IMAFF 93/13
August 1993

Spectrum of an Elliptic Free Fermionic Corner Transfer Matrix Hamiltonian*

R. Cuerno

Instituto de Matemáticas y Física Fundamental, CSIC
Serrano 123, E–28006 Madrid, SPAIN
e-mail:IMTRC59@CC.CSIC.ES

Abstract

The eigenvalues of the Corner Transfer Matrix Hamiltonian associated to the elliptic R matrix of the eight vertex free fermion model are computed in the anisotropic case for magnetic field smaller than the critical value. An argument based on generating functions is given, and the results are checked numerically. The spectrum consists of equally spaced levels.

*PACS: 05.50.+q, 02.90.+p, 75.10Jm

In recent times a great attention has been increasingly paid to the computation of correlation functions for integrable models in two dimensions. Following the approach of the Kyoto school it has been possible not only to diagonalize hamiltonians (as the XXZ model [1] and its higher spin analogs [2] in the thermodynamic limit for the corresponding massive regimes) by means of the representation theory of quantum affine algebras, but also to extend the arguments to cases for which no quantum group like structure has been identified yet, namely the zero field eight–vertex (8V) model [3]. A prominent role is played in this approach by the Corner Transfer Matrix Hamiltonian (CTMH), which in the quantum group invariant cases acts as a derivation in the quantum affine algebra (its physical interpretation is to be the lattice analog of the generator of Lorentz boosts [4]). A salient feature of this operator is that its spectrum consists of equally spaced levels [5]. Remarkably there exists an 8V model (the elliptic free fermionic solution of the eight vertex Yang–Baxter equation [6, 7, 8]) for which a sort of affine quantum group structure $\widehat{CH}_q(2)$ has been identified, in which the R matrix acts as an intertwiner [9]. The associated row–to–row hamiltonian is an (anisotropic) XY model in a magnetic field. This letter is a first step in the program of employing quantum group invariance to compute correlation functions in this 8V–like model. Namely it is shown here that the spectrum of the corresponding CTMH in the anisotropic regime[1] is given essentially by integer numbers, which makes it reasonable to try for it an interpretation as a derivation–like operator in $\widehat{CH}_q(2)$. This CTMH has already been studied in the context of the so called master symmetries [10] and its spectrum has been studied through the use of orthogonal polynomials [11, 12]. However in the latter references only the trigonometric (isotropic) and hyperbollic (disorder line) limits have been worked out, since the more general case leads to 5–term recurrence relations whose link to the 3–term recurrences characteristic of orthogonal polynomials [13] is unclear. In the present letter the use of both a convenient parametrization of the model and an argument following Davies [14], dealing with generating functions enables us to identify the correct form of the eigenvalues of the CTMH also in a more general case. We compare the results to some numerical computations and then end with some comments and suggestions.

The most general solution for the elliptic free fermionic 8V R matrix is, in the parametrization of ref. [8]:

$$R^{00}_{00}(u) \;=\; 1 - e(u)e(\psi_1)e(\psi_2) \qquad R^{11}_{11}(u) = e(u) - e(\psi_1)e(\psi_2)$$

[1] for values of the magnetic field smaller than the critical one. See below.

$$
\begin{aligned}
R_{01}^{01}(u) &= e(\psi_2) - e(u)e(\psi_1) \qquad R_{10}^{10}(u) = e(\psi_1) - e(u)e(\psi_2) \\
R_{01}^{10}(u) &= R_{10}^{01}(u) = (e(\psi_1)\mathrm{sn}(\psi_1))^{1/2}(e(\psi_2)\mathrm{sn}(\psi_2))^{1/2}(1-e(u))/\mathrm{sn}(u/2) \\
R_{00}^{11}(u) &= R_{11}^{00}(u) = -\mathrm{i}k(e(\psi_1)\mathrm{sn}(\psi_1))^{1/2}(e(\psi_2)\mathrm{sn}(\psi_2))^{1/2}(1+e(u))\mathrm{sn}(u/2)
\end{aligned} \tag{1}
$$

with $e(u)$ the elliptic exponential:

$$e(u) = \mathrm{cn}(u) + \mathrm{i}\ \mathrm{sn}(u) \tag{2}$$

k the elliptic modulus and $\mathrm{cn}(u)$, $\mathrm{sn}(u)$ the Jacobi elliptic functions of modulus k. In the sequel we will set

$$\psi_1 = \psi_2 \equiv \psi \tag{3}$$

since this is the interesting case from the hamiltonian point of view. Up to a normalization constant the row–to–row hamiltonian acting on the j–th and $j+1$–th spins is

$$H_{j,j+1} = \mathrm{i}\ \frac{dR_{j,j+1}(u)}{du}\bigg|_{u=0} = (1+\Gamma)\sigma_j^x\sigma_{j+1}^x + (1-\Gamma)\sigma_j^y\sigma_{j+1}^y + h(\sigma_j^z + \sigma_{j+1}^z) \tag{4}$$

The σ's are the spin 1/2 Pauli matrices and

$$\Gamma \equiv k\ \mathrm{sn}(\psi) \ \ , \ \ h \equiv \mathrm{cn}(\psi) \tag{5}$$

For real ψ and $0 < k \leq 1$, we stay in the Lieb–Schultz–Mattis (LSM) unit square [15, 16]

$$0 < \Gamma \leq 1 \ \ , \ \ 0 \leq h < 1 \tag{6}$$

We will take h and k as our physical independent parameters, (see Figure 1) though some expressions will be left in terms of ψ. A crucial point is the elliptic function dependence on this parameter shown in (5).

To define the CTM first construct a finite row of vertices by (see for instance [17])

$$G_j^{(n)}(u) \equiv R_n(u)R_{n-1}(u)\cdots R_j(u) \tag{7}$$

where R_j is the Boltzmann weight matrix correponding to the j–th vertex in the row. Then construct a wedge shaped region by piling up rows of vertices

$$A_N(u) = G_1^{(N)}(u)G_2^{(N)}(u)\cdots G_N^{(N)}(u) \tag{8}$$

We shall consider the thermodynamic limit $N \to \infty$ for which Baxter has shown [18] that the normalized (by its highest eigenvalue) CTM $A_N(u)$ is a one parameter commuting family written as

$$A_n(u) = \exp(-uH_{CTM}) \tag{9}$$

Expanding (8) for small u one finds for the CTMH

$$H_{CTM} = \sum_{j=1}^{\infty} j H_{j,j+1} \tag{10}$$

This is the operator which we intend to diagonalize. Since the model is free fermionic we find it convenient to search for eigenvectors of (10) which are linear combinations of the Jordan–Wigner fermions [19]

$$\begin{gathered} \tau_j^{x,y} \equiv \frac{1}{\sqrt{2}} e^{\frac{\pi \mathrm{i}}{2} \sum_{k=1}^{j-1} (\sigma_k^z - 1)} \sigma_j^{x,y} \\ \left\{ \tau_j^a, \tau_k^b \right\} = \delta_{jk} \delta_{ab} \end{gathered} \tag{11}$$

In terms of these

$$H_{CTM} = -2\mathrm{i} \sum_{j=1}^{\infty} j \left\{ (1+\Gamma) \tau_j^y \tau_{j+1}^x - (1-\Gamma) \tau_j^x \tau_{j+1}^y + h(\tau_j^x \tau_j^y + \tau_{j+1}^x \tau_{j+1}^y) \right\} \tag{12}$$

So look for states

$$\psi(l) \equiv \sum_{j=1}^{\infty} \left\{ A_{l,j}^- \tau_j^x + A_{l,j}^+ \tau_j^y \right\} \tag{13}$$

such that

$$[H_{CTM}, \psi(l)] = \lambda_l \psi(l) \tag{14}$$

l is some index which labels eigenstates. We get 2 sets of equations for the numerical coefficients in (13) ($\tilde{\lambda}_l \equiv -\lambda_l/2$, $\gamma_\pm \equiv 1 \pm \Gamma$):

$$\gamma_\pm (m-1) A_{l,m-1}^\pm - h(2m-1) A_{l,m}^\pm + \gamma_\mp m A_{l,m+1}^\pm = \pm \mathrm{i} \tilde{\lambda}_l A_{l,m}^\mp \tag{15}$$

Note that if we substitute any of the (15) into the other we get a 5–term recurrence for either $\{A_{l,j}^\pm\}$. Such relation readily becomes 3–term like if we set $\Gamma^2 = 1$ (Ising model [14, 20]) or $h = 0$ (doubled Ising model [17, 20]). Also the trigonometric ($k = 0$) and the hyperbollic ($k = 1$, disorder line $\Gamma^2 + h^2 = 1$) limits of this model correspond in essence to three term recurrences which can be solved by means of orthogonal polynomials [11]. We want to solve for the anisotropic case (6), so define the functions

$$A_l^\pm(t) \equiv \sum_{j=1}^{\infty} t^j \; A_{l,j}^\pm \tag{16}$$

We get the differential equations

$$(\gamma_{\pm}t^2 - 2ht + \gamma_{\mp})\frac{dA_l^{\pm}(t)}{dt} + (h - \gamma_{\mp}t^{-1})A_l^{\pm}(t) = \pm \mathrm{i}\tilde{\lambda}_l A_l^{\mp}(t) \tag{17}$$

Introducing the integrating factors

$$f_{\pm}(t) = \frac{t}{(\gamma_{\pm}t^2 - 2ht + \gamma_{\mp})^{1/2}} \tag{18}$$

such that

$$A_l^{\pm}(t) = \alpha_l^{\pm}(t) f_{\pm}(t) \tag{19}$$

we are left with the system

$$(\gamma_+ t^2 - 2ht + \gamma_-)^{1/2}(\gamma_- t^2 - 2ht + \gamma_+)^{1/2}\,\frac{d\alpha_l^{\pm}}{dt} = \pm \mathrm{i}\tilde{\lambda}_l \alpha_l^{\mp}(t) \tag{20}$$

Changing variables to

$$s = \int \frac{dt}{(\gamma_+ t^2 - 2ht + \gamma_-)^{1/2}(\gamma_- t^2 - 2ht + \gamma_+)^{1/2}} \tag{21}$$

the most general solution to (20) becomes

$$\begin{aligned} \alpha_l^{+}(s) &= a_1^{(l)} e^{s\tilde{\lambda}_l} + a_2^{(l)} e^{-s\tilde{\lambda}_l} \\ \alpha_l^{-}(s) &= -\mathrm{i}(a_1^{(l)} e^{s\tilde{\lambda}_l} - a_2^{(l)} e^{-s\tilde{\lambda}_l}) \end{aligned} \tag{22}$$

where $a_{1,2}^{(l)}$ are arbitrary integration constants. Due to the freedom in an overall constant we will be interested in the value of their ratio $\gamma^{(l)} = a_2^{(l)}/a_1^{(l)}$. So the whole issue resides in solving for the integral (21). Notice that in the generic case the integrand has four different poles (they appear as complex conjugate pairs) at

$$\frac{\mathrm{cn}(\psi) \pm \mathrm{i}k'\mathrm{sn}(\psi)}{1 + k\mathrm{sn}(\psi)} \quad , \quad \frac{\mathrm{cn}(\psi) \pm \mathrm{i}k'\mathrm{sn}(\psi)}{1 - k\mathrm{sn}(\psi)} \tag{23}$$

where k' is the conjugate elliptic modulus $k^2 + k'^2 = 1$. We recall the reader that these values coincide precisely with those of the central elements for quantum Clifford–Hopf $CH_q(2)$ invariant open chain Hamiltonians [21].

An adequate change of variables is [22]

$$y = \frac{\gamma_+ t^2 - 2ht + \gamma_-}{\gamma_- t^2 - 2ht + \gamma_+} \tag{24}$$

Notice that it breaks down in the trigonometric limit, which should be treated separately in this approach. Being $y_\pm = \frac{1\pm k}{1\mp k}$ the maximum and the minimum value of y respectively we get

$$s = \frac{1}{2k'\mathrm{sn}(\psi)} \int_y^{y_+} \frac{dy}{(y(y_+ - y)(y - y_-))^{1/2}} \tag{25}$$

and then [23]

$$s = \frac{1}{(1+k)|\mathrm{sn}(\psi)|} \mathrm{tn}^{-1} \left[\left(\frac{1+k}{1-k} \right)^{1/2} \frac{t\, r_- - r_+}{t\, r_+ - r_-}, \frac{2k^{1/2}}{1+k} \right] \tag{26}$$

with $r_\pm = (1 \pm \mathrm{sn}(\psi))^{1/2}$ and $\mathrm{tn}^{-1}(u,k)$ an inverse Jacobian elliptic function of modulus k. Now we want that the functions (16) generate normalizable states when $N \to \infty$, so we require that they are free of singularities inside the unit circle $|t| = 1$ [14]. This completely fixes the values of $\tilde{\lambda}_l$ and $\gamma^{(l)}$, with the result

$$\lambda_l = \frac{2\pi l}{K'(k)} |\mathrm{sn}(\psi)| \qquad l = 0, 1, 2, \ldots \tag{27}$$

where $K'(k)$ is the complete elliptic integral of the first kind of modulus k'. This spectrum reduces to its known values in the zero field ($h = 0$, $\psi = K(k)$) [17] and disorder line [11] cases.

Finally we present some numerical confirmation of the above result for the spectrum. Since our CTMH is a quadratic form in fermionic operators we can try to diagonalize it by finding the eigenvalues λ_l^2 of the $N \times N$ LSM matrix [15]

$$(A - B)(A + B) = MM^t \tag{28}$$

where M is the following tridiagonal matrix

$$M = \begin{pmatrix} h(0+1) & \gamma_- & 0 & \cdots & 0 & 0 & 0 \\ \gamma_+ & h(1+2) & 2\gamma_- & \cdots & 0 & 0 & 0 \\ 0 & 2\gamma_+ & h(2+3) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (N-2)\gamma_+ & h(N-1+N-2) & (N-1)\gamma_- \\ 0 & 0 & 0 & \cdots & 0 & (N-1)\gamma_+ & h(N-1) \end{pmatrix} \tag{29}$$

(28) is a pentadiagonal matrix as mentioned in [11], and the eigenvalue equation for it leads to the same 5–term recurrence as (15). We have diagonalized it numerically for

k	h	$C(\psi,k)_{theor}$	$C(\psi,k)_{num}$
10^{-2}	0.5	0.908	0.91
0.1	0.5	1.47238	1.47238
1	0.5	3.46410	3.46410
0.5	0.1	2.89898	2.89898
0.5	0.5	2.52324	2.52324
0.5	0.999	0.1302	0.1304

Table 1: Numerical vs. theoretical values of $C(\psi, k)$.

dimensions up to $N = 500$, and found perfect agreement of the lowest eigenvalues with the formula (27) for any values of k, h in the unit square (6). In Fig. 2 we have plotted the lowest eigenvalues for fixed magnetic field h and let the anisotropy Γ vary with the elliptic modulus k; we find good fit of the theoretical coefficient $C(\psi, k) \equiv 2\pi|\mathrm{sn}(\psi)|/K'(k)$. This is also shown in Table 1. We have found poorer convergence in the cases of small ψ or k (for fixed value of the other parameter), since they both correspond to approaching the isotropic limit $\Gamma = 0$ (see Fig. 1) which is critical in this parametrization [11].

In conclusion we have found that the CTMH for the elliptic free fermionic eight vertex model has a spectrum given by integer values also in an anisotropic range of its parameters. This leads us to think of its relation to some derivation–like operator for $\widehat{CH}_q(2)$ (see [10] for its effective role as a derivation in the context of master symmetries), and makes it interesting to consider the problem of the computation of correlation functions in this 8V model using quantum group representation theory. This will be the subject of further study. It would be also interesting to complete the proof of orthogonality and completeness of the basis of eigenstates that has been generated here [14], which would involve the use of identities among elliptic functions, and to think of the pole structure of the quasimomentum–rapidity like transformation (21) in connection with quantum group invariance. Other interesting formal aspect is the relation (if any) of the 5–term recurrence with the 3–term recurrences satisfied by orthogonal polynomials. The theory of q–deformed orthogonal polynomials (see for instance [24]) might be useful in this. Finally since the trigonometric limit of the R matrix (1) is related to $N = 2$ SUSY (see [25] for the more general $\widehat{CH}_q(D)$ case, D even) we are led to speculate whether the study of the algebraic properties of the CTMH we are considering could lead to the definition of a lattice analog of an $N = 2$ superconformal algebra, much in the same way as [17] constructed a lattice analog of Virasoro ($N = 0$) for the free fermionic point of Baxter's solution of the zero field 8V model [18] (our $h = 0$ case).

Acknowledgments

The author is pleased to thank A. Berkovich, and G. Sierra for many discussions, suggestions, and for encouragement, A. González–Ruiz for comments and V. R. Velasco for access to the computing facilities of his group. This research has been supported by the Spanish Ministry of Education and Science through predoctoral fellowship PN89–11798388.

References

[1] O.Foda and T.Miwa, Int. Jour. Mod. Phys. **A7** Suppl. **1A** (1992) 279.

B.Davies, O.Foda, M.Jimbo, T.Miwa and A.Nakayashiki, Commun. Math. Phys. **151** (1993) 89.

M.Jimbo, K.Miki, T.Miwa and A.Nakayashiki, Phys. Lett. **A168** (1992) 256.

[2] M.Idzumi, T.Tokihiro, K.Iohara, M.Jimbo, T.Miwa and T.Nakashima, Int. Jour. Mod. Phys. **A8** (1993) 1479.

[3] M.Jimbo, T.Miwa and A.Nakayashiki, "Difference equations for the correlation functions of the eight vertex model", preprint (1992).

[4] M.G.Tetel'man, Sov. Phys. JETP **55** (1982) 306.

H.B.Thacker, Physica **D18** (1986) 348.

[5] R.J.Baxter, Jour. Stat. Phys. **15** (1976) 485,

R.J.Baxter, Jour. Stat. Phys. **17** (1977) 1.

[6] B.U.Felderhof, Physica **66** (1973) 279.

[7] K.Sogo, M.Uchinami, Y.Akutsu and M.Wadati, Prog. Theor. Phys. **68** (1982) 508.

[8] V.V.Bazhanov and Yu.G.Stroganov, Teor. Mat. Fiz. **62** (1985) 37.

[9] R.Cuerno, C.Gómez, E.López and G.Sierra, Phys. Lett. **B307** (1993) 56.

[10] H.Araki, Commun. Math. Phys. **132** (1990) 155.

[11] T.T.Truong and I.Peschel, Int. Jour. Mod. Phys. **B4** (1990) 895.

[12] H.–P.Eckle and T.T.Truong, Phys. Lett. **A177** (1993) 81.

[13] T.S.Chihara,"An Introduction to Orthogonal Polynomials", Gordon and Breach, New York (1978).

[14] B.Davies, Physica **A154** (1988) 1.

B.Davies, Physica **A159** (1989) 171.

[15] E.Lieb, T.Schultz and D.Mattis, Ann. Phys. **16** (1961) 407.

[16] S.Katsura, Phys. Rev. **127** (1962) 1508.
E.Barouch and B.McCoy, Phys. Rev. **A3** (1971) 786.

[17] H.Itoyama and H.B.Thacker, Nucl. Phys. **B320** (1989) 541.

[18] R.J.Baxter, “Exactly solved models in statistical mechanics”, Academic Press, New York (1982).

[19] H.Hinrichsen and V.Rittenberg, Phys. Lett. **B275** (1992) 350.

[20] T.T.Truong and I.Peschel, Z. Phys. **B75** (1989) 119.

[21] R.Cuerno and A.González–Ruiz, Jour. Phys. A:Math. Gen. in press.

[22] A.–G.Greenhill, “The Applications of Elliptic Functions”, Macmillan, New York (1892).

[23] P.F.Byrd and M.D.Friedman, “Handbook of Elliptic Integrals for Engineers and Scientists”, Springer–Verlag, Heidelberg (1971).

[24] T.H.Koornwinder, “Orthogonal Polynomials in Connection with Quantum Groups”, in “Orthogonal Polynomials: Theory and Practice” P.Nevai ed. NATO ASI Series Vol. C 294, Kluwer, Dordrecht (1990).

[25] E.López, “Quantum Clifford–Hopf Algebras for Even Dimensions, Extended Supersymmetry and Generalized XY Spin Chains”, preprint IMAFF–12/93.

Figure Captions.

Fig. 1. (h,k) parameter space. Our computation applies away from the two isotropic border lines $h=1$, $k=0$.

Fig. 2. Lowest eigenvalues of MM^t for fixed $h=0.5$. Lines are guides for the eye.