ROTATION AXIS VARIATION DUE TO SPIN ORBIT RESONANCE

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Let $\mathcal{E}$ be a planet modeled by a homogeneous rigid ellipsoid with symmetry axis NS, with polar inertia moment J_3, equatorial moment J and mechanical flattening coefficient $\eta = (J_3 - J)/J_3$. [1]

The planet center of mass T is supposed to revolve on a keplerian orbit $t \to \vec{r}_T(t)$ about a focus S: the orbit plane will be called the *ecliptic* plane and $\vec{\vec{k}}$ will denote its unit normal vector (*celestial north*) which sees the planet revolving counterclockwise.

The longitude λ_T of $\vec{r}_T$ on the ecliptic will be reckoned from the aphelion of the ellipse; hence $\lambda_T = 0$ is the *aphelion* position, *i.e.* where $r_T \equiv |\vec{r}_T|$ is maximal: $r_T(0) = a(1+e)$, a being the major semiaxis of the keplerian ellipse and e its eccentricity.

With the above conventions, r_T and λ_T are related by: $r_T \equiv |\vec{r}_T| = p\cdot(1 - e\cos\lambda_T)^{-1}$, $p \equiv a(1-e^2)$ and the Kepler's laws imply that if λ is the keplerian *average anomaly* then: $\lambda \equiv \lambda_T + 2e\sin\lambda_T + \frac{3}{4}e^2\sin 2\lambda_T + \ldots$. and the motion is $\lambda \to \lambda + \omega_T t$, where $2\pi/\omega_T = 2\pi a^{3/2} g_N^{-1/2}$ is the year of the planet, $g_N \equiv k(m_S + m_T)$ if k is Newton's constant and m_T, m_S are the masses of the planet and of its star.

The planet is described by means of the following coordinates:

1) The total angular momentum, or *spin*, M and its projections $K \equiv M\cos i$ on the axis orthogonal to the ecliptic ("celestial north–south axis").

2) The angles γ, φ, λ where γ is the angle between an axis fixed on the ecliptic (*Aries line*) and the spin–ecliptic node (*i.e.* the intersection of the ecliptic and the plane orthogonal to the spin: *equinox line*); φ is the angle between the spin ecliptic node and the analogous spin–equator node; and λ is the average anomaly of the planet, which rotates uniformly at angular velocity ω_T.

It is a well known theorem in classical mechanics (Andoyer–Deprit) that the pairs (M, φ) and (K, γ) are canonically conjugate variables (see [G], p.318) and, introducing an auxiliary variable B canonically conjugate to λ, the energy of the system can be written:

$$H = \omega_T B + \frac{M^2}{2J} + V = \omega_T B + \frac{M^2}{2J} - \int_{\mathcal{E}} \frac{k m_T m_S}{|\vec{r}_T + \vec{x}|}\frac{d\vec{x}}{|\mathcal{E}|} \tag{1}$$

One could avoid introducing B: but one would then have a time dependent hamiltonian, which I do not like.

The complete description of the rigid body configuration would require an extra pair of canonically conjugated varibles, namely (L, ψ) where ψ is the angle between a comoving axis fixed on the equator and the spin–equator node and L is the projection $L = M\cos\vartheta$ of the spin on the NS axis. However, by symmetry, the hamiltonian does not depend on ψ; hence L is a constant of motion and one can see that it only gives an additive contribution to the energy, which is dropped in (1).

I shall consider the following approximation for V:

a) the integral is developed in powers of the ratio R/a and the expansion will be truncated to the order 2 included neglecting the orders 4 and higher. [2]

b) the expression thus obtained will be developed in powers of the eccentricity and the expansion will be truncated to 2d order.

The model thus obtained will be called the *D' Alembert precession-nutation model.* The $e = 0$ model was in fact used by D'Alembert, [L], to deduce his celebrated theory of the equinox precession for the Earth and the theory of the lunar plane precession.

It is not interesting to limit ourselves to the D'Alembert's case $e = 0$ because the phenomenon of large variations of the inclination axis are in this approximation not possible.

Define $\omega_D \equiv M/J$, calling it conventionally the *daily rotation*, and consider the line $\mathcal{L}$ in the plane of the action variables $\vec{A} = (M, K)$ defined by:

$$M = J\omega_D \equiv 2J\omega_T = const, \qquad K_0 \le K \le K_1 \tag{2}$$

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[1] this means that if R is the equatorial radius then the polar radius is $R/(1+2\eta)^{1/2}$.

[2] only the 2d order is non trivial among the first three orders.

and call $\vec{A}^0$ and $\vec{A}^1$ its extremes. Since $K = M\cos i$ one realizes that proceeding on the above line from $\vec{A}^0$ to $\vec{A}^1$ the spin inclination angle changes between i_0 and i_1 and the system is locked in a $1:2$ spin orbit resonance.

A motion in which the projection in the (M,K) plane closely follows the line $\mathcal{L}$ is therefore a motion during which a variation of the inclination axis is observed. Along such line, which will be fixed in the following, the ratio $\cos\vartheta \equiv L/M$ is a constant which will therefore also be fixed: and which will be supposed different from ± 1, (*i.e.* the angle ϑ, *nutation constant*, between the spin and the NS axis is supposed different from 0 or π).

The eccentricity will be taken $e = \eta^c$ for some $c > 1$ so that setting η small implies that the eccentricity is also (much) smaller.

The question that will be investigated, see [CG], is whether there exists a trajectory $X_{e,\eta}(t)$ starting at $t = 0$ in $\vec{A}_{e,\eta} = (M_{e,\eta}, K_{e,\eta})$ near $\vec{A}^0$, with phases $(\gamma_{e,\eta}, \varphi_{e,\eta}, \lambda_{e,\eta})$, and reaching after a long enough time the vicinity of $\vec{A}^1$.

If such a trajectory exists for all η small enough, *but different from* 0, one says that there is a *Arnold diffusion* phenomenon. [3]

What is remarkable, when the diffusion phenomenon happens, is that it happens in spite of the fact that K, M are adiabatic invariants, *i.e. they are variables which in any averaging approximation could only change by quantities which are infinitesimal as* $e, \eta \to 0$: while in the above situation K or i change by a quantity $\sim i_1 - i_0$ *independent on* e, η.

The existence of Arnold diffusion says therefore that no averaging approximation can be even approximately correct over very long time scales or for infinite time.

This is not at all a knell for the widely used averaging methods: but it shows the interest of estimating the time scales over which the phenomenon happens: it is becoming well understood that the correctness of the averaging methods give very accurate results over time scales that are often of the order of the age of the universe, [CeG].

The result, [CG], on the above question, in the case of the described precession nutation model is the following.

Theorem: *Suppose that* $[i_0, i_1]$ *does not contain the values* $0, \frac{\pi}{2}, \pi$ *and suppose that the angle* ϑ *between the* NS *axis and the spin axis does not vanish in the configurations in* $\mathcal{L}$ *line.* [4] *There are constants* c, b, d *such that if* $e = \eta^c$ *then for all* η *small enough, but non zero, there are phases* $\Phi_\eta = (\gamma_\eta, \varphi_\eta, \lambda_\eta)$ *and actions* $\vec{A}_\eta = (M_\eta, K_\eta)$ *such that:*
1) the trajectory $X_\eta(t) = (\vec{A}(t), \Phi(t))$*, starting in the configuration* $\vec{A}(0) = \vec{A}_\eta, \Phi(0) = \Phi_\eta$ *stays close, in the* $\vec{A}$*–variables, within* η^b *to the line* $\mathcal{L}$ *during a suitable time* $\ge T_\eta$*.*
2) the time T_η *is such that* $\vec{A}(T_\eta)$ *is close (i.e. within* η^b*) to* $\vec{A}^1$ *while* $\vec{A}(0)$ *is close to* $\vec{A}^0$*, (i.e. within* η^b*).*

Therefore in the above precession problem Arnold's diffusion is possible. The angle of spin inclination can change by an order of magnitude $O(1)$ in a long enough time, no matter how small is the spin orbit coupling (*i.e.* η), provided it is non zero and provided the eccentricity is small enough compared to the flattening. And it becomes interesting to get an idea of the size of the time T_η.

In [CG] an explicit estimate for T_η is derived:

Theorem: *in the situation of the previous theorem there is a constant* f *such that for* η *small enough the diffusion time* T_η *can be estimated to be not longer than:*

$$\bar{T}_\eta = \omega_T^{-1} e^{(e^2\eta^{3/2})^{-2} g(\mathcal{L})^{-2} f}, \qquad g(\mathcal{L}) = \sin\vartheta \min_{i_0 \le i \le i_1} \Big(\frac{1 - cosi}{1 + \cos i} \cos^2 i \, \sin i\Big) \tag{3}$$

Hence the estimate is a very very long time, compared to the times over which one expects the averaging methods to have some validity, which have scale of order $\sim \exp f\eta^{-1/2}$. Also the constants b, c, d, f, whose values can actually be computed, see [CG], turn out to be very poor for any practical application.

But to get to practical applications a lot of estimates would have to be refined and it is not completely obvious that this is really impossible. One can recall, on this respect, that for a long time it was stated by some people that the KAM estimates were too bad for any practical use: and this turned out to be grossly incorrect, [CeG],[CC],[LR]. In the case of diffusion the situation looks much harder, perhaps desperately so: but we shall see.

[3] this refers to one possible definition of Arnold's diffusion, [A]: it is a special case of a general definition proposed in [CG]. But there are other definitions: different and, often, not very precise.

[4] it has been remarked above that the angle ϑ is constant on the line $\mathcal{L}$.

This shows that the estimate of the time scale for the diffusion, *i.e.* essentially the time one has to wait to see a violation of the predictions of the averaging approximations, diverges if one tries to bridge gaps in inclination containing $i = 0, \frac{\pi}{2}, \pi$.

In particular the above estimate diverges if one tries to find a trajectory in which the spin sign is reversed (*i.e.* i goes through $\frac{\pi}{2}$). The difficulty in constructing such trajectories might be a manifestation of a physical phenomenon and not just a defect of the techniques of proof; it seems, indeed, that the chaotic motions of the planetary axes are unable, in absence of dissipative effects, to change the spin sign, *i.e.* the sign of K, [LRo].

Formula (3) is suggestive as it leads to the conjecture that (in general) diffusion along a path $\mathcal{L}$ in action space might really take place over a time scale dependinng exponentially on some power of the coupling constant *times a coefficient determined my maximising a suitable functional defined on* $\mathcal{L}$: therefore I call (3), by analogy, a *large deviation formula.*

I conclude with a technical comment: the analysis is based on the fact that the hamiltonian system under consideration has three very different time scales: namely the daily time scale (ω_D^{-1}, coinciding with the year time scale because the free system is in a $1:2$ resonance), the resonance time scale of order $O(\eta^{-1/2}\omega_D^{-1})$ (describing the characteristic time of oscillation transversal to the resonance) and the equinox precession scale of order $O(\eta^{-1}\omega_D^{-1})$ (arising from an application of the method of averaging and describing the precession of the equinox line, when $e = 0$). The three time scales become very different from each other when $\eta \to 0$: this is due to the degeneracy inherent in the problem where the unperturbed hamiltonian *does not depend on the variable K*. And it implies the phenomenon of *large angles at the homoclinic intersections*, see [CG], which in turn implies the existence of diffusion. Deeply different is what happens near resonances of non degenerate systems, where all the action variables appear non linearly in the unperturbed hamiltonian: in such cases there are only two distinct time scales describing the motions near the resonances and the theory of diffusion is much harder (because the homoclinic angles are exponentially small, see [CG], [G2]).

Thus the above theory is only one more instance of the (well known) fact that the degeneration intrinsically present in all celestial mechanics problems is very often, in fact, making the theory easier rather than harder (as sometimes naively claimed).

For connections between the techniques used in the theory of the above problem and the KAM theorem see also [G2],[G3].

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