Stable Non-Gaussian Diffusive Profiles

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Abstract

We prove two stability results for the scale invariant solutions of the nonlinear heat equation $\partial_t u = \Delta u - |u|^{p-1}u$ with $1 < p < 1+\frac{2}{n}$, n being the spatial dimension. The first result is that a small perturbation of a scale invariant solution vanishes as $t \to \infty$. The second result is global, with a positivity condition on the initial data.

1 Introduction

In [1], we investigated the long-time asymptotics of the solutions of nonlinear heat equations of the type

$$\partial_t u = \Delta u + F(u, \nabla u, \nabla\nabla u). \tag{1}$$

and proved that, for a large class of nonlinearities F, the solution behaved, as $t \to \infty$,

$$u(x,t) \sim At^{-\frac{1}{2}} f^*(\frac{x}{\sqrt{t}}) \tag{2}$$

with f^* Gaussian, $f^*(x) = e^{-\frac{x^2}{4}}$. The asymptotics was thus proven to be "universal", i.e. independent on the initial data $u(x,0)$ and the nonlinearity F, within some class. The only dependence of them occurs in the constant A. We explained this universality, following the work of Barenblatt [2] and of Goldenfeld et al [3, 4], in terms of the Renormalization Group, which also was an important ingredient in the proof (see Section 3).

The assumptions for the data in [1] was smallness in a suitable norm implying falloff at infinity (for a similar approach with initial data not decaying at infinity, i.e. the formation of fronts and patterns, see [5, 6] and for large data blowing up in a finite time,

*Supported by EC grant SC1-CT91-0695
†Supported by NSF grant DMS-8903041

see [7]). The assumptions for F excluded terms such as u^p, for $1 < p < 1 + \frac{2}{n}$, where n is the dimension of space ($p = 1 + \frac{2}{n}$ was considered and in this case the constant A in (2) is replaced by $A(\log t)^{-\frac{1}{2}}$).

Here we want to consider the asymptotics of (1) in the presence of these "relevant" (in the Renormalization Group terminology) terms in F. Although again rather general F may be considered, we formulate below our results for the nonlinear heat equation

$$\partial_t u = \Delta u - |u|^{p-1} u \tag{3}$$

where $u = u(x,t), x \in {}^n$, $1 < p < 1 + \frac{2}{n}$, and we discuss generalizations in Section 3.

Equation (3) has a one-parameter family of scale invariant solutions (see Section 2)

$$u(x,t) = t^{-\frac{1}{p-1}} f_\gamma(x t^{-\frac{1}{2}}) \tag{4}$$

where $\gamma \geq \gamma_p > 0$, provided $1 < p < 1+\frac{2}{n}$ and f_γ is non-Gaussian, having the asymptotics

$$f_\gamma(x) \sim |x|^{-\frac{2}{p-1}} \tag{5}$$

as $|x| \to \infty$ if $\gamma > \gamma_p$, while, for $\gamma = \gamma_p$, it decays at infinity as

$$f_{\gamma_p}(x) \sim |x|^{\frac{2}{p-1}-n} e^{-\frac{x^2}{4}}. \tag{6}$$

We prove here that these solutions are stable in two senses: first, there exists a ball in a Banach space of initial data such that the corresponding solutions tend, in the appropriate norm, as $t \to \infty$, to (4). Secondly, any initial data satisfying a suitable positivity condition will give rise to a solution again tending to (4).

More precisely, let $q > \frac{2}{p-1}$ and consider the Banach space B of L^∞ functions h equipped with the norm (with some abuse of notation!)

$$\|h\|_\infty = \operatorname*{ess\,sup}_\xi |h(\xi)(1 + |\xi|^q)|. \tag{7}$$

We consider the initial data (taken at time 1 for later convenience)

$$u(x,1) = f_\gamma(x) + h(x) \tag{8}$$

with $h \in B$. We prove the

Theorem *Let $1 < p < 1 + \frac{2}{n}$. There exist $\varepsilon > 0, C < \infty$ and $\mu > 0$ such that, if the initial data $u(x,1)$ of* (3) *is given by* (8) *with $h \in B$ and satisfies either*

$$\|h\|_\infty \leq \varepsilon$$

or

$$h(x) \geq 0$$

(a.e.) then, (3) *has a unique classical solution and, for all t,*

$$\|t^{\frac{1}{p-1}} u(\cdot t^{\frac{1}{2}}, t) - f_\gamma(\cdot)\|_\infty \leq C t^{-\mu} \|h\|_\infty$$

2 Proof

Before going to the proof of the Theorem, we will briefly discuss the scale invariant solutions (4). These are given by $f_\gamma(x) = \phi_\gamma(|x|)$ and ϕ_γ solves the ordinary differential equation

$$\phi'' + \left(\frac{n-1}{\eta} + \frac{\eta}{2}\right)\phi' + \frac{\phi}{p-1} - \phi^p = 0 \tag{9}$$

for $\eta = |x| \in [0, \infty[$.

The theory of positive solutions of (9) has been developped in [8, 10, 13]. The main result is that, for any $p > 1$, there exists smooth, everywhere positive solutions, ϕ_γ, of (9) with $\phi'_\gamma(0) = 0$ and $\phi_\gamma(0) = \gamma$ for γ larger than a certain critical value γ_p (but not too large). Actually, for $p < 1 + \frac{2}{n}$, $\gamma_p > 0$ while $\gamma_p = 0$ for $p \geq 1 + \frac{2}{n}$. The decay at infinity of these solutions is given in (5, 6).

The existence of a critical γ_p can be understood intuitively by viewing (9) as Newton's equation for a particle of mass one, whose "position" as a function of "time" is $\phi(\eta)$. The potential is then $U(\phi) = \frac{\phi^2}{2(p-1)} - \frac{\phi^{p+1}}{p+1}$ and the "friction term" $\left(\frac{n-1}{\eta} + \frac{\eta}{2}\right)\phi'$ depends on the "time" η. Hence, if $\phi'_\gamma(0) = 0$ and $\phi_\gamma(0) = \gamma$ is large enough, the time it takes to approach zero is long and, by then, the friction term has become sufficiently strong to prevent "overshooting". However, as p increases, the potential becomes flatter and one therefore expects γ_p to decrease with p.

Given the initial data (8), it is convenient to rewrite (3) in terms of the variables $\xi = xt^{-\frac{1}{2}}$ and $\tau = \log t$; so, define $v(\xi, \tau)$ by:

$$u(x,t) = t^{-\frac{1}{p-1}}(f_\gamma(xt^{-\frac{1}{2}}) + v(xt^{-\frac{1}{2}}, \log t)) \tag{10}$$

where now

$$v(\xi, 0) = h(\xi). \tag{11}$$

Then, (3) is equivalent to the equation

$$\partial_\tau v = \mathcal{L}v - \left(|f_\gamma + v|^{p-1}(f_\gamma + v) - f_\gamma^p - pf_\gamma^{p-1}v\right) \equiv \mathcal{L}v + N(v) \tag{12}$$

where we used the fact that (4) solves (3) and gathered the linear terms in

$$\mathcal{L} = \mathcal{L}_0 + V_\gamma,$$

with

$$\mathcal{L}_0 = \Delta + \frac{\xi}{2} \cdot \nabla + \frac{1}{p-1}, \tag{13}$$

and

$$V_\gamma(\xi) = -pf_\gamma^{p-1}(\xi). \tag{14}$$

To prove that the solution $t^{-\frac{1}{p-1}}(f_\gamma(xt^{-\frac{1}{2}}))$ is stable means to find a class of initial data $v(\xi, 0)$ such that the corresponding solution of (12) goes to zero as $\tau \to \infty$, in a suitable norm.

The Theorem of Section 1 reads now in terms of v as

Proposition 1. *With the assumptions of the Theorem,* (12), *with initial data (11) ,has a unique classical solution and*

$$\|v(\cdot,\tau)\|_\infty \le Ce^{-\mu\tau}\|h\|_\infty$$

The main input in the proof is the following estimate on the semigroup $e^{\tau\mathcal{L}}$:

Proposition 2. *The operator $e^{\tau\mathcal{L}}$ is a bounded operator in the Banach space B, and its norm satisfies*

$$\|e^{\tau\mathcal{L}}\| \le Ce^{-\mu\tau}$$

for some $\mu > 0$, $C < \infty$.

There are two important ingredients in the proof of Proposition 2. The first is the fact that $e^{\tau\mathcal{L}}$ is a contraction in a suitable Hilbert space of rapidly decreasing functions. To see this, note first that $\mathcal{L}_0$ is conjugate to the Schrödinger operator

$$e^{\frac{\xi^2}{8}}\mathcal{L}_0 e^{-\frac{\xi^2}{8}} = \Delta - \frac{\xi^2}{16} - \frac{n}{4} + \frac{1}{p-1} \tag{15}$$

i.e. the harmonic oscillator. Thus $\mathcal{L}_0$ is self-adjoint on its domain $\mathcal{D}(\mathcal{L}_0) \subset L^2(\mathbf{R}^n, d\mu)$, where

$$d\mu(\xi) = e^{\frac{\xi^2}{4}} d\xi.$$

$\mathcal{L}_0$ has a pure point spectrum $\{\frac{1}{p-1} - \frac{n}{2} - \frac{m}{2} \mid m = 0, 1, \dots\}$ and the largest eigenvalue $\frac{1}{p-1} - \frac{n}{2}$ is *positive* if $1 < p < 1 + \frac{2}{n}$. Thus $e^{\tau\mathcal{L}_0}$ is *not* contractive and, for $e^{\tau(\mathcal{L}_0 + V_\gamma)}$ to contract, we need to use the potential in a non-trivial way (this is the reason why $1 < p < 1 + \frac{2}{n}$ is harder than the $p > 1 + \frac{2}{n}$ case).

Remarkably, it is possible to prove that $\mathcal{L} < -E < 0$ without a detailed study of the function f_γ, but only using equation (9). We have the

Lemma 1. *The operator $e^{\tau\mathcal{L}}$ is a bounded operator in the Hilbert space $L^2(\mathbf{R}^n, d\mu)$ and its norm satisfies*

$$\|e^{\tau\mathcal{L}}\| \le e^{-E\tau}$$

for some $E > 0$.

Proof. Since V_γ is bounded, $\mathcal{L}$ is self-adjoint and, as for $\mathcal{L}_0$, its resolvent is compact and, therefore, its spectrum is pure point. Let $-E_\gamma$ be the largest eigenvalue.

First note that $-E_\gamma \le -E_{\gamma_p}$. Indeed, this holds since $V_\gamma \le V_{\gamma_p}$, because $f_\gamma \ge f_{\gamma_p}$, which in turn follows from the fact that γ_p is the smallest allowed value of $\phi_\gamma(0) = \gamma$ in (9), and that two solutions of (9), both with initial conditions $\phi'_\gamma(0) = 0$, will not cross. Hence it suffices to prove the claim for $\gamma = \gamma_p$. Let us write $E \equiv E_{\gamma_p}$.

Next we note that, by the Feynman-Kac formula [15], $e^{\tau\mathcal{L}}$ has a strictly positive kernel; indeed, since $-C < V_\gamma(\xi) < 0$, we have

$$e^{-\tau C} e^{\tau\mathcal{L}_0}(\xi, \xi') \le e^{\tau\mathcal{L}}(\xi, \xi') \le e^{\tau\mathcal{L}_0}(\xi, \xi') \tag{16}$$

and $e^{\tau\mathcal{L}_0}(\xi,\xi')$ is explicit, see (18) below. Thus, by the Perron-Frobenius theorem [11], $\mathcal{L}$ has a unique eigenvector Ω with eigenvalue $-E$ and Ω can be chosen to be strictly positive. That $-E$ is strictly negative, is now shown by the following argument.

Assume for a moment that $f \equiv f_{\gamma_p} \in \mathcal{D}(\mathcal{L})$, and write (9) as

$$\mathcal{L}f = -(p-1)f^p.$$

So,

$$(\Omega, \mathcal{L}f) = -(p-1)(\Omega, f^p)$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(^n, d\mu)$. By the self-adjointness of $\mathcal{L}$ and the definition of Ω, i.e. $\mathcal{L}\Omega = -E\Omega$, we have

$$-E = -(p-1)\frac{(\Omega, f^p)}{(\Omega, f)} < 0$$

since Ω and f are strictly positive.

Finally, to see that $f_{\gamma_p} \in \mathcal{D}(\mathcal{L})$ we use the fact that, since f_{γ_p} decays at infinity like (6), we have, for any $\delta > 0$, and any $n \in$,

$$|\nabla_\xi^n f_{\gamma_p}(\xi)| \le C(\delta, n) e^{-(\frac{1}{4}-\delta)\xi^2}. \tag{17}$$

This follows easily from (6) and the differential equation (9), and implies that $f_{\gamma_p} \in \mathcal{D}(\mathcal{L}_0) = \mathcal{D}(\mathcal{L})$. □

Notice that functions in $L^2(\mathbf{R}^n, d\mu)$ have essentially a Gaussian decay at infinity, which is much faster than what is allowed in our Banach space B, see (7). This brings us to the other crucial ingredient in the proof of Proposition 2, which is that $e^{\tau\mathcal{L}_0}$ contracts functions in B *pointwise* for ξ large. This follows from the explicit formula (Mehler's formula [15]):

$$(e^{\tau\mathcal{L}_0})(\xi,\xi') = (4\pi(1-e^{-\tau}))^{-\frac{n}{2}} e^{\tau\left(\frac{1}{p-1}-\frac{n}{2}\right)} \exp\left(-\frac{|\xi - e^{-\tau/2}\xi'|^2}{4(1-e^{-\tau})}\right) \tag{18}$$

Hence, if a function v satisfies

$$|v(\xi)| \le C(1+|\xi|^q)^{-1}, \tag{19}$$

for some constant C, we have

$$|(e^{\tau\mathcal{L}_0}v)(\xi)| \le C' e^{\frac{\tau}{p-1}}(1+|\xi|^q e^{\frac{\tau q}{2}})^{-1} \tag{20}$$

for $|\xi|$ large enough (of order $\sqrt{\tau}$) and another constant C'. Hence, the operator $e^{\tau\mathcal{L}_0}$ contracts, for large $|\xi|$ and large τ, any function that decays as in (19) with $q > \frac{2}{p-1}$. By (16), we see that $\mathcal{L}$ behaves similarly.

The idea of the proof of Proposition 2 is the following. For $|\xi|$ small (20) seems to expand by $e^{\frac{\tau}{p-1}}$: the potential V is important in this region and we want to use the information we obtained in the Hilbert space, Lemma 1 (recall that these functions have rapid decay, so that this bound should be used to capture the contraction only in the small ξ region). For large ξ , we shall use (20). This small-large ξ interplay is however slightly subtle, and we need to resort to an inductive argument to control the large τ behaviour in Proposition 2 (this is actually just the Renormalization Group idea applied to the linear problem).

Proof of Proposition 2. It is convenient to introduce the characteristic functions

$$\chi_s = \chi(|\xi| \le \rho)$$

$$\chi_\ell = \chi(|\xi| > \rho)$$

where ρ will be chosen suitably below. The properties of $\mathcal{L}$ that we need are summarized in the following

Lemma 2. *There exist constants $C < \infty$, $E > 0$, and $\delta > 0$, such that*

i) For $g \in B$,

$$\|e^{\tau\mathcal{L}} g\|_\infty \le C e^{\frac{\tau}{p-1}} \|g\|_\infty. \tag{21}$$

ii) For $g \in L^2({}^n, d\mu)$,

$$\|e^{\mathcal{L}} g\|_\infty \le C\|g\|_2, \tag{22}$$

where $\|\cdot\|_2$ is the norm in $L^2({}^n, d\mu)$.

iii) For g such that $\chi_s g \in L^2({}^n, d\mu)$,

$$\|\chi_\ell e^{\rho\mathcal{L}} \chi_s g\|_\infty \le e^{-\frac{\rho^2}{5}} \|\chi_s g\|_2, \tag{23}$$

for ρ large enough.

iv) For $g \in B$,

$$\|\chi_\ell e^{\rho\mathcal{L}} g\|_\infty \le e^{-\delta\rho} \|g\|_\infty, \tag{24}$$

for ρ large enough.

Let $\|g\|_\infty = 1$. Given Lemma 2, we set $\tau_n = n\rho$, and prove inductively that there exists $\alpha > 0$ such that $v(\tau_n) = e^{\tau_n\mathcal{L}} g$ satisfies, for ρ large,

$$\|\chi_s v(\tau_n)\|_2 + \|\chi_s v(\tau_n)\|_\infty \le e^{\frac{\rho^2}{6}} e^{-\alpha n}, \tag{25}$$

and

$$\|\chi_\ell v(\tau_n)\|_\infty \le e^{-\alpha n}. \tag{26}$$

Proposition 2 follows from (25,26), by taking $\mu = \frac{\alpha}{\rho}$ (for times not of the form $\tau = n\rho$, use (21)). The bounds (25,26) hold for $n = 0$, for ρ large enough, using $\|g\|_\infty = 1$ and the obvious inequality

$$\|\chi_s g\|_2 \le \rho^{\frac{1}{2}} e^{\frac{\rho^2}{8}} \|g\|_\infty. \tag{27}$$

So, let us assume (25, 26) for some $n \geq 0$ and prove it for $n+1$. Let $v = v(\tau_n)$ and write

$$v = \chi_s v + \chi_\ell v \equiv v_s + v_\ell.$$

We have from Lemma 1 and (25)

$$\|e^{\rho\mathcal{L}} v_s\|_2 \leq e^{-\rho E} e^{\frac{\rho^2}{6}} e^{-\alpha n} \tag{28}$$

and, combining (22) and Lemma 1

$$\|e^{\rho\mathcal{L}} v_s\|_\infty \leq C\|e^{(\rho-1)\mathcal{L}} v_s\|_2 \leq Ce^{-(\rho-1)E} e^{\frac{\rho^2}{6}} e^{-\alpha n}. \tag{29}$$

Finally, from (21) and (26), we have

$$\|e^{\rho\mathcal{L}} v_\ell\|_\infty \leq Ce^{\frac{\rho}{p-1}} e^{-\alpha n} \tag{30}$$

and, from this and (27), we get

$$\|\chi_s e^{\rho\mathcal{L}} v_\ell\|_2 \leq C\rho^{\frac{1}{2}} e^{\frac{\rho^2}{8}} e^{\frac{\rho}{p-1}} e^{-\alpha n}. \tag{31}$$

Combining (28-31), one gets (25), with n replaced by $n+1$ for ρ large enough and α small. On the other hand, (26), with n replaced by $n+1$, follows immediately from (25,26) and (23,24), taking $\alpha < \delta\rho$. □

We are left with the

Proof of Lemma 2. Part (i) follows immediately from (16) and (20).

For (ii), we use Schwartz' inequality and the bound

$$\sup_\xi \int |e^{\mathcal{L}}(\xi, \xi')|^2 d\xi' < \infty, \tag{32}$$

which follows from (16) and (18).

For (iii) proceed as in (ii) by using Schwartz' inequality, but replace (32) by

$$\sup_{|\xi|>\rho} \left(\int |e^{\rho\mathcal{L}}(\xi, \xi')|^2 \chi(|\xi'| \leq \rho) d\xi' \right)^{\frac{1}{2}} \leq e^{-\frac{\rho^2}{5}} \tag{33}$$

which again follows from (16) and (18) (we can replace $\frac{1}{5}$ in (33) by $\frac{1}{4} - \delta$ for any $\delta > 0$, if ρ is large enough).

Finally, (iv) follows from (20) and $q > \frac{2}{p-1}$. □

Proof of Proposition 1. The proof is straightforward, given Proposition 2. We consider the integral equation corresponding to (12):

$$v(\tau) = e^{\tau\mathcal{L}} h + \int_0^\tau ds e^{(\tau-s)\mathcal{L}} N(v(s)) \equiv \mathcal{S}(v, \tau) \tag{34}$$

with $v(\tau) \equiv v(\cdot, \tau)$.

First, let $\|h\|_\infty \leq \epsilon$; (34) is solved by the contraction mapping principle in the Banach space $\mathcal{B}$ of functions $v = v(\xi, \tau)$, where $v(\cdot, \tau) \in B$ for $\tau \in [0, \infty)$, and where the norm is

$$\|v\| = \sup_{\tau \in [0,\infty)} \|v(\cdot, \tau)\|_\infty e^{\tau\mu}$$

We show that $\mathcal{S}$ defined by (34) maps the ball $\mathcal{B}_0 = \{v \in \mathcal{B} | \ \|v\| \leq C\varepsilon\}$ into itself, for a suitable constant C, and is a contraction there . This follows, since we get, using (12),

$$|N(v)| \leq C'|v|^{\tilde{p}}$$

where $\tilde{p} = \min(2, p) > 1$, and C' is a constant; therefore

$$\|N(v(s))\|_\infty \leq C'\|v(s)\|_\infty^{\tilde{p}} \leq C'\epsilon^{\tilde{p}} e^{-\tilde{p}s\mu} \tag{35}$$

and so, Proposition 2 gives the claim for ε small. The proof that $\mathcal{S}$ is a contraction is similar. Finally, from the Feynman-Kac formula and the smoothness and boundedness of V one deduces that $e^{\tau\mathcal{L}}(\xi, \xi') = e^{\tau\mathcal{L}_0}(\xi, \xi')K(\xi, \xi')$ where K is smooth and bounded. The regularity of the kernel of $e^{\tau\mathcal{L}}$ (for short times, it behaves like the heat kernel), implies that the solution of (34) is actually the unique classical solution of (12) (for details of such arguments, see [7]).

Let now $h \geq 0$. It is standard that equation (3) has unique solution for all times, with positive initial data such as ours [9]. So, equation (12) also has a unique solution for all times. Moreover, by a comparison inequality ($v = 0$ is a solution of (12)), we have $v(\xi, \tau) \geq 0$ for all times. Thus, in (34), the second term is negative ((12) shows that $N(v) \leq 0$, and (16, 18) show that the kernel of $e^{\tau\mathcal{L}}$ is positive). Therefore, we have the pointwise inequality

$$v(\xi, \tau) \leq (e^{\tau\mathcal{L}} v(0))(\xi)$$

and Proposition 2 proves the claim. □

3 Extensions and concluding remarks

1. In Theorems 1 and 2, we could use a norm defined by

$$\|g\|_\infty = \operatorname{ess\,sup}_{\xi} |g(\xi)w(\xi)|. \tag{36}$$

where w is any bounded positive function decaying at infinity faster than $|\xi|^{-\frac{2}{p-1}}$. Indeed, the norm (7) was used only in the proof of (20). However, we would not necessarily get exponential decay of v as a function of τ.

2. In [10] and also in [14], results similar to ours were obtained on the stability of the self-similar solutions for $p < 1 + \frac{2}{n}$ and $\gamma = \gamma_p$. However, our results apply to a ball in a Banach space (and to any γ) while in [10] the initial data is assumed to satisfy a pointwise inequality (but not to be small). This is similar to the $h \geq 0$ case in our theorem, but with another inequality. For the first part of our Theorem, $u(x, 1)$, given

by (8), does not even have to be positive. On the other hand, very general results on the stability of the self-similar solutions, decaying as in (5), were obtained in [13], for $p > 1 + \frac{2}{n}$. Basically, it is shown there that any positive initial data decaying at infinity like $|\xi|^{-\frac{2}{p-1}}$ will give rise to a solution whose asymptotic behaviour in time is given by $f_\gamma(\xi)$.

3. With little extra work, the small $\|v\|$ part of the Theorem generalizes to more general nonlinearities, e.g. equations of the form

$$\partial_t u = \Delta u - |u|^{p-1}u + F(u, \nabla u) \tag{37}$$

whereby we need to add to (12) the term

$$\tilde{F}_\tau(v, \nabla v) = e^{\frac{p\tau}{p-1}} F\left(e^{-\frac{\tau}{p-1}}(f_\gamma + v), e^{-\frac{p+1}{2(p-1)}\tau}\nabla(f_\gamma + v)\right). \tag{38}$$

In order to define "small" initial data, we introduce the Banach space of C^1 functions with the norm:

$$\|h\| = \|h\|_\infty + \max_{1\leq i\leq n} \|\partial_i h\|_\infty \tag{39}$$

where $\|h\|_\infty$ is defined in (7).

We assume that F in (37) is C^1 and satisfies:

$$|F(a,b)| + |a\partial_a F(a,b)| + \max_{1\leq i\leq n} |b_i \partial_{b_i} F(a,b)| \leq \lambda |a|^{q_1} \left(\max_{1\leq i\leq n} |b_i|\right)^{q_2} \tag{40}$$

for $|a|, |b_i| \leq 1$, where

$$q_1 + \frac{p+1}{2} q_2 > p \tag{41}$$

and λ is taken small. Using (40, 41), and the fact that f_γ, $\partial_i f_\gamma$, belong to the space B, one gets

$$\|\tilde{F}_s(v, \nabla v)\|_\infty \leq C\lambda e^{-\delta s}$$

for some $\delta > 0$, which is like the RHS of (35) for λ small. The last two terms in the LHS of (40) are used to prove that the $\tilde{F}$ term defines a contraction. The only extra difficulty is to show that the solution of the fixed point problem (corresponding to (34)) is a classical solution. However, using the regularity of the kernel of $e^{\tau\mathcal{L}}$ discussed in the proof of Proposition 1, one shows first that v is $C^{2-\alpha}$ for any $\alpha > 0$ (its derivative is Hölder continuous of exponent $1-\alpha$); then, one uses that information, the integral equation and the regularity of the kernel to show that v is actually C^2. See [7] for details.

4. Finally we want to end with some comments on the RG picture behind these results. In [1] the RG map $\mathcal{R}_L$, for $L > 1$ was defined in a suitable Banach space of initial data $f(x) = u(x, 1)$ and a suitable space of nonlinearities F (taken to be holomorphic functions in [1]). $\mathcal{R}_L$ consisted simply of solving (1) up to the finite time L^2 and performing a scale transformation on the solution and on F:

$$\mathcal{R}_L(f, F) = (f_L, F_L) \tag{42}$$

where $f_L(x) = L^n u(Lx, L^2)$ and $F_L(u, v, w) = L^{2+n} F(L^{-n}u, L^{-n-1}v, L^{-n-2}w)$. This scaling assures the semigroup property $\mathcal{R}_{L^k} = \mathcal{R}_L^k$ on a common domain. The limit (2) was then shown to follow from

$$\mathcal{R}_L^k(f, F) \to (Af^*, 0) \tag{43}$$

as $k \to \infty$, where $(Af^*, 0)$ is a one parameter family of Gaussian ($f^*(x) = e^{-\frac{x^2}{4}}$) fixed points for $\mathcal{R}_L$. Universality, i.e. independence on f and F was then explained in terms of a dynamical systems picture: if (f, F) lie on the stable manifold of the line of fixed points, all the corresponding equations and data have the same asymptotics.

The Theorem of Section 1 is a statement on the stability of a family of *non-Gaussian* fixed points of the RG (which are, moreover, non perturbative, i.e. we do not use any "ε-expansion"). Equation (1) is invariant under the scaling $u \to u_L$ with

$$u_L(x, t) = L^{\frac{2}{p-1}} u(Lx, L^2 t)$$

which suggests setting

$$f_L(x) = L^{\frac{2}{p-1}} u(Lx, L^2)$$

and a correspondig definition of F_L. Then $\mathcal{R}_L(f_\gamma, F^*) = (f_\gamma, F^*)$ for $F^*(u) = -u^p$, and our Theorem constructs the stable manifold of this fixed point (in the f variable; for F, see the previous remark). The reason one needs an iterative approach to the limit (43), i.e. one controls the iteration of $\mathcal{R}_L$ rather than $\mathcal{R}_L^k$ directly, is the existence of the neutral direction: A is a nontrivial function of the data and of the equation. Here there is no neutral direction (this is the content of Lemma 1), and no iteration is needed (although we needed to resort to one in the proof of Proposition 2! But that was for a different purpose, namely the analysis of the linear operator $e^{\tau\mathcal{L}}$).

Combining our results with those of [1], we obtain the following p-dependence of the asymptotics of the solution of

$$\partial_t u = \partial^2 u - u^p,$$

for a suitable class of initial data. For $p > 1 + \frac{2}{n}$,

$$u(\cdot t^{1/2}, t) \simeq A t^{-\frac{n}{2}} f^*(\cdot)$$

where $f^*(\xi) = e^{-\xi^2/4}$ is independent of p; the prefactor A depends on p and on the initial data. We have

$$\int |u(x, t)| dx = \mathcal{O}(1)$$

as $t \to \infty$. For $p = 1 + \frac{2}{n}$,

$$u(\cdot t^{1/2}, t) \simeq (At \log t)^{-\frac{1}{2}} f^*(\cdot),$$

i.e. it is as before, but with a logarithmic correction and

$$\int |u(x, t)| dx = \mathcal{O}((\log t)^{-1/2}).$$

For $1 < p < 1 + \frac{2}{n}$,

$$u(\cdot t^{1/2}, t) \simeq t^{-\frac{1}{p-1}} f_\gamma(\cdot)$$

where f_γ is non-Gaussian, varies with p but is independent of the initial data (satisfying the hypotheses of the Theorem). Moreover,

$$\int |u(x,t)| dx = \mathcal{O}\left(t^{\frac{n}{2}-\frac{1}{p-1}}\right) \to 0$$

as $t \to \infty$.

Acknowledgments

It is a pleasure to thank J. Miekisz and U. Moschella for interesting discussions. During this work, J. Bricmont benefited from the hospitality of the University of Helsinki and A. Kupiainen from the one of the University of Louvain. This work was supported by NSF Grant DMS-8903041 and by EC Grant SC1-CT91-0695.

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