \epsilon v)$ and $(c - \epsilon u, 1 - c - \epsilon v)$, where $v = 1 - 2c, u = v(r(1 - 2\epsilon)v - 1)$.

The explanation of this phenomenon is that if the expanding constant r is less than 2 in some neighborhood of the fixed (or periodic) point, then the mapping f may be not expanding in a sense, because it may expand the both halves of the neighborhood U of this fixed point and then fold them (as in the example above), hence $\mathrm{Leb}(fU) \le r * \mathrm{Leb}(U)/2 < \mathrm{Leb}(U)$, where $\mathrm{Leb}(U)$ is the Lebesgue measure of the set U.

This result shows the existence of the localization phenomena in a sense that in the perturbed system (when $\varepsilon > 0$) it appears new stable ergodic measures with the support on sets of small Lebesgue measure. It seems that there are only two ways for chaotic dynamical systems to produce the localization phenomena under the weak coupling. The first way is described above and its cause is a folding of the mapping around the periodic trajectory. The second way is a shifting of points of trajectories (due to the interaction) to regions with smaller derivatives of the map. We shall also demonstrate this on the example. Let us consider for $3 < r$ and $c = 1/(1 + r)$ the following one-dimensional map:

$$f(x) = \begin{cases} 1 - rx, & \text{if } 0 \le x \le c; \\ 2/(r - 1)(1/2 - x), & \text{if } c \le x \le 1/2; \\ 1 - f(1 - x), & \text{if } 1/2 \le x \le 1. \end{cases}$$

This is a piecewise expanding (when we consider its second power f^2) strongly chaotic map in a sense that it has a stochastic attractor with a smooth invariant measure with a positive density on the whole interval $[0, 1]$. However for all sufficiently small $\epsilon > 0$ the map $F_\epsilon^{(2)}$ has two smooth invariant measures, that are delta-measures, concentrated in points closed to the points $(c, 1 - c)$ and $(1 - c, c)$. The explanation is that the interaction shifts trajectories which should normally go around the fixed points to the domain, where the derivative of the map is less than 1 on modulus. Therefore the stabilization takes place.

Note, that this stabilization of unstable fixed points here occurs as a result of the weak interaction of two equal strongly unstable dynamical systems.

Now let us consider the influence of small random perturbations to considered phenomena. By the random perturbation of the mapping $F : X^d \to X^d$ we mean a random Markov chain with the following transition probability densities: $p_\sigma(x, y) = q_\sigma(F(x), y)$, where $x, y \in X^d$ and $q_\sigma(x, y)$ is transition probability densities for random perturbations with the "intensity" σ. The latter means that $q_\sigma(x, y) = 0$ for $\mid x - y \mid > \sigma$. The stability of SBR invariant measure of piecewise expanding mappings under the same assumptions as in Theorem 1 and with very weak assumptions about densities $q_\sigma(x, y)$ was proved in ref. 4. Therefore, because the coupled map lattice system in Theorem 1 remains to be a piecewise expanding mapping for sufficiently small ϵ, the stability assertion is true here. The stability of the first example of periodic behavior may be proved in the same manner, restricting the system to small squares, considered in the proof. The second case is even simpler, because there under the action of coupling we have a stable cycle.

The condition that the expanding constant has to be sufficiently larger than 1 is evidently does not valid for quadratic mappings $f_a(x) = ax(1 - x)$, considering as a model in ref. 2. The existence of the SBR measure may be proved for such coupled lattice systems only for a few number of cases, when f_a is nonsingularly conjugated to a piecewise expanding map (for example when $a = 4$) and for more special not pure diffusional interactions. Figure 3 shows the bifurcation diagram for the coupled map lattices, consisted of typical chaotic quadratic mappings with respect to the interaction parameter ϵ. When $\epsilon < 0.18$ the behavior seems chaotical and we see also on average periodic behavior, but then the stabilization of periodic orbits begins (the usual picture with the inverse period doubling). The explanation of this stabilization is just the same, as in the last example - the shifting of points of trajectories to positions with smaller values of derivatives. It is interesting that when at $\epsilon \approx 0.295$ the stabilization finishes and the local behavior again becomes chaotic, the same occurs with the space average function. Indeed for this system and so large interaction values neighboring subsystems become strongly correlated (practically equal one to another) and therefore we cannot apply the central limit theorem to the averaged function. So it is more natural to suppose real periodic behavior (as it was described in Theorem 2) for such systems, rather than periodic on average one.

References

1. Chate H., Manneville P. Evidence of collective behavior in cellular automata. Europhys. Lett. 14:5 (1991), 409-413.
2. Pomeau Y. Periodic behavior of cellular automata. J. Stat. Phys. 70:5/6 (1993), 1379-1382.
3. Bunimovich L.A., Pesin Ya.G., Sinai Ya.G., Jacobson M.V. Ergodic theory of smooth dynamical systems. Modern problems of mathematics. Fundamental trends. v.2, 1985, 113-231.
4. Blank M.L. Small perturbations of chaotic dynamical systems. Uspekhi Matem. Nauk., 44:6 (1989), 3-28.
5. Blank M.L. Chaotic mappings and stochastic Markov chains. Abstracts of Congress IAMP-91, 1992.
6. Keller G, Kunzle M. Transfer operators for coupled map lattices, Ergod. Theor. Dynam. Syst., to appear.

FigureCaptions

Fig. 1. Graph of the transposing map and bifurcation diagrams for local and average dynamics of the corresponding couple maps system with respect to the interaction.

Fig. 2. Graph of the tent map and bifurcation diagrams.

Fig. 3. Graph of the quadratic map and bifurcation diagrams.