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Recursion Operators: Meaning and Existence for Completely Integrable Systems [*]

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Abstract

We show that any non resonant integrable system admits infinitely many Hamiltonian descriptions and recursion operators.

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1 Introduction

It is by now well known that completely integrable Hamiltonian dynamical systems may admit more than one Hamiltonian description (see for instance [Ma], [GD]). Usually, with these alternative descriptions one associates a $(1,1)$ tensor field which can be used (under suitable conditions) as a recursion operator, namely as an operator which generate enough constants of the motion in involution. It seems to be an open question whether it is possible to find a recursion operator for any completely integrable system.

In the hypotesis of non resonance (see later for the meaning of this term) it has been shown that a recursion operator can always be constructed (even for some infinite dimensional systems) [DMSV]. A recent paper claims however that this is not the case [Br].

It seems to us that it is of some interest to comment on possible meanings of recursion operators and to show that in condition of non resonance any integrable system can be reduced to a linear normal form via a nonlinear non-canonical transformation. For these normal forms it is straightforward to construct recursion operators. In particular, we constract one such an operator for the (counter)example given in [Br].

2 Integrable Systems (ISs)

Let M be a smooth $2n$-dimensional manifold. Let us suppose we can find n vector fields $X_1, \ldots, X_n \in \mathcal{X}(M)$ and n functions $F_1, \ldots, F_n \in \mathcal{F}(M)$ with the following properties

$$[X_i, X_j] = 0 \ , \tag{2.1}$$

$$L_{X_i} F^j = 0 \ . \quad i, j \in \{1, \ldots n\} \ . \tag{2.2}$$

Let us suppose also that on an open dense submanifold of M we have that

$$X_1 \wedge \cdots \wedge X_n \neq 0 \ , \tag{2.3}$$

$$dF^1 \wedge \cdots \wedge dF^n \neq 0 \ . \tag{2.4}$$

We shall show that any dynamical system Γ on M which is of the form

$$\Gamma = \sum_{i=1}^{n} \nu^i X_i \ , \quad \nu^i = \nu^i(F^1, \ldots F^n) \ , \tag{2.5}$$

is explicitely integrable on the submanifold on which (2.3) and (2.4) are satisfied.

We assume finally, that the level sets of the submersion

$$\mathbf{F} : M \to \mathbf{R}^n \ , \quad \mathbf{F} = (F^1, \ldots, F^n) \ , \tag{2.6}$$

are compact. Then the vector fields X_i are complete on each leaf $\mathbf{F}^{-1}(\mathbf{a})$, $\mathbf{a} \in \mathbf{R}^n$, and they integrate to a locally free action of the abelian group $\mathbf{R}^n$. Moreover, each leaf is parallelizable and we can find closed 1-forms $\alpha^1, \ldots, \alpha^n$, $d\alpha^i = 0$, such that

$$\alpha^i(X_j) = \delta^i_j \ , \quad i, j \in \{1, \ldots n\} \ . \tag{2.7}$$

With all previous construction, the vector field Γ in (2.5) can be explicitely integrated in a neighbourhood of each leaf $\mathbf{F}^{-1}(\mathbf{a})$ where we take as coordinates the functions $\{F^i, \phi^j\}$ with $d\phi^j = \alpha^j$. The equations of motion of Γ are given by

$$\begin{aligned} \dot{\phi}^i &= \nu^i(F^1, \ldots, F^n) \ , \\ \dot{F}^i &= 0 \ . \end{aligned} \tag{2.8}$$

Therefore, the corresponding solutions are

$$\begin{aligned} \phi_i(t) &= t\nu^i(\mathbf{F}(m_0)) + \phi^i(m_0) \ , \\ F_i(t) &= F_i(m_0) \ , \end{aligned} \tag{2.9}$$

with $m_0 \in M$ the initial point. We see that the functions ν^i play the rôle of frequencies.

We stress the fact that up to now we have not used any Hamiltonian structure.

2.1 Alternative Hamiltonian Descriptions for ISs

We shall now investigate under which conditions a dynamical system which is integrable in the sense stated before admits infinitely many alternative Hamiltonian descriptions.

With n-functions $f^1, \ldots, f^n$ obeying the condition $df^1_i \wedge dF^1 \wedge \cdots \wedge dF^n = 0$, $\forall i \in \{1, \ldots, n\}$, we can associate a closed 2-form by setting

$$\omega_f = \sum_i df_i \wedge \alpha^i \ , \tag{2.10}$$

which is non degenerate as long as $df_1 \wedge \cdots \wedge df_n \neq 0$. Any one of these symplectic 2-forms makes the action of $\mathbf{R}^n$ a Hamiltonian one. Indeed, by construction of ω_f,

$$i_{X_j}\omega_f = -df_j \ , \quad j \in \{1, \ldots, n\} \ . \tag{2.11}$$

As for the vector field Γ in (2.5) we shall have that

$$i_\Gamma \omega_f = -\sum_i \nu^i df_i \ . \tag{2.12}$$

A necessary condition for $i_\Gamma \omega_F$ to be exact is that it is closed, namely that

$$\sum_i d\nu^i \wedge df_i = 0 \ . \tag{2.13}$$

All sets of solutions of this equation for $f^1, \ldots, f^n$ satisfying $df_1 \wedge \cdots \wedge df_n \neq 0$ will give alternative Hamiltonian descriptions for the dynamical systems Γ in (2.4). Moreover, any such Γ will be completely integrable in the Liouville-Arnold sense, the functions $f_1, \ldots, f_n$ being constants of the motion (by assumption (2.2)) in involution,

$$\{f_i, f_j\}_A = \omega_f(X_i, X_j) = L_{X_i} f_j = 0 \ . \tag{2.14}$$

There are two limiting case where it is easy to exibit solutions of (2.13).

1. *The constant case.* All the frequencies ν^i are constant numbers so that $d\nu^i = 0$ and (2.13) is automatically satisfied.

Any 2-form in (2.10) is an admissible symplectic structure and the corresponding Hamiltonian function is given by

$$\omega_f = \sum_i \nu^i f_i \ . \tag{2.15}$$

An example of system for which this happens is given by the n-dimensional harmonic oscillator written as

$$\begin{aligned} \Gamma &= \sum_i \nu^i \Gamma_i \ , \\ \Gamma_i &= \frac{1}{\sqrt{m_i k_i}} p_i \frac{\partial}{\partial q_i} - \sqrt{m_i k_i} q_i \frac{\partial}{\partial p_i} \ , \qquad \text{no sum} \ , \\ \nu^i &= \frac{k_i}{m_i} \ . \end{aligned} \tag{2.16}$$

Here m_i and k_i are the mass and the elastic constant of the i-th oscillator. Now the functions F^i are just given by the partial Hamiltonians

$$F^i = \frac{1}{2}\left(\frac{p_i^2}{m_i} + k_i q_i^2\right) \ , \quad i \in \{1, \ldots, n\} \ . \tag{2.17}$$

2. *The non resonant case.* None of the frequencies ν^i is constant and we have that $d\nu^1 \wedge \ldots \wedge d\nu^n \neq 0$. In this case we may think of the ν^i as 'coordinates' and of the f^j as functions of the ν^i.

In this second case, very simple solutions of (2.13) are given by linear functions $f_i = \sum_j A_{ij} \nu^j \ , \ \ i \in \{1, \ldots n\} \ , \ A_{ij} \in \mathbf{R}$. The corresponding Hamiltonian description for Γ can given with quadratic Hamiltonian functions by

$$\omega_A = \sum_{ij} A_{ij} d\nu^i \wedge \alpha^j \ , \tag{2.18}$$

$$H_A = \frac{1}{2} \sum_{ij} A_{ij} \nu^i \nu^j \ . \tag{2.19}$$

Moreover, any other symplectic structure of the form

$$\omega_f = \sum_i df_i(\nu^i) \wedge \alpha^j \ , \tag{2.20}$$

in which any f_i depends only on the corresponding frequency ν^i, will be admissible as long as ω_f is non degenerate, i.e. as long as $df_1 \wedge \cdots \wedge df_n \neq 0$. The associated Hamiltonian functions depend on the explicit form of the functions f_i. For instance, if $f_i = \frac{\partial G_i}{\partial \nu^i}(\nu^i)$, the corresponding Hamiltonian can be written as

$$H_G = \sum_i (G_i - \nu^i \frac{\partial G_i}{\partial \nu^i}) \ . \tag{2.21}$$

A simple example for these case is given again by the n-dimensional harmonic oscillator written as

$$\Gamma = \sum_i F^i \Gamma_i \ , \tag{2.22}$$

where F^i and Γ_i are given by (2.17) and (2.16) respectively. Now the partial Hamiltonians F^i play the rôle of frequencies.

Remark. The intermediate cases are more involved. For further comments on them we refer to [DMSV].

Remark. It is worth stressing that there may be admissible Hamiltonian structures for Γ which cannot be derived by using the previous construction.

2.2 Recursion Operators for ISs

We shall now show how to construct recursion operators for the ISs which we have considered in previous sections. As we have seen, given the dynamical system (2.5) we can costruct infinitely many Hamiltonian structures given for instance by (2.10) or (2.20).

1. *The constant case.* $d\nu^i = 0, \ \forall \ i \in \{1, \ldots, n\}$.

Two possible alternative symplectic structures are obtained from (2.10) as

$$\omega_1 = \sum_{ij} \delta_{ij} dF^i \wedge \alpha^j \ = \sum_k \omega_k, \tag{2.23}$$

$$\omega_f = \sum_{ij} \delta_{ij} f^i(F^i) dF^i \wedge \alpha^j \ = \sum_k f^k(F^k) \omega_k \ , \tag{2.24}$$

with the condition $df_1 \wedge \cdots \wedge df_n \neq 0$. Given them, we can construct a $(1,1)$ tensor field T on M by

$$T = \omega_f \circ \omega_1^{-1} = \sum_k f^k(F^k)\ \mathbf{I}_k\ , \tag{2.25}$$

where $\mathbf{I}_k$ is the identity operator on the k-th two dimensional 'plane' of T^*M with 'coordinates' (dF^k, α^k).

2. *The non resonant case.* $d\nu^1 \wedge \ldots d\nu^n \neq 0$.

In this case two possible alternative symplectic descriptions are obtained from (2.20) as

$$\omega_1 = \sum_{ij} \delta_{ij} d\nu^i \wedge \alpha^j \ = \sum_k \omega_k, \tag{2.26}$$

$$\omega_f = \sum_{ij} \delta_{ij} f^i(\nu^i) d\nu^i \wedge \alpha^j \ = \sum_k f^k(\nu^k)\omega_k\ , \tag{2.27}$$

with the condition $df_1 \wedge \cdots \wedge df_n \neq 0$. Given these structures we can construct a $(1,1)$ tensor field T on M by

$$T = \omega_f \circ \omega_1^{-1} = \sum_k f^k(\nu^k)\ \mathbf{I}_k\ , \tag{2.28}$$

where $\mathbf{I}_k$ is the identity operator on the k-th two dimensional 'plane' of T^*M with 'coordinates' $(d\nu^k, \alpha^k)$.

¿From the way they are constructed, one sees that T in (2.24) and (2.28) are invariant under Γ, have double degenerate spectrum with eigenfunctions without critical points, and vanishing Nijenhuis torsion $\mathcal{N}_T$ *. Therefore they are recursion operators for the dynamical system Γ.

2.3 Liouville-Arnold ISs

Assume the dynamical vector field Γ on the symplectic manifold (M, ω_0) has n constants of the motion $H^1, \ldots, H^n$ which are in involution (with respect to the Poisson structure associated with ω_0), functionally independent, $dH^1 \wedge \cdots \wedge dH^n \neq 0$, and generate complete vector fields $X_1, \ldots, X_n$. We have then an action of $\mathbf{R}^n$ on M which is locally free and fibrating

In this situation one finds '*angle*' 1-forms $\alpha^1, \ldots, \alpha^n$ such that $\alpha^i(X_j) = \delta^i_j$ and $d\alpha^i = 0$. Given any function F of the H^j, (or $dF \wedge dH^1 \wedge \cdots \wedge dH^n = 0$) such that $det||\frac{\partial^2 F}{\partial H^i \partial H^j}|| \neq 0$, the 2-form

$$\omega_F = d(\frac{\partial F}{\partial H^i}\alpha^i) \tag{2.29}$$

*We remind that the tensor $\mathcal{N}_T$ is defined by $\mathcal{N}_T(X,Y) = [TX, TY] - T[TX, Y] - T[X, TY] + T^2[X,Y]\ ,\ \forall\ X, Y \in \mathcal{X}(M)$.

is an admissible symplectic structure for the $\mathbf{R}^n$ action. In particular, if

$$F = \frac{1}{2}\sum_i H_i^2 \tag{2.30}$$

we just get back the $\{H^i\}$ as Hamiltonian functions.

With a set of *action-angles* variables (J_k, ϕ^k) we have that

$$\Gamma = \nu^k \frac{\partial}{\partial \phi^k} \ , \tag{2.31}$$

$$\omega_0 = dJ_k \wedge d\phi^k \ , \tag{2.32}$$

$$i_\Gamma \omega = \nu^k dJ_k = -\frac{\partial H}{\partial J_k} dJ_k = -dH \ , \tag{2.33}$$

where $\nu^k = \frac{\partial H}{\partial J_k}$, $k \in \{1, \ldots, n\}$ are the frequencies. In the *non resonant case* when $d\nu^1 \wedge \cdots \wedge d\nu^n \neq 0$ or equivalently, $det||\frac{\partial \nu^k}{\partial J_l}|| \neq 0$, † we can use the ν^k as variables and write the admissible symplectic structure

$$\omega_\nu = \sum_k d\nu^k \wedge d\phi^k \ , \tag{2.34}$$

with Hamiltonian a quadratic function

$$H_\nu = \frac{1}{2}\sum_k (\nu^k)^2. \tag{2.35}$$

By using the analysis of section 2.1 we obtain that a not resonant complete integrable system has infinitely many admissible symplectic structures, some of them having the form

$$\omega_f = \sum_i df_i(\nu^i) \wedge d\phi^i \ , \tag{2.36}$$

with the condition $df^1 \wedge \cdots df^n \neq 0$. However, in general, we may not obtain ω_0 in this way. Moreover, such systems do admit recursion operators given by expression (2.28).

2.4 The example of Brouzet

In [Br] the following 2-degrees of freedom, completely integrable system is considered. Take $M = \mathbf{R}^2 \times \mathbf{T}^2 = \{(x, y, \theta, \eta)\}$ with symplectic structure $\omega_0 = dx \wedge d\theta + dy \wedge d\eta$. The dynamical system is described by the Hamiltonian $H = x^3 + y^3 + xy$. The corresponding dynamical vector field is given by

$$\begin{aligned} \Gamma &= \nu_\theta \frac{\partial}{\partial \theta} + \nu_\eta \frac{\partial}{\partial \eta} \ , \\ \nu_\theta &= 3x^2 + y \ , \\ \nu_\eta &= 3y^2 + x \ . \end{aligned} \tag{2.37}$$

†This is also equivalent to the notion of nondegeneracy of the Hamiltonian function used in [Br].

¿From what we have said before this system admits infinitely may alternative Hamiltonian descriptions in the dense open submanifold characterized by $d\nu_\theta \wedge d\nu_\eta \neq 0$, namely by $36xy - 1 \neq 0$, which coincides with the submanifold on which H is nondegenerate. Two such structures are given by

$$\omega_1 = d\nu_\theta \wedge d\theta + d\nu_\eta \wedge d\eta \ , \tag{2.38}$$

$$\omega_2 = f(\nu_\theta) d\nu_\theta \wedge d\theta + g(\nu_\eta) d\nu_\eta \wedge d\eta \ , \tag{2.39}$$

where f and g are any two functions such that $df \wedge dg \neq 0$. The corresponding recursion operators are given by

$$T = \omega_2 \circ \omega_1^{-1} = f(\nu_\theta) \left(d\nu_\theta \otimes \frac{\partial}{\partial \nu_\theta} + d\theta \otimes \frac{\partial}{\partial \theta} \right) + g(\nu_\eta) \left(d\nu_\eta \otimes \frac{\partial}{\partial \nu_\eta} + d\eta \otimes \frac{\partial}{\partial \eta} \right) \ . \tag{2.40}$$

We stress the fact that ω_0 is not among the symplectic structures constructed in (2.39) and that our recursion operators (2.40) cannot be ‘factorized’ through ω_0.

¿From this exemple it is clear that there is some ambiguity on what is a recursion operator for a given dynamical system. In the coming section we would like to make more clear this point.

3 Recursion operators

We notice that in studying the integrability of a given system Γ we could start directly with a $(1,1)$ tensor field T satisfying $L_\Gamma T = 0$, with double degenerate spectrum, with eigenfunctions without critical points, and vanishing Nijenhuis torsion $\mathcal{N}_T$. It is not necessary to require that T be factorizable via symplectic structures. Indeed this non unique decomposition can be constructed afterwards [DMSV].

In this section we shall comment some more on the meaning of recursion operators and on their use in the analysis of complete integrability [ZC], [Mar].

Therefore let us suppose we have a dynamical vector field $\Gamma \in \mathcal{X}(M)$ and a compatible $(1,1)$ tensor T , namely $L_\Gamma T = 0$, so that the functions $trT^k \ , \ k \geq 1$ are constants of the motion. By applying powers of T we get vector fields $\Gamma_k = T^k(\Gamma)$ which are symmetries for Γ. The Lie algebra $\{\Gamma_k \ , \ k \geq 0\}$ is abelian if $\mathcal{N}_T = 0$.

If $F \in \mathcal{F}(M)$ is a constant of the motion for Γ, we say that T is an *F-weak recursion operator* if $\mathcal{N}_T = 0$ and $d(T(dF)) = 0$ (we use the same symbol for T and for its dual). If T is an F-weak recursion operator, one can prove that $d(T^k(dF)) = 0 \ , \ \forall \ k > 1$. Locally, one finds functions $F_k \in \mathcal{F}(M)$ by $dF_k = T^k(dF)$ which are constants of the motion for Γ.

It is worth stressing that a given operator T may be a recursion operator for the constant of the motion F and not a recursion for another constant of the motion G. Moreover, it may also happen that the tensor T is an F-recursion operator but $T^k(dF) \wedge dF = 0 \ , \ \forall \ k \geq 1$, so that one cannot use T and F to generate new constants of the motion. This is what happens for instance with the Kepler problem if one starts with the standard Hamiltonian function [MV]. However, it is always true that $T(d(\frac{1}{k}trT^k)) = d(\frac{1}{k+1}trT^{k+1})$.

If ω is an admissible symplectic structure for Γ, namely $L_\Gamma\omega = 0$, we say that T is an *ω-weak recursion operator* if $\mathcal{N}_T = 0$ and $d(T(\omega)) = 0$ (again, we use the same symbol for the extension of T to forms). If T is an ω-weak recursion operator one proves that $d(T^k(\omega)) = 0 \ , \ \forall \ k > 1$. All 2-forms $\omega_k = T^k(\omega)$ are then admissible symplectic structures for Γ.

It is worth stressing that given any two admissible symplectic structures ω_1 and ω_2 for Γ, it need not be true that they are connected by a recursion operator. Moreover, it may happen that $T^k(\omega) \wedge \omega = 0 \ , \ \forall \ k \geq 1$ so that one does not generate new symplectic structures.

If Γ is Hamiltonian with respect to the couple (ω, H), namely $i_\Gamma\omega = -dH$, we say that T is a *strong recursion operator* if it is both an H-recursion operator and an ω-recursion operator. If this is the case, any vector field Γ_k is an Hamiltonian one with respect to ω with Hamiltonian function H_k as well as with respect to ω_k with Hamiltonian function H. Moreover, the constants of the motion H_k are pairwise in involution with respect to the Poisson structure constructed by inverting anyone of the symplectic structures $\omega_k \ , \ k \geq 0$.

4 Conclusions

We have shown that any non resonant integrable system admits infinitely many alternative symplectic structures and strong recursion operators. This class of systems include Hamiltonian ones with non degenerate Hamiltonian functions.

The result proven in [Br] seems rather to exlude the possibility of constructing recursion operators while *keeping fixed* one symplectic structure ω_0, namely to construct ω_0-recursion operator. But this not surprising result seems less relevant for the analysis of complete integrable systems.

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