Sept 26/93

# Transitivity of Locality and Duality in Quantum Field Theory. Some Modular Aspects

H.J. Borchers

Institut für Theoretische Physik
Universität Göttingen
Bunsenstrasse 9, D 3400 Göttingen

and

Jakob Yngvason

Science Institute
University of Iceland
Dunhaga 3, IS 107 Reykjavik, Iceland

## ABSTRACT

Duality conditions for Wightman fields are formulated in terms of the Tomita conjugations $S$ associated with algebras of unbounded operators. It is shown that two fields that are relatively local to an irreducible field fulfilling a condition of this type are relatively local to each other. Moreover, a local net of von Neumann algebras associated with such a field satisfies (essential) duality. These results do not rely on Lorentz covariance but follow from the observation that two algebras of (un)bounded operators with the same Tomita conjugation have the same (un)bounded weak commutant if one algebra is contained in the other.

## 1. Introduction

The modular theory of Tomita and Takesaki [To, Ta] is (in its simplest version) the study of the antilinear map $S : A\Omega \mapsto A^*\Omega$ where $A$ runs through the elements of a von Neumann algebra $\mathcal{M}$ having $\Omega$ as a cyclic and separating vector. In this paper we shall call $S$ a *Tomita conjugation*. The basic result of the theory is Tomita's theorem which states that the polar decomposition of $S$ gives rise to an antiunitary isomorphism between $\mathcal{M}$ and its commutant $\mathcal{M}'$, and a one-parameter group of automorphisms of $\mathcal{M}$. Quantum field theory in the algebraic setting of Haag, Kastler and Araki [H1] is a natural client of Tomita-Takesaki theory because analytic vectors for the energy (in particular the vacuum) are cyclic and and separating for the local von Neumann algebras. Indeed, several recent developments in algebraic quantum field theory are based on modular theory, see, e.g., [B2-3,BGL,BAL1-2,BSch,BSu,GF,Wie1-2].

In their pioneering work [BW1-2] Bisognano and Wichmann analyzed the modular structure of algebras generated by Lorentz covariant Wightman field operators localized in wedge shaped regions. They showed that the modular conjugation is essentially given by the PCT operator and the modular group is the one parameter group of Lorentz-boosts defined by the wedge. Using this result they then proved essential duality [Ro] of local nets of von Neumann algebras generated by Lorentz covariant Wightman fields. The connection between modular structure and duality was further studied in [Ri] within the Haag-Kastler-Araki framework. In two space-time dimensions a geometrical interpretation of the modular operators of wedge algebras has recently been derived from the spectrum condition for the translation group alone [B2]. This result has been used to investigate duality properties in conformal field theory by algebraic methods [BGL, GF], without reference to Wightman fields.

Coming back to the work of Bisognano and Wichmann, one may distinguish between three separate issues in their papers. The first is the identification of modular operators associated with the wedge algebras of the *unbounded* field operators. This part relies on the Lorentz covariance of the quantum field. The second is the use of the modular structure to derive criteria for the existence of local nets of *bounded* operators associated with the field. This aspect has been further studied by Driessler, Summers and Wichmann [DSW], Buchholz [Bu1] and Inoue [I3]. The third is the derivation of duality properties of local nets of von Neumann algebras associated with the field, also discussed in [W], [DSW], [B] and [I3].

In a sequence of papers [BY1-4] we have studied the connection between Wightman fields and local nets of von Neumann algebras from a somewhat different point of view, stressing the relation to noncommutative moment problems. In [BY4] we formulated a criterion for the existence of a local net of von Neumann algebras associated with a given Wightman field in terms of a certain positivity property of the Wightman functions (see Theorem 4.4 below). This criterion does not rely on any modular structure, a fact that was somewhat hidden in the original exposition of this result in [BY1]. In particular, it applies to fields without Lorentz covariance, and can even be extended in a straightforward way to fields with more general localization properties than Wightman fields (e.g., fields localized in space-like cones [BF] and to fields on curved space-time).

In view of this result it appears natural to regard the modular structure of algebras of unbounded field operators as a separate ingredient and ask which additional properties of the field and associated nets of von Neumann algebras can be derived from informations about this structure. In the present paper we consider two closely related properties of this sort. The first is the transitivity of relative locality, that was established in [B1] for Lorentz covariant Wightman fields. The second is the condition of (essential) duality referred to above.

It turns out that only the elementary part of modular theory is needed for the present purpose, namely the part which does not make use of the polar decomposition of the Tomita conjugation $S$. Such an operator may be defined for linear $*$-invariant families (not only algebras) of unbounded operators with a cyclic vector, provided the vector is also cyclic for the unbounded weak commutant of the family. In the next section we discuss this general mathematical framework. The main observation, on which all subsequent results are based, is the following one: If one operator family is contained in another and both have the same Tomita conjugation, then the (un)bounded weak commutants of the families are identical. In the case of von Neumann algebras and bounded commutants, this statement follows immediately from Tomita's theorem. However, it is in fact an elementary result with a simple proof that applies also to unbounded operators. Because of slight complications due to domain questions we bring this result in two versions below (Propositions 2.7 and 2.9).

In Section 3 we consider nets of unbounded operator algebras generated by Wightman fields. In order to avoid domain problems we require mild energy bounds for the field operators. We formulate general duality conditions for such nets in terms of the Tomita conjugations, and show how they lead to the concept of equivalence classes of relatively local fields (Theorem 3.5). We also point out that duality itself is really not the essential point, but rather the identity of the Tomita conjugations that follows from it (Theorem 3.6). Since applications to more general objects than Wightman fields are also of potential interest (e.g., to fields with different localization properties, or fields on curved space time), we have strived to isolate those aspects of the Wightman framework that are really needed to obtain these results.

In Section 4 we discuss the connection between modular properties of an unbounded field net and a net of von Neumann algebras associated with it, extending results of Bisognano and Wichmann and of Driessler, Summers and Wichmann to our general framework. In particular we show that duality of the unbounded net leads to duality of the net of von Neumann algebras (Theorem 4.5), and that if an irreducible field within an equivalence class of relatively local fields can be associated with a local net of von Neumann algebras, then the whole equivalence class can be associated with such a net (Theorems 4.6 and 4.7). In the final Section 5 we generalize the previous considerations to include Fermi fields as well as Bose fields.

## 2. Tomita conjugations and unbounded commutants

We begin by fixing the terminology and introducing some notation. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}$ a linear subdomain of $\mathcal{H}$. The family of all closable linear operators $X$, $\mathcal{H} \supset D(X) \to \mathcal{H}$, such that $\mathcal{D} \subset D(X) \cap D(X^*)$ will as in [Sch] be denoted by $\mathcal{C}^\dagger(\mathcal{D}, \mathcal{H})$. More generally, if $\mathcal{D}_1 \subset \mathcal{H}$, we denote by $\mathcal{C}^\dagger(\mathcal{D}, \mathcal{D}_1)$ the subfamily of all operators in $\mathcal{C}^\dagger(\mathcal{D}, \mathcal{H})$ with range in $\mathcal{D}_1$. A *$*$-operator family* on $\mathcal{D}$ is a family $\mathcal{F}$ of closable, linear operators $\mathcal{D} \to \mathcal{H}$ such that if $X \in \mathcal{F}$, then $X^* \mid \mathcal{D} \in \mathcal{F}$. A $*$-operator family $\mathcal{F}$ is called *linear*, if all linear combinations of elements in $\mathcal{F}$ belong to $\mathcal{F}$. A linear $*$-operator family $\mathcal{P}$

is a $*$-operator *algebra* if the operators $X \in \mathcal{P}$ leave $\mathcal{D}$ invariant and $\mathcal{P}$ is also closed under operator multiplications. The largest $*$-operator algebra on $\mathcal{D}$ consists of all operators $X$ with domain $\mathcal{D}$ such that $\mathcal{D} \subset D(X^*)$ and $X$ and $X^*$ leave $\mathcal{D}$ invariant. This algebra is denoted by $\mathcal{L}^\dagger(\mathcal{D})$. If $\mathcal{D}_0 \subset \mathcal{D}$ is a linear subdomain of $\mathcal{D}$ we use the notation $(\mathcal{F}, \mathcal{D}_0)$ for the family consisting of the restrictions of the operators in $\mathcal{F}$ to $\mathcal{D}_0$. It is clear that $(\mathcal{F}, \mathcal{D}_0)$ is a $*$-operator family on $\mathcal{D}_0$.

The *graph topology* induced on a domain $\mathcal{D}$ by a family $\mathcal{F}$ of operators on $\mathcal{D}$ is defined by the set of all seminorms of the form

$$\|\psi\|_{X_1,\dots,X_N} = \|\psi\| + \sum_{i=1}^{N} \|X_i\psi\| \tag{2.1}$$

with $X_1, \dots, X_N \in \mathcal{F}$. If $\mathcal{D}_0 \subset \mathcal{D}$ is dense in $\mathcal{D}$ in the graph topology induced by $\mathcal{F}$, then it follows in particular that $\mathcal{D}_0$ is a *core* for each operator $X \in \mathcal{F}$, i.e., the closure $\bar{X} = X^{**}$ of $X$ as an operator on $\mathcal{D}$ is equal to the closure of the restriction $X \mid \mathcal{D}_0$. In fact, the domain of $(X \mid \mathcal{D}_0)^{**}$ is the closure of $\mathcal{D}_0$ with respect to the graph topology induced by the single operator $X$. If $\mathcal{F}$ is an *algebra*, and $\mathcal{D}_0$ is a core for each $X \in \mathcal{F}$, then it follows conversely that $\mathcal{D}_0$ is dense in $\mathcal{D}$ in the graph topology induced by $\mathcal{F}$ [Sch]. For general families $\mathcal{F}$, however, this need not be true, i.e., $\mathcal{D}_0$ may be a core for each single operator $X \in \mathcal{F}$ without being dense in the graph topology induced by the whole family.

Weak and strong commutants of $*$-operator families play an important role in subsequent considerations. For weak commutants we use the following terminology:

**2.1 Definition.** Let $\mathcal{F}$ be a $*$-operator family on $\mathcal{D}$ and let $\mathcal{D}_0$ be a linear subdomain of $\mathcal{D}$. The *unbounded weak commutant* of $\mathcal{F}$ on $\mathcal{D}_0$ consists of all operators $Y \in \mathcal{C}^\dagger(\mathcal{D}_0, \mathcal{H})$, such that the equality

$$\langle X^*\phi, Y\psi\rangle = \langle Y^*\phi, X\psi\rangle \tag{2.2}$$

holds for all $\phi$, $\psi \in \mathcal{D}_0$. The unbounded weak commutant is denoted by $(\mathcal{F}, \mathcal{D}_0)^{uw}$. The *bounded part* of $(\mathcal{F}, \mathcal{D}_0)^{uw}$ is called the *weak commutant* and denoted by $(\mathcal{F}, \mathcal{D}_0)^{w}$.

For later reference we state without proofs some elementary properties of the unbounded and bounded weak commutants.

**2.2 Lemma.**

(i) *The restriction of the unbounded weak commutant to* $\mathcal{D}_0$*, i.e.,* $((\mathcal{F},\mathcal{D}_0)^{uw},\mathcal{D}_0)$*, is a linear* $*$*-operator family on* $\mathcal{D}_0$*.*

(ii) $(\mathcal{F},\mathcal{D}_0)^{w}$ *is a weakly closed and* $*$*-invariant linear subspace of the algebra* $\mathcal{B}(\mathcal{H})$ *of bounded operators on* $\mathcal{H}$*.*

(iii) *If* $\mathcal{P}$ *is a* $*$*-algebra on* $\mathcal{D}$ *and* $\mathcal{G}$ *is a* $*$*-family of algebraic generators*† *for* $\mathcal{P}$*, then* $(\mathcal{G},\mathcal{D})^{uw}=(\mathcal{P},\mathcal{D})^{uw}$*.*

(iv) *If* $\mathcal{F}$ *is a* $*$*-operator family on* $\mathcal{D}$ *and* $\mathcal{D}_0\subset\mathcal{D}$ *is a core for every operator in* $\mathcal{F}$*, then* $(\mathcal{F},\mathcal{D}_0)^{w}=(\mathcal{F},\mathcal{D})^{w}$*.*

(v) *Let* $\mathcal{F}$ *be a* $*$*-operator family on* $\mathcal{D}$*. If* $\mathcal{D}_0\subset\mathcal{D}$ *and* $Y\in(\mathcal{F},\mathcal{D}_0)^{uw}$ *are such that for each* $X\in\mathcal{F}$ *the domain* $\mathcal{D}_0$ *is dense in* $\mathcal{D}$ *in the graph topology induced by* $\{Y,Y^*,X,X^*\}$*, then* $Y\in(\mathcal{F},\mathcal{D})^{uw}$*.*

Besides weak commutants we also consider strong commutants:

**2.3 Definition.** Let $\mathcal{F}$ be a family of closable linear operators on $\mathcal{D}$. The *unbounded strong commutant* of $\mathcal{F}$, denoted $(\mathcal{F},\mathcal{D})^{us}$, consist of those closable operators $Z$ that satisfy

(a) $\mathcal{D}\subset D(Z)$, $Z\mathcal{D}\subset\mathcal{D}$ and $\mathcal{F}\mathcal{D}\subset D(Z)$,

(b) $[X,Z]\psi=0$ for all $X\in\mathcal{F}$, $\psi\in\mathcal{D}$.

The bounded part of $(\mathcal{F},\mathcal{D})^{us}$ is called the *strong commutant* and denoted by $(\mathcal{F},\mathcal{D})^{s}$.

The following properties are easily verified, using Lemma 2.2 (iii) and (v):

**2.4 Lemma.**

(i) *Let* $\mathcal{P}$ *be a* $*$*-algebra on a domain* $\mathcal{D}$*. If* $Y\in(\mathcal{P},\mathcal{D})^{uw}$ *and* $Y$ *leaves* $\mathcal{D}$ *invariant, then* $Y\in(\mathcal{P},\mathcal{D})^{us}$*. Conversely, if* $Y\in(\mathcal{P},\mathcal{D})^{us}$ *and* $\mathcal{D}\subset D(Y^*)$*, then* $Y\in(\mathcal{P},\mathcal{D})^{uw}$*.*

(ii) *Let* $\mathcal{P}$ *be a* $*$*-algebra on a domain* $\mathcal{D}$ *and* $\mathcal{G}\subset\mathcal{P}$ *a* $*$*-family of algebraic generators for* $\mathcal{P}$*. Let* $Y\in\mathcal{L}^\dagger(\mathcal{D})$*. If* $Y\in(\mathcal{G},\mathcal{D}_0)^{uw}$ *for some* $\mathcal{D}_0\subset\mathcal{D}$*, and for each* $X\in\mathcal{G}$ *the domain* $\mathcal{D}_0$ *is dense in* $\mathcal{D}$ *in the graph topology induced by* $\{Y,Y^*,X,X^*\}$*, then* $Y\in(\mathcal{P},\mathcal{D})^{us}$*.*

Next we consider the definition of Tomita conjugations for $*$-families of unbounded operators.

---

† By this we simply mean that $\mathcal{P}$ consists of finite sums of products of elements in $\mathcal{G}$. It is, however, clearly sufficient to require that these linear combination are dense in the topology on $\mathcal{P}$ generated by the seminorms $X\mapsto\|X\psi\|$ and $X\mapsto\|X^*\psi\|$ with $\psi\in\mathcal{D}$.

Let $\mathcal{F}$ be a linear $*$-operator family on a dense domain $\mathcal{D} \subset \mathcal{H}$ and let $\Omega$ be a vector in $\mathcal{D}$ that is cyclic for $\mathcal{F}$ as well as for $(\mathcal{F}, \mathcal{F}\Omega)^{uw}$. We consider only such families containing the unit operator, so $\Omega \in \mathcal{F}\Omega$. For every $X \in \mathcal{F}$ and $Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}$ we have

$$\langle Y\Omega, X^*\Omega\rangle = \langle X\Omega, Y^*\Omega\rangle. \tag{2.3}$$

Because $\{Y\Omega \mid Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}\}$ is dense in $\mathcal{H}$, we can define an antilinear operator $S^{(0)} : \mathcal{F}\Omega \to \mathcal{H}$ by

$$S^{(0)} X\Omega = X^*\Omega.$$

It follows also from (2.3) that ${S^{(0)}}^* Y\Omega = Y^*\Omega$ if $Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}$. In particular, $S^{(0)}$ has a densely defined adjoint and is hence closable. We denote its closure by $S$, or by $S_{\mathcal{F}}$ when the dependence on $\mathcal{F}$ is important. The dependence of $S$ on the vector $\Omega$ can be suppressed in the notation because $\Omega$ will be fixed in each case considered in the sequel.

The closed, antilinear operator $S$ satisfies $S^2 = \mathrm{id}$. It has the polar decomposition

$$S = J\Delta^{1/2} = \Delta^{-1/2}J \quad \text{with } \Delta = (S^*S) \geq 0 \quad \text{and } J = J^*, \quad J^2 = 1.$$

We shall not make use of this decomposition of $S$, however, except for the trivial remark that since $J$ is bounded, the domain $D(S)$ is the closure of $\mathcal{F}\Omega$ with respect to the graph topology induced by $\Delta^{1/2}$. The papers of Inoue [I1-2] contain several results on the action of the modular group generated by $\Delta$ in the case $\mathcal{F}$ is an algebra satisfying some additional hyptheses.

The domain of the adjoint $S^* = J\Delta^{-1/2}$ contains $(\mathcal{F}, \mathcal{F}\Omega)^{uw}\Omega$, and

$$S^* Y\Omega = Y^*\Omega \tag{2.4}$$

if $Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}$. If $\mathcal{F}$ is an algebra of bounded operators, then $S^*$ is equal to the Tomita conjugation of the von Neumann algebra $\mathcal{F}'$, see, e.g., [KR]. For unbounded operator families the domain of $S^*$ is in general strictly larger than $(\mathcal{F}, \mathcal{F}\Omega)^{uw}\Omega$. There are important cases in quantum field theory, however, where equality holds as discussed in the next setion.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two linear $*$-operator families on a common domain $\mathcal{D}$ with $\mathcal{F}_1 \subset \mathcal{F}_2$ and thus $(\mathcal{F}_2, \mathcal{D})^{uw} \subset (\mathcal{F}_1, \mathcal{D})^{uw}$. If $\Omega$ is cyclic for both $\mathcal{F}_1$ and $(\mathcal{F}_2, \mathcal{D})^{uw}$, then $S_{\mathcal{F}_1}$ and

$S_{\mathcal{F}_2}$ are well defined and $S_{\mathcal{F}_1} \subset S_{\mathcal{F}_2}$. Since $S_{\mathcal{F}_i}$ is, by definition, the closure of $S_{\mathcal{F}_i}|\mathcal{F}_i\Omega$, $i = 1, 2$, one immediately shows:

**2.5 Lemma.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be as above. The following are equivalent:*

(i) $S_{\mathcal{F}_1} = S_{\mathcal{F}_2}$.

(ii) $\mathcal{F}_2\Omega \subset D(S_{\mathcal{F}_1}) = D(\Delta_{\mathcal{F}_1}^{1/2})$.

(iii) $\mathcal{F}_1\Omega$ *is a core for* $\Delta_{\mathcal{F}_2}^{1/2}$.

A useful sufficient criterion for (iii) is that $\mathcal{F}_1\Omega \subset D(\Delta_{\mathcal{F}_2}^{1/2})$ and $\mathcal{F}_1\Omega$ is invariant under the unitary group $\Delta_{\mathcal{F}_2}^{it}$ generated by $\Delta_{\mathcal{F}_2}$. For later use we also note the following simple result.

**2.6 Lemma.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be as above. Assume there exists a $*$-operator family $\mathcal{Q} \subset (\mathcal{F}_2, \mathcal{F}_2\Omega)^{uw}$ such that $\Omega$ is cyclic for $\mathcal{Q}$ and $S_{\mathcal{F}_1}^* = S_{\mathcal{Q}}$. Then $S_{\mathcal{F}_1} = S_{\mathcal{F}_2}$.*

*Proof.* Since $\mathcal{Q} \subset (\mathcal{F}_2, \mathcal{F}_2\Omega)^{uw}$ we have $S_{\mathcal{Q}} \subset S_{\mathcal{F}_2}^*$ by (2.4) and thus $S_{\mathcal{F}_2} \subset S_{\mathcal{Q}}^* = S_{\mathcal{F}_1}$. □

We now assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same Tomita conjugation and want to draw some conclusions about the unbounded weak commutants of the operator families $\mathcal{F}_1$ and $\mathcal{F}_2$. Our first result applies to the case that the larger family is an *algebra*.

**2.7 Proposition.** *Let $\mathcal{P}$ be a $*$-operator algebra on a dense domain $\mathcal{D}$ and $\mathcal{F} \subset \mathcal{P}$ a linear $*$-operator subfamily containing the unit operator. Suppose $\Omega \in \mathcal{D}$ is cyclic for $\mathcal{F}$ and $(\mathcal{P}, \mathcal{P}\Omega)^{uw}$ (and hence also for $\mathcal{P}$ and $(\mathcal{F}, \mathcal{P}\Omega)^{uw}$). If the corresponding Tomita conjugations for $\mathcal{F}$ and $\mathcal{P}$ are equal, $S_{\mathcal{F}} = S_{\mathcal{P}}$, then $(\mathcal{F}, \mathcal{P}\Omega)^{uw} \subset (\mathcal{P}, \mathcal{F}\Omega)^{uw}$. If in addition $\mathcal{F}\Omega$ is a core for each $X \in \mathcal{P}$, then $(\mathcal{F}, \mathcal{P}\Omega)^{uw} = (\mathcal{P}, \mathcal{P}\Omega)^{uw}$.*

*Proof.* Let $Y \in (\mathcal{F}, \mathcal{P}\Omega)^{uw}$, $X \in \mathcal{P}$ and $Z_1$, $Z_2 \in \mathcal{F}$. Then

$$\langle Y^* Z_1\Omega, X Z_2\Omega\rangle = \langle Z_1\Omega, Y X Z_2\Omega\rangle = \langle Y^*\Omega, Z_1^* X Z_2\Omega\rangle,$$

where we have used that $\mathcal{P}$ is an algebra and $Y \in (\mathcal{F}, \mathcal{P}\Omega)^{uw}$. If we insert $1 = S_{\mathcal{P}}^2 = S_{\mathcal{F}} S_{\mathcal{P}}$ and use that $S_{\mathcal{F}}^* Y^*\Omega = Y\Omega$, we obtain

$$\langle Y^*\Omega, Z_1^* X Z_2\Omega\rangle = \langle Z_2^* X^* Z_1\Omega, Y\Omega\rangle = \langle Y^* X^* Z_1\Omega, Z_2\Omega\rangle = \langle X^* Z_1\Omega, Y Z_2\Omega\rangle.$$

Hence $Y$ commutes weakly with $X$ on $\mathcal{F}\Omega$ and the first part of the proposition is established.

Since $X\Omega \in \mathcal{P}\Omega \subset D(Y)$, weak commutativity of $X$ and $Y$ on $\mathcal{F}\Omega$ implies also that

$$\langle X^* Z\Omega, Y\Omega\rangle = \langle Y^* Z\Omega, X\Omega\rangle = \langle Z\Omega, YX\Omega\rangle$$

for all $Z \in \mathcal{F}$. This means that $Y\Omega \in D((X^\dagger|\mathcal{F}\Omega)^*)$, where we have denoted $X^*|\mathcal{P}\Omega = X^\dagger$. If $\mathcal{F}\Omega$ is a core for $X^\dagger$ we thus conclude, using that $\mathcal{F}\Omega$ is dense,

$$[Y, (X^\dagger)^*]\Omega = 0 \qquad \text{for all } X \in \mathcal{P},\ Y \in (\mathcal{F}, \mathcal{P}\Omega)^{uw}. \tag{2.5}$$

Next, pick $X_1,\ X_2 \in \mathcal{P}$. Since $\mathcal{P}$ is an algebra, we obtain, using (2.5) twice,

$$\langle X^* X_1\Omega, YX_2\Omega\rangle = \langle X^\dagger X_1\Omega, (X_2^\dagger)^* Y\Omega\rangle = \langle (XX_2)^\dagger X_1\Omega, Y\Omega\rangle$$
$$= \langle X_1\Omega, ((XX_2)^\dagger)^* Y\Omega\rangle = \langle X_1\Omega, YXX_2\Omega\rangle = \langle Y^* X_1\Omega, XX_2\Omega\rangle.$$

Hence $X$ commutes weakly with $Y$ on $\mathcal{P}\Omega$ if $\mathcal{F}\Omega$ is a core for each operator in $\mathcal{P}$. □

**2.8 Remark.** In the case of two von Neumann algebras $\mathcal{M}_1 \subset \mathcal{M}_2$ with a common cyclic and separating vector, Proposition 2.7 expresses the well known fact (see, e.g., [KR] Theorem 9.2.36) that $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $\Delta_{\mathcal{M}_1} = \Delta_{\mathcal{M}_2}$. For two Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{N} \subset \mathcal{M}'$ this means that $\mathcal{N} = \mathcal{M}'$ if and only if $\Delta_{\mathcal{M}} = \Delta_{\mathcal{N}}^{-1}$, or equivalently, if and only if $S_{\mathcal{N}}^* = S_{\mathcal{M}}$.

We now state and prove a variant of Proposition 2.7. In this version the larger family is not necessarily an algebra, and the relevant domain is generated by a subfamily of its strong commutant. We employ the following notation: If $\mathcal{Q}$ is a $*$-operator family, then $D^*(\mathcal{Q})$ denotes the intersection of the domains of the adjoints, $D(Y^*)$, with $Y \in \mathcal{Q}$.

**2.9 Proposition.** *Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two linear $*$-operator families (containing the unit operator) on a dense domain $\mathcal{D}$. Let $\mathcal{Q}$ be another $*$-operator family on $\mathcal{D}$ with $\mathcal{Q} \subset (\mathcal{F}_2, \mathcal{D})^{us}$. Suppose $\Omega \in \mathcal{D}$ is cyclic for $\mathcal{F}_1$ and $\mathcal{Q}$ (and hence also for $\mathcal{F}_2$ and $(\mathcal{F}_1, \mathcal{D})^{us}$). If the corresponding Tomita conjugations of $\mathcal{F}_1$ and $\mathcal{F}_2$ are equal, $S_{\mathcal{F}_1} = S_{\mathcal{F}_2}$, then $(\mathcal{F}_1, \mathcal{Q}\Omega)^{uw} \cap \mathcal{C}^\dagger(\mathcal{Q}\Omega, D^*(\mathcal{Q})) = (\mathcal{F}_2, \mathcal{Q}\Omega)^{uw} \cap \mathcal{C}^\dagger(\mathcal{Q}\Omega, D^*(\mathcal{Q}))$. In particular,*

$$(\mathcal{F}_1, \mathcal{Q}\Omega)^{uw} \cap \mathcal{L}^\dagger(\mathcal{D}) \subset (\mathcal{F}_2, \mathcal{Q}\Omega)^{uw}$$

*and*

$$(\mathcal{F}_1, \mathcal{Q}\Omega)^{w} \cap (\mathcal{Q}, \mathcal{Q}\Omega)^{w} \subset (\mathcal{F}_2, \mathcal{Q}\Omega)^{w}.$$

*Proof.* Pick $Y \in (\mathcal{F}_1, \mathcal{Q}\Omega)^{uw} \cap \mathcal{C}^\dagger(\mathcal{Q}\Omega, D^*(\mathcal{Q}))$ and $Z_1,\ Z_2 \in \mathcal{Q}$. We first show that $Z_2^* Y^* Z_1 \Omega \subset D(S_{\mathcal{F}_1}^*)$, and

$$S_{\mathcal{F}_1}^* Z_2^* Y^* Z_1 \Omega = Z_1^* Y Z_2 \Omega. \tag{2.6}$$

Indeed, if $X \in \mathcal{F}_1$ we have

$$\begin{aligned} \langle Z_2^* Y^* Z_1 \Omega, S_{\mathcal{F}_1} X \Omega \rangle &= \langle Z_2^* Y^* Z_1 \Omega, X^* \Omega \rangle = \langle Y^* Z_1 \Omega, X^* Z_2 \Omega \rangle \\ &= \langle X Z_1 \Omega, Y Z_2 \Omega \rangle = \langle Z_1 X \Omega, Y Z_2 \Omega \rangle = \langle X \Omega, Z_1^* Y Z_2 \Omega \rangle, \end{aligned}$$

which proves the statement. Next, take $X \in \mathcal{F}_2$. Since $S_{\mathcal{F}_1} = S_{\mathcal{F}_2}$ by assumption, we have $S_{\mathcal{F}_1} X \Omega = X^* \Omega$. Using this and (2.6) we now obtain for $Z_1,\ Z_2 \in \mathcal{Q}$ and $Y$ as above:

$$\begin{aligned} \langle Y^* Z_1 \Omega, X Z_2 \Omega \rangle &= \langle Y^* Z_1 \Omega, Z_2 X \Omega \rangle = \langle Z_2^* Y^* Z_1 \Omega, X \Omega \rangle = \langle Z_2^* Y^* Z_1 \Omega, S_{\mathcal{F}_1}^2 X \Omega \rangle \\ &= \langle X^* \Omega, Z_1^* Y Z_2 \Omega \rangle = \langle Z_1 X^* \Omega, Y Z_2 \Omega \rangle = \langle X^* Z_1 \Omega, Y Z_2 \Omega \rangle. \end{aligned}$$

Hence $Y$ commutes weakly with $\mathcal{F}_2$ on $\mathcal{Q}\Omega$. The rest of the statement is a consequence of the fact that $\mathcal{L}^\dagger(\mathcal{D})$ and $(\mathcal{Q}, \mathcal{Q}\Omega)^w$ are both contained in $\mathcal{C}^\dagger(\mathcal{Q}\Omega, D^*(\mathcal{Q}))$. □

## 3. Transitivity of locality and duality for quantum fields

Let $\{\Phi_\iota\}_{\iota \in I}$ be a family of translationally covariant Bose-Wightman fields defined on a common invariant domain *cd* in a Hilbert space $\mathcal{H}$. Each field $\Phi_\iota$ is an operator valued tempered distribution on Minkowski space $\mathbf{R}^d$; the index set $I$ may be finite or infinite. If $\mathcal{O}$ is an open subset of $\mathbf{R}^d$ we denote by $\mathcal{P}(\mathcal{O})$ the $*$-algebra generated algebraically by all smeared field operators $\Phi_\iota(f)$, $\iota \in I$, with support of the test functions $f$ in $\mathcal{O}$.

The algebras $\mathcal{P}(\mathcal{O})$ have certain properties that follow from the Wightman axioms. Not all of these properties will be needed here, and we list only those that are important for the present purpose.

(A) The $\mathcal{P}(\mathcal{O})$ are $*$-operator algebras (containing the unit operator) on a common dense domain $\mathcal{D} \subset \mathcal{H}$.

(I) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{P}(\mathcal{O}_1) \subset \mathcal{P}(\mathcal{O}_2)$.

(L) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then $\mathcal{P}(\mathcal{O}_1)$ and $\mathcal{P}(\mathcal{O}_2)$ commute on $\mathcal{D}$.

For convenience, we shall refer to a net of $*$-algebras $\mathcal{P}(\mathcal{O})$, satisfying (A) and (I) as a *field net*, and if (L) is also fullfilled we speak of a field net satisfying locality or a *local field net*.

Translational covariance and spectrum condition imply in Wightman theory the Reeh-Schlieder property: If $\Omega$ denotes the vacuum vector and $\mathcal{O}$ a nonempty open set, then one has

(RS) For all $\mathcal{O}$, $\mathcal{P}(\mathcal{O})\Omega$ is dense in $\mathcal{H}$ .

Under the same premises one can even derive a considerably stronger property for any translationally covariant subnet $\mathcal{P}_0(\cdot)$ such that $\mathcal{P}_0(\mathbf{R}^d)$ is generated by $\cup_{x\in\mathbf{R}^d}\mathcal{P}_0(\mathcal{O}+x)$ for all $\mathcal{O}$:

(BZ) $\mathcal{P}_0(\mathcal{O})\Omega$ is dense in $\mathcal{P}_0(\mathbf{R}^d)\Omega$ in the graph topology induced by $\mathcal{P}(\mathbf{R}^d)$.

This result was essentially proved in [BZ], Theorem 1. Since the version above is slightly stronger than the statement in [BZ], we prove it in the Appendix.

Instead of requiring that the algebras $\mathcal{P}(\mathcal{O})$ are from the outset defined for all open sets $\mathcal{O}$, it is often natural to consider to begin with some particular class $\mathcal{K}$ of such subsets, e.g. all (open) double cones. For the purpose of this paper a specific geometrical interpretation of these sets is not needed, but only that the class $\mathcal{K}$ is sufficiently rich in the following sense:

(K) For each pair of space-like separated points $x$, $y\in\mathbf{R}^d$, there are space-like separated $K_1$, $K_2\in\mathcal{K}$ with $x\in K_1$ and $y\in K_2$. Morover, if $K\in\mathcal{K}$, then $K'$ contains some $K_1\in\mathcal{K}$.

When such a class $\mathcal{K}$ has been singled out one may extend the net to arbitrary regions $\mathcal{O}$ by defining $\mathcal{P}(\mathcal{O})$ as the algebra generated by $\mathcal{P}(K)$ with $K\subset\mathcal{O}$, $K\in\mathcal{K}$:

$$\mathcal{P}(\mathcal{O})=\bigvee_{K\subset\mathcal{O},K\in\mathcal{K}}\mathcal{P}(K) \tag{3.1}$$

with the understanding that $\mathcal{P}(\emptyset)=\mathbf{C}\cdot 1$. The operators in $\mathcal{P}(\mathcal{O})$ are in general unbounded and we prefer not to close the algebras in any particular topopolgy. Hence (3.1) is meant in the purely algebraic sense. If $\mathcal{K}$ is the class of open double cones and the net $\mathcal{P}(\cdot)$ is generated by Wightman fields smeared with test functions of compact support, then the definition (3.1) agrees with the previous one, i.e. $\mathcal{P}(\mathcal{O})$ is the algebra generated by field operators whose test functions have support in $\mathcal{O}$. This follows by partition of unity. One can envisage more general cases, however, where (3.1) has to be included as a separate definition, e.g., when $\mathcal{K}$ is the class of space-like cones and $\mathcal{P}(K)$ is generated by cone-localized operators, or when $\mathcal{K}$ contains only double cones of some minimal size. In these cases (RS) and (BZ) should not be required for all nonempty, open sets but only for sets containing a (nonempty) $K \in \mathcal{K}$.

Besides $\mathcal{K}$ we shall consider a further class of sets, $\mathcal{W}$, related to (K) in the following way:

(KW) Each $K \in \mathcal{K}$ is contained in some $W \in \mathcal{W}$, and if $K_1$, $K_2 \in \mathcal{K}$ are space-like separated, then there are space-like separated $W_1$, $W_2 \in \mathcal{W}$, such that $K_i \subset W_i$, $i = 1, 2$. Moreover, each $W \in \mathcal{W}$ contains some $K \in \mathcal{K}$ and the same holds for $W'$.

For concreteness sake one might think of $\mathcal{W}$ as the class of all (open) space-like wedges, but again such a specific interpretation is not necessary. In fact, the possibility $\mathcal{W} = \mathcal{K}$ is not excluded.

If conditions (K) and (KW) are fulfilled, then locality of the whole field net $\mathcal{P}(\cdot)$ is equivalent to locality of the net restricted to either one of the classes $\mathcal{K}$ or $\mathcal{W}$. Denoting the space-like complement of the closure of $\mathcal{O}$ by $\mathcal{O}'$, locality can thus be stated as the condition

$$\mathcal{P}(\mathcal{O}') \subset (\mathcal{P}(\mathcal{O}), \mathcal{D})^{us} \tag{3.2}$$

for all $\mathcal{O} \in \mathcal{K}$, or equivalently, all $\mathcal{O} \in \mathcal{W}$.

Next we want to discuss the concept of *duality* for local field nets. If the algebras $\mathcal{P}(\mathcal{O})$ are von Neumann algebras on $\mathcal{H} = \mathcal{D}$ the usual duality condition [H2] means that the inclusion in (3.2) is an equality. Motivated by the considerations in the preceeding section we generalize this in the following way.

Let $\mathcal{P}(\cdot)$ be a local field net with a vector $\Omega$ satisfying (RS) for the class $\mathcal{K}$. Then for each $\mathcal{O}$ belonging to $\mathcal{K}$ or $\mathcal{W}$ the vector $\Omega$ is cyclic for $\mathcal{P}(\mathcal{O})$ and $\mathcal{P}(\mathcal{O}')$. From (3.2) it follows that the Tomita conjugations $S_{\mathcal{P}(\mathcal{O})}$ and $S_{\mathcal{P}(\mathcal{O}')}$ are well defined with $S_{\mathcal{P}(\mathcal{O}')} \subset S^*_{\mathcal{P}(\mathcal{O})}$.

**3.1 Definition.** The local field net $\mathcal{P}(\cdot)$ satisfies duality for the class $\mathcal{K}$, or $\mathcal{K}$-*duality*, if

$$S_{\mathcal{P}(K')} = S^*_{\mathcal{P}(K)} \tag{3.3}$$

for all $K \in \mathcal{K}$. In the same way, we say that net satisfies duality for the class $\mathcal{W}$, or $\mathcal{W}$-*duality*, if

$$S_{\mathcal{P}(W')} = S^*_{\mathcal{P}(W)} \tag{3.4}$$

holds for all $W \in \mathcal{W}$.

If the algebras $\mathcal{P}(\mathcal{O})$ consist of bounded operators this definition is equivalent to the usual one, i.e., $\mathcal{P}(\mathcal{O})'' = \mathcal{P}(\mathcal{O}')'$ by Remark 2.8. It appears here as a natural generalization, circumventing the problem of extending von Neumann's commutator theorem to the case of algebras of unbounded operators. For Wightman fields transforming covariantly with respect to a finite dimensional representation of the Lorentz group the duality condition is always fulfilled for the class $\mathcal{W}$ of space-like wedges in Minkowski space [BWI-II]. In this case $S_{\mathcal{P}(W)}$ can be explicitly computed for all $W \in \mathcal{W}$ and by inspection one sees that $S^*_{\mathcal{P}(W)} = S_{\mathcal{P}(W')}$.

Duality may be interpreted as a kind of maximality condition for local field nets. In fact, as we shall now show, a local field net satisfying $\mathcal{W}$-duality leads to a unique equivalence class of fields that are relatively local to each other.

To be precise, we must distinguish between two kinds of relative locality, strong and weak. Let $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ be two field nets with a common invariant dense domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$. We say that the nets are *strongly relatively local* on $\mathcal{D}$ if $\mathcal{P}_1(\mathcal{O}_1)$ and $\mathcal{P}_2(\mathcal{O}_2)$ commute on $\mathcal{D}$ for space like separated $\mathcal{O}_1$ and $\mathcal{O}_2$. If $\mathcal{D}_0 \subset \mathcal{D}$ and $\mathcal{P}_1(\mathcal{O}_1)$, $\mathcal{P}_2(\mathcal{O}_2)$ commute weakly on $\mathcal{D}_0$ in the sense of (2.1), we say that the nets are *weakly relatively local* on $\mathcal{D}_0$. When checking (weak) relative locality, it is as above sufficient to take $\mathcal{O}_1 = W \in \mathcal{W}$ and $\mathcal{O}_2 = W'$, because of (K) and (KW).

A connection between the two notions of relative locality follows immediately from Lemma 2.4 (ii):

**3.2 Lemma.** *Suppose $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ are weakly local to each other on $\mathcal{D}_0 \subset \mathcal{D}$. If there exist $*$-families of generators, $\mathcal{G}_1(W)$ and $\mathcal{G}_2(W')$, for the algebras $\mathcal{P}_1(W)$ and $\mathcal{P}_2(W')$, $W \in \mathcal{W}$, such that $\mathcal{D}_0$ is dense in $\mathcal{D}$ in the graph topology induced by $\mathcal{G}_1(W) \cup \mathcal{G}_2(W')$, then $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ are strongly relatively local to each other on $\mathcal{D}$.*

A sufficient condition for the hypotheses of Lemma 3.2 to hold can be stated in terms of mild energy bounds for the generators.

**3.3 Definition.** Let $X$ be a closable operator and $H$ a self-adjoint operator on $\mathcal{H}$. Let $E_I$ denote the spectral projector of $H$ correspondinging to $I \subset \mathbf{R}$. We say that $X$ obeys *compact $H$-bounds*, if $E_I\mathcal{H} \subset D(\bar{X})$ and $\bar{X}E_I$ is a bounded operator for all bounded $I$.

An equivalent condition is that $\bar{X}F(H)$ is a bounded operator for *some* measurable function $F$ which vanishes nowhere on the spectrum of $H$.

**3.4 Proposition.** *Suppose $\mathcal{G}$ is a family of operators on a dense domain $\mathcal{D} \subset \mathcal{H}$. Let $H$ be self adjoint and suppose that*
(i) *The operators in $\mathcal{G}$ obey compact $H$-bounds*
(ii) *$\mathcal{D}$ is invariant under $e^{itH}$, $t \in \mathbf{R}$, and for each $\psi \in \mathcal{D}$, $X \in \mathcal{G}$, the function $t \mapsto Xe^{itH}\psi$ is continuous and polynomially bounded.*
*Then every dense domain $\mathcal{D}_0 \subset \mathcal{D}$ that is invariant under the unitary group $e^{itH}$, $t \in \mathbf{R}$, is dense in $\mathcal{D}$ in the graph topology induced by $\mathcal{G}$.*

This result is a generalization of Lemma 5.2(b) in [DSW]. We prove it in the Appendix.

After this preparation we now come to the main result of this section. We consider three field nets, $\mathcal{P}_0(\cdot)$, $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ on a common dense domain $\mathcal{D}$ containing a vector $\Omega$ such that $\mathcal{D}_0 = \mathcal{P}_0(\mathbf{R}^d)\Omega$ is dense and (BZ) holds for $\mathcal{O} \in \mathcal{K}$ if $\mathcal{P}(\cdot)$ is defined as the net generated by $\mathcal{P}_0(\cdot)$, $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$. We denote by $\mathcal{P}_{0i}(\mathcal{O}) = \mathcal{P}_0(\mathcal{O}) \vee \mathcal{P}_i(\mathcal{O})$ the $*$-algebra generated by $\mathcal{P}_0(\mathcal{O})$ and $\mathcal{P}_i(\mathcal{O})$, and define $\mathcal{D}_{0i} = \mathcal{P}_{0i}(\mathbf{R}^d)\Omega$, $i = 1, 2$. It is assumed that $\mathcal{D}$, $\mathcal{D}_0$ and $\mathcal{D}_{0i}$ are invariant under time translations $e^{itH}$. This holds automatically if the field nets transform covariantly under time translations and $\Omega$ is invariant.

**3.5 Theorem.** *Assume*
(a) *$\mathcal{P}_0(\cdot)$ is local and satisfies $\mathcal{W}$-duality.*
(b) *$\mathcal{P}_i(\cdot)$ is weakly relatively local to $\mathcal{P}_0(\cdot)$ on $\mathcal{D}_{0i}$, $i = 1, 2$.*
*Then $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ are weakly relatively local to each other on $\mathcal{D}_0$.*
*If the algebras $\mathcal{P}_i(W)$, $i = 0, 1, 2$, $W \in \mathcal{W}$, possess families of generators, $\mathcal{G}_i(W)$, satisfying the conditions of Proposition 3.4, then the field nets are strongly relatively local to each other on the common invariant domain $\mathcal{D}$.*

*Proof.* Pick $W \in \mathcal{W}$. We use Lemma 2.6 and Proposition 2.7 with $\mathcal{F}_1 \equiv \mathcal{F} = \mathcal{P}_0(W)$, $\mathcal{F}_2 \equiv \mathcal{P} = \mathcal{P}_{02}(W)$ and $\mathcal{Q} = \mathcal{P}_0(W')$. Since $\mathcal{P}_0(\cdot)$ is local and weakly relatively local to $\mathcal{P}_2(\cdot)$ we know that $\mathcal{F}_2$ and $\mathcal{Q}$ commute weakly on $\mathcal{D}_{02}$. Duality of $\mathcal{P}_0(\cdot)$ thus implies

$S^*_{\mathcal{F}_1} = S_{\mathcal{Q}}$, so $S_{\mathcal{F}_1} = S_{\mathcal{F}_2}$ by Lemma 2.6. Since $\mathcal{P}_1(W') \subset (\mathcal{P}_0(W), \mathcal{P}_{01}(W')\Omega)^{uw}$, it follows by Proposition 2.7 that $\mathcal{P}_1(W') \subset (\mathcal{P}_2(W), \mathcal{P}_0(W')\Omega)^{uw}$. From (BZ) we know that $\mathcal{P}_0(W')\Omega$ is dense in $\mathcal{D}_0$ in the graph topology induced by $\mathcal{P}_1(W') \cup \mathcal{P}_2(W)$. Hence, by Lemma 2.2 (v), $\mathcal{P}_1(W')$ and $\mathcal{P}_2(W)$ are weakly relatively local to each other on $\mathcal{D}_0$. The last statement of the theorem follows from Lemma 3.2. □

It is clear from the proof that the only role of the duality assumption is to ensure that $\mathcal{P}_0(W)$ and $\mathcal{P}_{0i}(W)$ have the same Tomita conjugation. If this is known by other means the conclusions of the theorem thus also hold. If one applies Proposition 2.9 instead of Proposition 2.7 one obtains the following variant of Theorem 3.5.

**3.6 Theorem.** *Assume*

(a) $\mathcal{P}_0(\cdot)$ *is local.*

(b) $\mathcal{P}_i(\cdot)$ *is strongly relatively local to* $\mathcal{P}_0(\cdot)$ *on* $\mathcal{D}$, $i = 1, 2$.

(c) *The algebras* $\mathcal{P}_i(W)$ , $i = 1, 2$, $W \in \mathcal{W}$, *have families of generators* $\mathcal{G}_i(W)$ *such that* $\mathcal{G}_i(W)\Omega \subset D(S_{\mathcal{P}_0(W)})$.

*Then* $\mathcal{P}_1(\cdot)$ *and* $\mathcal{P}_2(\cdot)$ *are weakly relatively local to each other on* $\mathcal{D}_0$.

*If the generators satisfy the conditions of Proposition 3.4, then the field nets are strongly relatively local to each other on the common invariant domain* $\mathcal{D}$.

*Proof.* As in the proof of Theorem 3.5 we put $\mathcal{F}_1 = \mathcal{P}_0(W)$ and $\mathcal{Q} = \mathcal{P}_0(W')$, but this time we define $\mathcal{F}_2 = \mathcal{P}_0(W) + \mathcal{G}_2(W)$. Hypothesis (c) and Lemma 2.5 imply that the Tomita conjugations of $\mathcal{F}_1$ and $\mathcal{F}_2$ are equal. Since $\mathcal{P}_1(W') \subset (\mathcal{F}_1, \mathcal{Q}\Omega)^{uw} \cap \mathcal{L}^\dagger(\mathcal{D})$ we conclude from Proposition 2.9 that $\mathcal{P}_1(W')$ and $\mathcal{P}_2(W)$ commute weakly on $\mathcal{P}_0(W')\Omega$. Property (BZ) then ensures commutativity on $\mathcal{D}_0$. The energy bounds then imply commutativity on $\mathcal{D}$ as in Theorem 3.5 □

# 4. Duality for local von Neumann algebras generated by quantum fields

In this section we start by reviewing some facts concerning duality and essential duality for local nets of von Neumann algebras. We then discuss the question when a Wightman field can be associated with such a net, and link duality properties of the Wightman field with those of the von Neumann net.

We consider a net of von Neumann algebras $\mathcal{M}(K)$, defined for some specified class $\mathcal{K}$ of bounded open subsets of $\mathbf{R}^d$ satisfying condition (K) of the last section, and assume that conditions (I), (A) and (RS) are fulfilled (with $\mathcal{M}(\cdot)$ in place of $\mathcal{P}(\cdot)$). For more general regions $\mathcal{O}$ (in particular for $\mathcal{O} = W$ or $W'$ with $W \in \mathcal{W}$, or $\mathcal{O} = K'$ with $K \in \mathcal{K}$) we *define*

$$\mathcal{M}(\mathcal{O}) := \Big( \bigcup_{K \subset \mathcal{O},\ K \in \mathcal{K}} \mathcal{M}(K) \Big)''. \tag{4.1}$$

(Since the algebras $\mathcal{M}(K)$ consist of bounded operators it is convenient to include the weak closure in the definition, which was not done in (3.1).) One says that the net satisfies *duality* (more precisely $\mathcal{K}$-*duality*) if

$$\mathcal{M}(K) = \mathcal{M}(K')' \tag{4.2}$$

for all $K \in \mathcal{K}$. As remarked in the last section, (4.2) is equivalent to the condition

$$S_{\mathcal{M}(K)} = S^*_{\mathcal{M}(K')} \tag{4.3}$$

for the Tomita conjugations, if the algebras are von Neumann algebras.

If duality does not hold for a net $\mathcal{M}(\cdot)$, it is natural to ask whether $\mathcal{M}(\cdot)$ can at least be embedded into a net fulfilling this condition. For a local net the answer is affirmative in the particular case that the net satisfies the *essential duality* condition introduced by Roberts [Ro]. This means that besides $\mathcal{M}$ the net

$$K \mapsto \mathcal{M}^d(K) := \mathcal{M}(K')' \tag{4.4}$$

is also local. In fact, locality of $\mathcal{M}$ means that $\mathcal{M}(K) \subset \mathcal{M}^d(K)$, which implies $\mathcal{M}(K') \subset \mathcal{M}^d(K')$, and thus $\mathcal{M}^d(K')' \subset \mathcal{M}(K')' = \mathcal{M}^d(K)$. If $\mathcal{M}^d$ is local the opposite inclusion holds and $\mathcal{M}^d$ satisfies duality. It is also easy to see that $\mathcal{M}^d$ is the only net containing $\mathcal{M}$ with this property: If $\mathcal{N}$ is a local net containing $\mathcal{M}$, then $\mathcal{M}(K') \subset \mathcal{N}(K')$, and thus $\mathcal{N}(K) \subset \mathcal{N}(K')' \subset \mathcal{M}(K')' = \mathcal{M}^d(K)$. This implies also $\mathcal{N}(K') \subset \mathcal{M}^d(K')$, so if $\mathcal{N}$ satisfies duality we have $\mathcal{M}^d(K) = \mathcal{M}^d(K')' \subset \mathcal{N}(K')' = \mathcal{N}(K)$. Hence any local extension of $\mathcal{M}$ lies between $\mathcal{M}$ and $\mathcal{M}^d$, and if essential duality holds for the net $\mathcal{M}$ then it has a unique extension to a net satisfying duality, namely $\mathcal{M}^d$.

In [BW1,BW2] Bisognano and Wichmann established a link between essential duality for the algebras localized in double cones and duality for algebras localized in wedge-shaped regions. For convenience of the reader and later reference we formulate this as a Lemma below. Our version involves only the general conditions (K) and (KW) of Section 3.

Let $\mathcal{K}$ and $\mathcal{W}$ be two families of open subsets of Minkowski space satisfying conditions (K) and (KW). Let $\{\mathcal{M}(K)\}_{K\in\mathcal{K}}$ be a net of von Neumann algebras and define $\mathcal{M}(W)$ and $\mathcal{M}(W')$ as in (3.1) for $W \in \mathcal{W}$. Moreover, for $K \in \mathcal{K}$ define

$$\widehat{\mathcal{M}}(K) := \bigcap_{W \supset K,\, W \in \mathcal{W}} \mathcal{M}(W). \tag{4.5}$$

By (KW) this is always a local net if $\mathcal{M}(\cdot)$ is local. It is in general unrelated to the other extension $\mathcal{M}^d(\cdot)$ defined in (4.3), but if $\mathcal{M}(\cdot)$ satisfies $\mathcal{W}$-duality both extensions coincide:

**4.1 Lemma.** *If the duality condition*

$$\mathcal{M}(W)' = \mathcal{M}(W') \tag{4.6}$$

*holds for all $W \in \mathcal{W}$, then the net $\{\mathcal{M}(K)\}_{K\in\mathcal{K}}$ satisfies essential duality and*

$$\mathcal{M}^d(K) = \widehat{\mathcal{M}}(K) \tag{4.7}$$

*for all $K \in \mathcal{K}$.*

*Proof.* The net $\widehat{\mathcal{M}}$ can be extended to general regions $\mathcal{O}$ in Minkowski space by defining $\widehat{\mathcal{M}}(\mathcal{O})$ in the same way as in (4.1) with $\mathcal{M}$ replaced by $\widehat{\mathcal{M}}$. We first show that

$$\widehat{\mathcal{M}}(W) = \mathcal{M}(W) \quad \text{and} \quad \widehat{\mathcal{M}}(W') = \mathcal{M}(W') \tag{4.8}$$

for all $W \in \mathcal{W}$. It is clear that $\mathcal{M}(K) \subset \widehat{\mathcal{M}}(K)$ for all $K \in \mathcal{K}$ and thus $\mathcal{M}(\mathcal{O}) \subset \widehat{\mathcal{M}}(\mathcal{O})$ for all regions $\mathcal{O}$. Hence, by (4.3), $\mathcal{M}(W) = \mathcal{M}(W')' \supset \widehat{\mathcal{M}}(W')'$. On the other hand, by locality of $\widehat{\mathcal{M}}$, $\widehat{\mathcal{M}}(W')' \supset \widehat{\mathcal{M}}(W) \supset \mathcal{M}(W)$, so $\mathcal{M}(W) = \mathcal{M}(W')' = \widehat{\mathcal{M}}(W')' = \widehat{\mathcal{M}}(W)$.

The next assertion is that

$$\mathcal{M}(K') = \widehat{\mathcal{M}}(K') \tag{4.9}$$

for all $K \in \mathcal{K}$. The inclusion $\mathcal{M}(K') \subset \widehat{\mathcal{M}}(K')$ is clear, and if $K_1 \subset K'$, there exist by assumption space like separated $W_1, W \in \mathcal{W}$ such that $K_1 \subset W_1$ and $K \subset W$. This implies $\widehat{\mathcal{M}}(K_1) \subset \widehat{\mathcal{M}}(W') = \mathcal{M}(W') \subset \mathcal{M}(K')$ for all $K_1 \subset K'$, and thus $\widehat{\mathcal{M}}(K') \subset \mathcal{M}(K')$.

Finally, we show that the net $\{\widehat{\mathcal{M}}(K)\}_{K\in\mathcal{K}}$ satisfies duality. In fact, by (4.6) and (4.8) we have

$$\widehat{\mathcal{M}}(K) = \bigcap_{W\supset K} \mathcal{M}(W) = \bigcap_{W\supset K} \widehat{\mathcal{M}}(W')' =$$
$$\Big(\bigcup_{W\supset K} \widehat{\mathcal{M}}(W')\Big)' \supset \Big(\bigcup_{K'\supset W'} \widehat{\mathcal{M}}(W')\Big)' \supset \widehat{\mathcal{M}}(K')',$$

and the opposite inclusion follows from locality of $\widehat{\mathcal{M}}$.

Altogether we have shown that $\mathcal{M}(K')' = \widehat{\mathcal{M}}(K')' = \widehat{\mathcal{M}}(K)$, so $\mathcal{M}^d$ is equal to $\widehat{\mathcal{M}}$ and satisfies duality. □

We now turn to the connection between local nets of von Neumann algebras and local nets of algebras of unbounded operators. As in [DSW] and [BY4] we employ the following notion.

**4.2 Definition.** A field net $\{\mathcal{P}(K)\}_{K\in\mathcal{K}}$ is associated with a net of von Neumann algebras, $\{\mathcal{M}(K)\}_{K\in\mathcal{K}}$, if every $X \in \mathcal{P}(K)$ has an extension to a closed operator $\widetilde{X}$ with $\widetilde{X^\dagger} \subset \widetilde{X}^*$, such that $\widetilde{X}$ is affiliated with the von Neumann algebra $\mathcal{M}(K)$.

Affiliation of $\widetilde{X}$ with $\mathcal{M}(K)$ means precisely that the von Neumann algebra $m(\widetilde{X})$ generated by the polar decomposition of the extended operator is a subalgebra of $\mathcal{M}(K)$. It is in fact sufficient to require this for the operators in some $*$-invariant family $\mathcal{G}(K)$ of algebraic generators for $\mathcal{P}(K)$; hence the field net is associated with some *local* net of von Neumann algebras if and only if these generators have closed extensions that commute strongly for space-like separated regions. By (3.1) and (4.1) it then follows that the operators in $\mathcal{P}(\mathcal{O})$ have extensions affiliated with $\mathcal{M}(\mathcal{O})$ for arbitray regions $\mathcal{O}$.

As pointed out in [BY4] the extensions $\tilde{X}$ might in general have to be defined in a larger Hilbert space than the original operators $X$. In this paper, however, we shall only consider fields with the property that the weak commutants $\mathcal{P}(K)^w$ are algebras for

$K \in \mathcal{K}$, and for such fields an enlargement of the Hilbert space is not necessary. In fact, $\mathcal{P}(\cdot)$ is then always associated with the net

$$\mathcal{M}_{\min}(K) := \mathcal{P}(K)^{w\prime}, \tag{4.10}$$

which is a subnet of any other net of von Neumann algebras with this property. Note also that $\mathcal{M}_{\min}(\mathcal{O}) = \mathcal{P}(\mathcal{O})^{w\prime}$ for arbitrary regions $\mathcal{O}$, by (3.1) and (4.1).

If the weak commutants $\mathcal{P}(K)^w$ are algebras the question whether $\mathcal{P}(\cdot)$ can be associated with a *local* net of von Neumann algebras is the same as asking whether $\mathcal{M}_{\min}(\cdot)$ is local. Our criterion for locality of $\mathcal{M}_{\min}(\cdot)$ involves the following notion of positivity [P]:

**4.3 Definition.** Let $\mathcal{A}$ be a $*$-algebra with unit over $\mathbf{C}$ and suppose $X_0 \in \mathcal{A}$ commutes with all elements in $\mathcal{A}$. A linear functional $\omega$ on $\mathcal{A}$ is said to be *centrally positive* with respect to $X_0$ if $\omega$ is positive on all elements of the form $\sum_{n=1}^N X_0^n Y_n$ with $Y_n \in \mathcal{A}$ such that $\sum_{n=1}^N \lambda^n Y_n \in \mathcal{A}^+$ for all $\lambda \in \mathbf{R}$. Here $\mathcal{A}^+$ denotes the cone of elements of the form $\sum_i Z_i^* Z_i$, $Z_i \in \mathcal{A}$.

The following result on the connection between bounded and unbounded nets was proved in [BY4]:

**4.4 Theorem.** *Let $\mathcal{P}(K)_{K\in\mathcal{K}}$ be a local field net satisfying (BZ) and assume $\mathcal{P}(K)^w$ is a von Neumann algebra for every $K \in \mathcal{K}$. Then the following conditions are equivalent.*

(1) *The field net is associated in the sense of Definition 4.2 with some local net of von Neumann algebras acting on the same Hilbert space.*

(2) *The minimal net $\mathcal{M}_{\min}(K) = \mathcal{P}(K)^{w\prime}$ is local.*

(3) *For every open set $\mathcal{O}$ and every hermitian operator $X \in \mathcal{P}(\mathcal{O})$ the state defined by the vacuum vector $\Omega$ on the algebra generated by $X$ and $\mathcal{P}(\mathcal{O}')$ is centrally positive with respect to $X$.*

(4) *If $K, K_1 \in \mathcal{K}$ with $K_1 \subset K'$ and $X$ belongs to a set of hermitian generators $\mathcal{G}(K)$ for $\mathcal{P}(K)$, then the state defined by the vacuum vector $\Omega$ on the algebra generated by $X$ and $\mathcal{P}(K_1)$ is centrally positive with respect to $X$.*

In [DSW] it is shown that $\mathcal{P}(\mathcal{O})^w$ is a von Neumann algebra for all open regions $\mathcal{O}$ if the net is generated by Wightman fieds satisfying *generalized H-bounds* of order $< 1$, i.e. if $\Phi(f)^{**}e^{-(1+H^2)^{\alpha/2}}$ is a bounded operator for all field operators $\Phi(f)$ and some $\alpha < 1$, where $H$ denotes the Hamiltonian. In fact, such generalized $H$-bounds imply that every operator in $\mathcal{P}(\mathcal{O})^w$ commutes *strongly* with the closures of the operators in $\mathcal{P}(\mathcal{O})$. Hence the extension $\tilde{X}$ in Def. 4.2 may in this case simply be taken to be the closure $\bar{X}$.

Let us now consider a field net $\mathcal{P}(\cdot)$ satisfying the premises of Theorem 4.4 and assume the positivity condition (4) is fulfilled so $\mathcal{M}_{\min}(\cdot)$ is local. It is natural to ask whether essential duality holds for $\mathcal{M}_{\min}(\cdot)$. If this is the case a partial answer can be given to the question of uniqueness: All local nets of von Neumann algebras, with which $\mathcal{P}(\cdot)$ is associated, lie between $\mathcal{M}_{\min}(\cdot)$ and the net

$$\mathcal{M}_{\max}(K) := \mathcal{M}^d_{\min}(K) = \mathcal{P}(K')^w. \tag{4.11}$$

In particular, $\mathcal{P}(\cdot)$ is associated with a unique net of von Neumann algebras (on the same Hilbert space) if and only if $\mathcal{P}(K)^{w\prime} = \mathcal{P}(K')^w$ for all $K$. We now relate these questions to properties of the Tomita conjugations of $\mathcal{P}(\cdot)$.

We first note that from the definition (3.1) it follows that for arbitrary regions $\mathcal{O}$

$$\mathcal{P}(\mathcal{O})^w = \bigcap_{K\subset\mathcal{O}, K\in\mathcal{K}} \mathcal{P}(K)^w.$$

Hence $\mathcal{P}(\mathcal{O})^w$ is an algebra if $\mathcal{P}(K)^w$ is an algebra for all $K \in \mathcal{K}$. Let $\mathcal{O}$ belong to $\mathcal{K}$ or $\mathcal{W}$, so the Tomita conjugations $S_{\mathcal{P}(\mathcal{O})}$ and $S_{\mathcal{M}_{\min}(\mathcal{O})}$ are defined. Since $S_{\mathcal{P}(\mathcal{O})^{w\prime}} = S^*_{\mathcal{P}(\mathcal{O})^w}$ and $S_{\mathcal{P}(\mathcal{O})} \subset S^*_{\mathcal{P}(\mathcal{O})^w}$, we have

$$S_{\mathcal{P}(\mathcal{O})} \subset S_{\mathcal{M}_{\min}(\mathcal{O})}. \tag{4.12}$$

Replacing $\mathcal{O}$ by $\mathcal{O}'$ and taking adjoints we obtain

$$S_{\mathcal{M}_{\min}(\mathcal{O}')'} \subset S^*_{\mathcal{P}(\mathcal{O}')}. \tag{4.13}$$

Locality of $\mathcal{P}(\cdot)$ implies that $S_{\mathcal{P}(\mathcal{O})} \subset S^*_{\mathcal{P}(\mathcal{O}')}$. If the stronger condition $S_{\mathcal{P}(\mathcal{O})} = S^*_{\mathcal{P}(\mathcal{O}')}$ holds we deduce from (4.12) and (4.13) that $S_{\mathcal{M}_{\min}(\mathcal{O}')'} \subset S_{\mathcal{M}_{\min}(\mathcal{O})}$. Locality of $\mathcal{M}_{\min}(\cdot)$ and Proposition 2.7 then imply that $\mathcal{M}_{\min}(\mathcal{O}')' = \mathcal{M}_{\min}(\mathcal{O})$. Combining this with Lemma 4.1 we have thus obtained the following sharpening of Theorem 4.4:

**4.5 Theorem.** *Let $\mathcal{P}(\cdot)$ satisfy the premises and the positivity condition (4) of Theorem 4.4. If $\mathcal{W}$-duality holds for $\mathcal{P}(\cdot)$, i.e. $S_{\mathcal{P}(W)} = S^*_{\mathcal{P}(W')}$ for all $W \in \mathcal{W}$, then essential* *duality holds for the minimal net* $\mathcal{M}_{\min}(K) = \mathcal{P}(K)^{w\prime}$*.* $\mathcal{P}(\cdot)$ *is then associated with a unique net of von Neumann algebras satisfying* $\mathcal{K}$*-duality, namely*

$$\mathcal{M}_{\max}(K) = \mathcal{M}^d_{\min}(K) = \widehat{\mathcal{M}}_{\min}(K) = \mathcal{P}(K')^w.$$

*If* $\mathcal{K}$*-duality holds for* $\mathcal{P}(\cdot)$*, then* $\mathcal{M}_{\min}$ *also satisfies* $\mathcal{K}$*-duality, i.e.* $\mathcal{M}_{\max} = \mathcal{M}_{\min}$.

As a last subject of this section we consider variants of Theorems 4.4 and 4.5, where the positivity and duality conditions are only made for a subnet $\mathcal{P}_0(\cdot)$ of $\mathcal{P}(\cdot)$. These results should be compared with earlier results of a similar kind by Driessler, Summers and Wichmann, see in particular Theorems 4.6 and 6.1 in [DSW]. The main point is that $\mathcal{P}_0(\cdot)$ already suffices to generate a net of von Neumann algebras for $\mathcal{P}(\cdot)$ if the Tomita conjugations of $\mathcal{P}_0(W)$ and $\mathcal{P}(W)$ are identical, $W \in \mathcal{W}$.

**4.6 Theorem.** *Let* $\{\mathcal{P}(K)\}_{K\in\mathcal{K}}$ *be a local net satisfying (BZ). Assume* $\mathcal{P}(K)^w$ *is a von Neumann algebra for every* $K \in \mathcal{K}$*, and that the algebras* $\mathcal{P}(K)$ *are generated by operators fulfilling the conditions of Proposition 3.4. Let* $\mathcal{P}_0(\cdot)$ *be a subnet satisfying the same conditions as* $\mathcal{P}(\cdot)$ *for the weak commutants and assume* $\Omega$ *is cyclic for* $\mathcal{P}_0(W)$ *and* $\mathcal{P}_0(W')$*,* $W \in \mathcal{W}$*. If moreover* $S_{\mathcal{P}_0(W)} = S_{\mathcal{P}(W)}$ *for all* $W \in \mathcal{W}$*, then the following conditions are equivalent:*

(1) *The field net* $\mathcal{P}(\cdot)$ *is associated with some local net of von Neumann algebras on the Hilbert space of the field.*

(2) *The field net* $\mathcal{P}_0(\cdot)$ *is associated with some local net of von Neumann algebras on the Hilbert space of the field.*

(3) *If* $K, K_1 \in \mathcal{K}$ *with* $K_1 \subset K'$*, and* $X$ *belongs to a set of hermitian generators* $\mathcal{G}_0(K)$ *for* $\mathcal{P}_0(K)$*, then the state defined by the vacuum vector* $\Omega$ *on the algebra generated by* $X$ *and* $\mathcal{P}_0(K_1)$ *is centrally positive with respect to* $X$*.*

*If this holds, then* $\mathcal{P}(\cdot)$ *is associated with the net* $\widehat{\mathcal{M}}_{0,\min}(\cdot)$ *with* $\mathcal{M}_{0,\min}(K) = \mathcal{P}_0(K)^{w\prime}$*.*

*Proof.* Equality of the $S$-operators implies that $\mathcal{P}(W)^w = \mathcal{P}_0(W)^w$ for all $W \in \mathcal{W}$ by Propositions 2.7 and 3.4. Hence $\widehat{\mathcal{M}}_{0,\min}(\cdot) = \widehat{\mathcal{M}}_{\min}(\cdot)$. But (2) is equivalent to locality of $\mathcal{M}_{0,\min}(\cdot)$, which is equivalent to locality of $\widehat{\mathcal{M}}_{0,\min}(\cdot)$ by condition (KW). Hence (1) and (2) are equivalent. The equivalence of (2) and (3) follows from Theorem 4.4. □

If the subnet $\mathcal{P}_0(\cdot)$ satisfies $\mathcal{W}$-duality, $S^*_{\mathcal{P}_0(W)} = S_{\mathcal{P}_0(W')}$, then $S_{\mathcal{P}_0(W)} = S_{\mathcal{P}(W)}$ for all $W \in \mathcal{W}$ by Lemma 2.6. (Take $\mathcal{F}_1 = \mathcal{P}_0(W)$, $\mathcal{Q} = \mathcal{P}_0(W')$ and $\mathcal{F}_2 = \mathcal{P}_0(W)$.) In particular, $\mathcal{P}(\cdot)$ is also $\mathcal{W}$-dual. Combining Theorems 4.5 and 4.6 we thus also have

**4.7 Theorem.** *Let $\mathcal{P}(\cdot)$ satisfy the premises of Theorem 4.6. Let $\mathcal{P}_0(\cdot)$ be a subnet satisfying the same condition for the weak commutants and in addition the positivity condition* (3)*. If $\mathcal{W}$-duality holds for $\mathcal{P}_0(\cdot)$, then the minimal net $\mathcal{M}_{\min}(K) = \mathcal{P}(K)^{w\prime}$ satisfies essential duality, and $\mathcal{M}_{\max}(\cdot) = \widehat{\mathcal{M}}_{0,\min}(\cdot)$.*

## 5. Fermi fields and twisted duality

In this last section we discuss briefly how the preceeding results can be generalized to include Fermi fields. The basic concept in this case is a field net $\mathcal{P}(\cdot)$ (in the sense of Section 3), where the locality condition (L) is replaced by the $\mathbf{Z}^2$-graded local structure described below. Using the same device as in [BW2] to change anticommutators into commutators all previous results carry over with minor modifications to the new situation. We shall therefore only describe the necessary changes without formulating the statements anew.

The graded structure is defined by a unitary operator $U_0$ ("rotation by $2\pi$"), such that $U_0^2 = I$, $\mathcal{D}$ is invariant under $U_0$ and $U_0\mathcal{P}(\mathcal{O})U_0^{-1} = \mathcal{P}(\mathcal{O})$ for all $\mathcal{O}$. Morever, for translationally covariant fields we assume that $U_0$ commutes with the translations and that $U_0$ leaves the vacuum $\Omega$ invariant. Every $X \in \mathcal{P}(\mathbf{R}^d)$ has a unique decomposition into a "Bose" and a "Fermi" part with respect to $U_0$:

$$X = X_b + X_f \quad \text{with} \quad U_0 X_b U_0^{-1} = X_b, \quad U_0 X_f U_0^{-1} = -X_f, \tag{5.1}$$

where

$$X_b := \tfrac{1}{2}(X + U_0 X U_0^{-1}) \quad \text{and} \quad X_f := \tfrac{1}{2}(X - U_0 X U_0^{-1}). \tag{5.2}$$

In this way each $\mathcal{P}(\mathcal{O})$ becomes a $\mathbf{Z}^2$-graded algebra with unit:

$$\mathcal{P}(\mathcal{O}) = \mathcal{P}(\mathcal{O})_b \oplus \mathcal{P}(\mathcal{O})_f \tag{5.3}$$

with

$$\mathcal{P}(\mathcal{O})_f \cdot \mathcal{P}(\mathcal{O})_f \subset \mathcal{P}(\mathcal{O})_b = \mathcal{P}(\mathcal{O})_b \cdot \mathcal{P}(\mathcal{O})_b \tag{5.4}$$

and

$$\mathcal{P}(\mathcal{O})_b \cdot \mathcal{P}(\mathcal{O})_f = \mathcal{P}(\mathcal{O})_f \cdot \mathcal{P}(\mathcal{O})_b = \mathcal{P}(\mathcal{O})_f. \tag{5.5}$$

As a generalization of local commutativity we assume *graded local commutativity*, i.e.

$$[\mathcal{P}(\mathcal{O})_b, \mathcal{P}(\mathcal{O}')_b] = [\mathcal{P}(\mathcal{O})_b, \mathcal{P}(\mathcal{O}')_f] = [\mathcal{P}(\mathcal{O})_f, \mathcal{P}(\mathcal{O}')_f]_+ = \{0\} \tag{5.6}$$

for all $\mathcal{O}$.

In the same way as in [BW2] one may avoid the use of anticommutators by introducing the operator

$$Z := (1+i)^{-1}(U_0 + iI)$$

with the property

$$ZX_bZ^{-1} = X_b \quad \text{and} \quad ZX_fZ^{-1} = iU_0X_f \tag{5.7}$$

for all $X \in \mathcal{P}(\mathbf{R}^d)$. If $\mathcal{Q}$ is a set of operators on $\mathcal{D}$ we employ the notation

$$\mathcal{Q}^z := \{ZXZ^{-1} \mid X \in \mathcal{Q}\}. \tag{5.8}$$

We remark that because $Z$ leaves $\mathcal{D}$ invariant, the operation of changing $\mathcal{Q}$ into $\mathcal{Q}^z$ commutes with the formation of weak and strong commutants, i.e.

$$(\mathcal{Q}^{uw})^z = (\mathcal{Q}^z)^{uw}, \quad (\mathcal{Q}^w)^z = (\mathcal{Q}^z)^{w,} \quad (\mathcal{Q}^{us})^z = (\mathcal{Q}^z)^{us}, \quad (\mathcal{Q}^s)^z = (\mathcal{Q}^z)^s. \tag{5.9}$$

Note also that in spite of the fact that $Z^2 \neq 1$, one has $(\mathcal{Q}^z)^z = \mathcal{Q}$, if $\mathcal{Q}$ is invariant under ad $U_0$, i.e. if $X_b$ and $X_f$ belong to $\mathcal{Q}$ for $X \in \mathcal{Q}$.

Because of (5.7) and (5.1), we can write (5.6) in analogy to (3.2) as

$$\mathcal{P}(\mathcal{O}')^z \subset (\mathcal{P}(\mathcal{O}), \mathcal{D})^{us}. \tag{5.10}$$

If (5.10) (or equivalently (5.6)) holds we call the net $\mathcal{O} \mapsto \mathcal{P}(\mathcal{O})$ a $\mathbf{Z}_2$*-graded local field net.*

As in Section 3 we single out two classes of subsets $\mathcal{K}$ and $\mathcal{W}$ satisfying (K) and (KW). If $\mathcal{P}(K)$ is to begin with only defined for $K \in \mathcal{K}$, and $\mathcal{P}(\mathcal{O})$ is defined by (3.1) for more general regions, then a $\mathbf{Z}_2$-graded local structure for $\{\mathcal{P}(K)\}_{K\in\mathcal{K}}$ extends naturally to the algebras $\mathcal{P}(\mathcal{O})$ for general $\mathcal{O}$.

If a field net $\mathcal{P}(\cdot)$ satisfies (RS), or the stronger condition (BZ), for a $U_0$-invariant vector $\Omega$, then the same holds for the net $\mathcal{P}(\cdot)^z$. The Tomita conjugations $S_{\mathcal{P}(\mathcal{O})}$ and $S_{\mathcal{P}(\mathcal{O}')^z}$ are then well defined for $\mathcal{O}\in\mathcal{K}$ or $\mathcal{O}\in\mathcal{W}$, and (5.10) implies that

$$S^*_{\mathcal{P}(\mathcal{O})} \supset S_{\mathcal{P}(\mathcal{O}')^z}\,. \tag{5.11}$$

The duality conditions of Section 3 for such a field net generalize to *twisted $\mathcal{K}$-duality* and *twisted $\mathcal{W}$-duality*, which by definition mean that

$$S^*_{\mathcal{P}(\mathcal{O})} = S_{\mathcal{P}(\mathcal{O}')^z} \tag{5.12}$$

for all $\mathcal{O}\in\mathcal{K}$ or all $\mathcal{O}\in\mathcal{W}$, respectively.

With these definitions Theorems 3.5 and 3.6 now carry over to the case of graded nets with the only change that "local" has to be repalced by "$\mathbf{Z}_2$-graded local" , and "dual" by "twisted dual" . Notice that because of (5.9) and the fact that $(\mathcal{P}(\mathcal{O})^z)^z = \mathcal{P}(\mathcal{O})$ there is a symmetry between the algebras $\mathcal{P}(\mathcal{O})$ and their $z$-transforms in all results involving these algebras and their strong or weak commutants. Thus one can interchange $\mathcal{P}(\mathcal{O})$ and $\mathcal{P}(\mathcal{O}')^z$ in equations like (5.10)-(5.12), and also in the generalizations of Theorems 3.5 and 3.6.

The concept of a $\mathbf{Z}_2$-graded local net of von Neumann algebras, $\mathcal{O}\mapsto\mathcal{M}(\mathcal{O})$, is defined in the same way as a $\mathbf{Z}_2$-graded local net with the additional requirement that $\mathcal{M}(\mathcal{O})$ is a von Neumann algebra for all $\mathcal{O}$. For the association of a $\mathbf{Z}_2$-graded net of unbounded operator algebras with such a net we have to require that the association is compatible with the graded structure:

**5.1 Definition.** A $\mathbf{Z}_2$-graded net $\mathcal{P}(\cdot)$ of (unbounded) operator algebras is associated with a $\mathbf{Z}_2$-graded net $\mathcal{M}(\cdot)$ of von Neumann algebras (with the same operator $U_0$), if the following holds:

Every $X\in\mathcal{P}(\mathcal{O})$ has an extension to a closed operator $\tilde{X}$ on $\mathcal{H}$ with $\tilde{X}^*\supset\widetilde{X^\dagger}$ and $U_0\tilde{X}U_0^{-1} = (U_0XU_0^{-1})^{\sim}$, such that $\tilde{X}$ is affiliated with the von Neumann algebra $\mathcal{M}(\mathcal{O})$.

Since all extensions considered are supposed to satisfy the condition $U_0\widetilde{X}U_0^{-1} = (U_0XU_0^{-1})^{\sim}$ it follows that the isometric operator $V$ and the absolute value $|\widetilde{X}|$ appearing in the polar decomposition $\widetilde{X} = V|\widetilde{X}|$ have the correct Bose or Fermi character, i.e. $V$ is Bose/Fermi if $X$ is Bose/Fermi and $|\widetilde{X}|$ is in both cases a Bose operator. This assertion is a simple consequence of the uniqueness of the polar decomposition of a closed operator.

The criterion for the the association of a graded local net of unbounded operator algebras with a graded local net of von Neumann algebras is an obvious modification of Theorem 4.4. As in the Section 4 we assume that the weak commutants $\mathcal{P}(\mathcal{O})^w$ are algebras. The unbounded net $\mathcal{P}(\cdot)$ is then associated with a unique minimal $\mathbf{Z}_2$-graded net of von Neumann algebras on the the same Hilbert space, namely

$$\mathcal{M}_{\min}(\mathcal{O}) := \mathcal{P}(\mathcal{O})^{w\prime}. \tag{5.13}$$

In fact, since $U_0$ leaves $\mathcal{D}$ invariant and $U_0^2 = 1$, one has $U_0\mathcal{M}_{\min}(\mathcal{O})U_0^{-1} = \mathcal{M}_{\min}(\mathcal{O})$, so $U_0$ defines a grading of this minimal net, and one checks that the extensions defined as in the proof of Lemma 4.3 in [BY4] satisfy the requirements of Definition 5.1. Moreover, by (5.9) and since $\mathcal{P}(\mathcal{O}')^w$ is an algebra, one has

$$\mathcal{M}_{\min}(\mathcal{O}')^{z\prime} = \mathcal{P}(\mathcal{O}')^{zw}. \tag{5.14}$$

Hence graded local commutativity for the net $\mathcal{M}_{\min}$ is equivalent to the requirement that

$$\mathcal{P}(\mathcal{O})^{w\prime} \subset (\mathcal{P}(\mathcal{O}')^z)^w \tag{5.15}$$

for all $\mathcal{O}$. We can now proceed in an analogous way as in [ BY4], with (5.15) replacing Eq. (4.1) in that paper. This leads to a generalization of Theorem 4.4 (Theorem 4.6 in [BY4]):

The net $\mathcal{P}(\cdot)$ is associated with a graded local net of von Neumann algebras if and only if the following generalization of the positivity condition (4) holds:

(4′) If $K, K_1 \in \mathcal{K}$ with $K_1 \subset K'$ and $X$ belongs to a set of hermitian generators $\mathcal{G}(K)$ for $\mathcal{P}(K)$, then the state defined by the vacuum vector $\Omega$ on the algebra generated by $X$ and $\mathcal{P}(K_1)^z$ is centrally positive with respect to $X$.

Analogs of Theorems 4.5, 4.6 and 4.7 also hold for $\mathbf{Z}_2$-graded nets if $\mathcal{M}^d$ is defined as $\mathcal{M}^d(K) = \mathcal{M}(K')^{z\prime}$. The modifications are as follows: Repalce "local" by "$\mathbf{Z}_2$-graded local" ,"dual" by "twisted dual" , and $\mathcal{P}_0(K_1)$ in Theorem 4.6 by $\mathcal{P}_0(K_1)^z$.

## Appendix A

We derive here property (BZ) for a net $\mathcal{P}(\cdot)$ of unbounded operator algebras generated by Wightman fields. The crucial fact is the following consequence of translational covariance, spectrum condition and temperedness of the Wightman fields: Suppose $A, B_1, \ldots, B_n \in \mathcal{P}(\mathbf{R}^d)$ and put $B_i(x_i) = U(x_i)B_iU(-x_i)$, with $x_i \in \mathbf{R}^d$, $i = 1, \ldots, n$, where $U(x)$ denotes the representation of the translation group. Then the function $\Phi(x_1, \ldots, x_n) = AB_1(x_1)\cdots B_n(x_n)\Omega$ has an analytic continuation $\Phi(z_1, \ldots, z_n)$ for $z_1, z_2 - z_1, \ldots, z_n - z_{n-1}$ in the forward tube, as a function with values in the Hilbert space $\mathcal{H}$ of the operators (Jost vector). Note that this property is not a consequence of the translational covariance and spectrum condition alone, for although $x \mapsto U(x)\psi$ has an analytic continuation into the forward tube as an $\mathcal{H}$-valued function $z \mapsto U(z)\psi$ for all $\psi \in \mathcal{H}$, it is not a priori clear that $U(z)\psi$ belongs to the domain of the closure of each of the unbounded operators in $\mathcal{P}(\mathbf{R}^d)$ if $\psi \in \mathcal{D}$. For tempered Wightman fields, however, $\Phi(x_1, \ldots, x_n)$ is a polynomially bounded, continuous function, whose Fourier transform in distributional sense, $\tilde{\Phi}(p_1, \ldots, p_n)$, has support for $p_i + \cdots + p_n$ in the forward light cone, $i = 1, \ldots, n$, and $\Phi(z_1, \ldots, z_n)$ can be defined by smearing $\tilde{\Phi}(p_1, \ldots, p_n)$ with $\exp(i \sum_j z_j \cdot p_j)$.

It is not difficult to extend this argument to any translationally covariant field net with spectrum condition, provided $\bar{A}U(x)\psi$ is polynomially bounded for all $A \in \mathcal{P}(\mathbf{R}^d)$, $\psi \in \bar{\mathcal{D}}$, where $\bar{\mathcal{D}}$ denotes the closure of the domain $\mathcal{D}$ in the graph topology induced by $\mathcal{P}(\mathbf{R}^d)$. In fact, the analytic continuation $\Phi(\zeta)$ of $\Phi(x) = \bar{A}U(x)\psi$ for $\zeta$ in the forward tube is obtained in the same way as for Wightman fields by integrating the Fourier transform $\tilde{\Phi}$ with $e^{i\zeta\cdot p}$. In order to be able to iterate this procedure and define $\bar{A}U(\zeta_1)\bar{B}_1 \cdots U(\zeta_n)B_n\Omega$, it suffices to show that $U(\zeta)\psi$ belongs to $\bar{\mathcal{D}}$ for all $\psi \in \bar{\mathcal{D}}$ and $\zeta$ in the forward tube, because the closures of the operators in $\mathcal{P}(\mathbf{R}^d)$ leave $\bar{\mathcal{D}}$ invariant. To prove $U(\zeta)\psi \in \bar{\mathcal{D}}$ we consider for $A \in \mathcal{P}(\mathbf{R}^d)$ the vector valued function $\Psi(\zeta) = \Phi(\zeta)/P(\zeta)$, where $\Phi(\zeta)$ is the analytic continuation of $\Phi(x) = \bar{A}U(x)\psi$ as before, and $P(\zeta)$ is a polynomial without zeros in the closure of the forward tube and of sufficiently high degree so that $\Phi(x)/P(x)$ is integrable over $\mathbf{R}^d$. The function $\Psi(\zeta)$ is in the same way as $\Phi(\zeta)$ the Fourier-Laplace transform of a distribution with support in the forward light cone $\bar{V}^+$. From this we obtain the integral representation

$$\Phi(\zeta) = P(\zeta) \int K(\zeta - x) \frac{\bar{A}U(x)\psi}{P(x)} dx$$

where $K(\zeta) = (2\pi)^{-d} \int_{\bar{V}^+} \exp(i\zeta \cdot p) dp$. An approximation of the integral by a Riemannian sum provides a sequence $\psi_n$ of vectors in $\bar{\mathcal{D}}$ such that $\psi_n \to U(\zeta)\psi$ and $\bar{A}\psi_n \to \Phi(\zeta)$. Hence $U(\zeta)\psi \in \bar{\mathcal{D}}$.

Given the analytic continuation of $\Phi(x_1, \ldots, x_n)$ the derivation of (BZ) follows the same pattern as in the Reeh-Schlieder Theorem: Pick $A_1, \cdots A_N \in \mathcal{P}(\mathbf{R}^d)$ and assume $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$ are such that

$$\sum_{j=1}^{N} \langle \varphi_j, A_j B_1(x_1) \cdots B_n(x_n) \Omega \rangle = 0 \tag{A1}$$

for all $B_i \in \mathcal{P}_0(\mathcal{O})$ and $x_i$ in some neigbourhood $\mathcal{N}$ of zero. Then (A1) holds by analytic continuation for all $x_i \in \mathbf{R}^d$. This implies that $\mathcal{P}_0(\mathcal{O} + \mathcal{N})\Omega$ and $(\vee_{x \in \mathbf{R}^d} \mathcal{P}_0(\mathcal{O} + x)\Omega$ have the same closure in the graph topology induced by $\mathcal{P}(\mathbf{R}^d)$. Now we may replace $\mathcal{O}$ by a smaller set $\mathcal{O}_0$ such that $\mathcal{O}_0 + \mathcal{N} \subset \mathcal{O}$, and by assumption $\vee_{x \in \mathbf{R}^d} \mathcal{P}_0(\mathcal{O}_0 + x) = \mathcal{P}(\mathbf{R})^d$, so (BZ) holds for all (nonempty, open) $\mathcal{O}$.

## Appendix B

*Proof of Proposition 3.4:* Let $\chi_n$ be the characteristic function of the interval $[-n, n]$, and define $h_n = \chi_n * \gamma$ where $\gamma$ is some fixed $C^\infty$-function of compact support with $\int \gamma(t) dt = 1$. If $\tilde{h}_n(t) = \frac{1}{2\pi} \int \exp(-ist) h_n(s) ds$ is the Fourier transform of $h_n$, then $\int \tilde{h}_n(t) g(t) \to g(0)$ for all continuous, polynomially bounded functions $g$. If $X \in \mathcal{G}$ and $\psi \in \mathcal{D}$, we thus obtain, using (i) and (ii):

$$X\psi = \lim_{n \to \infty} \int \tilde{h}_n(t) X \exp(itH)\psi \, dt = \lim_{n \to \infty} \bar{X} h_n(H)\psi.$$

Now because $\mathcal{D}_0$ is dense and $\bar{X} h_n(H)$ bounded, we can for each $n$ approximate $\bar{X} h_n(H)\psi$ arbitrarily closely in norm with vectors of the form $\bar{X} h_n(H)\varphi$ with $\varphi \in \mathcal{D}_0$. Writing

$\bar{X}h_n(H)\varphi = \int \tilde{h}_n(t)X\exp(itH)\varphi\,dt$ and approximating the integral by a Riemannian sum we see that $\bar{X}h_n(H)\psi$, and hence $X\psi$, can be approximated arbitrarily well by a finite linear combination of translates of $\varphi$, i.e. by vectors of the form $\sum_j c_j \exp(it_jH)\varphi$ with $c_j \in \mathbf{C}$, $t_j \in \mathbf{R}$. Because $\mathcal{D}_0$ is invariant by assumption, we can thus for any $\varepsilon > 0$ obtain an $\varepsilon$-approximation of $X\psi$ by $X\xi$ for some $\xi \in \mathcal{D}_0$. It is clear that the same argument applies to any finite set of operators satisfying conditions (i) and (ii), so $\mathcal{D}_0$ is dense in $\mathcal{D}$ in the graph topology induced by $\mathcal{G}$.

**Acknowledgements** This work was partially supported by the Heraeus-Stiftung and the Icelandic Science Foundation.

## References

[BW1] J. Bisognano and E.H. Wichmann, *On the duality condition for a Hermitean scalar field*, J. Math. Phys. **16**, 985-1007 (1975).

[BW2] J. Bisognano and E.H. Wichmann, *On the duality condition for quantum fields*, J. Math. Phys. **17**, 303-321 (1976).

[B1] H.-J. Borchers, *Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix*, Il Nuovo Cimento **15**, 784-794 (1960).

[B2] H.-J. Borchers, *The CPT-Theorem in Two-dimensional Theories of Local Observables*, Commun. Math. Phys. **143**, 315-332 (1992).

[B3] H.-J. Borchers, *On Modular Inclusion and Spectrum Condition*, Lett.Math.Phys. to appear (1993).

[BY1] H.J. Borchers and J. Yngvason, *Positivity of Wightman Functionals and the Existence of Local Nets*, Commun. Math. Phys. **127**, 607-615 (1990).

[BY2] H.J. Borchers and J. Yngvason, *Partially Commutative Moment Problems*, Math. Nachr. **145**, 111-117 (1990).

[BY3] H.J. Borchers and J. Yngvason, *Local Nets and Self-Adjoint Extensions of Quantum Field Operators*, Lett. Math. Phys. **21**, 151-155 (1991).

[BY4] H.J. Borchers and J. Yngvason, *From Quantum Fields to Local von Neumann Algebras* , Rev. Math. Phys., Special Issue, 15-47 (1992).

[BZ] H.J. Borchers and W. Zimmermann, *On the Self-Adjointness of Field Operators*, Nuovo Cimento **31**, 1047-1059 (1964).

[BGL] R. Brunetti, D. Guido and R. Longo, *Modular structure and duality in conformal quantum field theory*, Preprint, Univ. di Roma, 1992.

[Bu1] D. Buchholz, *On Quantum Fields that Generate Local Algebras*, J. Math. Phys. **31**, 1839-1846 (1990).

[BAL1] D. Buchholz, C. D.'Antoni and R. Longo, *Nuclear Maps and Modular Structures* I. *General Properties*, Jour. Func. Analysis **88**, 233 (1990).

[BAL2] D. Buchholz, C. D.'Antoni and R. Longo, *Nuclear Maps and Modular Structures* II. *Application to Quantum Field Theory*, Commun. Math. Phys. **129**, 115-138 (1990).

[BF] D. Buchholz and K. Fredenhagen, *Locality and the structure of particle states*, Commun. Math. Phys. **84**, 1-54 (1982).

[BSch] D. Buchholz and H. Schulz-Mirbach, *Haag duality in conformal field theory*, Rev. Math. Phys. **2**, 105-125 (1990).

[BSu] D. Buchholz and S.J. Summers, *An Algebraic Characterization of Vacuum States in Minkowski Space*, Preprint, 1992.

[DSW] W. Driessler, S. Summers and E.H. Wichmann, *On the Connection between Quantum Fields and von Neumann Algebras of Local Observables*, Commun. Math. Phys. **105**, 49-84 (1986).

[GF].F. Gabbiani and J. Fröhlich, *Operator Algebras and Conformal Field Theory*, preprint ETH Zürich, (1992).

[H1]. R. Haag, *Local quantum field theory*, Berlin-Heidelberg-New York: Springer Verlag, 1992.

[H2] R.Haag, *Les problèmes mathématiques de la théorie quantique des champs*, in *Lille Conference 1957*, Paris: Editions de CNRS, 1959.

[I1] A. Inoue, *An Unbounded Generalization of the Tomita-Takesaki Theory*, Publ. R.I.M.S. Kyoto **22**, 725-765 (1986).

[I2] A. Inoue, *An Unbounded Generalization of the Tomita-Takesaki Theory* II, Publ. R.I.M.S. Kyoto **23**, 673-726 (1987).

[I3] A. Inoue, *Modular Structure of Algebras of Unbounded Operators*, Math. Proc. Camb. Phil. Soc. **111**, 369-386 (1992).

[KR] R.V. Kadison and J.R. Ringrose,*Fundamentals of the Theory of Operator Algebras* II, New York: Academic press, 1986.

[P] R.T. Powers, *Self-Adjoint Algebras of Unbounded Operators* II, Trans. Am. Math. Soc. **187**, 261-293 (1974).

[Ri] C. Rigotti, *Remarks on the Modular Operator and Local Observables*, Commun. Math. Phys. **61**, 267-273 (1978).

[Ro] J.Roberts, *Spontanously broken gauge symmetries and superselection rules*, in *International School of Mathematical Physics*, Universita' di Camerino, 1974.

[Sch] K. Schmüdgen, *Algebras of Unbounded Operators*, Berlin: Akademie-Verlag, 1989.

[Ta] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Lecture Notes in Math. **128**, Berlin-Heidelberg-New York: Springer-Verlag, 1970.

[To] M. Tomita, *Standard forms of von Neumann algebras*, in *Fifth functional analysis symposium of Math. Soc. of Japan*, Sendai, 1967.

[W] E.H. Wichmann, *On Systems of Local Operators and the Duality Condition*, J. Math. Phys. **24**, 1633-1644 (1983).

[Wie1] H.-W. Wiesbrock, *A comment an a recent work of Borchers*, Lett.Math.Phys. **25**, 157-159 (1992).

[Wie2] H.-W. Wiesbrock, *Half-Sided Modular Inclusions of von Neumann Algebras*, Preprint, FU Berlin (1992).