A Critical Phenomenon in Growth Systems

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Abstract

The paper treats of interacting infinite or finite systems whose components' states are in the set $\{0, 1, 2, 3, \ldots\}$. All components' initial states are zeroes. Components interact with each other in a local deterministic way, in addition to which every component's state grows by one with a constant probability θ at every moment of the discrete time.

Our main question about infinite systems is whether the average value of components tends to infinity or remains bounded as $t \to \infty$. The analogous question about finite systems is how long the system's average remains less than a constant: this time may be bounded or tend to infinity as the system' size tends to infinity.

Both in the infinite and finite cases sufficient conditions for both ways of behavior are given here. It is shown that the different ways of behavior may occur with one and the same deterministic interaction depending on the value of θ.

The combinatorial method applied here to growth systems, was developed [7, 8, 9] for systems with a finite set of states of every point. Our introductory examples (1 $\boxed{\textbf{median-NEC}}$) and (2 $\boxed{\textbf{median-line}}$) are growing analogues of Votings which were first proposed in [10]. The first of these Votings is now known as the NEC-system [1]. The techniques used here is similar to that of [7], explained for the case of the NEC-system in [4] (where it is called the Toom model). The finite case (of the NEC-system) was first treated of in [2]. The importance of non-symmetry was pointed at in [5]. An alternative method was proposed in [3].

1. Introductory Examples

Throughout the paper $\mathbf{R}$ stands for the set of real numbers, $\mathbf{R}_+$ is the set of non-negative real numbers, $\mathbf{Z}$ is the set of integers, $\mathbf{Z}_+$ is the set of non-negative integers. $\mathbf{E}(\cdot)$ means expectation.

Let us compare random processes with the state space $\mathbf{R}_+^{\mathbf{Z}^2}$. To specify a state means to assign a real number $x(i,j) \geq 0$ to every $(i,j) \in \mathbf{Z}^2$. Take the initial state 'all zeroes' and assume that at every moment of the discrete time every $x(i,j)$ undergoes the following two steps:

Step 1 : $x(i,j)$ takes the value of some deterministic function $d(\cdot)$ of its and its neighbors' previous states.

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Step 2 : $x(i,j)$ is incremented by 1 with probability θ independently of the others' increments and of the prehistory.

The only difference between processes in question is in the TRANSITION FUNCTION $d(\cdot)$ used in Step 1. Let $\mathbf{E}_\theta(i,j,t)$ stand for the expectation of $x(i,j)$ at time t. These expectations are the same for all $(i,j) \in \mathbf{Z}^2$ if the initial condition and the transition rule are uniform, and in these cases we call them $\mathbf{E}_\theta(t)$. We are interested in the behavior of $\mathbf{E}_\theta(t)$ as $t \to \infty$. The introductory Theorem 1 shows that in a fairly general case $\mathbf{E}_\theta(t)$ tends to infinity provided θ is close enough to 1.

It is very easy to present transition functions for which $\mathbf{E}_\theta(t)$ tends to infinity with any $\theta > 0$, for example if Step 1 does not change the value of $x(i,j)$ at all, or $d(\cdot)$ equals the arithmetical mean of its arguments.

By definition, median of $2k-1$ arguments equals the k-th number among them if they are ordered in the increasing order. Note that median does not favor greater or smaller values, i.e.

$$\mathsf{median}(-x_1, \ldots, -x_{2k-1}) = -\mathsf{median}(x_1, \ldots, x_{2k-1}).$$

However, in the process in which

$$d(\cdot) = \mathsf{median}(x(i,j), x(i+1,j), x(i,j+1)), \qquad \boxed{\textbf{median-NEC}} \tag{1}$$

$\mathbf{E}_\theta(t)$ remains bounded by a constant, provided θ is small enough. This follows from our Theorem 2. (In fact, this process takes values in $\mathbf{Z}_+^{\mathbf{Z}^2}$.) Thus, for some functions $d(\cdot)$, including (1 $\boxed{\textbf{median-NEC}}$), there is a critical value of θ, which separates two modes of behavior – with bounded and growing $\mathbf{E}_\theta(t)$.

Theorem 3 proves the analogous phenomenon for finite systems in the same assumptions about the deterministic interaction. Of course, in finite systems $\mathbf{E}_\theta(t)$ can not remain bounded forever: it remains bounded during time which grows exponentially in the system's size.

Geometrical positions of neighbors are essential. For example, it is fairly evident that the process with

$$d(\cdot) = \mathsf{median}(x(i,j), x(i+1,j), x(i-1,j)) \qquad \boxed{\textbf{median-line}} \tag{2}$$

has $\mathbf{E}_\theta(t) \to \infty$ for any $\theta > 0$. Theorem 4 generalizes this fact and states the class of transition functions with which the condition of Theorems 2 and 3 is not only sufficient, but also necessary for the statements of these theorems to be true. Theorem 5 applies Theorem 4 to the special case when $d(\cdot)$ is a median.

2. Definitions and Formulations

$\mathbf{Z}^{\dim}$ is the dim-dimensional discrete space. $\mathbf{Z}_M$ stands for the ring of residues modulo M and $\mathbf{Z}_M^{\dim}$ is the dim-dimensional torus. *Space* stands for $\mathbf{Z}^{\dim}$ if it infinite and for $\mathbf{Z}_M^{\dim}$ if it is finite.

$|S|$ stands for the cardinality of any finite set S. A family means a set whose elements are sets. For any functions $F \prec G$ means $F \leq \text{const} \cdot G$.

A natural m is fixed and called MEMORY. The set $\{-m, \ldots, -1, 0, 1, \ldots\}$ is called $Time$, and

$$V = Volume = Space \times Time = \{(s,t) : s \in Space,\ t \in Time\}.$$

Elements of V are called POINTS and subsets of V are called POINT-SETS. If $v = (s,t)$, then $s(v)$ means s and $t(v)$ means t. Points with $t < 0$ are called INITIAL, those with $t \geq 0$ are called INNER. Every inner point (s,t) has n NEIGHBORS

$$N_i(s,t) = (s,t) + (\Delta s_i, \Delta t_i), \quad i = 1, \ldots, n, \qquad \boxed{\boxed{\textbf{neighbors}}} \tag{3}$$

where the sequence of NEIGHBOR VECTORS

$$\Delta = ((\Delta s_1, \Delta t_1), \ldots, (\Delta s_n, \Delta t_n)) \qquad \boxed{\boxed{\textbf{vectors}}} \tag{4}$$

is one and the same for all inner points and

$$\forall i = 1, \ldots, n \ : \ -m \leq \Delta t_i \leq -1.$$

For every inner point v its neighborhood $N(v)$ is the set of its neighbors. Initial points have empty neighborhoods. To represent the system of neighborhoods, we shall use the directed graph ∇ which has V as its set of vertices, and edges that go to every inner point (s,t) from $(s,t) + (\Delta s_i, \Delta t_i),\ i = 1, \ldots, n$.

Every point v has one and the same set $X_v \equiv X_0 = \mathbf{Z}_+$ of its possible states and every inner point has one and the same transition function

$$d \ : \ X_{N(v)} \mapsto X_v, \quad \text{where} \quad X_{N(v)} = X_0^n \quad \text{and} \quad X_v = X_0. \qquad \boxed{\boxed{\textbf{d}}} \tag{5}$$

To every point $(s,t) \in V$ there corresponds a variable $x(s,t) \in X_0$ which represent the state of this point, that is the state of component s at time t. The set $X = X_0^V$ is the configuration space whose elements x are called CONFIGURATIONS.

Generally, to any subvolume $W \subset V$ there corresponds its configuration space $X_W = X_0^W$ whose elements $x(W)$ are called configurations on W. For any measure P on X, any $W \subset V$ and any $v \in V$, $P|_W$ stands for the projection of P to X_W and $P|_v$ stands for the projection of P to X_v.

We shall define RANDOM PROCESSES, i.e. normed measures on X using (every time the same) mutually independent HIDDEN VARIABLES $h(v)$ for all inner points v, everyone of which equals 1 with probability θ and 0 with probability $1 - \theta$. Thus we have the hidden product-measure η on the configuration space $H = \{0,1\}^V$ of the hidden variables.

To any sequence Δ (4 $\boxed{\textbf{vectors}}$) of neighbor vectors, any function $d(\cdot)$ (5 $\boxed{\textbf{d}}$) and any value of θ there corresponds the process, i.e. the normed measure on X induced by η with the map $D \ : \ H \mapsto X$ defined in the following inductive way:

$$x(v) = \begin{cases} 0 & \text{if } t(v) < 0, \\ d(x(N(v))) + h(v) & \text{if } t(v) \geq 0. \end{cases} \qquad \boxed{\boxed{\textbf{D}}} \tag{6}$$

Denote the expectation of $x(s,t)$ in this process by $\mathbf{E}_\theta^M(t)$. In all our theorems the system's size M is any natural or substituted by ∞ in the infinite case.

Theorem 1. If $d(x_1,\ldots,x_n) \geq \min(x_1,\ldots,x_n)$, then

$$\forall\ t,\ M \in \mathbf{Z}_+ \cup \{\infty\}\ :\ \mathbf{E}_\theta^M(t) \geq C(\theta)\cdot t$$

for θ large enough, where $C(\theta) \to 1/m$ when $\theta \to 1$. $\boxed{\textbf{unbounded-T}}$

Theorems 2 and 3, which constitute the main result of our paper, give a sufficient condition for $\mathbf{E}_\theta^M(v)$ to be uniformly bounded during an exponential or infinite time. Since our systems are uniform (i.e. neighbor vectors and the transition function are the same for all inner points), it is sufficient to speak about the neighborhood and transition function of the origin $\mathcal{O}$.

Definition 1. A subset $S \subset N(\mathcal{O})$ is called a VETO-SET if the value of $d(x(N(\mathcal{O})))$ can not be greater than $\max\{x(w),\ w \in S\}$.

Let us consider the infinite volume $V = \mathbf{Z}^{\dim}\cdot Time$ as a subset of the continuous space $R^{\dim+1}$ with the same origin $\mathcal{O}$. Let $conv(S)$ stand for the convex hull of any $S \subset R^{\dim+1}$. For any set $S \subset R^{\dim+1}$ and any number k we define:

$$kS = \{ks : s \in S\}, \quad ray(S) = \bigcup\{kS :\ k \geq 0\}.$$

We call $ray(S)$ RAY of S. A RAY means a closed half-line emanating from the $\mathcal{O}$.

Definition 2.

$$\sigma(\Delta,d) = \cap\{ray(conv(S))\ :\ S\quad \text{is a veto-set}\quad\}. \quad \boxed{\textbf{sigma}}$$

The following condition plays important role in this paper:

(7) $$\sigma(\Delta,d) = \{\mathcal{O}\}. \quad \boxed{\textbf{sigma=O}}$$

Theorem 2. Given condition (7 $\boxed{\textbf{sigma=O}}$). Then for $Space = \mathbf{Z}^{\dim}$ and for θ small enough, $\boxed{\textbf{bounded-inf-T}}$

(8) $$\forall\ t\ :\ \mathbf{E}_\theta^\infty(t) \leq \theta + O(\theta^2). \quad \boxed{\textbf{bounded-inf-F}}$$

More exactly, there is such a function $f(\theta) = \theta + O(\theta^2)$ that

$$\forall\ s,\ t,\ k \in Z_+\ :\ Prob(x(s,t) \geq k) \leq f^k(\theta).$$

Theorem 3. Given condition (7 $\boxed{\textbf{sigma=O}}$). Then for $Space = \mathbf{Z}_M^{\dim}$ with M large enough and for θ small enough, there is such $C > 1$ that $\boxed{\boxed{\textbf{bounded-fin-T}}}$

$$\forall\ t \le C^M\ :\ \mathbf{E}_\theta^M(t) \le \theta + O(\theta^2). \quad \boxed{\boxed{\textbf{bounded-fin-F}}} \tag{9}$$

More exactly, there is such a function $f(\theta) = \theta + O(\theta^2)$ that

$$\forall\ s,\ t \le C^M,\ k \in Z_+\ :\ Prob(x(s,t) \ge k) \le f^k(\theta).$$

Definition 3. Call a function MINIMAX if it can be expressed through minima and maxima. $\boxed{\boxed{\textbf{minimax-D}}}$

Theorem 4. If the function $d(\cdot)$ is minimax, then the condition (7 $\boxed{\textbf{sigma=O}}$) is not only sufficient, but also necessary for the statements of Theorems 2 and 3 to be true. More exactly, if condition (7 $\boxed{\textbf{sigma=O}}$) is wrong, then $\boxed{\boxed{\textbf{minimax-T}}}$

$$\forall\ t,\ M \in \mathbf{Z}_+ \cup \{\infty\}\ :\ \mathbf{E}_\theta^M(t) \ge C(\theta) \cdot t$$

where $C(\theta) > 0$ for all $\theta > 0$.

Theorem 5. If the function $d(\cdot)$ is a median, then the set $\sigma(\Delta, d)$ either equals $\{\mathcal{O}\}$, or is a ray. The latter case takes place iff there exists such a ray $\rho \subset R^{\dim+1}$ that for any hyperplane Π which contains this ray, $\boxed{\boxed{\textbf{plane-T}}}$

$$BEL > DIF_{out} + DIF_{in}.$$

Here BEL is the number of neighbor vectors that belong to ρ, $DIF_{out} > 0$ is the difference between the number of neighbor vectors on one side of Π and the number of neighbor vectors on the other side of Π, $DIF_{in} > 0$ is the difference between the number of neighbor vectors that belong to Π and are on one side of ρ and the number of neighbor vectors that belong to Π and are on the other side of ρ.

Note 1. Let all the neighbor vectors be mutually non-collinear. Then the latter case of Theorem 5 takes place iff there is such a neighbor vector $\delta \in R^{\dim+1}$ that all the other neighbor vectors form pairs, the two vectors of each pair having the opposite δ-direction projections to $Space$.

This statement can be applied to our examples (1 $\boxed{\textbf{median-NEC}}$) and (2 $\boxed{\textbf{median-line}}$). In our first example (1 $\boxed{\textbf{median-NEC}}$) such a neighbor vector δ evidently does not exist. In our second example (2 $\boxed{\textbf{median-line}}$) such a neighbor vector exists and equals $\delta = (0, 0, -1)$.

Now to prove our theorems. Sign $\square$ marks ends of proofs.

3. Proof of Theorem 1

It is sufficient to prove Theorem 1 for the case

(10) $$d(x_1,\ldots,x_n) = \min(x_1,\ldots,x_n). \qquad \boxed{\textbf{d=min}}$$

Reconstruct the directed graph ∇ into another directed graph $\bar{\nabla}$ by stretching every its vertex v into a directed edge and assign to this edge v a random time delay that equals $h(v)$. Old edges, that come from ∇, have zero delay. Define the time delay of a path in $\bar{\nabla}$ as the sum of delays along it. Define the time delay of an inner point as the minimum of delays of paths that lead to it from initial points.

Note 2. For any point v and any realization of hidden variables, $x(v) = Delay(v)$. $\boxed{\textbf{x=Delay}}$

Thus, $x(v)$ equals the minimum of delays $Delay(\Pi)$ that pertain to directed paths Π from initial points to v. For every path Π its length $l(\Pi)$ is no less than $t(v)/m$. The number of these paths is no more than $n^{t(v)}$. Delay of every path Π is the sum of delays along this path, that is the sum of $l(\Pi)$ independent random variables, everyone of which equals 1 with probability θ and 0 with probability $1-\theta$. This reduces the proof of Theorem 1 to an easy calculation that pertains to sums of independent random variables □

Note 3. For the particular case (10 $\boxed{\textbf{d=min}}$), this first passage percolation representation allows to prove assertions of our Theorems 2 and 3 in a more simple way.

4. Proof of Theorem 2

4.1. Outline

We shall present a function

(11) $$f(\theta) = \theta + O(\theta^2) \qquad \boxed{\textbf{f}}$$

and for every inner point v we shall define a finite family $\mathcal{F}(v)$ of point-sets such that

(12) $$\forall v \; : \; \sum\{\theta^{|S|}, \;\; S \in \mathcal{F}(v)\} \le f(\theta) \qquad \boxed{\textbf{sum}}$$

and if $x(v) \ge k > 0$, there are k disjoint point-sets $S_1,\ldots,S_k \in \mathcal{F}(v)$ such that

(13) $$h_w = 1 \quad \text{for all} \quad w \in S_1 \cup \ldots \cup S_k. \qquad \boxed{\textbf{h=1}}$$

Lemma 1. If there is a function $f(x)$ and for every inner point v there is a finite family $\mathcal{F}(v)$ for which (11 $\boxed{\textbf{f}}$), (12 $\boxed{\textbf{sum}}$) and (13 $\boxed{\textbf{h=1}}$) hold, then Theorem 2 holds. $\boxed{\textbf{function+family}}$

Proof: From (13 $\boxed{\mathbf{h{=}1}}$), then (12 $\boxed{\mathbf{sum}}$):

$$\begin{aligned}
& Prob(x(s,t) \geq k) \leq \\
& \quad \sum\{Prob(h_w = 1 \ \text{ for all } \ w \in S_1 \cup \cdots \cup S_k), \\
\sum\{\theta^{|S_1 \cup \ldots \cup S_k|}, \quad & \quad \sum\{\theta^{|S_1|+\ldots+|S_k|}, \ \ S_1, \ldots, S_k \in \mathcal{F}\} = \\
& \quad (\sum\{\theta^{|S|}, \ S \in \mathcal{F}\})^k \leq f^k(\theta) \ \ \square
\end{aligned}$$

4.2. Preliminaries

4.2.1. Graphs

When treating of graphs, we assume that they have no loops and every two vertices a and b are connected with at most one edge, which is denoted by $a - b$ if it is non-directed, and by $a \to b$ if it is directed from a to b. $G' \subseteq G$ means that G' is a subgraph of G.

4.2.2. Sets

Any two elements of a set are CONNECTED by this set. For any family F of sets, $Union(F)$ stands for the union of its elements. For any family F of sets, $Graph(F)$ is the non-directed graph whose set of vertices is $Union(F)$ and two vertices are connected with an edge iff they are connected by some element of F. Call a family F of sets CONNECTED iff $Graph(F)$ is connected. Two families F^1 and F^2 given, $Graph(F^1/F^2)$ is the non-directed graph whose set of vertices is $Union(F^1)$ and two vertices are connected with an edge iff they are connected by some element of $F^1 \cup F^2$. Family F^1 is called F^2-CONNECTED iff $Graph(F^1/F^2)$ is connected.

Lemma 2. Families F^1, F^2, F^3 are given. Assume

1) F^1 is $(F^2 \cup F^3)$-connected, $\boxed{\boxed{\mathbf{F1}}}$

2) F^2 is F^3-connected, and $\boxed{\boxed{\mathbf{F2}}}$

3) $Union(F^1) \cap Union(F^2) \neq \varnothing$. $\boxed{\boxed{\mathbf{FU}}}$

Then $F^1 \cup F^2$ is F^3-connected. $\boxed{\boxed{\mathbf{F\text{-}F\text{-}F}}}$

Proof: From 3 $\boxed{\mathbf{FU}}$) , there is some $z \in Union(F^1) \cap Union(F^2)$. Let us prove that all elements of $Union(F^1 \cup F^2)$ are $(F^1 \cup F^2 \cup F^3)$-connected by some paths with z. For elements of $Union(F^2)$ it follows from 2 $\boxed{\mathbf{F2}}$) . For elements of $Union(F^1)$ it follows from 1 $\boxed{\mathbf{F1}}$) $\square$

4.2.3. Neighborhoods

Given any directed graph G, those vertices whence edges go to a vertex, are called G-NEIGHBORS or just NEIGHBORS of this vertex. (In all notations that pertain to any graph G we may omit the index G if there is no danger of confusion.) The set of neighbors of a vertex v is called its NEIGHBORHOOD and denoted $N_G(v)$. Two different G-neighbors of one vertex are called G-PEERS. For example, neighborhoods $N(\cdot)$ which we defined in Section 2 for all points in V, are ∇-neighborhoods now, where the graph ∇ was also defined in Section 2.

Neighborhood of a set S of vertices is the union of neighborhoods of elements of S. Further, $N^k(S)$ is defined for every $k = 0, 1, 2, \ldots$ and every set S by the inductive rule:

$$N^0(S) = S, \qquad N^{k+1}(S) = N(N^k(S)),$$

and

$$N^{all}(S) = \bigcup_{k=0}^{\infty} N^k(S), \qquad N^+(S) = \bigcup_{k=1}^{\infty} N^k(S),$$

$$N^{end}(S) = \{A \in N^{all}(S) \; : \; N(A) = \varnothing\}.$$

Call $N^{all}(S)$ extended neighborhood of S and call its elements extended neighbors of S.

4.2.4. Linear Functions and Polars

Lemma 3. The condition (7 $\boxed{\textbf{sigma=O}}$) is equivalent to the following: There is such a natural $r \le \dim +1$ and such r homogeneous linear functions $L_1, \ldots, L_r$ on $R^{\dim +1}$ that $\boxed{\boxed{\textbf{linear}}}$

(14) $\quad L_1 + \ldots + L_r \equiv 0,$ and $\boxed{\boxed{\textbf{sum=0}}}$

(15) For all $i = 1, \ldots, r$ the set $\{w \in N(\mathcal{O}) \; : \; L_i(w) \ge 1\}$ is a veto-set. $\boxed{\boxed{\textbf{veto-set}}}$

Proof:

First, assume that there are such homogeneous lineat functions $L_1, \ldots, L_r$ on $R^{\dim +1}$ that (14 $\boxed{\textbf{sum=0}}$) and (15 $\boxed{\textbf{veto-set}}$) are true. Assume that there is some non-zero point $v \in \sigma(\Delta, d)$ and come to a contradiction. For $i = 1, \ldots, n$ denote

$$S_i = \{w \in N(\mathcal{O}) \; : \; L_i(w) \ge 1\}.$$

¿From (15 $\boxed{\textbf{veto-set}}$), S_i is a veto-set, whence $v \in ray(conv(S_i))$, which means that there is $k > 0$ for which $kv \in conv(S_i)$. Since values of L_i on all elements of S_i are not less than 1, its value on kv also is not less than 1. Therefore, the value of L_i on v is positive. Thus, the values of all our functions $L_1, \ldots, L_r$ on v are positive, which contradicts (14 $\boxed{\textbf{sum=0}}$).

Now assume that ρ is empty and prove existence of functions L_i in question. For any finite $S \subset R^{\dim +1}$, the set $ray(conv(S))$ can be represented as an intersection of several halfspaces. (A halfspace is a set in $R^{\dim+1}$, where some homogeneous linear function is non-negative.) Apply this to veto-sets, and you have several veto-halfspaces whose intersection is $\{\mathcal{O}\}$. (A veto-halfspace is a halfspace whose intersection with $N(\mathcal{O})$ is a veto-set.) For everyone of these veto-halfspaces we introduce a homogeneous linear function f_i which is non-positive on it and only on it. We know that the origin is the only point in $R^{\dim+1}$ where all f_i are non-positive. This allows us to apply Theorem 21.3 of [6] (a variant of Helly's theorem) to the hyperplane $\Pi = \{(s,t) \ : \ t = -1\}$, restrictions $f_i|_\Pi$ of our functions to it and any non-empty closed convex set $C \subseteq \Pi$. We take $C = \Pi$. Then, according to Theorem 21.3 of [6], there exist such non-negative real numbers λ_i, at most $\dim+1$ of which are positive, that for some $\varepsilon > 0$

$$\forall w \in C : \sum_i \lambda_i f_i(w) \geq \varepsilon.$$

Since the left part is linear and bounded from below by a positive constant on C, it has to be a positive constant on C:

$$\forall w \in C : \sum_i \lambda_i f_i(w) = \delta = \text{const} \geq \varepsilon.$$

Hence

$$\forall w \in R^{\dim+1} : \sum_i \lambda_i f_i(w) = -\delta t.$$

Then the functions $L_i = -(t + n\lambda_i f_i/\delta)$ fit (14 **sum=0**) and (15 **veto-set**) □

We denote $R = \{1, \dots, r\}$ where r is the number of functions $L_1, \dots, L_r$ and term any map $\pi : R \mapsto S$ a POLAR on a set S. The image $\pi(k)$ of a number $k \in R$ is called the k-th pole of π. Similarly $\pi(S) = \{\pi(i),\ i \in S\}$ for any $S \subseteq R$. In particular, $\pi(R) = \mathit{Range}(\pi)$ is the range of π. We say that $a, b \in \mathit{Range}(\pi)$ are CONNECTED by π.

4.2.5. Trusses

Let us claim two sequences of polars equivalent if one turns into the other by some permutation, and call the resulting classes of equivalence TRUSSES. We designate a truss by any sequence of its polars: $\mathcal{T} = (\pi_1, \dots, \pi_k)$. There is the empty truss which corresponds to the empty sequence of polars. $|\mathcal{T}|$ stands for the number of polars in the truss $\mathcal{T}$. For any truss $\mathcal{T}$ its range $\mathit{Range}(\mathcal{T})$ is defined as the union of ranges of its polars. For any two trusses $\mathcal{A}$ and $\mathcal{B}$ their concatenation $\mathcal{A} * \mathcal{B}$ results from writing one sequence after the other. $\mathcal{U}(v)$ stands for the truss that consists of one polar, whose all poles coincide with v.

Given a truss $(\pi_1, \dots, \pi_k)$ on any set S, we term it p-EVEN or just EVEN on a subset $Q \subseteq S$ if for any $i = 1, \dots, r$ the sequence $\pi_1(i), \dots, \pi_k(i)$ has exactly p members belonging to Q. We term a truss on S OVERALL-EVEN on Q if it is even on all elements of Q and OVERALL-EVEN if it is even on all elements of S.

For any truss $\mathcal{T}$, $\mathit{Graph}(\mathcal{T})$ stands for the non-directed graph whose set of vertices is $\mathit{Range}(\mathcal{T})$, two vertices being connected iff they are connected by some polar in $\mathcal{T}$. Call $\mathcal{T}$ connected if $\mathit{Graph}(\mathcal{T})$ is connected.

Given a truss $\mathcal{T}$ and a family F of sets, $\mathit{Graph}(\mathcal{T}/F)$ stands for the graph whose set of vertices is $\mathit{Range}(\mathcal{T})$, and two vertices are connected iff they are connected by some polar in $\mathcal{T}$ or by some set which belongs to F. Say that $\mathcal{T}$ is F-CONNECTED if $\mathit{Graph}(\mathcal{T}/F)$ is connected.

Note 4. If a truss $\mathcal{T}$ is F-connected and every element of F consists of one element, then $\mathcal{T}$ is connected. **one**

A truss $\mathcal{T}$ given, $Family(\mathcal{T})$ stands for the family of ranges of polars in $\mathcal{T}$.

Lemma 4. Given trusses $\mathcal{T}^1$ and $\mathcal{T}^2$ and a family F. Assume

1) $\mathcal{T}^1$ is $(Family(T^2) \cup F)$-connected,

2) $\mathcal{T}^2$ is F-connected, and

3) $Range(\mathcal{T}^1) \cap Range(\mathcal{T}^2) \neq \varnothing$.

Then $\mathcal{T}^1 * \mathcal{T}^2$ is F-connected. **T-T-F**

Proof: It follows from Lemma 2 **F-F-F** $\square$

4.2.6. Polars and Trusses on Graphs

A polar on a graph G means a polar on the set of vertices of G. A polar π on a graph G is called an EDGER on G if $Range(\pi)$ consists of two vertices connected with an edge, and we say that this edge corresponds to π.

Lemma 5. For any polar π_0 on a connected non-directed graph G, there is a tree $\gamma \subseteq G$, and a connected truss $\mathcal{T}$ on γ, which consists of edgers, such that: **polar-truss**

1) the truss $(\pi_0) * \mathcal{T}$ is connected,

2) the truss $(\pi_0) * \mathcal{T}$ is 0-even or 1-even on every vertex of G, and

3) $|Range((\pi_0) * \mathcal{T})| = |Range(\mathcal{T})| = |\mathcal{T}| + 1$.

Proof: Let γ be a minimal connected subgraph of G, whose set of vertices contains $Range(\pi_0)$. Then γ is a tree and every edge of it is a bridge. Based on this, for every edge $a - b$ of γ let us form an edger π_{a-b} on γ whose k-th pole for all k coincides with a or b, namely with that one of the two, which needs the edge $a - b$ to be connected with the k-th pole of π_0. The truss $\mathcal{T}$ that consists of these edgers, is the one sought. The last assertion of our lemma is true because $|\mathcal{T}|$ equals the number of edges of γ, ranges of $\mathcal{T}$ and $((\pi_0) * \mathcal{T})$ coincide with the set of vertices of γ and because in γ (like in any tree) the number of vertices is one more than the number of edges $\square$

4.2.7. Polars and Trusses on V. Arrows and Bows.

For any polar π on V, the value

$$L_1(\pi(1)) + \ldots + L_r(\pi(r))$$

is termed the STRETCH of π.

Note 5. If a truss on V is overall-even on V, then the sum of stretches of its polars equals zero. $\boxed{\textbf{zero}}$

The following two kinds of polars on V are important for us:

A polar α on V is called a k-ARROW or just an ARROW if $Range(\alpha) = \{u, v\}$, where $u \in N_\nabla(v)$, and $\alpha(k) = u$, and all the other poles of α coincide with v, and $L_k(u) - L_k(v) \geq 1$. Call u the HEAD and v the TAIL of this arrow.

A polar β on V is called a BOW if $Range(\beta) = \{u, v\}$ where u and v are ∇-peers.

Lemma 6. Given an overall-even truss $\mathcal{A} * \mathcal{B}$ on V, where $\mathcal{A}$ consists of arrows and $\mathcal{B}$ consists of bows. Then $|\mathcal{A}| \leq |\mathcal{B}| \cdot \text{const}$. $\boxed{\textbf{arrows-bows}}$

Proof: Denote

$$\mathcal{L} = \max\{L_i(\Delta s_j,\ \Delta t_j)\ :\ i = 1, \ldots, r,\quad j = 1, \ldots, n\}.$$

¿From (14 $\boxed{\textbf{sum=0}}$) the stretch of any bow is no more than $2r\mathcal{L}$ by modulo, hence no less than $-2r\mathcal{L}$. Also from (14 $\boxed{\textbf{sum=0}}$), the stretch of any arrow is no less than 1. Thus, the sum of stretches of polars in our truss (which equals zero from Note 5 $\boxed{\textbf{zero}}$), is no less than

$$0 \geq |\mathcal{A}| - 2r\mathcal{L} \cdot |\mathcal{B}|,$$

whence

$$|\mathcal{A}| \leq |\mathcal{B}| \cdot 2r\mathcal{L} \quad \square$$

Lemma 7. Consider all pairs of trusses $\mathcal{A}$, $\mathcal{B}$ such that: $\boxed{\textbf{Euler}}$

1) $\mathcal{A}$ consists of arrows

2) $\mathcal{B}$ consists of bows

3) $\mathcal{A} * \mathcal{B}$ is connected

4) $Range(\mathcal{A} * \mathcal{B})$ contains a given point

5) $|\mathcal{A} * \mathcal{B}| = k$

The number of these pairs does not exceed const^k.

Proof: Introduce the non-directed graph ∇' with V as its set of vertices, in which two vertices are connected if one is the the other's ∇-neighbor or they are ∇-peers. The degree of any vertex of ∇' does nor exceed $n^2 + 2n$. Note that all arrows and bows are edgers on ∇'. To every pair of trusses in question there corresponds a circuit on ∇' which starts and ends at the given point and makes $2k$ steps, every edger represented by two steps in the two opposite directions along the corresponding edge. At every step we must choose the edge, distribute the poles between its two ends and decide whether the edger belongs to $\mathcal{A}$ or to $\mathcal{B}$. Therefore the number of trusses in question does not exceed $(2^{r+1}(n^2 + 2n))^{2k}$ $\square$

4.3. Description of the family $\mathcal{F}(\cdot)$

Now for any given inner point v we can describe the family $\mathcal{F}(v)$ as follows: A point-set S belongs to $\mathcal{F}(v)$ if there are such trusses $\mathcal{A}$ and $\mathcal{B}$ that the following **Conditions F** are fulfilled:

Conditions F:

(F1) $\mathcal{A}$ consists of arrows. **arrows-F**

(F2) $\mathcal{B}$ consists of bows. **bows-F**

(F3) The truss $\mathcal{A} * \mathcal{B}$ is overall-even. **overall-even-F**

(F4) The truss $\mathcal{T} = \mathcal{U}(v) * \mathcal{A} * \mathcal{B}$ is connected. **connected-F**

(F5) S consists of those points in $Range(\mathcal{T})$ which are not tails of arrows in $\mathcal{A}$. **tails-F**

(F6) $|S| = |\mathcal{B}| + 1$. **+1-F**

Lemma 8. **Conditions F** given, there is $f(\theta)$ for which (11 **f**) and (12 **sum**) hold. **F-f**

Proof: Let us evaluate the cardinalities of sets $\mathcal{F}_k(v) = \{S \in \mathcal{F}(v) \ : \ |S| = k\}$.

$|\mathcal{F}_0(v)| = 0$, because the empty set does not belong to $\mathcal{F}(v)$, due to **(F6)**.

$|\mathcal{F}_1(v)| = 1$ because $\{v\}$ is the only element of $\mathcal{F}(v)$ whose cardinality equals 1. To prove it, assume $S \in \mathcal{F}(v)$ and $|S| = 1$. Then **(F6)** $\mathcal{B}$ is empty, whence (Lemma 6 **arrows-bows**) $\mathcal{A}$ is empty also, and $\mathcal{T} = \mathcal{U}(v)$, and **(F5)** $S = \{v\}$.

Now evaluate $|\mathcal{F}_k(v)|$ for $k \geq 2$. ¿From **(F5)**, S is uniquely determined by $\mathcal{A}$ and $\mathcal{B}$. Therefore $|\mathcal{F}_k(v)|$ is no more than the number of different corresponding pairs of trusses. If $S \in \mathcal{F}(v)$ and $|S| = k$, then the corresponding trusses have (**(F6)**, then **(F1)**, **(F2)** and **(F3)** allow to apply Lemma 6 $\boxed{\textbf{arrows-bows}}$)

$$|\mathcal{B}| = k-1 \implies |\mathcal{A}| \leq \text{const} \cdot (k-1) \implies k-1 \leq |\mathcal{A} * \mathcal{B}| \leq \text{const} \cdot (k-1).$$

Due to **(F1)**, **(F2)** and **(F4)** we can apply Lemma 7 $\boxed{\textbf{Euler}}$ to obtain

$$|\mathcal{F}_k(v)| \leq \sum_{i=k-1}^{\text{const}\cdot(k-1)} \text{const}^i \leq \text{const}^{k-1}.$$

Thus,

$$\sum\{\theta^{|S|}, \quad S \in \mathcal{F}(v)\} = \sum_{k=0}^{\infty} |\mathcal{F}_k(v)| \cdot \theta^k \leq \theta + \sum_{k=2}^{\infty} \text{const}^{k-1} \cdot \theta^k = \theta + O(\theta^2),$$

which may be taken for $f(\theta)$ $\square$

4.4. Inquest

$\boxed{\textbf{Inquest}}$

Inquest[2] is the central part of the proof. Its purpose is to present sets $S_1, \ldots, S_k \in \mathcal{F}(hub)$ which satisfy (13 $\boxed{\textbf{h=1}}$) for any inner point hub and any configuration x such that $x(hub) \geq k$. These sets will be obtained as

$$S(hub,\ x,\ rub), \quad rub = 1,\ \ldots,\ k.$$

Throughout Inquest, hub, x, rub are fixed. Call a point v a rub-UPSTART if

$$x(w) \geq rub \quad \text{and} \quad d(x(N_\nabla(v))) \leq rub - 1.$$

Note 6. If v is a rub-upstart, then $x(v) = rub$ and $d(x(N_\nabla(v))) = rub - 1$ and $h(v) = 1$. $\boxed{\textbf{upstart}}$

4.4.1. Graph G

To obtain a directed graph G, we construct inductively a sequence of graphs G_0, G_1, G_2, ... and at some step we stop. Some vertices of these graphs belong to V, others will be newly introduced. At every step we prove that $x(\cdot) \geq rub$ for all those vertices of G_i which belong to V.

Initial condition:

[2]Constructions of this sort are so awful that Janos Simon proposed to add some detective fun into them. Imagine that it is a crime that $x(hub) > 0$, and we look into its causes.

We start with a graph G_0 which has one vertex hub and no edges. Note that $x(hub) \geq rub$.

When the induction process stops:

When every vertex of G_i either has a non-empty G_i-neighborhood or is a rub-upstart. As soon as this happens, we stop and define $G = G_i$.

Induction step:

Suppose that the graph G_i has a vertex v which has no G_i-neighbors and is not an rub-upstart. Denote $d(x(N_\nabla(v))) = e$ and prove that $e \geq rub$. By our assumption, $x(v) \geq rub$. If $x(v) > rub$ then $e > rub - 1$ whence $e \geq rub$. If $x(v) = rub$ then $e \geq rub$ because otherwise v would be a rub-upstart. Thus, we have proved that $d(x(N_\nabla(v))) = e \geq rub$. Hence, every veto-set for v contains a point where $x(\cdot) \geq rub$. Thus, from (15 **veto-set**) for every $j = 1, \dots, r$ there exists $n_j \in N_\nabla(v)$ where

$$x(n_j) \geq rub \quad \text{and} \quad L_j(n_j) - L_j(v) \geq 1. \qquad \boxed{\textbf{n-j}} \tag{16}$$

Let this n_j be chosen in some standard way, say the first in the numeration (3 **neighbors**) of those which fit the conditions (16 **n-j**).

For every $j = 1, \dots, r$ we call $\tau_j = t(v) - t(n_j) - 1 \geq 0$ and define a 'chain'. If $\tau_j = 0$, the 'chain' is $n_j \to v$. Otherwise we introduce τ_j intermediate vertices $u_1, \dots, u_{\tau_j}$ and $\tau_j + 1$ intermediate edges connected in the form of a 'chain':

$$n_j \to u_1 \to u_2 \to \cdots \to u_{\tau_j} \to v.$$

We define the time function on the newly introduced vertices by the rule

$$t(u_k) = t(n_j) + k, \quad k = 1, \dots, \tau_j.$$

The newly defined graph G_{i+1} results from G_i by including these 'chains' for all $j = 1, \dots, r$ into it.

4.4.2. Tree T

Now classify the vertices of G into equivalence classes called BRANCHES as follows:

- If $t(a) = t(b)$ and $N_G^{all}(a) \cap N_G^{all}(b) \neq \varnothing$ then a and b are equivalent.
- Only those vertices of G are equivalent for which it follows from the previous rule and transitivity.

For any branch A define $t(A) = t(a)$, $a \in A$. Now define a directed graph T as follows:

- The set of vertices of T is the set of branches.
- An edge $A \to B$ is present in T iff $\exists\ a \in A,\ b \in B\ :\ a \in N(b)$.

Note 7. The point hub is equivalent to nothing else, and forms a branch $\{hub\}$. T is a tree in which all edges are directed towards the vertex $\{hub\}$. **tree**

4.4.3. Graphs $\Gamma(\cdot)$

For any branch A for which $N_T(A)$ is non-empty, define a non-directed graph $\Gamma(A)$ as follows:

- The set of vertices of $\Gamma(A)$ is $N_T(A)$.
- Vertices B^1 and B^2 are connected with an edge in $\Gamma(A)$ iff they are different and $\exists\ \ a \in A, \quad b^1 \in N_G^{all}(B^1) \cap N_G(a), \quad b^2 \in N_G^{all}(B^2) \cap N_G(a)$.

Lemma 9. Every $\Gamma(A)$ is connected. **Gamma**

Proof: Take any different vertices B^1 and B^2 of $\Gamma(A)$ and prove that they are connected with a path. By definitions,

$$\exists\ a^1,\ a^2 \in A, \quad b^1 \in B^1 \cap N_G(a^1), \quad b^2 \in B^2 \cap N_G(a^2).$$

If $a^1 = a^2$ then B^1 and B^2 are connected with an edge. Now let $a^1 \neq a^2$. Since a^1 and a^2 belong to one branch A, they are equivalent, which means that there is such a sequence $c_0 = a^1,\ c_1,\ \ldots,\ c_l = a^2$ of different elements of A that

$$\forall i = 1, \ldots, l\ :\ N_G^{all}(c_{i-1}) \cup N_G^{all}(c_i) \neq \varnothing\ .$$

This means that for every $i = 1, \ldots, l$ there are such

$$e_{i-1} \in N_G(c_{i-1}) \quad \text{and} \quad e'_{i-1} \in N_G(c_i)$$

that

$$N_G^{all}(e_{i-1}) \cap N_G^{all}(e'_i) \neq \varnothing\ .$$

Besides that, $t(e_{i-1}) = t(e'_i)$. Therefore e_{i-1} and e'_i are equivalent; let E_i stand for the branch to which they belong. Now, every two next terms of the sequence $B^1,\ E_1,\ \ldots,\ E_l,\ B^2$ either coincide or are connected with an edge $\square$

4.4.4. Two Trusses and a Set

Now i is an integer parameter which grows from 0 to some maximal value. For every value of i we shall form two trusses $\mathcal{A}_i$ and $\mathcal{B}_i$ on $N^{all}(hub)$ and a set σ_i of branches. Thereby we shall also define a truss

$$\mathcal{T}_i = \mathcal{U}(\text{hub}) * \mathcal{A}_i * \mathcal{B}_i,$$

a family

$$\Sigma_i = \{N_G^{all}(B),\ B \in \sigma_i\}$$

and a set of branches

$$OK_i = N_T^{all}(hub) - N_T^{all}(\sigma_i).$$

(On branches that belong to OK_i our construction is OK, that is need no more amendment.)

We shall construct $\mathcal{A}_i$, $\mathcal{B}_i$, σ_i, and thereby $\mathcal{T}_i$, Σ_i and OK_i, inductively, the parameter i increasing from zero. Along with constructing them, we shall prove by induction that they satisfy the following **Conditions $\mathbf{I}_i$**:

Conditions $\mathbf{I}_i$:

(I1$_i$) $\mathcal{A}_i$ consists of arrows. $\boxed{\textbf{arrows-I}}$

(I2$_i$) $\mathcal{B}_i$ consists of bows. $\boxed{\textbf{bows-I}}$

(I3$_i$) $\mathcal{T}_i$ is overall-even on OK_i. $\boxed{\textbf{overall-even-I}}$

(I4$_i$) $\mathcal{T}_i$ is Σ_i-connected. $\boxed{\textbf{connected-I}}$

(I5$_i$) $Union(OK_i) \cap Range(\mathcal{T}_i)$ equals the set of tails of arrows of $\mathcal{A}_i$. $\boxed{\textbf{tails-I}}$

(I6$_i$) $|\sigma_i| = |\Sigma_i| = |\mathcal{B}_i| + 1$. $\boxed{\textbf{+1-I}}$

(I7$_i$) $\mathcal{T}_i$ is 1-even on every element of Σ_i. $\boxed{\textbf{1-even-I}}$

(I8$_i$) $Range(\mathcal{T}_i) \subseteq N_G^{all}(hub)$. $\boxed{\textbf{Range-I}}$

(I9$_i$) If $A, B \in \Sigma_i$ and $A \neq B$ then $A \notin N_T^{all}(B)$ and $B \notin N_T^{all}(A)$. $\boxed{\textbf{uncomparable-I}}$

The initial case $i = 0$:

$\mathcal{A}_0$ and $\mathcal{B}_0$ are empty. σ_0 consists of one branch $\{hub\}$. It is evident that all the eight properties hold in this case.

When the induction process ends:

As soon as $\forall A \in \sigma_i\ :\ N_T(A) = \varnothing$, we stop.

The induction step:

Assume that $\mathcal{A}_i$, $\mathcal{B}_i$, σ_i (and thereby $\mathcal{T}_i$, Σ_i and OK_i) have been constructed and **Conditions** $\mathbf{I}_i$ are proved. Choose a branch $\Delta OK_i \in \sigma_i$ for which $N_T(\Delta OK_i) \neq \varnothing$. The two trusses to construct are

$$\mathcal{A}_{i+1} = \mathcal{A}_i * \Delta\mathcal{A}_i, \qquad \mathcal{B}_{i+1} = \mathcal{B}_i * \Delta\mathcal{B}_i$$

where it remains to define $\Delta\mathcal{A}_i$ and $\Delta\mathcal{B}_i$.

To define $\Delta\mathcal{A}_i$:

If $\Delta OK_i \cap Range(\mathcal{T}_i) = \varnothing$ then $\Delta\mathcal{A}_i$ is empty also.

Now let $\Delta OK_i \cap Range(\mathcal{T}_i)$ be non-empty. **(I7$_i$)** implies that for every $j = 1, \ldots, r$ there is at most one polar in $\mathcal{T}_i$ whose j-th pole belongs to ΔOK_i. Let S_i be the set of those values of j for which ΔOK_i contains the j-th pole of some polar in $\mathcal{T}_i$. Denote this pole a_j and form a j-arrow π_j whose head is $n_j(a_j)$ and whose tail is a_j. The truss $\Delta\mathcal{A}_i$ consists of these arrows $\pi_j, \quad j \in S_i$.

Note 8. $Range(\Delta\mathcal{A}_i) \subset N_G^{all}(\Delta OK_i)$. **Range-DA**

Note 9. Truss $\mathcal{T}_i * \Delta\mathcal{A}_i$ is overall-even on $OK_i \cup \Delta OK_i$. **T+DA-OK**

Note 10. Truss $\mathcal{T}_i * \Delta\mathcal{A}_i$ is 1-even on $N_G^{+}(\Delta OK_i)$. **T+DA-1-even**

To define $\Delta\mathcal{B}_i$:

Note 10 **T+DA-1-even** allows us to define a polar π_0 on $N_T(\Delta OK_i)$ by the following rule applied to all $j = 1, \ldots, r$: $\pi_0(j)$ is that element of $N_T(\Delta OK_i)$ whose N_G^{all} contains the j-th pole of some polar in $\mathcal{T}_i * \Delta\mathcal{A}_i$. ¿From Lemma 9 **Gamma** , $\Gamma(\Delta OK_i)$ is connected. Then Lemma 5 **polar-truss** provides us with a truss of edgers on a tree $\gamma(\Delta OK_i) \subseteq \Gamma(\Delta OK_i)$ which satisfy assertions of that lemma. For everyone ε of these edgers, we form a bow β as follows. Denote

$$Range(\varepsilon) = \{B, B'\} \subset N_T(\Delta OK_i).$$

Since branches B and B' are connected with an edge in $\Gamma(\Delta OK_i)$, there are points

$$a \in V, \quad b \in N_G^{all}(B) \cap N_G(a), \quad b' \in N_G^{all}(B') \cap N_G(a).$$

Now define β :

$$\forall\, j = 1, \ldots, r \;:\; \beta(j) = \begin{cases} b & \text{if } \varepsilon(j) = B, \\ b' & \text{if } \varepsilon(j) = B'. \end{cases} \tag{17}$$ **beta**

The truss $\Delta\mathcal{B}_i$ consists of these bows.

Note 11. $Range(\Delta\mathcal{B}_i) \subset N_G^{+}(\Delta OK_i)$. **Range-DB**

Denote $\Delta\mathcal{T}_i = \Delta\mathcal{A}_i * \Delta\mathcal{B}_i$ and $\Delta\sigma_i = \{S \in N_T(\Delta OK_i) \ : \ N_G^{all}(S) \cap Range(\Delta\mathcal{T}_i) \neq \varnothing \}$.

To define σ_{i+1} **:** it results from σ_i by excluding ΔOK_i and including all the elements of $\Delta\sigma_i$.

The induction step is described.

Note 12. $\mathcal{T}_{i+1} = \mathcal{T}_i * \Delta\mathcal{T}_i$. **T=T+DT**

Lemma 10. $Range(\Delta\mathcal{T}_i) \subset N_G^{all}(\Delta OK_i)$. **Range-DT**

Proof: It follows form Notes 8 **Range-DA** and 11 **Range-DB** □

Note 13. $\mathcal{T}_{i+1}$ is overall-even on ΔOK_i. **T+-DOK**

Note 14. $\Delta\mathcal{T}_i$ is 1-even on $N_G^{all}(A)$ for every $A \in \Delta\sigma_i$. **DT-1-even**

Note 15. $\Delta\mathcal{T}_i$ is 0-even on $N_G^{all}(A)$ for every $A \in N_T(\Delta OK_i) - \Delta\sigma_i$. **DT-0-even**

Denote $\Delta\Sigma_i = \{N_G^{all}(B),\ B \in \Delta\sigma_i\}$.

Lemma 11. $\Delta\mathcal{T}_i$ is $\Delta\Sigma_i$-connected. **DT-connected**

Proof: First, prove that $\Delta\mathcal{B}_i$ is $\Delta\Sigma_i$-connected. Take any $a^1,\ a^2 \in Range(\Delta\mathcal{B}_i)$. They belong to some branches $A^1,\ A^2 \in N_T(\Delta OK_i)$ Remember that $A^1,\ A^2$ are vertices of the tree $\gamma(\Delta OK_i)$. Let $E_0 = A^1,\ E^1, \ldots,\ E_l = A^2$ be the path in this tree from A^1 to A^2. According to (17 **beta**), every next terms E_{j-1} and E_j in this sequence are connected with an edger ε_j to which there corresponds a bow β_j which connects the point b_j with the point b'_j. Thus, in the sequence $b_1,\ b'_1, \ldots, b_l,\ b'_l$ every two next terms are connected either with a bow or with a set.

The statement of the lemma follows from the proven one and the fact that every arrow in $\Delta\mathcal{A}_i$ connects its tail with head, and has a head in $Range(\Delta\mathcal{B}_i)$. □

Note 16. $Range(\mathcal{T}_i) \cap Range(\Delta\mathcal{T}_i) \neq \varnothing$. **T-DT**

Note 17. $\Delta\Sigma_i \subseteq \Sigma_{i+1}$. **DSigma**

Lemma 12. $\Delta\mathcal{T}_i$ is Σ_{i+1}-connected. $\boxed{\boxed{\textbf{DT-S}}}$

Proof: It follows from Lemma 11 $\boxed{\textbf{DT-connected}}$ and Note 17 $\boxed{\textbf{DSigma}}$ $\square$

Lemma 13. $\mathcal{T}_i$ is $(Family(\Delta\mathcal{T}_i) \cup \Sigma_{i+1})$-connected. $\boxed{\boxed{\textbf{T-connected}}}$

Proof: From $(\mathbf{I4}_i)$, $\mathcal{T}_i$ is Σ_i-connected. Note that all the branches of Σ_i belong to Σ_{i+1} also, with the only exception of ΔOK_i. Thus, the only chance of our lemma's statement to be false is where connection was provided by ΔOK_i. But all the elements of $Range(\mathcal{T}_i) \cap \Delta OK_i$ are tails of arrows in $\Delta\mathcal{A}_i$, all of which are Σ_{i+1}-connected together from Lemma 12 $\boxed{\textbf{DT-S}}$ $\square$

Now let us prove **Conditions** $\mathbf{I}_{i+1}$.

Proofs of $(\mathbf{I1}_{i+1})$ and $(\mathbf{I2}_{i+1})$ are evident.

Proof of $(\mathbf{I3}_{i+1})$: Note that OK_{i+1} consists of OK_i, ΔOK_i and extended G-neighborhoods of those T-neighbors of ΔOK_i where there are no poles of our polars at all (Note 15 $\boxed{\textbf{DT-0-even}}$). Hence our statement follows from $(\mathbf{I3}_i)$ and Notes 9 $\boxed{\textbf{T+DA-OK}}$, 10 $\boxed{\textbf{Range-DT}}$ and 13 $\boxed{\textbf{T+-DOK}}$ $\square$

Proof of $(\mathbf{I4}_{i+1})$: It follows from Notes 12 $\boxed{\textbf{T=T+DT}}$ and 16 $\boxed{\textbf{T-DT}}$ and Lemmas 4 $\boxed{\textbf{T-T-F}}$ and 13 $\boxed{\textbf{T-connected}}$ $\square$

Proof of $(\mathbf{I5}_{i+1})$: The set of tails of arrows in $\Delta\mathcal{A}_i$ coincides with $Range(\mathcal{T}_i) \cap \Delta OK_i$ $\square$

Proof of $(\mathbf{I6}_{i+1})$: This is because

$$|\sigma_{i+1}| - |\sigma_i| = |\mathcal{B}_{i+1}| - |\mathcal{B}_i|.$$

Let us prove it. The right part equals the number of bows (17 $\boxed{\textbf{beta}}$). The left part is one less than the number of those branches in $N_T(\Delta OK_i)$ whose N_G^{all} intersects $Range(\mathcal{T}_{i+1})$. The number of these branches (see definition of $\Delta\mathcal{B}_i$) equals range of the truss provided by Lemma 5 $\boxed{\textbf{polar-truss}}$ which is one more than the number of bows, from the last statement of that lemma $\square$

Proof of $(\mathbf{I7}_{i+1})$: Take any $A \in \sigma_i$ and prove that $\mathcal{T}_i$ is 1-even on $N_G^{all}(A)$.

If $A \notin N_T(\Delta OK_i)$, this just follows from $(\mathbf{I7}_i)$, because in this case $A \in \sigma_i$.

If $A \in N_T(\Delta OK_i)$ then all the poles of $\mathcal{T}_{i+1}$ in $N_G^{all}(A)$ are those of $\Delta\mathcal{T}_i$. According to its construction (see Notes 14 $\boxed{\textbf{DT-1-even}}$ and 15 $\boxed{\textbf{DT-0-even}}$), $\Delta\mathcal{T}_i$ is 0-even or 1-even on $N_G^{all}(A)$ for every $A \in N_T(\Delta OK_i)$, but only the latter are included into σ_{i+1} $\square$

Proof of $(\mathbf{I8}_{i+1})$ is evident.

Proof of $(\mathbf{I9}_{i+1})$: Take any two different branches A and B in σ_{i+1} and prove that each of them does not belong to N_T^{all} of the other. Consider three cases:

1) $A,\ B \in \Sigma_i$. In this case our statement follows from **(I9$_i$)**.

2) $A,\ B \in N_T(\Delta OK_i)$. In this case $t(A) = t(B)$ which makes it impossible for one of them to belong to N_T^{all} of the other.

3) $A \in \Sigma_i$ and $B \in N_T(\Delta OK_i)$.

If $A \in N_T^{all}(B)$ then $A \in N_T^{all}(\Delta OK_i)$ which contradicts **(I9$_i$)**.

If $B \in N_T^{all}(A)$ then $N_T^{all}(A) \cap N_T^{all}(\Delta OK_i) \neq \varnothing$ which contradicts Note 7 $\boxed{\textbf{tree}}$ $\square$

4.4.5. Result of Induction

The induction process certainly will have an end, because $|OK_i|$ increases at every step. When it is over, we obtain the trusses $\mathcal{A}_i$ and $\mathcal{B}_i$ and the set Σ_i for which **Conditions I$_i$** are true and such that every element of σ_i has empty T-neighborhood. Then we denote $\mathcal{A} = \mathcal{A}_i,\ \ \mathcal{B} = \mathcal{B}_i,\ \ \sigma = \sigma_i$.

Note 18. $N_T(A) = \varnothing$ for every $A \in \sigma$. $\boxed{\textbf{empty-T}}$

Note 19. Every element of σ consists of one point. $\boxed{\textbf{in-Sigma}}$

Note 19 $\boxed{\textbf{in-Sigma}}$ allows us to introduce a point-set $S = \{v\ :\ \{v\} \in \sigma\}$.

Note 20. $|S| = |\sigma|$. $\boxed{\textbf{S=Sigma}}$

Note 21. $S \subseteq N_G^{end}(hub) \subseteq N_G^{all}(hub)$. $\boxed{\textbf{S-belongs}}$

Note 22. All the elements of S are rub-upstarts. $\boxed{\textbf{S-upstarts}}$

Now let us prove **Conditions F** for $\mathcal{A}$, $\mathcal{B}$ and S thus obtained.

(F1) follows from **(I1$_i$)**.

(F2) follows from **(I2$_i$)**.

Proof of **(F3)**: Outside S our truss is overall-even from **(I3$_i$)**. On every element of S our truss is 1-even from **(I7$_i$)** and Notes 18 $\boxed{\textbf{empty-T}}$ and 19 $\boxed{\textbf{in-Sigma}}$ □

Proof of **(F4)**: this follows from **(I4$_i$)** and Notes 4 $\boxed{\textbf{one}}$ and 19 $\boxed{\textbf{in-Sigma}}$ □

Proof of **(F5)**: It is evident that $S \subset Range(\mathcal{T})$. Elements of S can not be tails of arrows of $\mathcal{A}$ because $N_G(S) = \varnothing$. It remains to prove that all the other points in $N_G^{all}(hub)$ are some arrows' tails. This follows from **(I5$_i$)** and definitions of OK_i and S □

Proof of **(F6)**: This follows from **(I6$_i$)** and Note 20 $\boxed{\textbf{S=Sigma}}$ □

4.5. Final

Now only the point hub and the number k remain fixed. For each $x \in X$ such that $x(hub) \geq k$ and each $rub = 1, \ldots, k$ we have constructed trusses $\mathcal{A}$ and $\mathcal{B}$ and a set S. Now we can specify the k disjoint sets $S_1, \ldots, S_k$ mentioned in SEction 4.1: these are the sets S resulting from our construction with $rub = 1, \ldots, k$.

Lemma 14. Given hub and k, the sets $S_1, \ldots, S_k$ are disjoint. $\boxed{\boxed{\textbf{disjoint}}}$

Proof: This follows from Notes 6 $\boxed{\textbf{upstart}}$ and 22 $\boxed{\textbf{S-upstarts}}$ □

Thus Theorem 2 $\boxed{\textbf{bounded-inf-T}}$ is proved □

5. Proof of Theorem 3

5.1. Outline

Proof of Theorem 3 is similar to and based on the proof of Theorem 2. In this case we construct for every point v two finite families $\mathcal{F}'(v)$ and $\mathcal{F}''(v)$ of point-sets which have the following three properties:

$$\begin{cases} \text{If } x(v) \geq k > 0, \text{ there are } k \text{ disjoint point-sets } S_1, \ldots, S_k \in \mathcal{F}'(v) \cup \mathcal{F}''(v) \\ \text{such that } h_w = 1 \text{ for all } w \in S_1 \cup \ldots \cup S_k. \end{cases} \quad \boxed{\boxed{\textbf{S-F+F}}} \tag{18}$$

$$\sum \{\theta^{|S|},\ S \in \mathcal{F}'(v)\} \leq f'(\theta) = \theta + O(\theta^2) \quad \boxed{\boxed{\textbf{S-in-F'}}} \tag{19}$$

$$\sum \{\theta^{|S|},\ S \in \mathcal{F}''(v)\} \leq f''(\theta, v, M) = \text{const} \cdot t(v) \cdot (\text{const} \cdot \theta)^{\text{const} \cdot M}. \quad \boxed{\boxed{\textbf{S-in-F''}}} \tag{20}$$

Lemma 15. Given $\mathcal{F}'(v)$ and $\mathcal{F}''(v)$ which satisfy (18 $\boxed{\textbf{S-F+F}}$), (19 $\boxed{\textbf{S-in-F'}}$) and (20 $\boxed{\textbf{S-in-F''}}$). Then Theorem 3 holds. $\boxed{\boxed{\textbf{Theorem-3}}}$

Proof: We can choose $C = \text{const}$ so that while $t(v) < C^M$, the right part of (20 $\boxed{\textbf{S-in-F''}}$) is still less than $\text{const} \cdot (\text{const} \cdot \theta)^{\text{const} \cdot M}$. Then we can choose such M_0 that for all $M \geq M_0$ this expression is $O(\theta^2)$, and therefore $f''(\cdot) \leq O(\theta^2)$. Thus,

$$\sum \{\theta^{|S|},\ S \in \mathcal{F}'(v) \cup \mathcal{F}''(v)\} \leq \theta + O(\theta^2) \quad \square$$

5.2. Norm, Distance, Diameter and Span

Choose any norm $\mathsf{norm}(v)$ in $\mathbf{R}^{\dim +1}$, define distance $\mathsf{dist}(v,w) = \mathsf{norm}(v-w)$, let distance between two finite sets $\mathsf{dist}(S_1, S_2)$ stand for the minimal distance between points of S_1 and points of S_2, and let $\mathsf{diam}(S)$ stand for the diameter of S, that is the maximal distance between its points.

Note 23. If a connected truss $\mathcal{T}$ consists of arrows and bows, then

$$\mathsf{diam}(Range(\mathcal{T})) \leq \mathsf{const} \cdot |\mathcal{T}|. \qquad \boxed{\boxed{\textbf{connected-diam}}}$$

Note 24. $\mathsf{diam}(S_1 \cup S_2) \leq \mathsf{diam}(S_1) + \mathsf{diam}(S_2) + \mathsf{dist}(S_1, S_2)$. $\boxed{\boxed{\textbf{2-diam}}}$

Denote

$$\lambda = \max\{\mathsf{norm}(\Delta s_i, \Delta t_i),\ i = 1, \dots, n\}, \qquad \mathsf{span}(S) = \mathsf{diam}(S) + \lambda.$$

Call two point-sets λ-close if the distance between them does not exceed λ. Call a family of point-sets λ-CONNECTED if it is possible to go from any its element to any other, every step made between λ-close sets.

Lemma 16. If a family Σ of point-sets is λ-connected, then $\boxed{\boxed{\textbf{connected-span}}}$

$$\mathsf{span}(Union(\Sigma)) \leq \sum\{\mathsf{span}(S),\ S \in \Sigma\}.$$

Proof: By induction, based on Note 24 $\boxed{\textbf{2-diam}}$ $\square$

Lemma 17. For any $v \in V$, $\boxed{\boxed{\textbf{span-neighbors}}}$

$$\mathsf{span}(N_G^{all}(v)) \leq \sum\{\mathsf{span}(N_G^{all}(w)),\ w \in N_G(v)\} + \lambda.$$

Proof: This follows from Lemma 16 $\boxed{\textbf{connected-span}}$ $\square$

5.3. Infinite Representation of The Finite Case

We are considering a process P_M on a finite volume $V_M = \mathbf{Z}_M^{\dim} \cdot Time$. Let us call elements of the finite volume M-points. Let $V_\infty = \mathbf{Z}_\infty^{\dim} \cdot Time$ stand for the corresponding infinite volume whose elements are called ∞-points now. The map $\mathsf{fin} \ : \ V_\infty \mapsto V_M$ is defined as follows: $\mathsf{fin}(s,t) = (s',t)$ where components of s' are residues of components of s modulo M. The opposite map $\mathsf{inf} \ : \ V_M \mapsto V_\infty$ is defined by

$$\forall \ (s,t) \in V_M \ : \ \mathsf{inf}(s,t) = (s,t) \in V_\infty.$$

Call elements of $X_0^{V_\infty}$ ∞-configurations. To every M-configuration x there corresponds a periodical ∞-configuration $y = \mathsf{INF}(x)$ defined by the rule

$$\forall \ v \in V_\infty \ : \ y(v) = x(\mathsf{fin}(v)).$$

We shall use the auxiliary infinite process P'_∞ on V_∞ induced by the finite hidden measure η_M with the map $D' \ : \ H_M \mapsto X_\infty$ defined as follows:

$$x(v) = \begin{cases} 0 & \text{if } t(v) < 0, \\ d(x(N(v))) + h(\mathsf{fin}(v)) & \text{if } t(v) \geq 0. \end{cases}$$

Note 25. $\forall \ v \in V_\infty \ : \ P'_\infty|_v \equiv P_M|_{\mathsf{fin}(v)}$. $\boxed{\textbf{inf=fin}}$

Say that a set $S \subset V_\infty$ OVERLAPS if $\exists \ v, v' \in S \ : \ \mathsf{fin}(v) = \mathsf{fin}(v')$.

Note 26. If $S \subset V_\infty$ does not overlap, then $|\mathsf{fin}(S)| = |S|$. $\boxed{\textbf{not-overlap}}$

Note 27. If $S \subset V_\infty$ overlaps, then $\mathsf{diam}(S) \geq M$. $\boxed{\textbf{overlap-diam}}$

5.4. Description of the families $\mathcal{F}'(\cdot)$ and $\mathcal{F}''(\cdot)$

To define $\mathcal{F}'(v)$: A point-set $S \subset V_M$ belongs to $\mathcal{F}'(v)$ if it is representable as $S = \mathsf{fin}(S_\infty)$ where S_∞ is such a subset of V_∞ for which there are such trusses $\mathcal{A}$ and $\mathcal{B}$ on V_∞, that the following **Conditions F'** hold:

Conditions F':

(F'1) $\mathcal{A}$ consists of arrows. $\boxed{\textbf{arrows-F'}}$

(F'2) $\mathcal{B}$ consists of bows. $\boxed{\textbf{bows-F'}}$

(F'3) The truss $\mathcal{A} * \mathcal{B}$ is overall-even. **overall-even-F'**

(F'4) The truss $\mathcal{T} = \mathcal{U}(\mathsf{inf}(v)) * \mathcal{A} * \mathcal{B}$ is connected. **connected-F'**

(F'5) S_∞ consists of those points in $Range(\mathcal{T})$ which are not tails of arrows in $\mathcal{A}$. **tails-F'**

(F'6) $|S_\infty| = |\mathcal{B}| + 1$. **+1-F'**

(F'7) S_∞ does not overlap. **not-overlap-F'**

Lemma 18. Family $\mathcal{F}'(v)$ satisfies (19 **S-in-F'**). **F'-f**

Proof: is the same as in the infinite case (see Lemma 8 **F-f**) $\square$

To define $\mathcal{F}''(v)$**:** A point-set $S \subset V_M$ belongs to $\mathcal{F}''(v)$ if it is representable as $S = \mathsf{fin}(S_\infty)$ where S_∞ is such a subset of V_∞ for which there are such trusses $\mathcal{A}$ and $\mathcal{B}$ on V_∞, and such a point $w \in V_\infty$ that the following **Conditions F"** hold:

Conditions F":

(F"1) $\mathcal{A}$ consists of arrows. **arrows-F"**

(F"2) $\mathcal{B}$ consists of bows. **bows-F"**

(F"3) The truss $\mathcal{A} * \mathcal{B}$ is overall-even. **overall-even-F"**

(F"4) The truss $\mathcal{T} = \mathcal{U}(w) * \mathcal{A} * \mathcal{B}$ is connected and $t(w) < t(v)$. **connected-F"**

(F"5) S_∞ consists of those points in $Range(\mathcal{T})$ which are not tails of arrows in $\mathcal{A}$. **tails-F"**

(F"6) $|S_\infty| = |\mathcal{B}| + 1$. **+1-F"**

(F"7) S_∞ does not overlap. **not-overlap-F"**

(F"8) $\mathsf{span}(Range(\mathcal{A} * \mathcal{B})) \geq (M - \lambda)/n$. **span-F"**

Lemma 19. Family $\mathcal{F}''(v)$ satisfies (20 **S-in-F"**). **F"-f"**

Proof: From **(F"8)** and definition of span

$$\mathsf{diam}(Range(\mathcal{A} * \mathcal{B})) \geq \frac{M - \lambda}{n} - \lambda.$$

Then from Note 23 $\boxed{\textbf{connected-diam}}$

$$|\mathcal{A} * \mathcal{B}| \geq \text{const} \cdot M - \text{const}.$$

Due to **(F"1)**, **(F"2)** and **(F"3)** we can apply Lemma 6 $\boxed{\textbf{arrows-bows}}$ which gives

$$|\mathcal{B}| \geq \text{const} \cdot M - \text{const}.$$

Then from **(F"6)**, **(F"7)** and Note 26 $\boxed{\textbf{not-overlap}}$

$$|S| = |S_\infty| \geq C_1 M - C_2,$$

where C_1 and C_2 are positive constants.

Now let us evaluate the cardinalities of the sets

$$\mathcal{F}''_k(v) = \{S \in \mathcal{F}''(v) \ : \ |S| = k\}.$$

As we have proved, $|\mathcal{F}''_k(v)| = 0$ for all $k < C_1 M - C_2$.

Now let $k \geq C_1 M - C_2$. From **(F"6)**, $|\mathcal{B}| < k$, and from Lemma 6 $\boxed{\textbf{arrows-bows}}$ $|\mathcal{A} * \mathcal{B}| \leq \text{const} \cdot k$, and from Lemma 7 $\boxed{\textbf{Euler}}$ $|\mathcal{F}''_k(v)| \leq \text{const}^k$. Thus,

$$\sum\{\theta^{|S|},\ S \in \mathcal{F}''(v)\} \prec M^{\dim} \cdot t(v) \cdot \sum_{k=C_1 M - C_2}^{\infty} (\text{const} \cdot \theta)^k \prec M^{\dim} \cdot t(v) \cdot (\text{const} \cdot \theta)^{\text{const} \cdot M}.$$

For small enough θ, this is less than the right part of (20 $\boxed{\textbf{S-in-F"}}$) $\square$

5.5. Final

In Section 5.3 we have represented the finite process P_M as a special infinite process P'_∞, to which we can apply constructions of Section 4. Given a ∞-point v, a ∞-configuration x such that $x(v) \geq k > 0$ and a number $rub \in [1,\ k]$, we make for them Inquest described in Section 4.4 $\boxed{\textbf{Inquest}}$ and obtain the triple $(\mathcal{A},\ \mathcal{B},\ S)$. Let $S_\infty(v, x, rub)$ stand for the last term S of this triple.

Lemma 20. If $S_\infty(v, x, rub)$ overlaps, then there exists such a point $\mathsf{ersatz}(v)$ that

$$(21) \begin{cases} \bullet\ \mathsf{ersatz}(v) \in N_G^+(v), \\ \bullet\ S_\infty(\mathsf{ersatz}(v), x, rub) \text{ does not overlap, and} \\ \bullet\ \mathsf{span}(Range(\mathcal{T}(\mathsf{ersatz}(v),\ x,\ rub))) \geq (M - \lambda)/n. \end{cases} \qquad \boxed{\boxed{\textbf{ersatz}}}$$

Proof: Take the set of those points $w \in N_G^{all}(v)$ for which $S_\infty(w, x, rub)$ overlaps, take a point in this set whose time is the smallest, and call this point *pit*. Let us prove by contradiction that among G-neighbors of *pit* there is at least one which fits (21 $\boxed{\textbf{ersatz}}$). Assume that

$$\forall\ w \in N_G(pit)\ :\ \mathsf{span}(Range(\mathcal{T}(w))) < (M - \lambda)/n.$$

Then from Note 21 $\boxed{\textbf{S-belongs}}$ and Lemma 17 $\boxed{\textbf{span-neighbors}}$,

$$\mathsf{span}(S_\infty(pit, x, rub)) \le \mathsf{span}(N_G^{all}(pit)) < M$$

which contradicts Note 27 $\boxed{\textbf{overlap-diam}}$ □

Now we can present a set $S_M \in \mathcal{F}(hub) \cup \mathcal{F}''(hub)$ for any M-point hub, M-configuration x and $rub \in [1,\ k]$, where $k \in [1, x(hub)]$:

$$S_M(hub,\ x,\ rub) = \mathsf{fin}(S_\infty(\mathsf{inf}(hub), \mathsf{INF}(x), rub)), \tag{22}$$ $\boxed{\boxed{\textbf{S-fin-dsnt}}}$

if $S_\infty(\mathsf{inf}(hub), \mathsf{INF}(x), rub)$ does not overlap, and

$$S_M(hub,\ x,\ rub) = \mathsf{fin}(S_\infty(\mathsf{ersatz}(\mathsf{inf}(hub)),\ \mathsf{INF}(x),\ rub)), \tag{23}$$ $\boxed{\boxed{\textbf{S-fin-does}}}$

if $S_\infty(\mathsf{inf}(hub), \mathsf{INF}(x), rub)$ overlaps.

Lemma 21. Every $S_M(\cdot)$ defined by (22 $\boxed{\textbf{S-fin-dsnt}}$) belongs to $\mathcal{F}'(hub)$ and every $S_M(\cdot)$ defined by (23 $\boxed{\textbf{S-fin-does}}$) belongs to $\mathcal{F}''(hub)$. $\boxed{\boxed{\textbf{S-in-F+F}}}$

Proof: The first assertion can be proved just as in the infinite case. Let us prove the second. All **Conditions F"** but the last one were proved when treating of the infinite case. The last condition follows from (21 $\boxed{\textbf{ersatz}}$) □

LEMMA Sets $S_M(hub,\ x,\ rub)$, $\quad rub = 1, \ldots, k$ are disjoint. $\boxed{\boxed{\textbf{fin-disjoint}}}$

Proof: is the same as in the infinite case □

Thus Theorem 3 $\boxed{\textbf{bounded-fin-T}}$ is proved □

6. Proof of Theorem 4

6.1. Preliminaries

For any $c \in \mathbf{R}^{\dim+1}$ and $A, B \subset \mathbf{R}^{\dim+1}$ denote

$$A + c = \{a + c\ :\ a \in A\}, \quad A + B = \{a + b\ :\ a \in A, \quad b \in B\}.$$

Say that a set $A \subset \mathbf{R}^{\dim+1}$ is OBTUSE for a set $B \subset \mathbf{R}^{\dim+1}$ if

$$\forall\ c \in \mathbf{R}^{\dim+1}\ :\ (A + c) \cap B = \varnothing \implies (A + c) \cap conv(B) = \varnothing\ .$$

Note 28. Given any sets $A, B, C \subset \mathbf{R}^{\dim+1}$ and any vectors $a,\ b \in \mathbf{R}^{\dim+1}$. If A is obtuse for B, then $A+C+a$ is obtuse for $B+b$. $\boxed{\boxed{\textbf{+obtuse}}}$

Lemma 22. For any $A \subset \mathbf{R}^{\dim+1}$ the set $-k \cdot conv(A)$ is obtuse for A provided $k \geq \dim+1$. $\boxed{\boxed{\textbf{-obtuse}}}$

Proof: Let it be proved for the case when A is the set of vertices of a simplex. The general proof is based on this and Carathéodory's theorem □

Lemma 23. For any finite family of bounded sets there is a bounded set which is obtuse for all of them and contains a sphere of any given radius. $\boxed{\boxed{\textbf{family-obtuse}}}$

Proof: It follows from Note 28 $\boxed{\textbf{+obtuse}}$ and Lemma 22 $\boxed{\textbf{-obtuse}}$ □

6.2. Tower

Note 29. Every minimax function can be presented as a minimum of several terms, everyone of which is a maximum of several arguments. $\boxed{\boxed{\textbf{normal-form}}}$

Assume that our function d is presented in the form described in Note 29 $\boxed{\textbf{normal-form}}$ and call TERM-SET (of $\mathcal{O}$) the set of neighbor vectors corresponding to arguments which enter one term. Term-sets of any point v are term-sets of $\mathcal{O}$ shifted by $\mathcal{O}$.

Call a set $S \subset V$ SELF-SUFFICIENT if for any non-initial $v \in S$ every term-set of v intersects S.

Lemma 24. There is such a self-sufficient set $Tower$, that $\boxed{\boxed{\textbf{self-sufficient}}}$

$$\forall\ t\ :\ 1 \leq |\{v \in Tower \cap \mathbf{Z}^{\dim+1}\ :\ t(v) = t\}| \leq \text{const}. \quad \boxed{\boxed{\textbf{self-const}}} \tag{24}$$

Proof: Take any ray $\rho \subseteq \sigma(\Delta, d)$. Take any large enough bounded set P which is obtuse for all term-sets (which exists from Lemma 23 $\boxed{\textbf{family-obtuse}}$), and so shifted that all its points have time coordinate less than $-m$. Define $Tower = P - \rho$ and prove that it fits the claim. The condition (24 $\boxed{\textbf{self-const}}$) is evidently fulfilled. It remains to prove that $Tower$ is self-sufficient. Take any inner point $v \in Tower$ and any term-set T. Since $\rho \subseteq \sigma$, $\rho \subseteq ray(conv(T))$. Thus, there is a point $p \in conv(T) \cap \rho$. Thus,

$$v + p \in conv(v+T) \cap (v+\rho).$$

Since $v \in Tower$ and $v+p \in \rho+p$, then $v+p \in Tower$. Thus, $Tower$ intersects $conv(v+T)$. Thus (Note 28 $\boxed{\textbf{+obtuse}}$), since $Tower$ is obtuse for $v+T$, $Tower$ intersects $v+T$ □

Note 30. If a point-set $Tower$ satisfies (24 $\boxed{\textbf{self-const}}$), then

$$\forall\, v \in V \quad \exists\, s \in Space \;:\; v \in Tower + s \qquad \boxed{\boxed{\textbf{any-in-S}}}$$

6.3. Final

Take any self-sufficient set $Tower$ which satisfies (24 $\boxed{\textbf{self-const}}$) and separate it into parts by the rule:

$$Floor_i = \{v \in Tower \;:\; m(i-1) \le t(v) < mi\}.$$

Note 31. $\forall\, i \;:\; 1 \le |Floor_i| \le C = \mathsf{const}$. $\boxed{\boxed{\textbf{Floor-const}}}$

Note 32. For any $i \ge 1$ and any $v \in Floor_i$ every term-set of v intersects $Floor_{i-1}$. $\boxed{\boxed{\textbf{intersects}}}$

Lemma 25. $\forall v \in V \;:\; \mathbf{E}_\theta(v) \ge \theta^C \cdot t(v)/m$, where C is that of Note 31 $\boxed{\textbf{Floor-const}}$. $\boxed{\boxed{\textbf{Theorem-4}}}$

Proof: First we prove that

$$\forall\, v \in Floor_i \;:\; E_\theta(\min\{x(v) \;:\; v \in Floor_i\}) \ge \theta^C \cdot i.$$

This is because

$$\begin{aligned}
&\min\{x(v),\ v \in Floor_{i+1}\} = \\
&\quad \min\{d(x(N(v))) + h(v),\ v \in Floor_{i+1}\} \ge \\
&\quad \min\{d(x(N(v))),\ v \in Floor_{i+1}\} + \min\{h(v),\ v \in Floor_{i+1}\} \ge \\
&\quad \min\{x(v),\ v \in Floor_i\} + \min\{h(v),\ v \in Floor_{i+1}\}.
\end{aligned}$$

(Expectation of the last term is no less than θ^C.)

Assertion of the lemma follows from this and Note 30 $\boxed{\textbf{any-in-S}}$ □

Theorem 4 follows from Lemma 25 $\boxed{\textbf{Theorem-4}}$ □

7. Proof of Theorem 5

Let $d = \mathsf{median}$.

First, assume that a ray ρ in question exists. Then $\sigma(\Delta, d) \supseteq \rho$ which proves Theorem 5 in one direction.

Now to prove Theorem 5 in the other direction by contradiction: If it is wrong, we can rotate Π at a small angle so that its new position will violate the following lemma:

Lemma 26. Let a ray α belong to σ. Let hyperplane Π contain α. Assume that all the neighbor vectors that do not belong to α, do not belong to Π also. Then the number of neighbor vectors that belong to α, is greater than the difference between the number of neighbor vectors on one side of Π and the number of neighbor vectors on the other side of Π. **plane-L**

Proof: Assume otherwise. Then there is another hyperplane, close to Π, such that ρ is on one side of Π and the majority of neighbor vectors is on the other side of Π, which makes a contradiction $\square$

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