Local Stability of Critical Fronts in Non-linear Parabolic Equations

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Abstract. For the Ginzburg-Landau equation and similar non-linear parabolic equations on the real line, we show the local stability of the slowest monotonic front solution by computing explicitly the leading term in the asymptotic behavior of a small perturbation as $t \to \infty$. The proof is based on the Renormalization Group method for parabolic equations [BKL].

1. Introduction and Main Result

We consider non-linear parabolic partial differential equations of the form

$$\partial_t u(x,t) \;=\; \partial_x^2 u(x,t) + F(u(x,t)) \;, \tag{1.1}$$

where $x \in \mathbf{R}$, $t \in \mathbf{R}_+$, and $F : \mathbf{R} \to \mathbf{R}$ is a polynomial (or analytic) map satisfying $F(0) = F(1) = 0$, $F'(0) > 0$, $F'(1) < 0$, and $F''(u) < 0$ for all $u \in (0,1)$. The examples we have in mind are the real Ginzburg–Landau (GL) equation, where $F(u) = u - u^3$, and the Kolmogorov–Petrovskii–Piskunov [KPP] equation, where $F(u) = u - u^2$.

Under these assumptions on the non-linearity F, Eq.(1.1) has *front solutions* of the form $u(x,t) = f_c(x-ct)$, with $f_c'(\xi) < 0$ for all ξ, $\lim_{\xi\to-\infty} f_c(\xi) = 1$, $\lim_{\xi\to\infty} f_c(\xi) = 0$, provided c is greater than the critical value $c^* = 2(F'(0))^{1/2}$ [AW]. Of course, due to the translation invariance in the variable $x \in \mathbf{R}$, Eq.(1.1) has in fact a one-parameter family of such front solutions for each $c \geq c^*$. These fronts turn out to be stable in some sense, and a lot is known about the basin of attraction of each of them [AW], [B]. For example, if $u(x,0) \in [0,1]$ satisfies $\liminf_{x\to-\infty} u(x,0) > 0$ and $u(x,0) = 0$ for all $x \geq 0$, then the solution $u(x,t)$ of Eq.(1.1) converges as $t \to \infty$ to a front solution moving with the critical speed c^*.

However, all these global stability results are obtained using comparison theorems based on the Maximum Principle for parabolic equations [F], [PW]. These techniques are very powerful here, but fail if Eq.(1.1) is slightly generalized (for example if u is complex or vector-valued, or if the linear operator is of higher order). For this reason, several contributions have been devoted to proving the *local* stability of the fronts by more general methods. In the non-critical case ($c > c^*$), this problem was solved by Sattinger [S] who showed that perturbations of the fronts decay to zero exponentially fast as $t \to \infty$ in appropriate weighted norms (spectral stabilisation method). In the critical case ($c = c^*$), the stability was shown by Kirchgässner [Ki] for the KPP equation, by Bricmont and Kupiainen [BK] for the GL equation (Renormalization Group method), and by Eckmann and Wayne [EW] in the general case (Lyapunov function method). In particular, it is known that perturbations of the critical front decay to zero faster than $t^{-1+\delta}$ (for all $\delta > 0$) in a suitable function space [BK].

In this paper, we present a more detailed stability analysis of the critical front solution $f_{c^*}(x - c^*t)$. We work in a frame moving to the right with constant velocity

c^*, so that Eq.(1.1) becomes

$$\partial_t u(x,t) \;=\; \partial_x^2 u(x,t) + c^* \partial_x u(x,t) + F(u(x,t)) \;, \tag{1.2}$$

and $f(x) \equiv f_{c^*}(x)$ is now a stationary solution of Eq.(1.2). Then, following [Ki], we look for solutions of Eq.(1.2) of the form $u(x,t) = f(x) + f'(x)v(x,t)$. As is easily verified, the equation for $v(x,t)$ is

$$\partial_t v(x,t) \;=\; \partial_x^2 v(x,t) + \gamma(x)\partial_x v(x,t) + v(x,t)\mathcal{N}(f(x), f'(x)v(x,t)) \;, \tag{1.3}$$

where

$$\gamma(x) \;=\; c^* + \frac{2f''(x)}{f'(x)} \;, \tag{1.4}$$

and

$$\mathcal{N}(\xi,\eta) \;=\; \frac{1}{\eta}(F(\xi+\eta) - F(\xi) - F'(\xi)\eta) \;. \tag{1.5}$$

In Theorem 1.1 below we prove that, in some appropriate function space, sufficiently small initial data for v decay to zero like $t^{-3/2}$ as $t \to \infty$. We also show that the leading term in the asymptotic expansion of $v(x,t)$ is $t^{-3/2}A^*\psi^*(xt^{-1/2})$, where $A^* \in \mathbf{R}$ depends on the initial data and ψ^* is the "universal" function

$$\psi^*(x) \;=\; \frac{1}{\sqrt{4\pi}} \cdot \begin{cases} 1 & \text{if } x \le 0 \\ e^{-x^2/4} & \text{if } x > 0 \end{cases} \;. \tag{1.6}$$

Before stating the theorem precisely, we briefly comment on the result and explain the function space we shall choose for $v(\cdot,t)$. From [AW], we know that the critical front f verifies

$$f(x) \;=\; \begin{cases} 1 - Ce^{\beta x} + \mathcal{O}(e^{2\beta x}) & \text{as } x \to -\infty \\ (A+Bx)e^{-c^* x/2} + \mathcal{O}(x^2 e^{-c^* x}) & \text{as } x \to +\infty \end{cases} \;, \tag{1.7}$$

where $B, C > 0$ and $\beta = \frac{1}{2}(-c^* + \sqrt{c^{*2} - 4F'(1)}) > 0$. Similar expressions hold for the derivatives f', f'', so that

$$\gamma(x) \;=\; \begin{cases} \alpha + \mathcal{O}(e^{\beta x}) & \text{as } x \to -\infty \\ \frac{2}{x+a} + \mathcal{O}(xe^{-c^* x/2}) & \text{as } x \to +\infty \end{cases} \;, \tag{1.8}$$

where $\alpha = c^* + 2\beta = \sqrt{c^{*2} - 4F'(1)}$ and $a = (A/B - 2/c^*)$. Now, consider the linearized equation

$$\partial_t v(x,t) \;=\; \partial_x^2 v(x,t) + \gamma(x)\partial_x v(x,t) \;, \quad v(x,0) = g(x) \;. \tag{1.9}$$

As $x \to +\infty$, the operator in the right-hand side becomes $\partial_x^2 + (2/x)\partial_x$, which is just the radial part of the Laplacian in 3 dimensions. So, provided $\int_1^\infty x^2 |g(x)|\, dx < \infty$, we expect that $v(x,t)$ decays in this region like $t^{-3/2}$, i.e., like a solution of the heat equation in 3 dimensions. On the other hand, as $x \to -\infty$, Eq.(1.9) becomes $\partial_t v = \partial_x^2 v + \alpha \partial_x v$, whose fundamental solution is

$$\mathcal{K}_t(x,y) \;=\; \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{1}{4t}(x+\alpha t-y)^2} \;=\; \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{\alpha^2}{4}t} e^{-\frac{1}{4t}(x-y)^2} e^{-\frac{\alpha}{2}(x-y)}\;.$$

Thus, provided $\int_{-\infty}^0 e^{\lambda x}|g(x)|\, dx < \infty$ for some $\lambda \in (0,\alpha)$, we expect an exponential decay of $v(x,t)$ as $t \to \infty$. In fact, due to the coupling with the region $x \geq 0$, the actual decay is again $t^{-3/2}$. These heuristic considerations remain valid for the full Eq.(1.3) under some additional conditions. In particular, we have to take $\lambda \leq \beta$ and to restrict ourselves to sufficiently small data $g(x)$ in the corresponding uniform norms.

In view of these remarks, we look for solutions of Eq.(1.3) in the Banach space X_1 of measurable functions $g : \mathbf{R} \to \mathbf{R}$ defined by the norm

$$\|g\|_1 \;=\; \int_{\mathbf{R}} W_\beta(x)|g(x)|\, dx + \operatorname*{ess\,sup}_{x\in\mathbf{R}} \left(W_\beta(x)|g(x)|\right)\;,$$

where W_β is the weight function

$$W_\beta(x) \;=\; \begin{cases} e^{\beta x} & \text{if } x \leq 0 \\ (1+x)^3 & \text{if } x > 0 \end{cases}\;. \tag{1.10}$$

Also, for any $L \geq 1$, we define the Banach space X_L by the rescaled norm

$$\|g\|_L \;=\; \int_{\mathbf{R}} W_{\beta L}(x)|g(x)|\, dx + \operatorname*{ess\,sup}_{x\in\mathbf{R}} \left(W_{\beta L}(x)|g(x)|\right)\;. \tag{1.11}$$

With these definitions, we now state our stability result for the critical front f:

Theorem 1.1. *Fix any $\delta \in (0,1/2)$. Then there exists an $\epsilon > 0$ such that, for all $g \in X_1$ verifying $\|g\|_1 < \epsilon$, there is a unique solution $v(x,t)$ of Eq.(1.3) such that $v(\cdot,t) \in X_1$ for all $t \geq 1$ and $v(x,1) = g(x)$. Moreover, there exists a constant $A^* = A^*(g)$ such that*

$$\|t^{3/2}v(xt^{1/2},t) - A^*\psi^*(x)\|_{t^{1/2}} \;=\; \mathcal{O}\left(\frac{1}{t^{1/2-\delta}}\right)\;, \tag{1.12}$$

as $t \to \infty$.

Remarks.

1) In Eq.(1.12), $\|\cdot\|_{t^{1/2}}$ means $\|\cdot\|_L$ (the norm in X_L) for $L = t^{1/2}$.

2) Here and in Section 4, it is notationally convenient to take $t = 1$ as initial time instead of $t = 0$, see [BKL]. Of course, this choice does not affect the conclusion of Theorem 1.1 in any essential manner, since we are dealing with the asymptotic behavior as $t \to \infty$.

3) The assertion of Theorem 1.1 is *translation invariant* in the sense that it does not depend on the choice of the origin in the moving frame, i.e., on the value of $f(0)$. In the course of the proof (Section 3), we shall use this property to make a convenient choice of the origin.

An immediate consequence of Theorem 1.1 is (see also [BK]):

Corollary 1.2. *Under the hypotheses of Theorem 1.1, the perturbation $f'v$ of the critical front solution f converges to zero as $t \to \infty$ in the sense that*

$$\sup_{x\in\mathbf{R}} \left(1 + \frac{e^{c^* x/2}}{1+|x|}\right) |f'(x)v(x,t)| \;=\; \mathcal{O}\left(\frac{1}{t^{3/2}}\right) \;.$$

Our proof of Theorem 1.1 consists of two well distinct parts. In Section 3, we show that Eq.(1.12) holds for the solutions of the linearized equation (1.9). In this case, one can choose $\delta = 0$, $\epsilon = \infty$ and the constant A^* is simply given by a weighted integral of the initial data g. We prove that by analyzing the resolvent operator $(H - s)^{-1}$ near $s = 0$, where $H = -\partial_x^2 - \gamma(x)\partial_x$. In particular, we split H into an explicit part and a relatively compact perturbation and we use the analytic Fredholm theorem [RS1] to obtain a corresponding factorization of the resolvent.

In Section 4, we treat the non-linearity $v\mathcal{N}$ in Eq.(1.3) by the so-called "Renormalization Group" (RG) method for partial differential equations [BKL]. This method rests on the fact that, very often, solutions of parabolic equations become asymptotically scale invariant as $t \to \infty$. The problem of finding the asymptotics is reduced to the iteration of the "RG map", which corresponds to evolving the solution during a fixed interval of time and then applying a scaling transformation. In our case, the non-linearity is "irrelevant" in the RG terminology [BKL], and thus does not affect very much the asymptotics of the solutions in a neighborhood of zero (it appears in Eq.(1.12) only in the constant A^*). This fact allows us to treat simultaneously a whole class of equations ("universality").

Since the whole proof of Theorem 1.1 is somewhat intricate, we first explain the main ideas and techniques on a simpler example (which has its own interest) in Section 2.

Acknowledgements. I thank J.-P. Eckmann for introducing me to the subject and for constant help. I am especially indebted to C.-A. Pillet for explaining me all the functional-analytic techniques used in Sections 2 and 3. This work was supported in part by the Fonds National Suisse.

2. A Preliminary Case

It is well known and easily verified that, for all $g \in \mathrm{L}^1(\mathbf{R})$, the solution $u(x,t)$ of the heat equation $\partial_t u = \partial_x^2 u$, $u(x,0) = g(x)$ satisfies

$$\lim_{t\to\infty} t^{1/2}\, u(xt^{1/2}, t) \;=\; \frac{1}{\sqrt{4\pi}}\, e^{-x^2/4} \int_{\mathbf{R}} g(y)\, dy \ . \tag{2.1}$$

In this section, we shall show a corresponding result for the linear equation

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + \gamma(x)\partial_x u(x,t) \ , \quad u(x,0) = g(x) \ , \tag{2.2}$$

where $\gamma : \mathbf{R} \to \mathbf{R}$ verifies

$$\operatorname*{ess\,sup}_{x\in\mathbf{R}} |\gamma(x)| < \infty \ , \quad \int_{\mathbf{R}} (1+|x|)^2 |\gamma(x)|\, dx < \infty \ . \tag{2.3}$$

In particular, we shall see that the "drift" $\gamma(x)\partial_x u$ in Eq.(2.2) has almost no effect on the asymptotic behavior of u as $t \to \infty$.

Warning. This section is *not* part of the proof of Theorem 1.1, but is only a preparation for the study of Eq.(1.9) in Section 3. Various symbols $(\gamma, p, \mathcal{H}, D_L, G_k, \Omega, \ldots)$ have here another meaning than in the rest of the paper. In particular, the drift $\gamma(x)$ in Eq.(2.2) is *not* given by Eq.(1.4)!

To formulate our result, we first introduce some notations. Let $\mathcal{H}$ be the Hilbert space $\mathrm{L}^2(\mathbf{R})$ equipped with the usual scalar product. Let Y^1, Y^∞ be the Banach spaces of (measurable) functions on $\mathbf{R}$ defined by the norms

$$\|g\|_{Y^1} = \int_{\mathbf{R}} (1+|x|)|g(x)|\,dx \ , \quad \|g\|_{Y^\infty} = \operatorname*{ess\,sup}_{x\in\mathbf{R}} (1+|x|)|g(x)| \ ,$$

and let Y be the intersection $Y^1 \cap Y^\infty$ equipped with the norm $\|g\|_Y = \|g\|_{Y^1} + \|g\|_{Y^\infty}$. Define also the weight function

$$p(x) = \sqrt{\frac{2}{1+\Gamma}} \exp\left(\frac{1}{2}\int_{-\infty}^{x} \gamma(y)\,dy\right) \ , \tag{2.4}$$

where $\Gamma = \exp(\int_{\mathbf{R}} \gamma(y)\,dy)$, and let $\phi^*(x) = (4\pi)^{-1/2} e^{-x^2/4}$. Then the following holds:

Proposition 2.1. *Let $g \in Y$. Then Eq.(2.2) has a unique solution $u(\cdot,t) \in Y$ for all $t \ge 0$, and there exists a constant $C > 0$ (independent of g) such that*

$$\|t^{1/2}\, u(xt^{1/2},t) - \phi^*(x)\int_{\mathbf{R}} p^2(y)g(y)\,dy\|_Y \le \frac{C}{t^{1/2}}\,\|g\|_Y \ , \tag{2.5}$$

as $t \to \infty$.

In other words, $u(x,t)$ behaves asymptotically like $t^{-1/2}A^*\phi^*(xt^{-1/2})$, where ϕ^* is a "universal" function (it does not depend on γ, g) and A^* is a weighted integral of the initial data g.

Proof of Proposition 2.1. Let $\mathcal{H}_p$ be the Hilbert space $\mathrm{L}^2(\mathbf{R})$ equipped with the weighted scalar product $(u,v)_p = \int_{\mathbf{R}} p^2(x)\bar{u}(x)v(x)\,dx$. Since $\gamma \in \mathrm{L}^1(\mathbf{R})$, there exist constants $c_-, c_+ > 0$ such that $c_- \le p(x) \le c_+$ for all $x \in \mathbf{R}$, so that the norm $\|\cdot\|_p = (\cdot,\cdot)_p^{1/2}$ is equivalent to the usual norm in $\mathrm{L}^2(\mathbf{R})$. Let $\mathcal{H}^2 \subset \mathrm{L}^2(\mathbf{R})$ be the Sobolev space of functions whose first and second derivatives exist and are square-integrable, and define the operator H on $\mathcal{H}_p$ by $\mathcal{D}(H) = \mathcal{H}^2$ and

$$H = -\partial_x^2 - \gamma(x)\partial_x = -\frac{1}{p^2(x)}\partial_x p^2(x)\partial_x \ .$$

H is selfadjoint and positive [Ka, §V-3.6]. In particular, the resolvent operator $(H - k^2)^{-1} \in \mathcal{L}(\mathcal{H}_p)$ exists for all $k \in \mathbf{C}_+ = \{z \in \mathbf{C}\,|\,\mathrm{Im}\,(z) > 0\}$, $k \to (H-k^2)^{-1}$ is analytic and

$$\|(H-k^2)^{-1}\|_{\mathcal{L}(\mathcal{H}_p)} \le \frac{1}{\mathrm{dist}(k^2,\mathbf{R}_+)} \le \frac{1}{|k|\mathrm{Im}\,(k)} \ . \tag{2.6}$$

(Here and in the sequel, $\mathcal{L}(\mathcal{H},\mathcal{H}')$ denotes the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$, and $\mathcal{L}(\mathcal{H}) \equiv \mathcal{L}(\mathcal{H},\mathcal{H})$.) Also, H generates on $\mathcal{H}_p$ a holomorphic semigroup of contractions given by the Dunford-Taylor integral [Ka, §IX-1.6]

$$e^{-Ht} = \frac{1}{i\pi}\int_{\mathcal{P}} k(H-k^2)^{-1}e^{-k^2 t}\,dk\ , \quad t>0\ , \tag{2.7}$$

where the path $\mathcal{P}$ goes from $\mathrm{Re}\,(k) = -\infty$ to ∞ in $\mathbf{C}_+$. In the rest of this section, we shall take $\mathcal{P} = \{\mathrm{Im}\,(k) = 1 + \frac{1}{2}|\mathrm{Re}\,(k)|\ , -\infty < \mathrm{Re}\,(k) < \infty\}$ (Fig. 1).

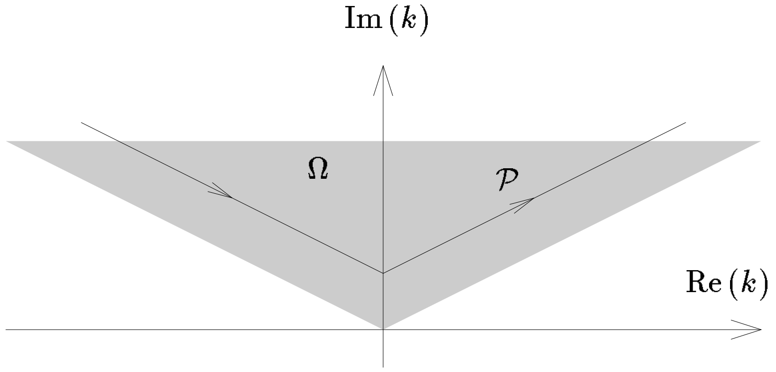

Fig. 1: The integration path $\mathcal{P}$ and the sector $\Omega \subset \mathbf{C}_+$ (shaded region)

Now, for all $g \in \mathcal{H}_p$, the unique solution of Eq.(2.2) in $\mathcal{H}_p$ is given by $u(\cdot\,,t) = e^{-Ht}g$. Thus, to prove Proposition 2.1, we have to compute the limit as $t \to \infty$ of $D_{t^{1/2}}e^{-Ht}g$, where D_L is the dilatation operator defined by $(D_L h)(x) = Lh(Lx)$. However, this limit does not exist for all $g \in \mathcal{H}_p$, so we shall restrict ourselves to the subspace $Y \subset \mathcal{H}_p$. In a first stage (Step 1), we show that the resolvent $(H-k^2)^{-1}$ is a bounded operator in Y. Then (Step 2) we study this operator in a neighborhood of $k=0$ and compute the limit of $D_{t^{1/2}}e^{-Ht}$ in $\mathcal{L}(Y)$ as $t \to \infty$.

Step 1. We use a classical method [RS4, §XIII.4] to factorize $(H-k^2)^{-1}$. Let H_0 be the selfadjoint positive operator in $\mathcal{H}$ defined by $\mathcal{D}(H_0) = \mathcal{H}^2$, $H_0 = -\partial_x^2 = H + \gamma(x)\partial_x$, and (for all $k \in \mathbf{C}_+$) define $G_k = (H_0 - k^2)^{-1} \in \mathcal{L}(\mathcal{H})$. As is well known, G_k is given by the integral kernel

$$G_k(x,y) = \frac{i}{2k}e^{ik|x-y|}\ , \quad k \in \mathbf{C}_+\ . \tag{2.8}$$

Then we have the factorization

$$(H-k^2)^{-1} \;=\; G_k(\mathbf{1}-\gamma\partial G_k)^{-1}\;, \quad k\in\mathbf{C}_+\;, \tag{2.9}$$

where $\gamma\partial G_k\in\mathcal{L}(\mathcal{H})$ is defined by $(\gamma\partial G_k h)(x)=\gamma(x)\partial_x(G_k h)(x)$.

The justification of Eq.(2.9) goes as follows. Using Eq.(2.3) and the integral kernel

$$(\gamma\partial G_k)(x,y) \;=\; -\frac{1}{2}\gamma(x)\mathrm{sgn}(x-y)e^{ik|x-y|}\;, \tag{2.10}$$

it is straightforward to verify that $\gamma\partial G_k$ is Hilbert-Schmidt (hence compact) in $\mathcal{H}$ for all $k\in\mathbf{C}_+$, that $k\to\gamma\partial G_k$ is analytic, and that $\|\gamma\partial G_k\|_{\mathcal{L}(\mathcal{H})}\le$ const.Im $(k)^{-1}$ as Im $(k)\to\infty$. Thus, $(\mathbf{1}-\gamma\partial G_k)$ is invertible if Im (k) is sufficiently large, and it follows from the analytic Fredholm theorem [DS, §VII.6.13] that $(\mathbf{1}-\gamma\partial G_k)^{-1}\in\mathcal{L}(\mathcal{H})$ exists and is analytic for all $k\in\mathbf{C}_+\backslash S$, where S is some discrete set which has no limit point in $\mathbf{C}_+$. In fact, $S=\emptyset$ in the present case, for if $k\in S$, then there exists an $u\in\mathcal{H}$, $u\neq 0$ such that $(\mathbf{1}-\gamma\partial G_k)u=0$, i.e., $(H-k^2)(G_k u)=0$. But this is impossible, since $G_k u\in\mathcal{H}^2$, $G_k u\neq 0$ and k^2 is not in the spectrum of H. Thus, $(\mathbf{1}-\gamma\partial G_k)^{-1}\in\mathcal{L}(\mathcal{H})$ for all $k\in\mathbf{C}_+$, and multiplying both sides of Eq.(2.9) on the left by $(H-k^2)$ (which is one-to-one), we see that equality holds.

Using this factorization, we now show that $(H-k^2)^{-1}\in\mathcal{L}(Y)$ for all $k\in\mathbf{C}_+$. Again, using Eqs.(2.3), (2.8), (2.10), it is straightforward to verify that

i) $G_k\in\mathcal{L}(Y^1,Y)$, $k\to G_k$ is analytic, and $\|G_k\|_{\mathcal{L}(Y^1,Y)}\le$ const.$|k|^{-1}$ as Im $(k)\to\infty$;

ii) $\gamma\partial G_k\in\mathcal{L}(Y^1\cap\mathcal{H})$, $k\to\gamma\partial G_k$ is analytic and $\|\gamma\partial G_k\|_{\mathcal{L}(Y^1\cap\mathcal{H})}\le$ const.Im $(k)^{-1}$ as Im $(k)\to\infty$.

Moreover, $\gamma\partial G_k$ is compact in $\mathcal{L}(Y^1\cap\mathcal{H})$, as can be verified using the Riesz criterion for compactness of bounded subsets in $\mathrm{L}^p(\mathbf{R})$ [RS4, §XIII.14]. Thus, applying the analytic Fredholm theorem in the Banach space $Y^1\cap\mathcal{H}$, we see as above that $(\mathbf{1}-\gamma\partial G_k)^{-1}\in\mathcal{L}(Y^1\cap\mathcal{H})$ exists and is analytic for all $k\in\mathbf{C}_+$. Since $Y\subset(Y^1\cap\mathcal{H})\subset Y^1$, we conclude that $(H-k^2)^{-1}=G_k(\mathbf{1}-\gamma\partial G_k)^{-1}\in\mathcal{L}(Y)$ for all $k\in\mathbf{C}_+$, and is analytic in k.

Step 2. In view of these results, Eq.(2.7) defines a holomorphic semigroup e^{-Ht} in $\mathcal{L}(Y)$ for $t>0$. (Note however that e^{-Ht} is no longer strongly continuous at $t=0$.) Applying the dilatation operator $D_{t^{1/2}}$ and using Eq.(2.9), we obtain after rescaling the integration variable

$$D_{t^{1/2}}e^{-Ht} \;=\; \frac{1}{2\pi}\int_{\mathcal{P}} S_k^{\epsilon}(\mathbf{1}-\gamma\partial G_{\epsilon k})^{-1}e^{-k^2}\,dk\;, \tag{2.11}$$

where $\epsilon = t^{-1/2}$ and $S_k^\epsilon = -2ik\epsilon^2 D_{1/\epsilon} G_{\epsilon k}$. Thus, to prove Proposition 2.1, it remains to study S_k^ϵ and $(\mathbf{1} - \gamma\partial G_{\epsilon k})^{-1}$ in the limit $\epsilon \to 0$.

First, from the integral kernel

$$S_k^\epsilon(x,y) \;=\; e^{ik|x-\epsilon y|} \;, \tag{2.12}$$

it is readily verified that $S_k^\epsilon \in \mathcal{L}(Y^1, Y)$ for all $k \in \mathbf{C}_+$ and all $\epsilon \geq 0$, that $k \to S_k^\epsilon$ is analytic, and that $\|S_k^\epsilon\|_{\mathcal{L}(Y^1,Y)} \leq$ const. uniformly for all $k \in \mathcal{P}$ and all $\epsilon \in [0,1]$. Moreover, $\|S_k^\epsilon - S_k^0\|_{\mathcal{L}(Y^1,Y)} \leq \text{const.}\epsilon|k|$ for all $k \in \mathcal{P}$.

On the other hand, one can see from Eq.(2.10) that $\gamma\partial G_k \in \mathcal{L}(Y^1)$ for all $k \in \bar{\mathbf{C}}_+$ and that $k \to \gamma\partial G_k$ is Lipschitz in this closed region. In particular, $\gamma\partial G_0 \in \mathcal{L}(Y^1)$ and $\|\gamma\partial G_k - \gamma\partial G_0\|_{\mathcal{L}(Y^1)} \leq \text{const.}|k|$ for all $k \in \bar{\mathbf{C}}_+$. Moreover, it is shown in Appendix B that $(\mathbf{1} - \gamma\partial G_0)$ is invertible in $\mathcal{L}(Y^1)$ and that

$$\Big((\mathbf{1} - \gamma\partial G_0)^{-1} g\Big)(x) \;=\; g(x) - \frac{\gamma(x)}{2p^2(x)} \int_{\mathbf{R}} \mathrm{sgn}_p(x-y) p^2(y) g(y)\, dy \;, \tag{2.13}$$

where sgn_p is the modified signum function

$$\mathrm{sgn}_p(\xi) \;=\; \frac{2}{1+\Gamma} \begin{cases} \Gamma & \text{if } \xi > 0 \\ -1 & \text{if } \xi < 0 \end{cases} \;, \tag{2.14}$$

and Γ is as in Eq.(2.4). In particular, we have the formula

$$\int_{\mathbf{R}} \Big((\mathbf{1} - \gamma\partial G_0)^{-1} g\Big)(x)\, dx \;=\; \int_{\mathbf{R}} p^2(y) g(y)\, dy \;, \tag{2.15}$$

for all $g \in Y^1$. So, if $\Omega = \{z \in \mathbf{C} \,|\, \mathrm{Im}\,(z) > \frac{1}{2}|\mathrm{Re}\,(z)|\}$ (see Fig. 1), we have

i) $\|(\mathbf{1} - \gamma\partial G_k)^{-1}\|_{\mathcal{L}(Y,Y^1)} \leq$ const., and

ii) $\|(\mathbf{1} - \gamma\partial G_k)^{-1} - (\mathbf{1} - \gamma\partial G_0)^{-1}\|_{\mathcal{L}(Y,Y^1)} \leq \text{const.}|k|$,

for all $k \in \bar{\Omega}$. Indeed, these inequalities are obvious in a neighborhood of 0, since $(\mathbf{1} - \gamma\partial G_0)^{-1} \in \mathcal{L}(Y^1) \subset \mathcal{L}(Y, Y^1)$ and $\|\gamma\partial G_k - \gamma\partial G_0\|_{\mathcal{L}(Y^1)} \leq \text{const.}|k|$. On the other hand, $(\mathbf{1} - \gamma\partial G_k)^{-1} \in \mathcal{L}(Y^1 \cap \mathcal{H}) \subset \mathcal{L}(Y, Y^1)$ is analytic in $\mathbf{C}_+$ and converges to $\mathbf{1}$ as $\mathrm{Im}\,(k) \to +\infty$. This proves the claim.

Using these estimates, it is now straightforward to conclude the proof of Proposition 2.1. Setting $\epsilon = t^{-1/2}$, we have from Eq.(2.11)

$$\begin{aligned} D_{t^{1/2}} e^{-Ht} &= \frac{1}{2\pi} \int_{\mathcal{P}} S_k^{\epsilon} \Big((\mathbf{1} - \gamma \partial G_{\epsilon k})^{-1} - (\mathbf{1} - \gamma \partial G_0)^{-1} \Big) e^{-k^2} \, dk \\ &+ \frac{1}{2\pi} \int_{\mathcal{P}} (S_k^{\epsilon} - S_k^0)(\mathbf{1} - \gamma \partial G_0)^{-1} e^{-k^2} \, dk \\ &+ \frac{1}{2\pi} \int_{\mathcal{P}} S_k^0 (\mathbf{1} - \gamma \partial G_0)^{-1} e^{-k^2} \, dk \; . \end{aligned} \tag{2.16}$$

Since $\epsilon k \in \Omega$ for all $k \in \mathcal{P}$ and all $\epsilon \in (0,1]$, the first two integrands in the right-hand side are bounded by const.$\epsilon |k| |e^{-k^2}|$ in $\mathcal{L}(Y)$, so that the corresponding integrals are $\mathcal{O}(\epsilon)$ as $\epsilon \to 0$. On the other hand, using Eqs.(2.12), (2.15) and the formula

$$\frac{1}{2\pi} \int_{\mathcal{P}} e^{ik|x|} e^{-k^2} \, dk \; = \; \frac{1}{\sqrt{4\pi}} e^{-x^2/4} \; \equiv \; \phi^*(x) \; ,$$

we see that the last term in Eq.(2.16) is just the rank one operator on Y defined by $g(x) \to \phi^*(x) \int_{\mathbf{R}} p^2(y) g(y) \, dy$. This proves Eq.(2.5). □

3. The Linear Equation

In this section, we study the asymptotic behavior of the solutions of the linearized equation (1.9)

$$\partial_t v(x,t) \; = \; \partial_x^2 v(x,t) + \gamma(x) \partial_x v(x,t) \; , \quad v(x,0) = g(x) \; , \tag{3.1}$$

where $\gamma(x)$ is given by Eq.(1.4). The result will be an essential step in the proof of Theorem 1.1 in Section 4. We proceed along the same lines as in the preliminary example of Section 2.

First of all, using the translation invariance of the problem (see the Remark after Theorem 1.1), we can assume without loss of generality that $\alpha = 2/a$ in Eq.(1.8); this amounts to fixing the origin in the moving frame. Thus, defining

$$\gamma_a(x) \; = \; \begin{cases} 2/a & \text{if } x \le 0 \\ 2/(x+a) & \text{if } x > 0 \end{cases} \; , \tag{3.2}$$

we see that $\gamma(x) = \gamma_a(x) + \hat{\gamma}(x)$, where $\hat{\gamma}(x)$ decays exponentially to 0 as $|x| \to \infty$. In particular, we have

$$|\hat{\gamma}| \stackrel{\text{def}}{=} \int_{-\infty}^{0} |\hat{\gamma}(x)| \, dx + \int_0^{\infty} (1+x)^3 |\hat{\gamma}(x)| \, dx + \sup_{x \in \mathbf{R}} |\hat{\gamma}(x)| \; < \; \infty \; . \tag{3.3}$$

For later use, we note that $1/a = c^*/2 + \beta > \beta$, see Eq.(1.8).

Next, we introduce the weight functions

$$p_a(x) \;=\; \begin{cases} ae^{x/a} & \text{if } x \le 0 \\ (x+a) & \text{if } x > 0 \end{cases} \;, \quad p(x) \;=\; p_a(x)\exp\left(-\frac{1}{2}\int_x^\infty \hat{\gamma}(y)\,dy\right) \;, \tag{3.4}$$

and the corresponding Hilbert spaces $\mathcal{H}_a = \mathrm{L}^2(\mathbf{R}, p_a^2\,dx)$, $\mathcal{H}_p = \mathrm{L}^2(\mathbf{R}, p^2\,dx)$ equipped with the scalar products

$$(u,v)_a \;=\; \int_{\mathbf{R}} p_a^2(x)\bar{u}(x)v(x)\,dx \;, \quad (u,v)_p \;=\; \int_{\mathbf{R}} p^2(x)\bar{u}(x)v(x)\,dx \;,$$

respectively. Since $p_a^2(x)e^{-|\hat{\gamma}|} \le p^2(x) \le p_a^2(x)e^{|\hat{\gamma}|}$ for all $x \in \mathbf{R}$, we see that $\mathcal{H}_a$, $\mathcal{H}_p$ coincide (as vector spaces) and that the norms $\|\cdot\|_a = (\cdot,\cdot)_a^{1/2}$, $\|\cdot\|_p = (\cdot,\cdot)_p^{1/2}$ are equivalent.

Finally, for any $L \ge 1$, we denote by X_L^1, X_L^∞ the Banach spaces defined by the norms

$$\|g\|_{L,1} \;=\; \int_{\mathbf{R}} W_{\beta L}(x)|g(x)|\,dx \;, \quad \|g\|_{L,\infty} \;=\; \operatorname*{ess\,sup}_{x\in\mathbf{R}} \left(W_{\beta L}(x)|g(x)|\right) \;, \tag{3.5}$$

respectively. Comparing Eqs.(1.11), (3.5), we see that $X_L = X_L^1 \cap X_L^\infty$ and $\|\cdot\|_L = \|\cdot\|_{L,1} + \|\cdot\|_{L,\infty}$. We also note that $X_1 \subset \mathcal{H}_a$, since $W_\beta(x) \ge \text{const.} p_a(x)$ for all $x \in \mathbf{R}$, see Eqs.(1.10), (3.4).

With these definitions, we can formulate the analogue of Proposition 2.1:

Proposition 3.1. *Let $g \in X_1$. Then Eq.(3.1) has a unique solution $v(\cdot,t) \in X_1$ for all $t \ge 0$, and there exists a constant $C > 0$ (independent of g) such that*

$$\|t^{3/2}v(xt^{1/2},t) - \psi^*(x)\int_{\mathbf{R}} p^2(y)g(y)\,dy\|_{t^{1/2}} \;\le\; \frac{C}{t^{1/2}}\,\|g\|_1 \;, \tag{3.6}$$

as $t \to \infty$, where ψ^* is given by Eq.(1.6) and p by Eq.(3.4).

Proof of Proposition 3.1. Let $\mathcal{H}^2 \subset \mathcal{H}_a$ be the Sobolev space of functions whose first and second derivatives exist and are in $\mathcal{H}_a$, and define the operator H on $\mathcal{H}_p$ by $\mathcal{D}(H) = \mathcal{H}^2$ and

$$H \;=\; -\partial_x^2 - \gamma(x)\partial_x \;=\; -\frac{1}{p^2(x)}\partial_x p^2(x)\partial_x \;.$$

Since $p'/p = \frac{1}{2}\gamma \in \mathrm{L}^\infty(\mathbf{R})$, $\mathcal{H}^2$ is the maximal domain for H in $\mathcal{H}_p$, and H is selfadjoint and positive. In particular, the formulas (2.6), (2.7) hold. To prove Proposition 3.1, we have to compute the limit as $t \to \infty$ of $D_{t^{1/2}} e^{-Ht} g$, where now D_L is the dilatation operator defined by

$$(D_L h)(x) \;=\; L^3 h(Lx) \;. \tag{3.7}$$

Again, this limit does not exist for all $g \in \mathcal{H}_p$, so we shall restrict ourselves to the subspace X_1.

As in the preceding section, we write H as the sum of a simpler operator H_0 and a relatively compact perturbation. We define H_0 by $\mathcal{D}(H_0) = \mathcal{H}^2$, and

$$H_0 \;=\; -\partial_x^2 - \gamma_a(x)\partial_x \;=\; -\frac{1}{p_a^2(x)}\partial_x p_a^2(x)\partial_x \;=\; H + \hat{\gamma}(x)\partial_x \;. \tag{3.8}$$

As an operator on $\mathcal{H}_a$, H_0 is selfadjoint and positive, and for all $k \in \mathbf{C}_+ = \{z \in \mathbf{C} \,|\, \mathrm{Im}\,(z) > 0\}$ the resolvent $G_k = (H_0 - k^2)^{-1} \in \mathcal{L}(\mathcal{H}_a)$ is an integral operator whose kernel $G_k(x,y)$ can be expressed in terms of elementary functions, see Appendix A.

Following the proof of Proposition 2.1, the first step is to factorize $(H - k^2)^{-1}$ as

$$(H - k^2)^{-1} \;=\; G_k(\mathbf{1} - \hat{\gamma}\partial G_k)^{-1} \;, \tag{3.9}$$

where $\hat{\gamma}\partial G_k \in \mathcal{L}(\mathcal{H}_a)$ is defined by $(\hat{\gamma}\partial G_k h)(x) = \hat{\gamma}(x)\partial_x(G_k h)(x)$. This is done exactly as in the preceding section, using the analytic Fredholm theorem and noting that $\hat{\gamma}\partial G_k$ is Hilbert-Schmidt for all $k \in \mathbf{C}_+$, that $k \to \hat{\gamma}\partial G_k$ is analytic, and that $\|\hat{\gamma}\partial G_k\|_{\mathcal{L}(\mathcal{H}_a)} \le$ const.Im$\,(k)^{-1}$ as $\mathrm{Im}\,(k) \to \infty$. All these properties of $\hat{\gamma}\partial G_k$ can be derived from the integral kernel (A.3) by straightforward calculations.

The next step is to show that $G_k(\mathbf{1} - \hat{\gamma}\partial G_k)^{-1} \in \mathcal{L}(X_1)$. However, this is not true for all $k \in \mathbf{C}_+$. The reason is that the space X_1 is essentially smaller than $\mathcal{H}_a$, in the sense that the weight $W_\beta(x)$ has a slower exponential decay as $x \to -\infty$ than $p_a(x)$, see Eqs.(1.10), (3.4). So we expect that restricting H to X_1 will change its essential spectrum. In fact, it is easy to see from the kernel (A.2) that $G_k \in \mathcal{L}(X_1^1, X_1)$ if and only if $k \in \mathcal{W} = \{z \in \mathbf{C}_+ \,|\, \mathrm{Re}\,(\sqrt{1/a^2 - z^2}) > 1/a - \beta\}$ (Fig. 2).

In this domain, one can check that $k \to G_k$ is analytic and that $\|G_k\|_{\mathcal{L}(X_1^1, X_1)} \le$ const.Im$\,(k)^{-1}$ as $\mathrm{Im}\,(k) \to \infty$. Similarly, $\hat{\gamma}\partial G_k \in \mathcal{L}(X_1^1 \cap \mathcal{H}_a)$ for all $k \in \mathcal{W}$, $k \to$

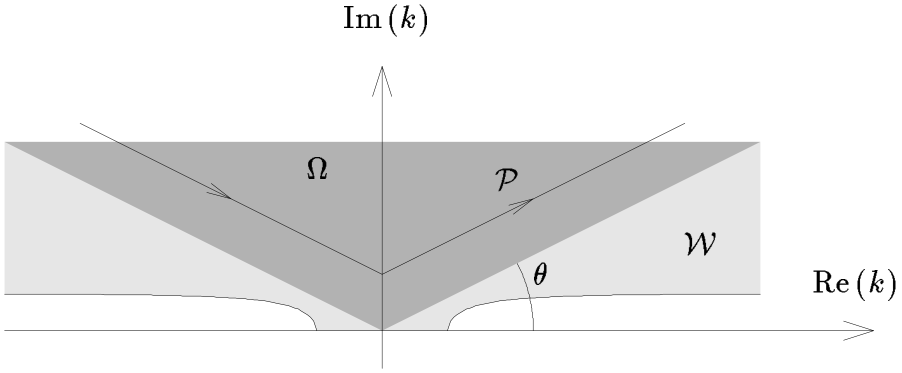

Fig. 2: The resolvent set $\mathcal{W}$, the sector $\Omega \subset \mathcal{W}$ and the integration path $\mathcal{P}$

$\hat{\gamma}\partial G_k$ is analytic, and $\|\hat{\gamma}\partial G_k\|_{\mathcal{L}(X_1^1\cap\mathcal{H}_a)} \leq \mathrm{const.Im}\,(k)^{-1}$ as $\mathrm{Im}\,(k) \to \infty$. Moreover, using again the Riesz criterion for compactness, one verifies that $\hat{\gamma}\partial G_k$ is compact in $\mathcal{L}(X_1^1 \cap \mathcal{H}_a)$ for all $k \in \mathcal{W}$. Thus, by the analytic Fredholm theorem, $(\mathbf{1} - \hat{\gamma}\partial G_k)^{-1} \in \mathcal{L}(X_1^1 \cap \mathcal{H}_a) \subset \mathcal{L}(X_1, X_1^1)$ exists and is analytic for all $k \in \mathcal{W}$, so that the same is true for $(H - k^2)^{-1} = G_k(\mathbf{1} - \hat{\gamma}\partial G_k)^{-1} \in \mathcal{L}(X_1)$.

On the other hand, let $\Omega = \{z \in \mathbf{C} \,|\, \mathrm{Im}\,(z) > \tan(\theta)|\mathrm{Re}\,(z)|\}$, for some $\theta \in (0, \pi/4)$ satisfying $\sin(2\theta) > 1 - a\beta$ (Fig. 2). Then $\Omega \subset \mathcal{W}$ (see Lemma A.1), and choosing as integration path $\mathcal{P} = \{\mathrm{Im}\,(k) = 1 + \tan(\theta)|\mathrm{Re}\,(k)|\,,\, -\infty < \mathrm{Re}\,(k) < \infty\} \subset \Omega$, we see from Eq.(2.7) that e^{-Ht} is a holomorphic semigroup in $\mathcal{L}(X_1)$ for $t > 0$. In particular, $D_{t^{1/2}}e^{-Ht} \in \mathcal{L}(X_1, X_{t^{1/2}})$ for all $t \geq 1$, and we have the analogue of Eq.(2.11)

$$D_{t^{1/2}}e^{-Ht} \;=\; \frac{1}{2\pi}\int_{\mathcal{P}} S_k^\epsilon(\mathbf{1} - \hat{\gamma}\partial G_{\epsilon k})^{-1} e^{-k^2}\, dk \;, \tag{3.10}$$

where again $\epsilon = t^{-1/2}$ and $S_k^\epsilon = -2ik\epsilon^2 D_{1/\epsilon}G_{\epsilon k} \in \mathcal{L}(X_1^1, X_{1/\epsilon})$.

Now, an essential difference with the preceding case is that $\lim_{\epsilon\to 0} S_k^\epsilon$ does not exist, see Appendix A. In fact, one can show from Eq.(A.5) that $\|S_k^\epsilon\|_{\mathcal{L}(X_1^1, X_{1/\epsilon})} \leq \mathrm{const.}|k|\epsilon^{-1}$ for all $k \in \mathcal{P}$ and all $\epsilon \in (0,1]$. On the other hand, $\lim_{k\to 0}\hat{\gamma}\partial G_k$ does exist in $\mathcal{L}(X_1^1)$, and we even have $\|\hat{\gamma}\partial G_k - \hat{\gamma}\partial G_0\|_{\mathcal{L}(X_1^1)} \leq \mathrm{const.}|k|^2$ for all $k \in \Omega$, $|k| \leq 1$. Moreover, $\hat{\gamma}\partial G_0$ is given by the very simple integral kernel (A.4), so that $(\mathbf{1} - \hat{\gamma}\partial G_0)$ can be inverted explicitly in $\mathcal{L}(X_1^1)$, see Appendix B. One finds

$$\Big((\mathbf{1} - \hat{\gamma}\partial G_0)^{-1} g\Big)(x) \;=\; g(x) - \frac{\hat{\gamma}(x)}{p^2(x)}\int_{-\infty}^{x} p^2(y)g(y)\, dy \;, \tag{3.11}$$

for all $g \in X_1^1$. In particular, we have the formula

$$\int_{\mathbf{R}} p_a^2(x)\Big((\mathbf{1} - \hat{\gamma}\partial G_0)^{-1} g\Big)(x)\, dx \;=\; \int_{\mathbf{R}} p^2(y) g(y)\, dy \ . \tag{3.12}$$

As in the preceding section, we conclude that $\|(\mathbf{1}-\hat{\gamma}\partial G_k)^{-1} - (\mathbf{1}-\hat{\gamma}\partial G_0)^{-1}\|_{\mathcal{L}(X_1, X_1^1)} \le$ const.$|k|^2$ for all $k \in \bar{\Omega}$, since this is true in a neighborhood of 0 and since $(\mathbf{1}-\hat{\gamma}\partial G_k)^{-1} \to \mathbf{1}$ as $\mathrm{Im}\,(k) \to \infty$.

Now, writing

$$\begin{aligned} D_{t^{1/2}} e^{-Ht} \;&=\; \frac{1}{2\pi} \int_{\mathcal{P}} S_k^\epsilon \Big((\mathbf{1} - \hat{\gamma}\partial G_{\epsilon k})^{-1} - (\mathbf{1} - \hat{\gamma}\partial G_0)^{-1} \Big) e^{-k^2}\, dk \\ &+\; \frac{1}{2\pi} \int_{\mathcal{P}} S_k^\epsilon (\mathbf{1} - \hat{\gamma}\partial G_0)^{-1} e^{-k^2}\, dk \ , \end{aligned}$$

we see that the first term in the right-hand side is $\mathcal{O}(\epsilon)$ in $\mathcal{L}(X_1, X_{1/\epsilon})$ since the integrand is bounded by const.$(|k|\epsilon^{-1})(\epsilon|k|)^2|e^{-k^2}|$ for all $k \in \mathcal{P}$. The second term is nothing but $D_{t^{1/2}} e^{-H_0 t}(\mathbf{1} - \hat{\gamma}\partial G_0)^{-1}$, since by construction $D_{t^{1/2}} e^{-H_0 t}$ can be obtained by setting $\hat{\gamma} = 0$ in Eq.(3.10). Thus, we have shown that

$$D_{t^{1/2}} e^{-Ht} \;=\; D_{t^{1/2}} e^{-H_0 t}(\mathbf{1} - \hat{\gamma}\partial G_0)^{-1} \;+\; \mathcal{O}\left(\frac{1}{t^{1/2}}\right) \ , \quad t \to \infty \ , \tag{3.13}$$

in $\mathcal{L}(X_1, X_{t^{1/2}})$. This result reduces the proof of Eq.(3.6) to the study of the simpler semigroup $K_t = e^{-H_0 t}$. Since the resolvent $G_k = (H_0 - k^2)^{-1}$ is explicitly known, it is a straightforward task to obtain an asymptotic expansion of the kernel $K_t(x, y)$ as $t \to \infty$. This calculation is outlined in Appendix C, and leads to the following result:

Lemma 3.2. *There exists a $C > 0$ such that, for all $g \in X_1^1$, one has*

$$\|D_{t^{1/2}} e^{-H_0 t} g - \psi^* \int_{\mathbf{R}} p_a^2(y) g(y)\, dy\|_{t^{1/2}} \;\le\; \frac{C}{t^{1/2}}\, \|g\|_{1,1} \ , \tag{3.14}$$

as $t \to \infty$.

Thus, combining Eqs.(3.13), (3.14), (3.12), we obtain Eq.(3.6). □

4. Proof of Theorem 1.1

In this section, we combine the linear result (3.6) with a Renormalization Group (RG) argument to obtain a proof of Theorem 1.1. We follow closely [BKL] and refer the reader to this paper for a thorough discussion of the method and for further applications.

We begin by explaining in an informal way how the RG works. First, we note that if $v(x,t)$ is a solution of Eq.(1.3) and if $w(x,t) = \Lambda^3 v(\Lambda x, \Lambda^2 t)$ for some $\Lambda \geq 1$, then w verifies the rescaled equation

$$\partial_t w(x,t) \;=\; \partial_x^2 w(x,t) + \Lambda\gamma(\Lambda x)\partial_x w(x,t) + \frac{1}{\Lambda}\left(\mathcal{M}_\Lambda w\right)(x,t) \;, \tag{4.1}$$

where

$$\left(\mathcal{M}_\Lambda w\right)(x,t) \;=\; \Lambda^3 w(x,t)\,\mathcal{N}\left(f(\Lambda x), f'(\Lambda x)\frac{1}{\Lambda^3}\,w(x,t)\right) \;. \tag{4.2}$$

Then, for any fixed $L \geq 1$, we define a non-linear map $R_{L,\Lambda} : X_\Lambda \to X_{L\Lambda}$ (see Eq.(1.11) for the definition of these Banach spaces) by the formula

$$(R_{L,\Lambda} g)(x) \;=\; L^3 w_g(Lx, L^2) \;, \quad g \in X_\Lambda \;, \tag{4.3}$$

where w_g is the unique solution of Eq.(4.1) in X_Λ satisfying the initial condition $w_g(x,1) = g(x)$. In other words, applying $R_{L,\Lambda}$ means evolving g according to Eq.(4.1) during a finite lapse of time (i.e., from $t = 1$ to $t = L^2$) and then applying the dilatation operator (3.7). We call $R_{L,\Lambda}$ the *RG map (on the scale Λ)*.

With this definition, Theorem 1.1 can be reformulated as follows: for all sufficiently small $g \in X_1$, there exists an $A^* \in \mathbf{R}$ such that

$$\|R_{L,1} g - A^*\psi^*\|_L \;=\; \mathcal{O}\left(L^{2\delta-1}\right) \;, \quad L \to \infty \;. \tag{4.4}$$

(To compare with Eq.(1.12), set $t = L^2$). The RG idea is to fix some $L \gg 1$ and to prove first a "discrete time" version of Eq.(4.4), namely

$$\|R_{L^n,1} g - A^*\psi^*\|_{L^n} \;=\; \mathcal{O}\left((L^n)^{2\delta-1}\right) \;, \quad n \to \infty \;. \tag{4.5}$$

In order to do that, one applies the "semigroup formula"

$$R_{L^n,1} \;=\; R_{L,L^{n-1}} \circ R_{L,L^{n-2}} \circ \ldots \circ R_{L,L} \circ R_{L,1} \;, \tag{4.6}$$

which is easily verified using the definition of $R_{L,\Lambda}$. According to Eq.(4.6), $R_{L^n,1}$ can be obtained by simply *iterating the RG map* $R_{L,\Lambda}$ for a sequence of values of the scaling parameter Λ. Now, computing $R_{L,\Lambda}$ requires solving Eq.(4.1) on a *finite* interval of time, and this is easily done (for sufficiently small initial data) using the Contraction Mapping Principle. In this way, one can control $R_{L^n,1}$ for all $n \in \mathbf{N}$ and prove Eq.(4.5) for small $g \in X_1$. The proof of Eq.(4.4) is then completed by a simple interpolation argument.

4.1. The linear RG map

Fix $\Lambda \geq 1$, and set $H_\Lambda = -\partial_x^2 - \Lambda\gamma(\Lambda x)\partial_x$. For any $L \geq 1$, we define the *linear RG map* $R^0_{L,\Lambda} : X_\Lambda \to X_{L\Lambda}$ by

$$R^0_{L,\Lambda} = D_L \, e^{-H_\Lambda(L^2-1)} , \tag{4.7}$$

where D_L is the dilatation operator (3.7). By construction, $R^0_{L,\Lambda}$ is linear and coincides with $R_{L,\Lambda}$ if we set $\mathcal{M}_\Lambda = 0$ in Eq.(4.1). For $L = \infty$, we define the rank one operator $R^0_{\infty,\Lambda} = \psi^* \Pi_\Lambda$, where ψ^* is given by Eq.(1.6) and $\Pi_\Lambda : X_\Lambda \to \mathbf{R}$ is the linear form

$$\Pi_\Lambda(g) = \frac{1}{\Lambda^2} \int_{\mathbf{R}} p^2(\Lambda y) g(y) \, dy , \tag{4.8}$$

with p as in Eq.(3.4). In this subsection, we study the properties of $R^0_{L,\Lambda}$, $R^0_{\infty,\Lambda}$ in a series of lemmas. The first one is a uniform bound on $R^0_{\infty,\Lambda}$:

Lemma 4.1. *$R^0_{\infty,\Lambda} \in \mathcal{L}(X_\Lambda, X_{\Lambda'})$ for all $\Lambda, \Lambda' \geq 1$, and there exists a $C_0 > 0$ such that $\|R^0_{\infty,\Lambda}\|_{\mathcal{L}(X_\Lambda, X_{\Lambda'})} \leq C_0$ uniformly in Λ, Λ'.*

Proof. From Eqs.(3.4), (1.11), it is easy to verify that $\Pi_\Lambda \in \mathcal{L}(X_\Lambda, \mathbf{R})$ and that $\|\Pi_\Lambda\|_{\mathcal{L}(X_\Lambda,\mathbf{R})} \leq C_0$ uniformly in Λ. Similarly, $\psi^* \in X_{\Lambda'}$ and $\|\psi^*\|_{\Lambda'} \leq \|\psi^*\|_1 \leq C_0$ for all $\Lambda' \geq 1$. $\square$

The next lemma is essentially a reformulation of Proposition 3.1:

Lemma 4.2. *$R^0_{L,\Lambda} \in \mathcal{L}(X_\Lambda, X_{L\Lambda})$ for all $L, \Lambda \geq 1$, and there exists a $C_1 > 0$ such that*

$$\|R^0_{L,\Lambda} - R^0_{\infty,\Lambda}\|_{\mathcal{L}(X_\Lambda, X_{L\Lambda})} \leq \frac{C_1}{L} , \tag{4.9}$$

for all $L \geq \sqrt{2}$ and all $\Lambda \geq 1$.

Remark. Without loss of generality, we shall assume that $C_1 > 1$.

Proof. For the purpose of the proof, we set $\mathcal{R}^0_{L,\Lambda} = D_L e^{-H_\Lambda L^2}$ and $\mathcal{R}^0_{\infty,\Lambda} = R^0_{\infty,\Lambda}$. From Eq.(3.6) with $t = L^2$, we have

$$\|\mathcal{R}^0_{L,1} - \mathcal{R}^0_{\infty,1}\|_{\mathcal{L}(X_1,X_L)} \leq \frac{C}{L} , \tag{4.10}$$

for some $C > 0$ and all $L \geq 1$. Now, since by construction $e^{-H_\Lambda L^2} = D_\Lambda e^{-H_1 (L\Lambda)^2} D_\Lambda^{-1}$, we see that

$$\mathcal{R}^0_{L,\Lambda} = \mathcal{R}^0_{L\Lambda,1}\, D_\Lambda^{-1} . \tag{4.11}$$

Moreover, one readily verifies that $\Pi_\Lambda = \Pi_1 D_\Lambda^{-1}$, so that Eq.(4.11) remains true for $L = \infty$. Finally, using Eqs.(1.11), (3.7), it is not difficult to show that

$$\|D_\Lambda\|_{\mathcal{L}(X_L,X_{L\Lambda})} \leq \Lambda^3 , \quad \|D_\Lambda^{-1}\|_{\mathcal{L}(X_{L\Lambda},X_L)} \leq \Lambda , \tag{4.12}$$

for all $L, \Lambda \geq 1$. Combining Eqs.(4.10), (4.11), (4.12), we obtain

$$\begin{aligned} \|\mathcal{R}^0_{L,\Lambda} - \mathcal{R}^0_{\infty,\Lambda}\|_{\mathcal{L}(X_\Lambda,X_{L\Lambda})} &\leq \|\mathcal{R}^0_{L\Lambda,1} - \mathcal{R}^0_{\infty,1}\|_{\mathcal{L}(X_1,X_{L\Lambda})} \|D_\Lambda^{-1}\|_{\mathcal{L}(X_\Lambda,X_1)} \\ &\leq \frac{C}{L\Lambda}\Lambda = \frac{C}{L} . \end{aligned} \tag{4.13}$$

This is nearly Eq.(4.9), except that $\mathcal{R}^0_{L,\Lambda}$ has to be replaced by $R^0_{L,\Lambda}$. But assuming $L^2 \geq 2$ and setting $\ell = \sqrt{L^2 - 1}$, we have $R^0_{L,\Lambda} = D_{L/\ell}\mathcal{R}^0_{\ell,\Lambda}$, and thus

$$R^0_{L,\Lambda} - R^0_{\infty,\Lambda} = D_{L/\ell}\left(\mathcal{R}^0_{\ell,\Lambda} - \mathcal{R}^0_{\infty,\Lambda}\right) + \left(D_{L/\ell}\psi^* - \psi^*\right)\Pi_\Lambda . \tag{4.14}$$

In view of Eqs.(4.12), (4.13), the first term in the right-hand side is bounded by $(L/\ell)^3 C/\ell$ in $\mathcal{L}(X_\Lambda, X_{L\Lambda})$. Moreover, it is easy to verify that $\|D_{L/\ell}\psi^* - \psi^*\|_{L\Lambda} \leq \|D_{L/\ell}\psi^* - \psi^*\|_1 = \mathcal{O}(L^{-2})$ as $L \to \infty$, and we already know that $\|\Pi_\Lambda\|_{\mathcal{L}(X_\Lambda,\mathbf{R})} \leq C_0$ for all $\Lambda \geq 1$. Thus, the right-hand side of Eq.(4.14) is $\mathcal{O}(L^{-1})$ uniformly in Λ, and this proves Eq.(4.9). □

The next result shows that ψ^* is an asymptotic fixed point for $R^0_{L,\Lambda}$:

Lemma 4.3. *There exists a $C_2 > 0$ such that, for all $L \geq \sqrt{2}$ and all $\Lambda \geq 1$,*

$$|\Pi_\Lambda(\psi^*) - 1| \leq \frac{C_2}{\Lambda} , \quad \|R^0_{L,\Lambda}\psi^* - R^0_{\infty,\Lambda}\psi^*\|_{L\Lambda} \leq \frac{C_2}{L\Lambda} . \tag{4.15}$$

Proof. The first part of Eq.(4.15) is proved by direct calculation, using Eqs.(1.6), (3.4). In fact, since $\Lambda^{-2}p^2(\Lambda x) \to x^2\theta(x)$ as $\Lambda \to \infty$ (θ is the Heaviside step function) and since $\int_{\mathbf{R}} x^2\theta(x)\psi^*(x)\,dx = 1$, it follows from the Lebesgue dominated convergence theorem that $\Pi_\Lambda(\psi^*) \to 1$ as $\Lambda \to \infty$. A little more work shows that $\Pi_\Lambda(\psi^*) - 1 = \mathcal{O}(\Lambda^{-1})$.

For the second part of Eq.(4.15), we fix L, Λ and note that, in view of Eq.(4.9), we have

$$R^0_{L\Lambda,1}\psi^* \;=\; R^0_{\infty,1}\psi^* + \rho_1 \;=\; \psi^*\Pi_1(\psi^*) + \rho_1 \ , \tag{4.16}$$

where $\rho_1 \in X_{L\Lambda}$ and $\|\rho_1\|_{L\Lambda} \le \|\psi^*\|_1 C_1(L\Lambda)^{-1} \le C_0C_1(L\Lambda)^{-1}$. On the other hand, since $R^0_{L\Lambda,1}\psi^* = R^0_{L,\Lambda}R^0_{\Lambda,1}\psi^*$, we also have

$$R^0_{L\Lambda,1}\psi^* \;=\; R^0_{\infty,\Lambda}R^0_{\Lambda,1}\psi^* + \left(R^0_{L,\Lambda} - R^0_{\infty,\Lambda}\right)\left(\psi^*\Pi_1(\psi^*) + \rho_2\right) \ , \tag{4.17}$$

where $\rho_2 \in X_\Lambda$ and $\|\rho_2\|_\Lambda \le C_0C_1\Lambda^{-1}$.

Now, we use the identity

$$\Pi_{L\Lambda} \circ R^0_{L,\Lambda} \;=\; \Pi_\Lambda \ , \tag{4.18}$$

which holds for all $L, \Lambda \ge 1$. Indeed, a short calculation shows that if $w(\cdot,t) \in X_\Lambda$ is a solution of the *linearized* equation Eq.(4.1) (i.e., with $\mathcal{M}_\Lambda = 0$), then $\Pi_\Lambda(w)$ is a conserved quantity; in other words, $\Pi_\Lambda e^{-H_\Lambda t} = \Pi_\Lambda$ for all $t \ge 0$. On the other hand, it is obvious that $\Pi_{L\Lambda}D_L \equiv \Pi_\Lambda$. Thus, $\Pi_{L\Lambda}D_L e^{-H_\Lambda(L^2-1)} = \Pi_\Lambda e^{-H_\Lambda(L^2-1)} = \Pi_\Lambda$.

According to Eq.(4.18), the first term in the right-hand side of Eq.(4.17) is nothing but $\psi^*\Pi_1(\psi^*)$. Thus, combining Eqs.(4.16), (4.17), we obtain

$$\left(R^0_{L,\Lambda} - R^0_{\infty,\Lambda}\right)\psi^* \;=\; \frac{1}{\Pi_1(\psi^*)}\left(\rho_1 - (R^0_{L,\Lambda} - R^0_{\infty,\Lambda})\rho_2\right) \ .$$

Now, using Eq.(4.9) and the definitions of ρ_1, ρ_2, we see that the right-hand side is bounded by const.$(L\Lambda)^{-1}$ in $X_{L\Lambda}$. This concludes the proof. $\square$

Finally, the last result is a uniform bound for the semigroup $e^{-H_\Lambda t}$ in $\mathcal{L}(X_\Lambda)$:

Lemma 4.4. *There exists a $C_3 > 0$ such that, for all $L, \Lambda \ge 1$,*

$$\sup_{t\in[1,L^2]} \|e^{-H_\Lambda(t-1)}\|_{\mathcal{L}(X_\Lambda)} \;\le\; C_3 L \ . \tag{4.19}$$

Remark. Eq.(4.19) implies that Eqs.(4.9), (4.15) hold in fact for all $L \geq 1$.

Proof. From Eq.(4.7), we have

$$e^{-H_\Lambda(L^2-1)} = D_L^{-1}\left(R^0_{L,\Lambda} - R^0_{\infty,\Lambda}\right) + D_L^{-1} R^0_{\infty,\Lambda} .$$

If $L \geq \sqrt{2}$, we see from Eqs.(4.9), (4.12) and Lemma 4.1 that the first term in the right-hand side is bounded by $L(C_1/L)$ in $\mathcal{L}(X_\Lambda)$ and the second one by LC_0. Thus, to prove Eq.(4.19), it remains only to show that

$$\sup_{t\in[0,1]} \|e^{-H_\Lambda t}\|_{\mathcal{L}(X_\Lambda)} \leq C_3 , \tag{4.20}$$

uniformly in $\Lambda \geq 1$.

To prove Eq.(4.20), we use a roundabout argument. First, we note that the assertion is *translation invariant*, in the sense that a translation of the origin only affects the value of C_3, but not the uniformity in Λ. So, if f is the critical front and F the nonlinearity in Eq.(1.1), we can assume (for the purpose of the proof) that $F'(f(0)) < 0$, since $f(x) \to 1$ as $x \to -\infty$ and $F'(1) < 0$. We set $1/b = \frac{1}{2}\sqrt{c^{*2} - 4F'(f(0))}$ and $\beta' = \beta + 1/b - 1/a \in (0, 1/b)$.

Next, we define the weight function

$$\sigma(x) = \exp\left(\frac{1}{2}\int_0^x (\gamma_b(y) - \gamma(y))\,dy\right) ,$$

where γ_b is given by Eq.(3.2) with a replaced by b. As is easily seen, $\sigma(x) \approx$ const. as $x \to \infty$ and $\sigma(x) \approx \text{const.}\exp(x/b - x/a)$ as $x \to -\infty$. As a consequence, there exists a $C > 0$ such that for all $x \in \mathbf{R}$

$$\frac{1}{C} W_{\beta'\Lambda}(x) \leq W_{\beta\Lambda}(x)\sigma(\Lambda x) \leq C W_{\beta'\Lambda}(x) , \tag{4.21}$$

where W_β is defined in Eq.(1.10). On the other hand, we have the identity

$$e^{-H_\Lambda t}(x,y) = \frac{\sigma(\Lambda x)}{\sigma(\Lambda y)}\, e^{-\hat{H}_\Lambda t}(x,y) , \tag{4.22}$$

where $\hat{H}_\Lambda = -\partial_x^2 - \Lambda\gamma_b(\Lambda x)\partial_x + \Lambda^2 V(\Lambda x)$ and

$$\begin{aligned} V(x) &= \tfrac{1}{2}\gamma'(x) + \tfrac{1}{4}\gamma^2(x) - \tfrac{1}{2}\gamma_b'(x) - \tfrac{1}{4}\gamma_b^2(x) \\ &= \tfrac{1}{4}c^{*2} - F'(f(x)) - \tfrac{1}{2}\gamma_b'(x) - \tfrac{1}{4}\gamma_b^2(x) . \end{aligned} \tag{4.23}$$

Indeed, Eq.(4.22) is obtained by commuting H_Λ with $\sigma(\Lambda x)$, and the last equality in Eq.(4.23) is easily verified using Eq.(1.4) and the differential equation satisfied by f.

Now, by construction, $V(x) \geq 0$ for all $x \in \mathbf{R}$. Indeed, if $x > 0$, then $\frac{1}{2}\gamma_b' + \frac{1}{4}\gamma_b^2 \equiv 0$, and $\frac{1}{4}c^{*2} = F'(0) \geq F'(f(x))$. If $x < 0$, then $\gamma_b' = 0$ and $\frac{1}{4}c^{*2} - \frac{1}{4}\gamma_b^2 = F'(f(0)) \geq F'(f(x))$. As a consequence, we have for all $x, y \in \mathbf{R}$

$$e^{-\hat{H}_\Lambda t}(x,y) \leq e^{-H^0_{b/\Lambda} t}(x,y) \ , \tag{4.24}$$

where $H^0_{b/\Lambda} = -\partial_x^2 - \Lambda\gamma_b(\Lambda x)\partial_x \equiv -\partial_x^2 - \gamma_{b/\Lambda}(x)\partial_x$. Combining Eqs.(4.21), (4.22), (4.24), we thus obtain

$$0 \leq \frac{W_{\beta\Lambda}(x)}{W_{\beta\Lambda}(y)}\, e^{-H_\Lambda t}(x,y) \leq C^2 \frac{W_{\beta'\Lambda}(x)}{W_{\beta'\Lambda}(y)}\, e^{-H^0_{b/\Lambda} t}(x,y) \ , \tag{4.25}$$

for all $x, y \in \mathbf{R}$ and all $t \in [0,1]$.

To complete the argument, we have to show that the right-hand side of Eq.(4.25) is a bounded operator in $\mathrm{L}^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ uniformly in $\Lambda \geq 1$ and $t \in [0,1]$. Equivalently, we have to show that the resolvent $(H^0_{b/\Lambda} - k^2)^{-1}$ is a bounded operator in the space X'_Λ defined by the weight $W_{\beta'\Lambda}$ as in Eq.(1.11), and that $\|(H^0_{b/\Lambda} - k^2)^{-1}\|_{\mathcal{L}(X'_\Lambda)} \leq$ const.$\mathrm{Im}\,(k)^{-2}$ as $\mathrm{Im}\,(k) \to \infty$, uniformly in Λ. But $(H^0_{b/\Lambda} - k^2)^{-1}$ is explicitly known: it is given by the kernel (A.2) with a replaced by b/Λ. Thus the desired property can be verified by direct calculations. □

4.2. The non-linear RG map

From now on, L will be a large, fixed number and Λ will be equal to L^n for some $n \in \mathbf{N}$. We shall assume that L has been chosen large enough so that

$$L^{2\delta} \geq C_1 \ , \tag{4.26}$$

where δ is as in Theorem 1.1 and C_1 as in Lemma 4.2.

To prove the existence of the RG map $R_{L,\Lambda}$, we rewrite Eq.(4.1) as an integral equation

$$\begin{aligned} w(t) &= e^{-H_\Lambda (t-1)} g + \frac{1}{\Lambda}\int_0^{t-1} e^{-H_\Lambda s}\,(\mathcal{M}_\Lambda w)\,(t-s)\,ds \\ &\equiv \quad w_0(t) \quad + \frac{1}{\Lambda}\,(N_\Lambda w)\,(t) \ , \end{aligned} \tag{4.27}$$

where $t \in [1, L^2]$, $w(t) = w(\cdot, t)$ and $g = w(0) \in X_\Lambda$. We look for solutions of Eq.(4.27) in the Banach space $Z_\Lambda = C_b^0((1, L^2], X_\Lambda)$ (the space of all bounded, continuous functions from $(1, L^2]$ to X_Λ) equipped with the norm

$$\|w\|_{Z_\Lambda} = \sup_{t \in (1, L^2]} \|w(t)\|_\Lambda \ .$$

In view of Eq.(4.19), $w_0 \in Z_\Lambda$ for all $g \in X_\Lambda$ and $\|w_0\|_{Z_\Lambda} \le C_3 L \|g\|_\Lambda$. Thus, denoting by B_ϵ the ball of center w_0 and radius $C_3 L\epsilon$ in Z_Λ, we have the following result:

Lemma 4.5. *There exists a constant $C_4 > 0$ (independent of L, Λ) such that, if $\epsilon > 0$ verifies*

$$2C_3 L\epsilon \le 1 \ , \quad 4C_3^2 C_4 L^4 \epsilon \le 1 \ , \tag{4.28}$$

and if $\|g\|_\Lambda \le \epsilon$, then the right-hand side of Eq.(4.27) is a contraction in B_ϵ, and thus has a unique fixed point in B_ϵ.

Remark. Strictly speaking, this lemma proves the local existence and uniqueness of solutions only for Eq.(4.27), and not for Eq.(4.1). Nevertheless, using the smoothing properties of the linear semigroup $e^{-H_\Lambda t}$, it is not difficult to show that a solution $w(x, t)$ of Eq.(4.27) does satisfy Eq.(4.1) for $t \in (1, L^2]$.

Proof. Let $h \in X_\Lambda$. According to Eqs.(1.5), (4.2), $(\mathcal{M}_\Lambda h)(x)$ is a finite sum (or a convergent series) of terms of the form

$$(T_{k,\ell} h)(x) = \Lambda^3 h(x) f(\Lambda x)^k \left(f'(\Lambda x)\Lambda^{-3} h(x)\right)^\ell \ ,$$

for some integers $k \ge 0$, $\ell \ge 1$. Since $|f(x)| \le 1$ and $\Lambda \ge 1$, it follows that

$$\left|(T_{k,\ell} h)(x) - (T_{k,\ell}\tilde{h})(x)\right| \le \left|f'(\Lambda x)\right|^\ell \left(\sum_{i=0}^{\ell} |h(x)|^i |\tilde{h}(x)|^{\ell-i}\right) |h(x) - \tilde{h}(x)| \ ,$$

for all $h, \tilde{h} \in X_\Lambda$. Thus, assuming that $\|h\|_\Lambda, \|\tilde{h}\|_\Lambda \le \eta \le 1$ and using the bound

$$\sup_{x \in \mathbf{R}} \frac{|f'(\Lambda x)|}{W_{\beta\Lambda}(x)} \le C \ ,$$

for some $C > 0$, we obtain

$$\|T_{k,\ell} h - T_{k,\ell}\tilde{h}\|_\Lambda \le C^\ell \ell \eta^\ell \|h - \tilde{h}\|_\Lambda \ .$$

Hence, summing over k, ℓ, we see that

$$\|\mathcal{M}_\Lambda h - \mathcal{M}_\Lambda \tilde{h}\|_\Lambda \;\le\; C_4 \eta \|h - \tilde{h}\|_\Lambda \ ,$$

for some $C_4 > 0$. Finally, combining this result with Eqs.(4.19), (4.27), we find

$$\|N_\Lambda w - N_\Lambda \tilde{w}\|_{Z_\Lambda} \;\le\; C_3 C_4 L^3 \eta \|w - \tilde{w}\|_{Z_\Lambda} \ , \tag{4.29}$$

for all $w, \tilde{w} \in Z_\Lambda$ with $\|w\|_{Z_\Lambda}, \|\tilde{w}\|_{Z_\Lambda} \le \eta \le 1$.

Now, if $\epsilon > 0$ verifies Eq.(4.28) and if $\|g\|_\Lambda \le \epsilon$, then every $w \in B_\epsilon$ verifies $\|w\|_{Z_\Lambda} \le 2C_3 L\epsilon \le 1$. Thus, using Eq.(4.29) and recalling that $\Lambda \ge 1$, it is straightforward to verify that the right-hand side of Eq.(4.27) maps B_ϵ into itself and is a contraction there. $\square$

According to Lemma 4.5, the RG map (4.3) is well defined for all $g \in X_\Lambda$ such that $\|g\|_\Lambda \le \epsilon$. In view of Eqs.(4.27), (4.7) we have

$$R_{L,\Lambda} g \;=\; D_L e^{-H_\Lambda(L^2-1)} g + \frac{1}{\Lambda}\, D_L(N_\Lambda w_g)(L^2) \;\equiv\; R^0_{L,\Lambda} g + \Delta_{L,\Lambda} g \ , \tag{4.30}$$

where $\Delta_{L,\Lambda} : X_\Lambda \to X_{L\Lambda}$ is defined on the same domain as $R_{L,\Lambda}$ and verifies

$$\|\Delta_{L,\Lambda} g\|_{L\Lambda} \;\le\; \frac{C_5(L)}{\Lambda}\, \|g\|^2_\Lambda \ , \tag{4.31}$$

with $C_5(L) = L^3 (C_3 C_4 L^3)(2C_3 L)^2$, see Eqs.(4.12), (4.29).

4.3. Iteration of the RG map

Using the preceding estimates, we now prove Eq.(4.5) for sufficiently small $g \in X_1$. Assuming that $L \gg 1$ verifies Eq.(4.26), we define $g_n \in X_{L^n}$ inductively by

$$g_{n+1} \;=\; R_{L,L^n} g_n \ , \quad g_0 = g \ , \tag{4.32}$$

so that $g_n = R_{L^n,1} g$, see Eq.(4.6). We also define $A_n, \pi_n \in \mathbf{R}$ and $\rho_n \in X_{L^n}$ by

$$A_n \;=\; \Pi_{L^n}(g_n) \ , \quad \pi_n^{-1} \;=\; \Pi_{L^n}(\psi^*) \ , \quad \rho_n = g_n - \pi_n A_n \psi^* \ , \tag{4.33}$$

so that $g_n = \pi_n A_n \psi^* + \rho_n$ and $\Pi_{L^n}(\rho_n) = 0$ for all $n \in \mathbf{N}$. Using Eq.(4.32) we shall derive recursion relations for A_n, ρ_n and show that $A_n \to A^*$ and $\rho_n \to 0$ as $n \to \infty$. To simplify the notations, we write $|g_n|, |\rho_n|$ instead of $\|g_n\|_{L^n}, \|\rho_n\|_{L^n}$.

First, using Eqs.(4.30), (4.18) with $\Lambda = L^n$, we find

$$\begin{aligned} A_{n+1} &= \Pi_{L^{n+1}}(g_{n+1}) = \Pi_{L^{n+1}}(R^0_{L,L^n} g_n) + \Pi_{L^{n+1}}(\Delta_{L,L^n} g_n) \\ &= \Pi_{L^n}(g_n) + \Pi_{L^{n+1}}(\Delta_{L,L^n} g_n) = A_n + \Pi_{L^{n+1}}(\Delta_{L,L^n} g_n) , \end{aligned}$$

so that

$$|A_{n+1} - A_n| \le \frac{C_0 C_5}{L^n} |g_n|^2 , \tag{4.34}$$

in view of Eq.(4.31). Similarly, we have

$$\begin{aligned} \rho_{n+1} &= g_{n+1} - \pi_{n+1} A_{n+1} \psi^* = R^0_{L,L^n} g_n + \Delta_{L,L^n} g_n - \pi_{n+1} A_{n+1} \psi^* \\ &= R^0_{L,L^n} \rho_n + \Delta_{L,L^n} g_n + \pi_n A_n R^0_{L,L^n} \psi^* - \pi_{n+1} A_{n+1} \psi^* \\ &= R^0_{L,L^n} \rho_n + \Delta_{L,L^n} g_n + \pi_n A_n (R^0_{L,L^n} - R^0_{\infty,L^n}) \psi^* \\ &\quad + A_n \psi^* (1 - \pi_{n+1}) + \pi_{n+1} \psi^* (A_n - A_{n+1}) . \end{aligned}$$

Recalling that $\Pi_{L^n}(\rho_n) = 0$ and using Eqs.(4.9), (4.15), (4.31), we thus obtain

$$|\rho_{n+1}| \le \frac{C_1}{L} |\rho_n| + \frac{C_5}{L^n} |g_n|^2 + \frac{2C_0 C_2}{L^{n+1}} |A_n| + C_0 |A_{n+1} - A_n| . \tag{4.35}$$

Finally, defining $\kappa = C_1/L \le L^{2\delta - 1} < 1$ (see Eq.(4.26)) and using the inequality $|g_n| \le C_0|A_n| + |\rho_n|$, we rewrite the recursion relations (4.34), (4.35) as

$$\begin{aligned} |A_{n+1} - A_n| &\le \frac{C_6}{L^n} (|A_n| + |\rho_n|)^2 , \\ |\rho_{n+1}| &\le \kappa |\rho_n| + \frac{C_6}{L^n} |A_n| + \frac{C_6}{L^n} (|A_n| + |\rho_n|)^2 , \end{aligned} \tag{4.36}$$

for some $C_6 = C_6(L) > 0$. It is now easy to prove the convergence of A_n, ρ_n:

Lemma 4.6. *Assume that A_n, ρ_n verify the recursion relations (4.36), and that $\eta \equiv |A_0| + |\rho_0| \le K^{-1}$, where*

$$K = \prod_{n=0}^{\infty} \left(1 + \frac{3C_6}{L^n}\right) < \infty .$$

Then $|A_n| + |\rho_n| \le K\eta \le 1$ *for all* $n \in \mathbf{N}$, *and there exist* $A^* \in \mathbf{R}$ *and* $C_7 = C_7(L) > 0$ *such that*

$$|A_n - A^*| \le \frac{C_7 \eta^2}{L^n} , \quad |\rho_n| \le C_7 \eta \kappa^n , \quad n \in \mathbf{N} . \tag{4.37}$$

Proof. Set $B_n = |A_n| + |\rho_n|$. Summing the two relations (4.36), we obtain

$$B_{n+1} \le \left(1 + \frac{C_6}{L^n}\right) B_n + \frac{2C_6}{L^n} B_n^2 , \quad n \in \mathbf{N} .$$

Thus, if $\eta \equiv B_0 \le K^{-1}$, then obviously $B_n \le K\eta \le 1$ for all $n \in \mathbf{N}$. Inserting this bound in Eq.(4.36), we find

$$|A_{n+1} - A_n| \le \frac{C_6}{L^n} K^2 \eta^2 , \quad |\rho_{n+1}| \le \kappa |\rho_n| + \frac{2C_6}{L^n} K\eta , \quad n \in \mathbf{N} .$$

In particular, $\{A_n\}$ is a Cauchy sequence and thus converges to some $A^* \in \mathbf{R}$. Recalling that $\kappa L = C_1 > 1$, it is now easy to show that Eq.(4.37) holds with, e.g., $C_7 = 1 + 2C_6 K^2 C_1/(C_1 - 1)$. $\square$

According to this lemma, if $g \in X_1$ is sufficiently small and if $g_n = R_{L^n,1} g$, then

$$\|g_n - A^* \psi^*\|_{L^n} \le |\pi_n A_n - A^*| \|\psi^*\|_{L^n} + \|\rho_n\|_{L^n} = \mathcal{O}\left((L^n)^{2\delta - 1}\right) ,$$

as $n \to \infty$, by Eqs.(4.15), (4.37). In other words, we have proved:

Proposition 4.7. *Assume that* L *verifies Eq.(4.26) with* δ *as in Theorem 1.1. Then there exists an* $\epsilon > 0$ *such that, if* $g \in X_1$ *verifies* $\|g\|_1 \le \epsilon$, *then Eq.(1.3) has a unique solution* $v(x,t)$ *such that* $v(\cdot,t) \in X_1$ *for all* $t \ge 1$ *and* $v(x,1) = g(x)$. *Moreover,* $R_{L^n,1} g \equiv L^n v(L^n \cdot, L^{2n}) \in X_{L^n}$ *for all* $n \in \mathbf{N}$, *and there exists an* $A^* \in \mathbf{R}$ *such that Eq.(4.5) holds.*

This proves Eq.(1.12) for the sequence of times $t_n = L^{2n}$, $n \in \mathbf{N}$. To conclude the proof of Theorem 1.1, we take L, g as in Proposition 4.7 and write any $t \ge 1$ as $(\tau L^n)^2$ for some $n \in \mathbf{N}$ and some $\tau \in [1, L)$. Using the same notations as above, we have

$$\begin{aligned} R_{\tau L^n,1} g &= R_{\tau,L^n} R_{L^n,1} g = R_{\tau,L^n} g_n \\ &= R^0_{\infty,L^n} g_n + (R^0_{\tau,L^n} - R^0_{\infty,L^n})(\pi_n A_n \psi^* + \rho_n) + \Delta_{\tau,L^n} g_n . \end{aligned}$$

Recalling that $R^0_{\infty,L^n}g_n = \psi^*\Pi_{L^n}(g_n) = \psi^* A_n$ and using Eqs.(4.37), (4.15), (4.9), (4.31), we easily find

$$\|R_{\tau L^n,1}g - A^*\psi^*\|_{\tau L^n} \leq \frac{C_0C_7}{L^n} + \frac{C_0C_2}{\tau L^n} + \frac{C_1C_7\kappa^n}{\tau} + \frac{C_5}{L^n} \leq \frac{C_8(L)}{(\tau L^n)^{1-2\delta}} ,$$

uniformly in $n \in \mathbf{N}$ and $\tau \in [1, L)$. This concludes the proof of Theorem 1.1. □

Appendix A. Integral kernels of G_k, $\hat{\gamma}\partial G_k$ and S_k^ϵ

Integral kernel of G_k. Let H_0 be the differential operator (3.8), and let $G_k = (H_0 - k^2)^{-1}$. Then the kernel $G_k(x,y)$ verifies the differential equation

$$(\partial_x^2 + \gamma_a(x)\partial_x + k^2)G_k(x,y) = -\delta(x-y) , \tag{A.1}$$

where $\gamma_a(x)$ is given by (3.2). For $x > 0$, $x \neq y$, this is a Bessel-type equation whose two solutions are $(x+a)^{-1}e^{\pm ikx}$. For $x < 0$, $x \neq y$, the solutions are simply $e^{(\pm\omega(k)-1/a)x}$, with $\omega(k) = \sqrt{1/a^2 - k^2}$. Thus, solving (A.1) for $x \neq 0, y$ and matching the solution at these points, we obtain

$$G_k(x,y) = \begin{cases} \dfrac{y+a}{x+a}\dfrac{i}{2k}\left\{e^{ik|x-y|} - \dfrac{\omega(k)+ik}{\omega(k)-ik}\,e^{ik(x+y)}\right\} & x>0,\ y>0 \\ \dfrac{a}{x+a}\,\dfrac{1}{\omega(k)-ik}\,e^{ikx}e^{(\omega(k)+1/a)y} & x>0,\ y<0 \\ \dfrac{y+a}{a}\,\dfrac{1}{\omega(k)-ik}\,e^{iky}e^{(\omega(k)-1/a)x} & x<0,\ y>0 \\ e^{(y-x)/a}\dfrac{1}{2\omega(k)}\left\{e^{-\omega(k)|x-y|} + \dfrac{\omega(k)+ik}{\omega(k)-ik}\,e^{\omega(k)(x+y)}\right\} & x<0,\ y<0 \end{cases} \tag{A.2}$$

To bound the various terms in $G_k(x,y)$, the following (elementary) properties of $\omega(k)$ are useful:

Lemma A.1. *For all* $k \in \mathbf{C}_+$, *we have* $|\omega(k)| \geq \mathrm{Re}\,(\omega(k)) \geq \mathrm{Im}\,(k)$, *and*

$$|\omega(k) - 1/a| \leq |k| , \quad |\omega(k) - ik| \geq \max(1/a, |k|) , \quad \left|\frac{\omega(k)+ik}{\omega(k)-ik}\right| \leq 1 .$$

If moreover $k \in \Omega$ (see Fig. 2), then $\mathrm{Re}\,(\omega(k)) \geq \sin(2\theta)/a$.

In Section 3, we use several properties of G_k which can all be obtained from the kernel (A.2) by straightforward calculations. For example, to show that $G_k \in \mathcal{L}(X_1)$ for all $k \in \mathcal{W}$, we use the bound

$$\|G_k\|_{\mathcal{L}(X_1)} \leq \left(\sup_{x \in \mathbf{R}} \int_{\mathbf{R}} dy + \sup_{y \in \mathbf{R}} \int_{\mathbf{R}} dx \right) \left(\frac{W_\beta(x)}{W_\beta(y)} \, |G_k(x,y)| \right) < \infty \, .$$

Integral kernel of $\hat{\gamma}\partial G_k$. The kernel of $\hat{\gamma}\partial G_k$ is obtained by differentiating (A.2) with respect to x and multiplying each term by $\hat{\gamma}(x)$. Denoting by (1), (2), (3), (4) the four cases as in (A.2), we find

$$(\hat{\gamma}\partial G_k)(x,y) = \begin{cases} \hat{\gamma}(x) \left(ik - \dfrac{1}{x+a} \right) G_k(x,y) + \hat{\gamma}(x)\,\theta(y-x) \dfrac{y+a}{x+a} \, e^{ik(y-x)} & (1) \\ \hat{\gamma}(x) \left(ik - \dfrac{1}{x+a} \right) G_k(x,y) & (2) \\ \hat{\gamma}(x) \left(\omega(k) - \dfrac{1}{a} \right) G_k(x,y) & (3) \\ \hat{\gamma}(x) \left(\omega(k) - \dfrac{1}{a} \right) G_k(x,y) - \hat{\gamma}(x)\theta(x-y) e^{(y-x)/a} e^{-\omega(k)(y-x)} & (4) \end{cases} \tag{A.3}$$

where $\theta : \mathbf{R} \to [0,1]$ is the Heaviside step function. In particular, letting $k \to 0$, we obtain the simple expression

$$(\hat{\gamma}\partial G_0)(x,y) = -\hat{\gamma}(x)\,\theta(x-y)\,\frac{p_a^2(y)}{p_a^2(x)} \, , \tag{A.4}$$

where p_a is defined in (3.4).

Integral kernel of S_k^ϵ. Let $S_k^\epsilon = -2ik\epsilon^2 D_{1/\epsilon} G_{\epsilon k}$, where D_L is the dilatation operator (3.7). Then

$$S_k^\epsilon(x,y) = \begin{cases} \dfrac{1}{\epsilon}\dfrac{y+a}{x+\epsilon a} \left\{ e^{ik|x-\epsilon y|} - \dfrac{\omega(\epsilon k)+i\epsilon k}{\omega(\epsilon k)-i\epsilon k} \, e^{ik(x+\epsilon y)} \right\} & (1) \\ \dfrac{a}{x+\epsilon a} \, \dfrac{-2ik}{\omega(\epsilon k)-i\epsilon k} \, e^{ikx} \, e^{(\omega(\epsilon k)+1/a)y} & (2) \\ \dfrac{y+a}{\epsilon a} \, \dfrac{-2ik}{\omega(\epsilon k)-i\epsilon k} \, e^{i\epsilon k y} \, e^{(\omega(\epsilon k)-1/a)x/\epsilon} & (3) \\ \dfrac{1}{\epsilon} \, e^{(y-x/\epsilon)/a} \, \dfrac{-2ik}{2\omega(\epsilon k)} \left\{ e^{-\omega(\epsilon k)|x/\epsilon - y|} + \dfrac{\omega(\epsilon k)+i\epsilon k}{\omega(\epsilon k)-i\epsilon k} \, e^{\omega(\epsilon k)(x/\epsilon+y)} \right\} & (4) \end{cases} \tag{A.5}$$

As is easily seen, $\lim_{\epsilon \to 0} S_k^\epsilon$ does not exist.

Appendix B. Formulas for $(\mathbf{1}-\gamma\partial G_0)^{-1}$ and $(\mathbf{1}-\hat{\gamma}\partial G_0)^{-1}$

Formula for $(\mathbf{1}-\gamma\partial G_0)^{-1}$ (Section 2). Let $\gamma\partial G_k \in \mathcal{L}(Y^1)$ be as in (2.9). For $h, g \in Y^1$, we want to solve $(\mathbf{1}-\gamma\partial G_0)h = g$. In view of (2.10), this amounts to solving the integral equation

$$h(x) + \tfrac{1}{2}\gamma(x)\int_{\mathbf{R}} \mathrm{sgn}(x-y)h(y)\,dy \;=\; g(x)\;,$$

which is just

$$\frac{1}{2p^2(x)}\left(p^2(x)\int_{\mathbf{R}} \mathrm{sgn}(x-y)h(y)\,dy\right)' \;=\; g(x)\;,$$

see (2.4). Multiplying both sides by $2p^2(x)$ and integrating, we obtain

$$p^2(x)\int_{\mathbf{R}} \mathrm{sgn}(x-y)h(y)\,dy \;=\; -C + 2\int_{-\infty}^{x} p^2(y)g(y)\,dy\;, \tag{B.1}$$

for some $C \in \mathbf{R}$. Moreover, evaluating (B.1) at $x = \pm\infty$ and using (2.4), we find

$$C \;=\; \frac{2}{1+\Gamma}\int_{\mathbf{R}} h(y)\,dy \;=\; \frac{2}{1+\Gamma}\int_{\mathbf{R}} p^2(y)g(y)\,dy\;. \tag{B.2}$$

Thus, inserting (B.2) in (B.1), we see that

$$\int_{\mathbf{R}} \mathrm{sgn}(x-y)h(y)\,dy \;=\; \frac{1}{p^2(x)}\int_{\mathbf{R}} \mathrm{sgn}_p(x-y)p^2(y)g(y)\,dy\;, \tag{B.3}$$

where sgn_p is defined in (2.14). Now, differentiating (B.3) with respect to x yields (2.13), and (2.15) follows immediately from (B.2).

Formula for $(\mathbf{1}-\hat{\gamma}\partial G_0)^{-1}$ (Section 3). Let $\hat{\gamma}\partial G_k \in \mathcal{L}(X_1^1)$ be as in (3.9). For $h, g \in X_1^1$, we want to solve $(\mathbf{1}-\hat{\gamma}\partial G_0)h = g$. In view of (A.4), this amounts to solving the integral equation

$$h(x) + \frac{\hat{\gamma}(x)}{p_a^2(x)}\int_{-\infty}^{x} p_a^2(y)h(y)\,dy \;=\; g(x)\;,$$

which is nothing but

$$\frac{1}{p^2(x)}\left(\frac{p^2(x)}{p_a^2(x)}\int_{-\infty}^{x} p_a^2(y)h(y)\,dy\right)' \;=\; g(x)\;,$$

see (3.4). Proceeding as above, we find

$$\frac{p^2(x)}{p_a^2(x)}\int_{-\infty}^{x} p_a^2(y)h(y)\,dy \;=\; \int_{-\infty}^{x} p^2(y)g(y)\,dy\;, \tag{B.4}$$

from where (3.12) follows immediately by taking $x \to \infty$. Now, multiplying both sides of (B.4) by $p_a^2(x)/p^2(x)$ and differentiating with respect to x, we obtain (3.11).

Appendix C. Integral kernel of $e^{-H_0 t}$

Let H_0 be the differential operator (3.8). In this section, we compute the kernel $K_t(x,y)$ of the semigroup $e^{-H_0 t}$ to leading order as $t \to \infty$, and use the result to prove Lemma 3.2. Our basic formula is, see (2.7)

$$K_t(x,y) = \frac{1}{i\pi} \int_{\mathcal{P}} k G_k(x,y) e^{-k^2 t} \, dk \ , \tag{C.1}$$

where $G_k(x,y)$ is given by (A.2) and the integration path $\mathcal{P}$ is as in Fig. 2. As far as we know, the integral in (C.1) cannot be carried out explicitly. For the sake of brevity, we just give here the main steps of the calculation, and leave the details to the reader.

When $x, y > 0$, we see from (A.2) that $G_k(x,y) = G_k^1(x,y) + G_k^2(x,y)$, with

$$G_k^1(x,y) = \frac{y+a}{x+a} \frac{i}{2k} \left(e^{ik|x-y|} - e^{ik(x+y)} \right) \ , \quad G_k^2(x,y) = \frac{y+a}{x+a} \frac{e^{ik(x+y)}}{\omega(k) - ik} \ .$$

For the first piece G_k^1, the integral in (C.1) can be performed explicitly and yields

$$K_t^1(x,y) = \frac{y+a}{x+a} \frac{1}{\sqrt{4\pi t}} \left(e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t} \right) \ . \tag{C.2}$$

For the second piece G_k^2, it is convenient to use the integration path defined by $k = i(x+y)/2t + zt^{-1/2}$, $-\infty < z < \infty$. Inserting in (C.1) and noting that $ik(x+y) - k^2 t = -(x+y)^2/4t - z^2$, we obtain

$$K_t^2(x,y) = \frac{y+a}{x+a} \frac{1}{\sqrt{4\pi t}} e^{-(x+y)^2/4t} \, \Phi\left(\frac{x+y}{2t}, 0, t \right) \ , \tag{C.3}$$

where

$$\Phi(\xi,\eta,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} \frac{-2ik}{\omega(k) - ik} e^{(1/a - \omega(k))\eta} e^{-z^2} \, dz \ , \quad k = i\xi + zt^{-1/2} \ .$$

We now expand $K_t = K_t^1 + K_t^2$ in powers of $t^{-1/2}$ using

$$\begin{aligned} e^{-(x-y)^2/4t} &= e^{-(x^2+y^2)/4t} \left\{ 1 + \frac{xy}{2t} + \frac{xy}{2t} S\left(\frac{xy}{2t} \right) \right\} \ , \\ e^{-(x+y)^2/4t} &= e^{-(x^2+y^2)/4t} \left\{ 1 - \frac{xy}{2t} + \mathcal{O}\left(\frac{xy}{2t} \right)^2 \right\} \ , \\ \Phi(\xi,\eta,t) &= 2a\xi + \frac{a^2}{t} - 2a^2\xi^2 + \mathcal{O}\left((\xi + t^{-1/2})^3 + \eta e^{\eta/a} Q(\xi + t^{-1/2}) \right) \ , \end{aligned} \tag{C.4}$$

where $S(z) = (e^z - 1 - z)/z$ and $Q(z) = z^3 + z^5$. Here and in the sequel, the order terms $\mathcal{O}(\cdot)$ hold *uniformly* in all variables $(x, y, t, \ldots)$. Substituting (C.4) into (C.2), (C.3), we obtain after straightforward manipulations

$$K_t(x,y) = \frac{(y+a)^2}{\sqrt{4\pi}\, t^{3/2}}\, e^{-(x^2+y^2)/4t} \left\{ 1 + S\left(\frac{xy}{2t}\right) + \mathcal{O}\left(P_1\left(\frac{xy}{2t}, \frac{x+y}{2t}, \frac{1}{\sqrt{t}} \right) \right) \right\} , \quad \text{(C.5)}$$

where $P_1(\xi, \eta, \zeta)$ is a polynomial in ξ, η, ζ without constant term (i.e., $P_1(0,0,0) = 0$).

When $x > 0$, $y < 0$, we use the integration path defined by $k = ix/2t + zt^{-1/2}$, $-\infty < z < \infty$. Proceeding as above, we obtain

$$K_t(x,y) = \frac{1}{\sqrt{4\pi t}}\, \frac{a}{x+a}\, e^{-x^2/4t}\, e^{2y/a}\, \Phi\left(\frac{x}{2t}, -y, t \right) .$$

Expanding Φ as in (C.4), we thus find

$$K_t(x,y) = \frac{a^2 e^{2y/a}}{\sqrt{4\pi}\, t^{3/2}}\, e^{-x^2/4t} \left\{ 1 + \mathcal{O}\left((1 + |y| e^{|y|/a}) P_2\left(\frac{x}{2t}, \frac{1}{\sqrt{t}} \right) \right) \right\} , \qquad \text{(C.6)}$$

where $P_2(\xi, \eta)$ is again a polynomial without constant term.

When $x < 0$, $y > 0$, we choose the integration path $\mathcal{P}$ as in Fig. 2. In particular, $\mathcal{P} \subset \Omega$ so that (by Lemma A.1) $\mathrm{Re}\,(\omega(k)) + \beta - 1/a \geq \Delta$ for all $k \in \mathcal{P}$, where $\Delta = \sin(2\theta)/a + \beta - 1/a > 0$. The various terms in $G_k(x,y)$ can be expanded as follows

$$\begin{aligned} \frac{1}{a(\omega(k) - ik)} &= 1 + aik + \mathcal{O}(|k|^2) , \\ e^{iky} &= 1 + iky + \mathcal{O}(|k|^2 y^2) , \\ e^{(\omega(k) - 1/a)x} &= 1 + \mathcal{O}\left(|k|^2 |x| e^{(\Delta - \beta)x} \right) . \end{aligned}$$

Substituting into (C.1) and using

$$\int_{\mathcal{P}} k^n e^{-k^2 t}\, dk = \begin{cases} 0 & \text{n odd} \\ t^{-(n+1)/2}\, \Gamma\left(\frac{n+1}{2}\right) & \text{n even} \end{cases}$$

we easily obtain

$$K_t(x,y) = \frac{(y+a)^2}{\sqrt{4\pi}\, t^{3/2}} \left\{ 1 + \mathcal{O}\left(e^{(\Delta - \beta)x} P_3\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, \frac{1}{\sqrt{t}} \right) \right) \right\} , \qquad \text{(C.7)}$$

where $P_3(\xi, \eta, \zeta)$ is a polynomial of degree 1 in ξ, η and without constant term.

Finally, when $x, y < 0$, $G_k(x,y)$ is again a sum of two terms

$$G_k^1(x,y) \;=\; \frac{e^{(y-x)/a}}{2\omega(k)} \left(e^{-\omega(k)|x-y|} - e^{\omega(k)(x+y)}\right) \;, \quad G_k^2(x,y) \;=\; \frac{e^{(y-x)/a}}{\omega(k)-ik}\, e^{\omega(k)(x+y)} \;.$$

For the first term, (C.1) can be performed explicitly and yields an exponentially small contribution to K_t:

$$K_t^1(x,y) \;=\; \frac{1}{\sqrt{4\pi t}}\, e^{-t/a^2}\, e^{(y-x)/a} \left(e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right) \;. \tag{C.8}$$

For the second term, we proceed as in the case $x < 0$, $y > 0$. The result is

$$K_t^2(x,y) \;=\; \frac{a^2 e^{2y/a}}{\sqrt{4\pi}\, t^{3/2}} \left\{1 + \mathcal{O}\left(e^{(\Delta-\beta)(x+y)} P_4\left(\frac{|x+y|}{\sqrt{t}}, \frac{1}{\sqrt{t}}\right)\right)\right\} \;, \tag{C.9}$$

where $P_4(\xi,\eta)$ is a polynomial of degree 1 in ξ and without constant term.

Now, for all $x, y \in \mathbf{R}$, we set

$$\mathcal{R}_t(x,y) \;=\; t^{3/2} K_t(xt^{1/2}, y) \;, \quad \mathcal{R}_\infty(x,y) \;=\; \psi^*(x) p_a^2(y) \;,$$

where ψ^* is defined in (1.6) and p_a in (3.4). To prove Lemma 3.2, we have to show that $\|\mathcal{R}_t - \mathcal{R}_\infty\|_{\mathcal{L}(X_1^1, X_{t^{1/2}})} = \mathcal{O}(t^{-1/2})$ as $t \to \infty$. In view of (1.10), (1.11), this amounts to showing that

$$\sup_{y\in\mathbf{R}} \left(\int_{\mathbf{R}} dx + \sup_{x\in\mathbf{R}}\right) \frac{W_{\beta t^{1/2}}(x)}{W_\beta(y)} \left|\mathcal{R}_t(x,y) - \mathcal{R}_\infty(x,y)\right| \;=\; \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \;.$$

But this follows from (C.5)–(C.9) by (more or less) straightforward calculations.

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