Spectral Estimates Around a Critical Level

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Contents

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1 Introduction and Main Statements

The semi-classical trace formula of [?], [?], [?], [?] is a rigorous version of the Gutzwiller trace formula which provides information about the spectral function of Schrödinger operators in the semi-classical regime. To be more specific, consider an operator of Schrödinger type on a compact manifold M, that is an operator of the form

$$A_\hbar = \sum_{j=0}^{N} \hbar^j A_j \tag{1}$$

where, for each j, A_j is a differential operator on M of order j. Define the Hamiltonian H on T^*M by

$$H(x,p) \;=\; \sum_{j=0}^{r} \sigma_{\text{prin}}(A_j)(x,p), \quad H \in C^\infty(T^*M), \tag{2}$$

the sum of the principal symbols of the A_j. We will assume that $A_\hbar$ is elliptic, in the sense that there exists a $c > 0$ such that $H(x,p) \geq c > 0$ on T^*M. The main example is a Schrödinger operator,

$$A_\hbar = -\hbar^2\Delta + V$$

with Δ the Laplace-Beltrami operator associated with a Riemannian metric on M and $V \in C^\infty(M)$ is strictly positive. $A_\hbar$ then has discrete spectrum, consisting of eigenvalues $\{E_j(\hbar)\}$ with finite multiplicities, and the trace formula alluded to above states that, under certain assumptions, the sums

$$\Upsilon_{\hbar,E}(\varphi) \;=\; \sum_j \varphi\left(\frac{E_j(\hbar)-E}{\hbar}\right) \tag{3}$$

where φ is a test function with compactly supported Fourier transform, have an asymptotic expansion in integer powers of $\hbar$ as $\hbar \to 0$, and computes the leading coefficients of these expansions. One of the assumptions of this Theorem is that the parameter E appearing in (3) be a regular value of the classical Hamiltonian $H(x,p)$, that is, that there are no equilibria with energy E. In that case the leading order term of (3) for φ's whose Fourier transform $\hat\varphi$ is supported in a sufficiently small interval around 0 is:

$$\frac{\hat\varphi(0)}{(2\pi)^n}\,\mathrm{LVol}\,(\Sigma_E)\,\hbar^{-n+1}\ , \tag{4}$$

where $\mathrm{LVol}\,(\Sigma_E)$ is the Liouville measure of the energy surface $\Sigma_E \;=\; \{H = E\}$. The trace formula also computes the leading order term of (3) in case zero is not in the support of $\hat\varphi(0)$, under suitable assumptions on the set of periodic trajectories of the Hamilton flow of H, but we won't recall that formula here. The goal of the present work is to study the behavior of (3) in case there are equilibria with energy E. As a consequence, we will obtain, in generic cases at least, the limit of the counting function

$$N(\hbar) \;=\; \sharp\,\{\, j\ ;\ |E_j(\hbar) - E| \leq c\hbar \,\} \tag{5}$$

as $\hbar \to 0$.

When one drops the assumption that E be a regular value, the behavior of (3) will depend on the nature of the singularities of H on Σ_E, which of course can be extremely complicated. Here we will only consider the simplest case: that of a Hamiltonian H which has a non-degenerate critical manifold Θ in the sense of Morse theory. More specifically, we will work under the following assumption:

Main Hypothesis

The set of critical points of the Hamiltonian $H(x,p)$ is a smooth submanifold, Θ, and H has a non-degenerate normal Hessian on Θ, that is,

$$Q(H)_z := d_z^2 H : (T_z(T^*M)/T_z(\Theta))^2 \to \mathbb{R} \tag{6}$$

is non-degenerate for all $z \in \Theta$. Moreover, we will assume that the multiplicities of the eigenvalues of the normal Hessian are independent of $z \in \Theta$.

Since, in particular, the differential of H is zero in directions tangent to Θ, the connected components of Θ are contained in level sets of H. Since we are interested in estimates that are local in the energy we may assume without loss of generality that

$$\Theta \subset H^{-1}(E_c) \ , \tag{7}$$

for some particular critical value of the energy, E_c. We will also assume, in order to simplify the statements of our main theorems, that Θ is connected. There is no essential loss of generality in doing this, since the contributions of the different components of Θ to the asymptotics of $\Upsilon_\hbar$ can just be added: this is so because we will analyze $\Upsilon_\hbar$ by microlocal techniques.

Theorem 1.1 *Under the Main Hypothesis, let $n = dim M$, $N = codim \Theta$, and let ν be equal to the number of negative eigenvalues of the Hessian of H on Θ. Let φ be a smooth function on $\mathbb{R}$ with Fourier transform supported in a sufficiently small neighborhood of the origin. Then, as $\hbar \to 0$:*

A. *If $\nu \geq 1$, $N - \nu \geq 1$ and both are odd, there is an asymptotic expansion*

$$\sum_j \varphi\left(\frac{E_j(\hbar) - E_c}{\hbar}\right) \sim \hbar^{-(n-1)} \sum_{j=0}^{\infty} \sum_{l=0,1} c_{j,l}\, \hbar^j\, [\log(1/\hbar)]^l \tag{8}$$

involving logarithms of $\hbar$. Moreover, the first-non-zero coefficient involving such a logarithm, $c_{j_0,1}$, occurs for

$$j_0 = \frac{N}{2} - 1 \, .$$

(Notice that N must be even in this case.)

B. *If $\nu \geq 1$, $N - \nu \geq 1$ and one of them is even, or if the Hessian is definite on Θ, there is an asymptotic expansion*

$$\sum_j \varphi\left(\frac{E_j(\hbar) - E_c}{\hbar}\right) \sim \hbar^{-(n-1)} \sum_{j=0}^{\infty} c_j\, \hbar^{j/2} \ , \tag{9}$$

involving half-integer powers of $\hbar$.

Remark. The condition that the support of $\hat{\varphi}$ be in a small neighborhood of zero can be relaxed to: $\hat{\varphi}$ of compact support, provided the linearized flow at every $z \in \Theta$ has no non-zero periodic vectors. In that case, write $\varphi = \varphi_1 + \varphi_2$ with $\hat{\varphi}_1$ small, and treat the φ_2 term as in Theorem 5.6 of [?].

We can compute some of the coefficients in the expansions of theorem 1.1, namely the coefficients of the leading term and of the leading log term (in part **A**). To state these formulae we need to recall a basic fact about clean fixed point manifolds of symplectic mappings. Let $\{f_t\}$ denote the Hamilton flow of H on T^*M. Then, if t is sufficiently close to 0 but not equal to 0, Θ is a clean fixed point manifold of f_t (equivalently, t is not a period of one of the linearized flows $\{d(f_s)_z\}_s$ on $T_z(T^*M)/T_z(\Theta)$ for z in Θ). Recall that by [?] we can in this situation associate to Θ and f_t a canonical measure μ_t on Θ; the construction of this important measure is recalled in the appendix to this paper. There we will also show that the family of measures $|t|^{N/2}\mu_t$ is smooth at $t = 0$. In that case, a key rôle is played by the measure

$$\mu_0 \;=\; \lim_{t\to 0^+} \frac{d^{N/2-1}}{dt^{N/2-1}}\, t^{N/2}\,\mu_t\,. \tag{10}$$

One should think of μ_0 as a regularization of Liouville measure on Θ.

Theorem 1.2 *Keeping the hypotheses of Theorem 1.1, the leading term of the asymptotics of Theorem 1.1 is of the order $h^{-(n-1)}\log h^{-1}$ if $N = 2$ and the normal Hessian of H is indefinite. In all other cases the leading term is of order $h^{-(n-1)}$. Furthermore, we have*

*(i) In case **A**, if $\hat{\varphi}$ is flat at 0 (i.e. $\hat{\varphi}^{(k)}(0) = 0$ for all $k \geq 1$), the coefficient of the leading logarithmic term is*

$$c_{N/2-1,1} \;=\; C_{N,n,\nu}\cdot\hat{\varphi}(0)\;\mu_0(\Theta)\,, \tag{11}$$

with

$$C_{N,n,\nu} \;=\; (2\pi)^{-n}(2i)^{N/2-1}\cdot\frac{\Gamma\left(\frac{\nu}{2}\right)\,\Gamma\left(\frac{N-\nu}{2}\right)}{\Gamma\left(\frac{N}{2}\right)}\cdot\Sigma_{\nu-1}\cdot\Sigma_{N-\nu-1}$$

where if $k > 1$, Σ_{k-1} is the surface measure of the unit sphere in $\mathbb{R}^k$: $\Sigma_{k-1} = 2\pi^{(k-1)/2}/\Gamma\left((k-1)/2\right)$, and where $\Sigma_0 := 2$.

(ii) If $N = 2$ but the normal Hessian is definite, say positive definite, then the coefficient of the leading term is

$$c_0 = (2\pi)^{-n}\;\left(LVol(\Sigma_{E_c}) \,+\, c(\Theta)\right), \tag{12}$$

where

$$c(\Theta) \;=\; \pi\hat{\varphi}(0)\,\mu_0(\Theta) \,+\, i\int_{\mathbb{R}} |t|\,\hat{\varphi}(t)\,\mu_t(\Theta)\,\frac{dt}{t}\;. \tag{13}$$

(iii) Finally, if $N > 2$, the leading coefficient $c_{0,0}$ is in both cases of Theorem 1.1 the same as in the regular case:

$$c_{0,0} \;=\; c_0 \;=\; \frac{\hat{\varphi}(0)}{(2\pi)^n}\;LVol(\Sigma_{E_c})\;. \tag{14}$$

Remarks. (a) The integral (13) should be interpreted as the principal value distribution $PV(1/t)$ paired with the smooth function $|t|\hat{\varphi}(t)\mu_t(\Theta)$.

(b) At first sight the coefficient (13) looks complex even if φ is real-valued, which would of course contradict the definition of $\Upsilon_\hbar$. However, using the fact that $\overline{\mu_t} = \mu_{-t}$ and that $\overline{\hat{\varphi}(t)} = \hat{\varphi}(-t)$ for real-valued φ, one easily shows that (13) is in fact real for such φ.
(c) Note that if the normal Hessian is definite, then Θ is itself already a component of Σ_{E_c}. In this case Θ contributes to the expansion of $\Upsilon_\hbar$ starting from the order $\hbar^{-n+N/2}$. One can compute the part of the coefficient of this term coming from Θ for arbitrary N, cf. section 3.4 below, but we have omitted this result, which did not seem very useful if $N > 2$.

Summary: The picture that emerges from theorems 1.1 and 1.2 is as follows. First, the Weyl law (4) for the top order asymptotics of $\Upsilon_{\hbar,E_c}$ changes if and only if $N = 2$. This is in agreement with the easily verified fact that the singularity of the Liouville measure on Σ_{E_c} is integrable if $N > 2$, but not if $N = 2$. Secondly, in case **A** of Theorem 1.1, the leading contribution of the critical manifold Θ to the asymptotics changes by a logarithmic term, with coefficient proportional to $\mu_0(\Theta)$.

Example. Consider a Schrödinger operator $-(\hbar^2/2)\Delta + V(x)$ on an n-dimensional compact Riemannian manifold M, with strictly positive potential V having a local maximum E_c at the point $x_0 \in M$ and suppose that in fact Θ consists of the single point $z_0 = (x_0, 0) \in T^*M$. Then $N = 2n$ and $\nu = n$ and if $\{f_t\}$ is the Hamilton flow of $H = p^2/2 + V(x)$ then

$$\mu_t(\{z_0\}) = |\det(I - d(f_t)_{z_0})|^{-1/2} \tag{15}$$

(cf. the appendix). If the dimension of M is odd, then logarithms of $\hbar$ will appear in the expansion of $\Upsilon_{\hbar,E_c}$, the first one with index $j_0 = n - 1$ and coefficient $c_{n-1,1}$ which can be read off from (11). A particularly interesting case here is that of a one-dimensional Schrödinger operator, since if $n = 1$ then $N = 2$ and the Weyl law (4) changes; in fact, this is the only case where this happens for Schrödinger operators. An easy computation shows that in this case, if $\hat{\varphi}(0) = 1$,

$$c_{0,1} = \frac{2}{\sqrt{|\det d^2_{z_0} H(z)|}} = \frac{2}{\sqrt{|V''(x_0)|}}. \tag{16}$$

Using Theorems 1.1 and 1.2 and a Tauberian argument, we will show:

Theorem 1.3 *Under the Main Hypothesis, let $f(\hbar)$ denote the leading order term in the expansion of $\Upsilon_\hbar$ in theorem 1.1 with φ such that $\hat{\varphi}(0) = 1$ and N and ν of any parity. Assume that the linearized flows $d(f_t)_z$ on $T_z(T^*M)/T_z(\Theta)$, $z \in \Theta$, do not have any non-zero periods. Furthermore, assume that if $N > 2$ then the set of non-trivial periodic trajectories of $\{f_t\}$ on $\Sigma_{E_c} \setminus \Theta$ has Liouville measure 0. Under these assumptions,*

$$N(\hbar) = 2c\, f(\hbar) + o(f(\hbar)) .$$

Example. If $N = 2$ and the normal Hessian $Q(H)$ is indefinite then the linearized flows in theorem 1.3 are hyperbolic, and the hypothesis of the theorem are automatically satisfied. Thus in this case

$$N(\hbar) \simeq \frac{4c}{(2\pi)^{n-1}}\, \mu_0(\Theta)\, \hbar^{-(n-1)} \log \hbar^{-1}$$

and inspection of the proof below shows that one can actually improve the $o(\hbar^{-(n-1)}\log\hbar^{-1})$-estimate for the error to $\mathcal{O}(\hbar^{-(n-1)})$. If $Q(H)$ is definite then of course the linearized flows are completely periodic and this theorem does not apply.

The hypothesis on the linearized flow in Theorem 1.3 can surely be weakened. An open problem in this direction is to find a full trace formula, i.e. an asymptotic expansion of $\Upsilon_\hbar(\varphi)$ without the restriction on the support of the Fourier transform in Theorems 1.1 and 1.2. One expects additional contributions, e.g. logarithmic terms, arising from the periods of the linearized flow along the critical manifold.

The result of Theorem 1.3 is precisely the one predicted by the uncertainty principle, a form of which says that one can put one semi-classical state per unit volume of phase space, measured in units of $(2\pi\hbar)^n$. Hence the function N should satisfy

$$\hbar^n\, N(\hbar) \ \sim (2\pi)^{-n}\,\text{Liouville Volume of } H^{-1}[E_c-\hbar c, E_c+\hbar c]\ . \tag{17}$$

This is justified by Theorem 1.3. If we take for example a one-dimensional Schrödinger operator whose potential V has a local maximum E_c at the point x_0 (and at no other points) then by the above Theorem: $N(\hbar)\simeq \text{const}\cdot\log(1/\hbar)$, and the $\log(1/\hbar)$ is predicted by (17).

Our next results are on the behavior of the eigenfunctions corresponding to the eigenvalues counted by $N(\hbar)$. In case E non-critical, and under the assumption that the Hamiltonian flow is ergodic on the energy surface $H^{-1}(E)$, it is known that almost all such eigenfunctions become uniformly distributed in a suitable sense, see [?], [?] [?] and specially [?]. In the present setting, and if $N>2$, the notion of ergodicity of the classical flow is still valid since the Liouville density, although singular, remains integrable, and it turns out that almost all the eigenfunctions are still uniformly distributed. If $N=2$ and the Hessian is indefinite then, as Theorem 1.5 below shows, the singularity of the Liouville measure "traps" the wave functions on Θ (without any considerations of ergodicity). This second result, in dimension one, has been found independently by Colin de Verdière and Parisse, [?], who use it to show that there can exist eigenfunctions of the Laplacian on a compact manifold that concentrate along an unstable periodic geodesic. We also mention earlier work of Duclos and Hogreve, on the eigenfunctions with eigenvalue precisely E_c, [?].

To state our Theorems, we need to quote a construction from the appendix of [?]. We denote by $\mathcal{S}(T^*M)$ the space of smooth functions that satisfy Schwartz estimates in the fiber directions, uniformly in the M variables. In [?], we defined a quantization procedure of functions in $\mathcal{S}(T^*M)$, $a\mapsto \mathrm{Op}(a,\hbar)$, where $\mathrm{Op}(a,\hbar)$ is an $\hbar$-dependent operator with semi-classical symbol a. The operators $\mathrm{Op}(a,\hbar)$ are a manifold version of the $\hbar$-admissible operators of Robert and Helffer, [?]. Roughly, the Schwartz kernel of $\mathrm{Op}(a,\hbar)$ has the local expression $\int e^{i\xi\cdot(x-y)}\,a(x,\hbar\xi)\,d\xi$. We first need a slight generalization of Theorem 1.3.

Theorem 1.4 *Let $\{\psi_j^\hbar\}$ be an orthonormal basis of $L^2(M)$ such that $A_\hbar\psi_j^\hbar\ = E_j(\hbar)\psi_j^\hbar$, and let $a\in\mathcal{S}(TM)$. Under the main hypothesis, the conclusions of Theorems 1.1 and 1.2 remain valid for the asymptotics as $\hbar\to 0$ of*

$$\sum_j <\psi_j^\hbar, Op(a,\hbar)\psi_j^\hbar>\ \varphi\left(\frac{E_j(\hbar)-E_c}{\hbar}\right)\ , \tag{18}$$

provided the formulae for the coefficients in the expansion are modified as follows:

(i) If $\nu \geq 1$, $N-\nu \geq 1$ and both are odd,

$$c_{N/2-1,1} \;=\; C_{N,n,\nu} \cdot \hat{\varphi}(0) \int_{\Theta} a\, d\mu_0 \,. \tag{19}$$

(ii) If $N > 2$,

$$c_{0,0} \;==\; c_0 \;=\; \frac{\hat{\varphi}(0)}{(2\pi)^n} \int_{\Sigma_{E_c}} a\, d\lambda \tag{20}$$

where λ denotes Liouville measure on Σ_{E_c}.

Using this result, the Egorov-type theorem of [**?**], a refined Tauberian theorem proved in §6 and a well-known argument of Colin de Verdière and Zelditch, we prove:

Theorem 1.5 *Let $\{\psi_j^\hbar\}$ denote a basis of $L^2(M)$ of eigenfunctions of $A_\hbar$ with associated eigenvalues $E_j(\hbar)$, and make the same assumptions as in Theorem 1.3. Then:*
(1) If $N > 2$ and the flow on Σ_{E_c} is ergodic, there exists a density-one subset

$$L(\hbar) \subset \{\, j \;;\; |E_j(\hbar) - E_c| < c\hbar \,\} \tag{21}$$

such that for all $a \in \mathcal{S}(T^*M)$ and all integer-valued functions $\hbar \to j(\hbar)$ for which $j(\hbar) \in L(\hbar)$ for all $\hbar$ we have

$$\lim_{\hbar\to 0} \langle \psi^\hbar_{j(\hbar)} \,,\, Op(a,\hbar)\psi^\hbar_{j(\hbar)} \rangle \;=\; \frac{1}{LVol\,\Sigma_{E_c}} \int_{\Sigma_{E_c}} a\, d\lambda \;, \tag{22}$$

λ denoting Liouville measure.
(2) If $N = 2$ and the Hessian d^2H is indefinite along Θ, there exists a density-one subset $L(\hbar)$ such that for all $a \in \mathcal{S}(T^*M)$ which restricted to Θ are constant, equal to a_Θ say, and for all $\hbar \to j(\hbar) \in L(\hbar)$,

$$\lim_{\hbar\to 0} \langle \psi^\hbar_{j(\hbar)} \,,\, Op(a,\hbar)\psi^\hbar_{j(\hbar)} \rangle \;=\; a_\Theta \tag{23}$$

uniformly with $j \in L(\hbar)$.

Remarks. (a) That $L(\hbar)$ have density one means that

$$\lim_{\hbar\to 0} \frac{\# L(\hbar)}{N(\hbar)} \;=\; 1\,.$$

The statements in (22) and (23) are equivalent with the following: $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\hbar \in (0,\delta)$ and $j \in L(\hbar)$ imply that the distance between $\langle \psi_j^\hbar \,,\, \mathrm{Op}(a,\hbar)\psi_j^\hbar \rangle$ and the corresponding right-hand side is less than ϵ. These follow easily from

$$\lim_{\hbar\to 0} \frac{1}{N(\hbar)} \sum_{j\,;\,|E_j(\hbar)-E_c|<c\hbar} \Big| \langle \psi_j^\hbar \,,\, \mathrm{Op}(a,\hbar)\psi_j^\hbar \rangle - m(a) \Big| \;=\; 0\,, \tag{24}$$

where $m(a)$ denotes the right-hand side of (22), (23), respectively.
(b) In case $N = 2$ and $n = 1$ (so that $\Theta = \{z_0\}$) the assumption of $a|_\Theta$ being constant is automatic, and the limit in (23) is simply $a(z_0)$.
(c) In case $N = 2$ this result implies localization of the eigenfunctions on Θ, since the right-hand side of (23) is zero if a vanishes on Θ.

Finally, we wish to note that all the above results immediately extend to Schrödinger operators on $\mathbb{R}^n$, by the technique of [?], section 5, provided we restrict attention to the discrete spectrum below $\liminf_{|x|\to\infty} V(x)$ and modify the definition of $\Upsilon_{\hbar,E}$, by summing only over $\{j : |E_j(\hbar) - E| \leq \hbar^{1-\varepsilon}\}$ in (3), for some fixed but arbitrary $\varepsilon \in (0,1)$.

The paper is organized as follows. In §2, we reduce the study of (3) to a stationary phase problem with a degenerate phase. Precisely, the phase in question is $\psi = t\,Q(w)$ in (t,w) space, where Q is a non-degenerate quadratic form related to the normal Hessian, t represents time and w is a variable normal to Θ. In §3 we do a careful study of such oscillatory integrals with main result a stationary phase lemma for them. We refer to the introduction to §3 for a discussion of the relationship of our results with those already in the literature. Together, §§2 and 3 prove Theorem 1.1, but do not compute any of the coefficients in terms of the Hamilton flow of H. This is because in §2 we use the Morse lemma with parameters, which introduces a Jacobian in the calculation over which we have no a priori control.

This is circumvented by the results of §4, where we show that a suitable Fourier transform of (3) is a singular Lagrangian distribution $\Upsilon(s,E)$ in two variables, associated to a pair of intersecting Lagrangians. We then compute the symbol of Υ, in the sense of the symbol calculus for singular Lagrangian distributions due to Guillemin, Melrose and Uhlmann. We then show that that symbol is intimately related to those coefficients of the asymptotic expansions of Theorem 1.1 that we compute in Theorem 1.2. As a by-product of this approach, we show in §5 that our estimates can be made uniform in a neighborhood of size $\hbar$ of E_c. In §6 we prove a Tauberian lemma that will demonstrate Theorem 1.3, and §7 is devoted to the proof of Theorems 1.4 and 1.5.

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2 Oscillatory Integral Representations of Spectral Functions

In this section we recall some results from [?] on oscillatory integral representations of various spectral objects. Let M be a compact manifold, and $A_\hbar$ an $\hbar$-dependent differential operator on M of the form (1). We will in fact assume $H(x,p) \geq c > 0, \forall (x,p) \in T^*M$. By ellipticity and compactness, for each positive value of $\hbar$ the operator $A_\hbar$ has discrete spectrum. We will denote by $E_1(\hbar) \leq E_2(\hbar) \leq \cdots$ the eigenvalues (repeated according to their multiplicity) and $\psi_j^\hbar$, $j = 1, 2, \ldots$, corresponding eigenfunctions of $A_\hbar$. We wish to obtain an oscillatory integral description of

$$\sum_j \varphi\Big(\frac{E_j(\hbar) - E}{\hbar}\Big) \tag{25}$$

where $\check{\varphi} \in C_0^\infty(\mathbb{R})$, E is a parameter, and the large parameter in the integral we are after is of course $\kappa = \hbar^{-1}$. Here $\check{\varphi}(t) = (2\pi)^{-1}\hat{\varphi}(-t) == (2\pi)^{-1} \int_{\mathbb{R}} \varphi(s) e^{ist}\, ds$ is the inverse Fourier transform of φ. In [?] it is shown that (25) is given, modulo $O(\hbar^\infty)$, by an integral of the form

$$
\frac{1}{2\pi}\int_0^{2\pi}\int e^{-iks}\check{\varphi}(t)\ \mathrm{Tr}\left[\chi(\tilde{P})\, e^{i\,(-tD_\theta\tilde{P}+(s+tE)D_\theta)}\right]\, ds\, dt \tag{26}
$$

where:

a) $\hbar^{-1}=k\in\mathbb{Z}^+$.

b) $\tilde{P}$ is a certain classical pseudodifferential operator on $M\times S^1$, of order zero, commuting with D_θ (θ the angular variable on $S^1=\mathbb{R}/2\pi\mathbb{Z}$).

c) $\chi\in C_0^\infty(\mathbb{R})$ a suitable cut off function.

The operator $\tilde{P}$ is a microlocalized version of the operator

$$
P=\sum_{j=0}^{r} D_\theta^{-j}A_j
$$

on $M\times S^1$, where D_θ^{-1} is the obvious relative parametrix of D_θ. We won't describe here in detail how $\tilde{P}$ is constructed; it will suffice to say that its principal symbol has the following property: there exist constants $C_1>C_2>0$ such that

$$
\sigma_{\tilde{P}}(x,\theta\,;\,\xi,\kappa)\ =\ \begin{cases} H(x,\xi/\kappa) & \text{if } |H(x,\xi/\kappa)|\ <\ C_2\\ 0 & \text{if } |H(x,\xi/\kappa)|\ \geq\ C_1\end{cases} \tag{27}
$$

In fact $\tilde{P}$ is microlocally supported in the cone $|H(x,\xi/\kappa)|\leq C_1$. The representation (26) of (25) is valid modulo $O(h^\infty)$ for $0\leq E\leq C_2-\varepsilon$ for every $\varepsilon>0$.

In the application of (26) presented in this paper, we will be in situations where the main term in the stationary phase expansion of (26) (for large k) arises from critical points at which $s=0$ and E is equal to a critical value E_c of the Hamiltonian H.

To proceed further, we need an oscillatory integral formula for the Schwartz kernel $\mathcal{K}(t,s,x,y,\theta,\theta')$ of the operator

$$
\mathcal{U}\ =\ e^{i(-tD_\theta\tilde{P}+sD_\theta)}\ .
$$

In local coordinates with $|t|$ small, $\mathcal{K}$ can be written in the form

$$
\mathcal{K}=\int e^{i\kappa[\theta-\theta'+s+S(t,x,p)-y\cdot p]}\ \kappa^n\ \alpha(x,y,t;\kappa p,\kappa)\ dp\, d\kappa \tag{28}
$$

where α is a classical symbol supported in the cone

$$
|H(x,\xi/\kappa)|\leq C_1.
$$

The function $S(t,x,p)$ is the solution of the Hamilton-Jacobi equation

$$
\begin{cases} \partial_t S+H(x,\nabla_x S)\ =\ 0\\ \qquad\qquad S|_{t=0}\ =\ x\cdot p\end{cases} \tag{29}
$$

All this follows from the standard construction of $e^{-itD_\theta\tilde{P}}$ as a Fourier integral operator. The Cauchy problem (29) corresponds to a different choice of phase function from that of Hörmander in [?]. The phase (29) has more physical significance in the present context.

Indeed it is well-known that S is the generating function of $\{\phi_t\}$, the Hamilton flow of $H(x,p)$:

$$\phi_t\left(\frac{\partial S}{\partial p}, p\right) = \left(x, \frac{\partial S}{\partial x}\right) . \tag{30}$$

One can solve the Cauchy problem (29) by the method of characteristics. The amplitude

$$\alpha = \alpha(x, y, t\,;\,\xi, \kappa)\ , \quad \xi = \kappa\, p,$$

α a classical symbol of order 0 with respect to (ξ, κ), of the FIO $\exp(itD_\theta \tilde{P})$ is obtained by solving the transport equations associated with the identity $D_t e^{itD_\theta \tilde{P}} = -D_\theta \tilde{P} e^{itD_\theta \tilde{P}}$, with initial condition

$$\alpha|_{t=0} = (2\pi)^{-(n+1)}.$$

Next, in formula (26) we need to compose with $\chi(\tilde{P})$. However, we are interested in (25) with E in a small interval $[E_1, E_2]$ and the cut-off function χ is chosen so that $\chi(\tilde{P})$ is microlocally the identity in a small cone

$$E_1 - \varepsilon \le H(x, \frac{\xi}{\kappa}) \le E_2 + \varepsilon\ . \tag{31}$$

In particular the symbol of $\chi(\tilde{P})$ is constant one in (31), and does not show up in the calculations below.

Plugging (28) back in (26) we get:

Lemma 2.1 *If the support of $\check{\varphi}$ is small enough, (25) is equal, modulo $O(\hbar^\infty)$, to a finite sum of integrals of the form*

$$2\pi \cdot \hbar^{-n} \int e^{i\hbar^{-1}[S(t,x,p) - x\cdot p + tE]}\ \check{\varphi}(t)\ \alpha(x, x, t\,;\, \hbar^{-1}p, \hbar^{-1})\ dx\, dt\, dp\ . \tag{32}$$

The estimates are locally uniform as E ranges over regular values of $H(x,p)$.

(The extra 2π in (32) comes from $\int_0^{2\pi} \mathcal{K}|_{\theta=\theta'}\, d\theta$).

We seek to apply the method of stationary phase to the integral (32), under the Main Hypothesis of §1. We'll analyze next the critical points of the phase. Recall that we are working under the assumption that the support of $\check{\varphi}$ is small, and so our reasoning will be for t in a neighborhood of zero which may not be the same at each occurrence.

Lemma 2.2 *The critical points of $S(t,x,p) - x\cdot p + tE$ (as a function of (t,x,p)) are precisely the solutions to the following equations:*

$$H(x,p) = E\ , \quad \textit{and} \quad \phi_t(x,p) = (x,p)\ . \tag{33}$$

The proof is an easy consequence of the Hamilton-Jacobi equation that S satisfies, and of (30). If $E \ne E_c$, then for small t the only solutions to (33) are $t = 0$ and (x,p) arbitrary on $H^{-1}(E)$. If $E = E_c$, the set of critical points of the phase is the union

$$\{\, t = 0,\ (x,p) \in H^{-1}(E_c)\,\} \cup \{\, t \text{ arbitrary}\,,\ (x,p) \in \Theta\,\}\ . \tag{34}$$

We will correspondingly smoothly cut-off the integrand in (32), and break the integral in two parts, Υ_1 and Υ_2, where Υ_1 is the integral (32) localized in the (x,p) variables to some neighborhood $\mathcal{U}$ of Θ in phase space. That is, define

$$(35)\ \Upsilon_1(\hbar,E) \ := \ \hbar^{-n}\int e^{i\hbar^{-1}[S(t,x,p)-x\cdot p+tE]}\ \check{\varphi}(t)\ \rho(x,p)\,\alpha(x,x,t\,;\hbar^{-1}p,\hbar^{-1})\ dx\,dt\,dp\ ,$$

where ρ is a smooth function supported near Θ and identically one in a smaller neighborhood of Θ. Then define Υ_2 to be the difference (32) minus Υ_1. The only critical points of the phase that contribute to the asymptotic behavior of Υ_2 are of the form $t=0$ and $(x,p)\in H^{-1}(E)$ but *away* from Θ. Since, by assumption, Θ is the set of all critical points of H, this means that the critical points away from Θ form a non-degenerate manifold, and hence *the asymptotics of* Υ_2 *can be handled by ordinary stationary phase.* In fact, the analysis of [?] applies, and shows that Υ_2 has a classical asymptotic expansion of the form

$$\Upsilon_2 \ \sim\ \hbar^{-n+1}\sum_{j=0}^{\infty}\hbar^j\,a_j\ , \tag{36}$$

locally uniformly in E, with

$$a_0 \ = \ (2\pi)^{-n+1}\ \check{\varphi}(0)\int_{\Sigma_E}\rho d\mathrm{LVol} \tag{37}$$

(compare with (4)). Next we concentrate on Υ_1, which contains the new phenomena. Let

$$W(t,x,p) = S(t,x,p) - x\cdot p\ , \tag{38}$$

which is the non-trivial part of the phase in (32). Since $W|_{t=0}=0$, there is a smooth function $F(t,x,p)$ such that

$$S(t,x,p)-x\cdot p \ = \ t\,F(t,x,p)\ , \tag{39}$$

and hence the phase in (32) is

$$t\ (\ F(t,x,p)+E\ )\ . \tag{40}$$

Moreover, the Hamilton-Jacobi equation implies that

$$F|_{t=0} \ = \ -H\ , \tag{41}$$

while property (30) implies that

$$d_{(x,p)}F \ = \ 0 \quad\Leftrightarrow\quad \forall t\ \phi_t(x,p) \ = \ (x,p) \quad\Leftrightarrow\quad (x,p)\in\Theta\ . \tag{42}$$

We will now apply the Morse lemma with parameters to the function F. Reasoning locally near a point of Θ, begin by introducing coordinates (z',z'') in phase space in such a way that Θ is the submanifold $\{z''=0\}$. Let $a=(t,z')$, which we regard as parameters in the function $F=F(a,z'')$. By (41) and the Main Hypothesis, the Hessian of $F|_{a=(0,z')}$ at $z''=0$ is non-degenerate. Hence, by Lemma 1.2.2. of [?], there exists a change of variables,

$$(a,z'')\mapsto(a,y)\ ,\quad y=y(a,z'')=y(t,z',z'') \tag{43}$$

such that

$$F(a,z'') \ = \ F(t,z',0)+\frac{1}{2}\,\langle d^2_{z''}F_{(t,z',0)}y,y\rangle\ . \tag{44}$$

The neighborhood $\mathcal{U}$ of Θ to which we have localized Υ_1 is chosen as the (x,p)-projection of the common domain of these changes of coordinates, the latter being a suitable small neighborhood of $\Theta\times\{0\}$. Note that here we may need to further shrink the support of $\hat{\varphi}$.

Lemma 2.3 *For all t and all $(x,p) \in \Theta$,*

$$S(t,x,p) \;=\; -t\,E_c + x\cdot p\;. \tag{45}$$

PROOF   This easily follows from an explicit formula for S, in terms of the Hamilton flow of H: we refer to D. Robert's book, [**?**], formula (39) page 213.

□

Corollary 2.4 *For all t and all z',*

$$F(t,z',0) \;=\; -E_c\;.$$

The quadratic forms appearing in (44) depend on (t,z'). However, by the second part of the Main Hypothesis, they can be smoothly diagonalized: there exists a smooth family of orthogonal matrices, $R(t,z')$, such that

$$R\,\left(d^2_{z''}F(t,z',0)\right)\,R^{-1} \;=\; \mathrm{diag}\,(\lambda_1(t,z'),\cdots,\lambda_N(t,z')). \tag{46}$$

Hence if we let $u = Ry$ and $w_j \;:=\; |\lambda_j(t,z')|^{-1}u_j$ the phase becomes, after perhaps a permutation of the w_j's:

$$t\,(E-E_c) + \frac{t}{2}\,\left(w_1^2+\cdots+w_\nu^2 - w_{\nu+1}^2-\cdots-w_N^2\right)\;. \tag{47}$$

We will write this as

$$t\,\left(E-E_c-\frac{1}{2}\,\langle Qw,w\rangle\right)\;. \tag{48}$$

The change of coordinates introduces of course a Jacobian

$$|Du/Dw| \;=\; |\det d^2_{z''}F_{(t,z',0)}| \tag{49}$$

which will complicate the computation of the coefficients in the expansions of theorem 1.1. Since $F|_{t=0} = -H$, we see that the Jacobian (49), at $t=0$, equals the absolute value of the determinant of the normal Hessian of H; however, we will also need the derivatives (up till large order in t) of this Jacobian. Note that the signature of Q equals the signature of the normal Hessian of H, with ν the number of its negative eigenvalues. We summarize:

Proposition 2.5 *Under the previous assumptions, if the support of $\check{\varphi}$ is in a small neighborhood of zero, the quantity*

$$\sum_{j=1}^{\infty}\varphi\left[\frac{E_j(\hbar)-E}{\hbar}\right]$$

is, modulo a classical asymptotic series of the form (36), equal to a finite sum of integrals of the form

$$\hbar^{-n}\int e^{i\,\hbar^{-1}\,t\,(E-E_c-\frac{1}{2}\langle Qw,w\rangle)}\;\check{\varphi}(t)\,\beta(t,w\,;\,\hbar^{-1})\,dt\,dw\;, \tag{50}$$

where

$$\beta \;=\; 2\pi\int\;\rho\,\alpha|_{x=y}\,J\,dz_1 \tag{51}$$

J being the Jacobian of the change of variables $(t,x,p)\mapsto(t,z',w)$.

3 A Generalized Stationary Phase Formula

In this section we will make a detailed study of the asymptotic behavior of the class of oscillatory integrals with cubic phases of the form:

$$I(\kappa) = I_{a,Q}(\kappa) = \int_{\mathbb{R}^{N+1}} a(t,z)\, e^{i\kappa t\langle Qz,z\rangle}\, dt\, dz\ ,\ \kappa \to \infty, \tag{52}$$

which arose in the previous section. Here $\langle Qz, z\rangle$ is a non-degenerate real quadratic from on $\mathbb{R}^N$, and $a \in C_c^\infty(\mathbb{R}^{N+1})$. More generally, we could have taken as amplitude a classical symbol $a(t, z; \kappa)$ of order 0, with fiber variable $\kappa \in \mathbb{R}$, compactly supported in the space variables (t, z).

The main work in analyzing (52) is for indefinite Q: the analysis in the definite case is much easier and will be postponed till section 3.4. We will first look in section 3.1 below at the case of an indefinite form on $\mathbb{R}^2$. The case of a general Q with both indices of inertia odd can immediately be reduced to this special case. If one of the indices of inertia of Q is even one needs to do some additional work, which occupies section 3.3. Finally, in section 3.5 we prove our main Theorem 1.1.

Before starting our analysis, we want to make some remarks on previously known results and general background, to put this section in some perspective. A large number of authors have studied over the years the asymptotic behavior of a general oscillatory integral

$$I_{a,f}(\kappa)\ =\ \int_{\mathbb{R}^n} a(y)\, e^{i\kappa f(y)} dy,\ \kappa \to \infty, \tag{53}$$

with analytic phases f and compactly supported smooth amplitudes a. We mention the work of Arnold, Bernstein, Malgrange and of Varchenko, and refer to the book [?] for more information and detailed references. Using Hironaka's theorem on the resolution of singularities one can very generally prove the existence of an asymptotic expansion

$$I_{a,f}(\kappa) \sim \sum_{\alpha\in A}\sum_{p\geq 0}\sum_{k=0}^{K(f)} C_{\alpha,p,k}[a]\, \kappa^{-\alpha-p}\, (\log \kappa)^k,\ \ \kappa \to \infty \tag{54}$$

cf. [?]. Here A will be some finite set of strictly positive rational numbers, determined by f. This general result does not, however, provide information about either the principal exponents $\beta(f) := -\min A$ and $K(f)$, or about the distribution coefficients $C_{\alpha,p,k}[\cdot]$.

Varchenko [?] has shown how to read off these principal exponents from the Newton polyhedrae of f, for a large class of phases f; cf. also [?], chapters 6 to 8. Unfortunately, his work does not apply to the phases $tQ(z)$ we are interested in here. Moreover, Varchenko's results only provide scant information about the distributions $C_{\alpha,p,k}$ in (54). These have to be known rather explicitly for the applications we have in mind, since in general one will only be able to detect the presence of a critical manifold from the lower order terms in the expansion of $\Upsilon_\hbar$.

While it perhaps might be possible to extend the treatment of [?], [?] to the integrals (52), we will use here a more elementary and direct method, by which we will obtain a complete expansion of these integrals which is as explicit as the classical stationary phase expansion for quadratic phases.

After completion of this section, J. Duistermaat informed us that it should also be possible to derive the asymptotics of (52) from the results of [?], which studies the singularities of fundamental solutions of the operator $Q(D)$.

3.1 The 2-dimensional case.

We begin by analyzing (52) if $N = 2$ and Q is indefinite. After a linear change of coordinates we may assume that $\langle Qz, z\rangle = z_1^2 - z_2^2$. Making the additional change of variables $u = z_1 + z_2$, $v = z_1 - z_2$, we obtain an integral of the form

$$I(\kappa) = \int_{\mathbb{R}^3} a(t, u, v)\, e^{i\kappa tuv} dt du dv \; . \tag{55}$$

We will first restrict ourselves to amplitudes of the form

$$a(t, u, v) = \varphi(t) f(u) g(v) \; , \quad \varphi, f, g \in C_c^\infty(\mathbb{R}). \tag{56}$$

The asymptotics for a general amplitude can easily be reduced to this case by the following simple lemma (alternatively, one could rewrite the proofs below for general a, at the cost of complicating the notations):

Lemma 3.1 *Suppose that one has an asymptotic expansion of (53) for product type amplitudes $a(x) = a_1(x_1)\cdots a_n(x_n)$:*

$$I_{a,f}(\kappa) - \sum_{j=1}^{K-1} C_j[a]\kappa^{n_j} = \mathcal{O}(\kappa^{n_K}||a||_{C^{L(K)}}), \tag{57}$$

with $K \in N$ arbitrary, $L(K) \in \mathbb{N}$ depending on K, $C_j \in \mathcal{D}'(\mathbb{R}^n)$ and $n_1 > n_2 > \cdots > n_j \to -\infty$. Then the same expansion (57) holds for general amplitudes in $C_c^\infty(\mathbb{R}^n)$.

Proof Clearly the expansion (57) remains valid for finite sums of amplitudes of product type. Now $C_c^\infty(\mathbb{R}) \otimes \cdots \otimes C_c^\infty(\mathbb{R})$ (n-fold tensor product) is dense in $C_c^\infty(\mathbb{R}^n)$. Hence we can approximate an arbitrary amplitude $b \in C_c^\infty(\mathbb{R})$ in any C^L-norm by such a finite sum $\tilde{a}$ of product amplitudes, up to an arbitrarily small error $\varepsilon > 0$. If we take L sufficiently large, write $b = \tilde{a} + (b - \tilde{a})$ and apply (57), we can conclude that the RHS of (57) with a replaced by b can be estimated by $C \cdot \left(\kappa^{n_K}||b||_{C^{L(K)}} + \varepsilon \cdot (1 + \sum_{j=1}^K \kappa^{n_j})\right)$. Since ε can be taken arbitrarily small, this proves the lemma.

□

We return to (55,), (56). Performing the t-integral, (55) becomes

$$\int_{\mathbb{R}} \hat{\varphi}(-\kappa uv)\, f(u)\, g(v)\, du dv$$

which equals

$$I(\kappa) = \int \hat{\varphi}(-\kappa\rho) \left(\int_{\{uv=\rho\}} f(u)\, g(v)\, \frac{du}{|u|} \right) d\rho \; , \tag{58}$$

since $u^{-1}du \wedge d(uv) = du \wedge dv$; $u^{-1}du$ is the Gelfand-Leray form of the function uv (cf. [?]). It is now natural to use convolution on the multiplicative group $\mathbb{R}_{>0}$:

$$(f * g)(\rho) = \int_0^\infty f(u)\, g(\frac{\rho}{u}) \frac{du}{u} \; . \tag{59}$$

Writing $f_-(u) = f(-u)$, $g_-(v) = g(-v)$, (58) equals

$$\begin{aligned} I(k) = \frac{1}{\kappa} \Big\{ \int_0^\infty \hat{\varphi}(-\rho)[(f * g) \; + (f_- * g_-)](\frac{\rho}{\kappa}) d\rho + \\ \int_0^\infty \hat{\varphi}(\rho)[(f * g_-) \; + \; (f_- * g)](\frac{\rho}{\kappa}) d\rho \Big\} \; . \end{aligned} \tag{60}$$

Hence it suffices to determine the asymptotics of (59) for $\rho \to 0^+$. This we will do using the Mellin transform $f \to \mathcal{M}(f)$ of f. Recall that if $f \in C_0^\infty([0,\infty))$ and $\mathrm{Re}\, s > 0$,

$$F(s) = \mathcal{M}(f)(s) := \int_0^\infty f(u) u^{s-1} du ,$$

Also recall that

$$\mathcal{M}[f * g] = \mathcal{M}[f]\, \mathcal{M}[g]$$

and that

$$f(u) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, u^{-s} ds, \quad \sigma > 0 \tag{61}$$

(the inversion formula). The following lemma is classical:

Lemma 3.2 *Let $f \in C_c^\infty([0,\infty))$. Then $F(s) := \mathcal{M}[f](s)$ extends to a meromorphic function on $\mathbb{C} \setminus \{0, -1, -2, \cdots\}$. Moreover, $F(s)$ has simple poles at $s = -k,\ k \in \mathbb{N}$, with residues $f^{(k)}(0)/k!$. Finally,*

$$F(\sigma + i\tau) u^{-(\sigma + i\tau)}$$

is in the Schwarz space $\mathcal{S}(\mathbb{R})$ as function of τ when $\sigma \notin -\mathbb{N}$.

Proof Repeated integration by parts shows that

$$F(s) = \frac{(-1)^{k+1}}{s(s+1)...(s+k)} \int_0^\infty f^{(k+1)}(u)\, u^{s+k}\, du, \quad \mathrm{Re} s > 0 \tag{62}$$

from which the first half of the lemma follows. To prove the final statement, observe that $F(\sigma + i\tau)$, as of a function of τ, is one over some polynomial in τ times the Fourier transform of $f^{(k+1)}(e^y) e^{y(\sigma+k+1)}$, which is a rapidly decreasing function of y as long as $\sigma > -k-1$.

□

In the sequel the following distributions $\Lambda_j(\cdot)$ will appear: define constants γ_j by

$$\gamma_j = \sum_{l=1}^{j} \frac{1}{l} \tag{63}$$

and let

$$\Lambda_j(f) = -\int_0^\infty f^{(j+1)}(u) \log u\ du + \gamma_j\, f^{(j)}(0) . \tag{64}$$

Lemma 3.3 *Let $f \in C_0^\infty([0,\infty))$, $F(s) = \mathcal{M}[f](s)$. The constant term in the Laurent expansion of F around $s = -j$ equals*

$$\lim_{s \to -j} \left(F(s) - \frac{f^{(j)}(0)}{j!} \frac{1}{s+j} \right) = \frac{1}{j!} \Lambda_j(f) .$$

Proof By (62),

$$F(s) - \frac{f^{(j)}(0)}{(s+j)j!} = \int_0^\infty f^{(j+1)}(u) \frac{H(s+j, u)}{s+j}\, du,$$

with

$$H(z,u) = \frac{(-1)^{j+1} u^z}{(z-1)\cdots(z-j)} + \frac{1}{j!} .$$

Note that $H(0,u)=0$, and that the coefficient of z in the power series expansion of $H(z,u)$ around $z=0$ is

$$-\frac{1}{j!}(\log u + \gamma_j).$$

This proves the lemma.

$\square$

The following proposition is the main step in our analysis of $I(k)$.

Proposition 3.4 *Let $f, g \in C_0^\infty([0,\infty))$. Then*

$$(f*g)(\rho) \sim \tag{65}$$
$$\sum_{j=0}^{\infty} \frac{1}{(j!)^2} \left(f^{(j)}(0)\, g^{(j)}(0) \rho^j \log \rho^{-1} + \ \left(\Lambda_j(f) g^{(j)}(0) + f^{(j)}(0) \Lambda_j(g) \right) \rho^j \right) \ ,$$

as $\rho \to 0^+$.

PROOF By Mellin's inversion formula (61), if $F = \mathcal{M}[f]$, $G = \mathcal{M}[g]$, then

$$(f*g)(\rho) \ = \ (2\pi i)^{-1} \int_{\sigma-\infty}^{\sigma+\infty} F(s) G(s) \rho^{-s} \, ds \ , \ \sigma > 0. \tag{66}$$

If we shift the path of integration to $\mathrm{Re} s = -K + \varepsilon$ ($K \in \mathbb{N}$, $0 < \varepsilon < 1$), using lemma 3.2, we obtain

$$(f*g)(\rho) = \sum_{j=0}^{K} \mathrm{Res}_{s=-j} (F(s) G(s) \rho^{-s}) + O(\rho^{K+\varepsilon})$$

Since $\rho^{-s} = \rho^j - (s+j)\rho^j \log(\rho) + O((s+j)^2)$, Lemma 3.3 implies that

$$\begin{aligned} &\mathrm{Res}_{s=-j}(FG\rho^{-s}) = \\ &\quad = \ (j!)^{-2} \left(f^{(j)}(0) g^{(j)}(0) \rho^j \log \rho^{-1} \ + \ (\Lambda_j(f) g^{(j)}(0) + f^{(j)}(0) \Lambda_j(g)) \rho^j \right), \end{aligned}$$

which completes the proof of (65).

$\square$

We now return to (60). Substitute the expansions (65) and rearrange terms. Then we obtain, after some computations,

$$I(\kappa) \sim \sum_{j=0}^{\infty} c_{j,1}\, \kappa^{-1-j} \log \kappa + c_{j,0}\, \kappa^{-1-j} \ = \tag{67}$$

$$= \ \sum_{j=0}^{\infty} \sum_{l=0,1} c_{j,l}\, \kappa^{-1-j} \, (\log \kappa)^l,$$

where

$$(j!)^2\, c_{j,1} = (4\pi)\, i^j\, f^{(j)}(0)\, g^{(j)}(0)\, \varphi^{(j)}(0) \tag{68}$$

and

$$(j!)^2\, c_{j,0} \ = \ -2 f^{(j)}(0) g^{(j)}(0) \int_{\mathbb{R}} \hat{\varphi}(-\rho)\, \rho^j \log|\rho| \, d\rho$$

(69) $-2\pi i^j \varphi^{(j)}(0)\left[g^{(j)}(0)\int_{\mathbb{R}} f^{(j+1)}\mathrm{sgn}u\log|u|du \;+\; f^{(j)}(0)\int_{\mathbb{R}} g^{(j+1)}(u)\mathrm{sgn}u\log|u|du\right]$

$$+\; 8\pi\, i^j\, \gamma_j\, \varphi^{(j)}(0)\, f^{(j)}(0)\, g^{(j)}(0)\ .$$

To write this in a more symmetric form, observe that since the Fourier transform of $\rho^{-1}\log|\rho|$ is equal to

(70) $$(\rho^{-1}\hat{\log}|\rho|)(x) = \pi i\,(\log|x| + \gamma)\,\mathrm{sgn}x$$

γ being Euler's constant (cf. [**?**], theorem 48 on page 103),

$$2\int \hat{\varphi}(-\rho)\rho^j\log|\rho|d\rho \;=\; (-1)^j\, 2\int \rho^{j+1}\,\hat{\varphi}(\rho)\,\rho^{-1}\log|\rho|\,d\rho \;=$$

$$=\; 2\pi i^j\int \varphi^{(j+1)}(x)(\log|x|+\gamma)\mathrm{sgn}x\,dx$$

(71) $$=\; 2\pi i^j\int \varphi^{(j+1)}(x)\log|x|\,\mathrm{sgn}x\,dx \;-\; 4\pi i^j\gamma\varphi^{(j)}(0)\ .$$

To lighten the notation a little bit, let us introduce the tempered distribution $\Gamma\in\mathcal{S}'(\mathbb{R})$, defined by

(72) $$\Gamma(\psi) \;=\; \int_{\mathbb{R}} \psi'(x)\log|x|\,\mathrm{sgn}x\,dx$$

We now combine (67)-(71), and finally replace the amplitude (56) by a general amplitude $a\in C_c^\infty(\mathbb{R}^3)$, using lemma 3.1. The result is the following asymptotic expansion: we denote by i_t, i_u, i_v the immersions $i_t(x) = (x,0,0)$, $i_u(x) = (0,x,0)$, etc.

Theorem 3.5 *Let*

(73) $$I(\kappa) \;=\; \int_{\mathbb{R}^3} a(t,u,v)\,e^{i\kappa tuv}\,dtdudv,\quad u\in C_c^\infty(\mathbb{R}^3).$$

Then

(74) $$I(\kappa)\sim\sum_{j=0}^{\infty}\sum_{l=0,1} C_{j,l}[a]\,\kappa^{-1-j}\,(\log\kappa)^l\ ,$$

where

(75) $$C_{j,1}[a] \;=\; 4\pi\,(j!)^{-2}\,i^j\,(\partial_t\partial_u\partial_v)^j a\,(0)$$

and

(76) $$C_{j,0}[a] \;=$$

$$-2\pi(j!)^{-2}i^j\left\{\Gamma(i_t^*(\partial_t\partial_u\partial_v)^j a) \;+\; \Gamma(i_u^*(\partial_t\partial_u\partial_v)^j a) \;+\; \Gamma(i_v^*(\partial_t\partial_u\partial_v)^j a)\right\} \;+$$

$$+\; 4\pi(j!)^{-2}i^j(\gamma+2\gamma_j)(\partial_t\partial_u\partial_v)^j a(0).$$

Here

$$\Gamma(i_t^*(\partial_t\partial_u\partial_v)^j(a)) = \int_{\mathbb{R}}(\partial_t^{j+1}\partial_u^j\partial_v^j a)\,(x,0,0)\,\log|x|\mathrm{sgn}x\,dx\ ,\mathrm{etc.}$$

Note the, of course necessary, symmetry of the formulas with respect to permutations of u, v and t.

In the Tauberian part of the proof of the modified Weyl law in the case codim $\Theta = 2$ we will need the following observation:

Corollary 3.6 *Let*

$$I_E(\kappa) \;=\; \int a(t,u,v)\, e^{i\kappa t(uv-E)}\, dudtdv.$$

Then

(77) $$\kappa\, I_E(\kappa) \;=\; 4\pi \log(\kappa - E)\, a(0) + O(1)\ ,\ \ \kappa \to \infty.$$

the $O(1)$-error being uniform both in $E \to 0$ and $\kappa \to \infty$.

Proof Specializing, as before, to amplitudes (56), we have

$$\kappa\, I_E(\kappa) \;=\; \int \hat{\varphi}(-\rho) \int_{\{uv-E=\rho/\kappa\}} f(u)g(v)\frac{du}{|u|}$$

Proceeding as before, we have determine the asymptotic behavior for $\rho \to 0$ of

$$\int_{uv=\rho+E} f(u)g(v)\frac{du}{u} \;=\; (f*g)(\rho+E) \;=\; (2\pi i)^{-1}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} F(s)G(s)(\rho+E)^{-s}ds\ ,$$

where $0<\varepsilon<1$. Shift the path of integration to $\varepsilon - 1$ (assuming $\varepsilon < 1$) and note that the residue in 0 is

$$\log(\rho+E)^{-1}f(0)g(0) \;+\; [\Lambda_0(f)g(0) + f(0)\Lambda_0(g)] \;=\; \log(\rho+E)^{-1}f(0)g(0) \;+\; O(1).$$

The line integral over $Re\, s = \varepsilon - 1$ is bounded by

$$(\rho+E)^{1-\varepsilon}\,\|F(\varepsilon-1+i\cdot)\|_2\,\|G(\varepsilon-1+i\cdot)\|_2\ ,$$

which is $O(1)$ as $(\rho, E) \to 0$. This proves the corollary.

□

3.2 Case 1: Odd indices of inertia

We next turn to $I_{a,Q}$ for general symmetric Q. We will first concentrate, in the following two sections, on the case of indefinite Q's, and postpone the definite case to section 3.4. Let $|Q|$ be the absolute value of Q (defined by functional calculus of symmetric matrices). Then for some $\nu \in \mathbb{N},\ \nu \geq 1$, after perhaps a permutation of the coordinates,

(78) $$\langle Q(|Q|^{-1/2}z), |Q|^{-1/2}z\rangle = |z'|^2 - |z''|^2\ ,\ z=(z',z'')\in \mathbb{R}^\nu \times \mathbb{R}^{N-\nu}\ .$$

If both indices of inertia ν and $N-\nu$ of Q are >1, we can introduce polar coordinates

$$z' = r\zeta,\ z'' = s\eta\ , \zeta \in S_{\nu-1}\ ,\ \eta \in S_{N-\nu-1}\ ,\ r,s>0\ ,$$

and $I = I_{a,Q}$ can then be written as

(79) $$I(\kappa) = \int_{\mathbb{R}}\int_0^\infty\int_0^\infty \alpha(t,r,s)e^{i\kappa t(r^2-s^2)}drdsdt$$

with

(80) $$\alpha(t,r,s) = |\det Q|^{-1/2}r^{\nu-1}\, s^{N-\nu-1} \int_{S_{\nu-1}}\int_{S_{N-\nu-1}} a(t, |Q|^{-1/2}(r\zeta, s\eta))\, d\sigma_{\nu-1}(\zeta)d\sigma_{N-\nu-1}(\eta)\ .$$

Here the $\sigma's$ denote the rotation invariant measures on the unit spheres $S_{\nu-1}$, $S_{N-\nu-1}$, with the usual normalization.

If $\nu = 1$ or $N - \nu = 1$, we have to slightly adapt the formula for α. If, for example, $N - \nu = 1$ but $\nu > 1$ then we have to replace (80) by

$$\alpha(t, r, s) = \sum_{\pm} |\det Q|^{-1/2} r^{\nu-1} \int_{S_{\nu-1}} a(t, |Q|^{-1/2}(r\zeta, \pm s)) d\sigma_{\nu-1}(\zeta), \tag{81}$$

while in the important special case $\nu = N - \nu = 1$,

$$\alpha(t, r, s) = \sum_{\pm}\sum_{\pm} |\det Q|^{-1/2} a(t, |Q|^{-1/2}(\pm r, \pm s)). \tag{82}$$

(In this last case we of course just have an integral like (79) but extended over $\mathbb{R}^3$, to which the results of the previous section immediately apply).

Now suppose that $\alpha(t, r, s)$ in (79), extends to an even C^∞-function of both r and of s on all of $\mathbb{R}^2$. Then the integral (79) equals 1/4 times the same integral extended over $\mathbb{R}^3$ and the change of variables $u = r + s, v = r - s$, reduces this last integral to (73). We now can apply theorem 3.5, which leads, after some computations, to the following result:

Theorem 3.7 *Suppose that $\alpha(t, r, s) \in C_c^\infty(\mathbb{R} \times [0, \infty)^2)$ extends to an even C^∞-function in both variables r and s separately. Then $I(\kappa)$ defined by (79) has an asymptotic expansion*

$$I_{a,Q}(\kappa) \sim \sum_{j=0}^{\infty} \sum_{l=0,1} c_{j,l}\, \kappa^{-1-j} (\log \kappa)^l\,,\ \kappa \to \infty\ , \tag{83}$$

with

$$2^{2j}\, i^{-j}\, (j!)^2\, c_{j,1} \ = \ \frac{\pi}{2}\, (\partial_t(\partial_r^2 - \partial_s^2))^j \alpha(0, 0, 0) \tag{84}$$

and

$$\begin{aligned} 2^{2j}\, i^{-j}\, (j!)^2\, c_{j,0} \ &= \ -\frac{\pi}{4} \int_{\mathbb{R}} (\partial_t(\partial_r^2 - \partial_s^2))^j \partial_t \alpha(t, 0, 0)\, sgn t \log |t|\, dt \\ &\quad -\frac{\pi}{2} \int_0^\infty (\partial_t(\partial_r^2 - \partial_s^2))^j (\partial_r \alpha + \partial_s \alpha)(0, \frac{r}{2}, \frac{r}{2}) \log r\, dr \\ &\quad +\frac{\pi}{2}\, (\gamma + 2\gamma_j)\, (\partial_t(\partial_r^2 - \partial_s^2))^j \alpha(0, 0, 0) \end{aligned} \tag{85}$$

where, as before, γ is Euler's constant and γ_j is given by (63).

This theorem applies in particular to $I_{a,Q}$ when Q has both indices of inertia odd, with α given by (80) - (82).

3.3 Case 2: At least one even index of inertia.

If ν or $N - \nu$ is even, or if both are, the nature of the asymptotic expansion of $I_{a,Q}$ changes: it will no longer contain log-terms. Instead, half-integer powers of κ will appear. We will derive this from a more general result, namely an asymptotic expansion of (79) for arbitrary $\alpha \in C_c^\infty(\mathbb{R} \times [0, \infty)^2)$: *this* expansion will in general contain both logarithms and half-integer powers. However, if α is odd in one of the two variables the coefficients of the log terms will disappear. Similarly, if α is even in both r and in s the coefficients of all half-integer powers will vanish. Note that the α's defined by (80) - (82) will have some definite parity both in r and in s.

The difference between these cases can heuristically be understood as follows: suppose for example that α is odd in r. Then we can write $\alpha = r\alpha'(r,s,t)$ with α' even in r. If we now make the change of variables $\rho = r^2$ in (80) and next introduce a new coordinate $w = \rho - s^2$, the phase becomes just $t \cdot w$, which is a non-degenerate phase function which can in principle be handled by classical stationary phase. The set of critical points now will just be $\{t = 0, \rho = s^2\} = \{t = 0, r = \pm s\}$. Contrast this with the critical set of $t(r^2 - s^2)$, which additionally includes $\mathbb{R} \times \{r = s = 0\}$. However, if one tries to actually carry out this approach, one finds that the new amplitude will contain singularities at $w = 0$, coming from the fact that our domain of integration contains a cusp at 0. Rather than dealing directly with this complication we have preferred to follow the slightly more roundabout route of analyzing (79) for arbitrary α without parity restrictions.

Let therefore α in $C_c^\infty(\mathbb{R} \times [0,\infty)^2)$ be arbitrary. If as before we make the change of variables $u = r + s, v = r - s$, (79) becomes

$$I(\kappa) = \frac{1}{2} \int_{u \geq |v|} \int_{\mathbb{R}} \alpha\left(t, \frac{u+v}{2}, \frac{u-v}{2}\right) e^{i\kappa t u v}\, dt du dv \,. \tag{86}$$

Specializing again to amplitudes of the form $\alpha\left(t, \frac{u+v}{2}, \frac{u-v}{2}\right) = \varphi(t) f(u) g(v)$ we can write this as

$$\begin{aligned} &\frac{1}{2\kappa} \int_0^\infty d\rho\, \hat{\varphi}(-\rho) \int_{(\rho/\kappa)^{\frac{1}{2}}}^\infty \frac{du}{u}\, f(u) g(\frac{\rho}{\kappa u}) + \\ &\frac{1}{2\kappa} \int_0^\infty d\rho\, \hat{\varphi}(\rho) \int_{(\rho/\kappa)^{\frac{1}{2}}}^\infty \frac{du}{u}\, f(u) g_-(\frac{\rho}{\kappa u}), \end{aligned} \tag{87}$$

where $g_-(v) := g(-v)$. We therefore need a half-sided version of Proposition 3.4:

Proposition 3.8 *There exist certain (computable) universal constants A_{jk} such that if we let*

$$q_j(f,g) = \sum_{k=0}^{j} A_{j,k} \left(f^{(k)}(0) g^{(j-k)}(0) - g^{(k)}(0) f^{(j-k)}(0) \right) \tag{88}$$

then, for $\rho \to 0+$,

$$\int_{\sqrt{\rho}}^\infty f(u) g(\frac{\rho}{u}) \frac{du}{u} \sim \tag{89}$$

$$\sum_{j=0}^\infty \frac{1}{2} (j!)^{-2} f^{(j)}(0) g^{(j)}(0)\, \rho^j \, \log \rho^{-1} \, + \, (j!)^{-2} g^{(j)}(0) \Lambda_j(f)\, \rho^j \, + \, q_j(f,g))\, \rho^{j/2}$$

Remark. The precise value of the constants A_{jk} won't be important for the sequel. Note that $q_j(f,g)$ is anti-symmetric in f and g, and that

$$\int_{\sqrt{\rho}}^\infty f(u) g(\frac{\rho}{u}) \frac{du}{u} + \int_{\sqrt{\rho}}^\infty g(u) f(\frac{\rho}{u}) \frac{du}{u} = \int_0^\infty f(u) g(\frac{\rho}{u}) \frac{du}{u} \tag{90}$$

Hence, in view of the anti-symmetry of the q_j, Proposition 3.8 implies Proposition 3.4. We can in fact use 3.4 to short-circuit the proof of 3.8 (which is completely different) at a certain point.

PROOF First expand $g(\rho/u)$ in the left hand side of (89) in a power series in ρ/u. Then

$$\int_{\sqrt{\rho}}^{\infty} f(u)\, g(\frac{\rho}{u}) \frac{du}{u} \sim \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!} \rho^j \int_{\sqrt{\rho}}^{\infty} f(u) u^{-(j+1)} du\ . \tag{91}$$

The error-term after breaking off the series at $j = K$ can be estimated by

$$C\, \|g^{(K+1)}\|_\infty\, \|f\|_1\, \rho^{\frac{K+1}{2}}$$

so this will indeed lead to an asymptotic expansion. Next, integrating by parts,

$$\int_{\sqrt{\rho}}^{\infty} f(u)\, u^{-(j+1)} \frac{du}{u} = \sum_{k=0}^{j-1} \frac{(j-k-1)!}{j!} f^{(k)}(\sqrt{\rho}) \rho^{\frac{k-j}{2}} - \tag{92}$$

$$-\frac{f^{(j)}(\sqrt{\rho})}{j!} \log\sqrt{\rho} - \frac{1}{j!} \int_{\sqrt{\rho}}^{\infty} f^{(j+1)}(u) \log u\, du.$$

Expand each of the $f^{(k)}(\sqrt{\rho})$ in a power series in $\sqrt{\rho}$, and replace the integral on the right hand side of (92) by

$$- \int_0^\infty f^{(j+1)}(u) \log u\, du + \int_0^{\sqrt{\rho}} f^{(j+1)}(u) \log u\, du \sim \tag{93}$$

$$- \int_0^\infty f^{(j+1)}(u) \log u\, du + \sum_{l=0}^{\infty} \frac{f^{(j+1+l)}(0)}{l!} \left(\frac{\rho^{(l+1)/2}}{l+1} \log\sqrt{\rho} - \frac{\rho^{(l+1)/2}}{(l+1)^2} \right) .$$

Now substitute (92) and (93) into (91) and rearrange terms. The coefficient of $\rho^j \log\sqrt{\rho}$ turns out to be a telescoping series, and the end result is an asymptotic expansion

$$\sum_{j\ge 0} -(j!)^{-2} g^{(j)}(0) f^{(j)}(0) \rho^j \log\sqrt{\rho} - \tag{94}$$

$$\sum_{j\ge 0} (j!)^{-2} g^{(j)}(0) (\int_0^\infty f^{(j+1)}(u) \log u\, du)\, \rho^j\ +$$

$$\sum_{n\ge 0} \left(\sum_{k+l+j=n, k\le j-1} \frac{(j-k-1)!}{(j!)^2 l!} g^{(j)}(0) f^{(k+l)}(0) - \sum_{2j+l+1=n} \frac{f^{(j+l+1)}(0) g^{(j)}(0)}{(j!)^2 (l+1)(l+1)!} \right) \rho^{\frac{n}{2}}$$

To finish the proof, we look at the coefficient of $\rho^{n/2}$ in (94). A term $f^{(p)}(0) g^{(p)}(0)$ will occur if either $j = k + l = p$, which implies that $n = j + k + l = 2p$ even, or if $j + l + 1 = j = p$, which is not possible since $l \ge 0$. Hence the sum $\sum_{n\ge 0}$ in (94) can be rewritten as (changing the summation index n into j)

$$\sum_j (\sum_{k\le j-1} \frac{(j-k-1)!}{(j!)^2 (j-k)!}) (g^{(j)}(0) f^{(j)}(0)) \cdot \rho^j\ + \sum_{j\ge 0} q_j(f, g) \rho^{j/2} \tag{95}$$

for suitable q_j. Since the coefficient of $g^{(j)}(0) f^{(j)}(0)$ in (95) is exactly $(j!)^{-2}\gamma_j$, the anti-symmetry of the q_j follows from (90) and Proposition 3.4. Finally, since q_j depends on products of derivatives $f^{(a)}(0) g^{(b)}(0)$ with $a + b = j$, it has to have the form (88). The precise values of the $A_{j,k}$ follow from a computation which we omit.

□

Note, for later use, that $q_0 = 0$ and that

$$q_1(f;g) \;=\; g'(0)f(0) - g(0)f'(0) \tag{96}$$

We can now substitute (89) in (87) to obtain the asymptotic expansion of (86) and hence of (79). We will leave the details to the reader, and will just note the following points. First, at a certain stage of the computation one again has to use (70). Next, in order to have a compact expression for the coefficients of the half-integer powers in the asymptotic expansion below, let us introduce the distributions:

$$q_j \;=\; (-1)^j \sum_{k=1}^{j} A_{j,k} \left(\partial_u^k \partial_v^{j-k} - \partial_v^k \partial_u^{j-k} \right) \delta(u,v), \tag{97}$$

supported at the origin, and also the distributions

$$F_j^{\pm} \;=\; \Gamma(\frac{j}{2}+1)\; e^{\mp i\pi(j+2)/4}\; (t \mp i0)^{-j/2-1}; \tag{98}$$

$F_j^{\pm}$ is the Fourier transform of $\rho_{\pm}^{j/2}$, where $\rho_{\pm} := \max(\pm\rho, 0)$, cf. [?]. With these notations we have the following result:

Theorem 3.9 *Let*

$$I(\kappa) \;=\; \int \alpha(t,r,s) e^{i\kappa t(r^2-s^2)} dtdrds,$$

α in $C_c^\infty(\mathbb{R}\times[0,\infty)^2)$. Then

$$I(\kappa) \sim \left(\begin{array}{c} \textit{asymptotic expansion (83)} \\ \textit{of Theorem 3.7} \end{array} \right) + \sum_{j\geq 1} b_j \kappa^{-1-j/2} \tag{99}$$

with

$$b_j = \langle F_j^-(t) \otimes q_j(u,v) + F_j^+(t) \otimes q_j(u,-v),\; \alpha(t, \frac{u+v}{2}, \frac{u-v}{2}) \rangle \tag{100}$$

Since any odd function differentiated an even number of times and evaluated at 0 gives 0, we immediately obtain the following corollary:

Corollary 3.10 *Suppose Q has at least one even index of inertia. Then $I = I_{a,Q}$ has an asymptotic expansion*

$$I(\kappa) \;\sim\; \sum_{j=0}^{\infty} c_j \kappa^{-1-j/2} \tag{101}$$

in integer and half-integer powers of κ, with coefficients which can be read off from (80) - (82), (85) and (100).

Remark. From (96), (98) and (99) we read off that for example the coefficient of $\kappa^{-3/2}$ is:

$$-\langle F_1^-(t), \partial_s \alpha(t,0,0)\rangle - \langle F_1^+(t), \partial_r \alpha(t,0,0)\rangle,$$

so half-integer powers will in general always occur, unless of course α is even in r and s. In the latter case all coefficients b_j in (99) will vanish.

Remark. If $N > 2$, the first non-zero term in the asymptotic expansions (83), (99) of $I_{a,Q}$ is:

$$-\frac{\pi}{\kappa}\int_0^\infty \partial_r[\alpha(0,\frac{r}{2},\frac{r}{2})]\log r\,dr = \frac{\pi}{\kappa}\int_0^\infty \alpha(0,r,r)\frac{dr}{r}, \tag{102}$$

the integration by parts being allowed since $\alpha(0) = 0$. We claim that this is precisely the coefficient in the Weyl law:

$$\frac{2\pi}{\kappa}\int_{\{Q=0\}} a(0,z)\,dL_Q(z), \tag{103}$$

where dL_Q is the Liouville measure on $\{Q = 0\}$, which is a well-defined locally finite measure if $N > 2$. One obtains (103) formally by introducing the local coordinate $q = Q(z)$ in (52) (strictly speaking not allowed at $z = 0$) and applying the classical stationary phase formula to the resulting 2-dimensional (t, q) integral. The equality of (102) and (103) follows from an easy computation: if we assume wlog that $Q(z) = |z'|^2 - |z''|^2 = r^2 - s^2$, then

$$dL_Q = \pm\frac{1}{2}\frac{r^k s^{N-k-1} d\sigma_{k-1}\wedge d\sigma_{N-k-1}\wedge ds - r^{k-1}s^{N-k}d\sigma_{k-1}\wedge d\sigma_{N-k-1}\wedge dr}{(r^2-s^2)},$$

(with $d\sigma_{k-1} = d\sigma_{k-1}(\zeta)$, $d\sigma_{N-k-1} = d\sigma_{N-k-1}(\eta)$), since $dL_Q \wedge dQ$ equals the volume form on $\mathbb{R}^N$. Hence

$$\int_{\{Q=0\}} a(0,z)\lambda_Q(z) = \frac{1}{2}\int_{\{r=s\}}\frac{r\alpha(0,r,s)ds - s\alpha(0,r,s)}{(r^2-s^2)}dr = \frac{1}{2}\int_0^\infty \alpha(0,r,r)\frac{dr}{r}.$$

3.4 Case 3: Q definite.

Finally we consider the easier case of definite Q. Assume for example that Q is negative definite. Replacing $a(z,t)$ by $|\det Q|^{-1/2}a(|Q|^{-1/2}z,t)$ we may assume that $\langle Qz, z\rangle = -|z|^2$. If we specialize again to amplitudes of the form $\varphi(t)a(z)$, (52) equals

$$I(\kappa) = \frac{1}{\kappa^{N/2}}\int_0^\infty \hat{\varphi}(r^2)\,\alpha(\frac{r}{\kappa^{1/2}})\,r^{N-1}dr \tag{104}$$

where

$$\alpha(r) = \int_{S_{N-1}} a(r\zeta)\,d\sigma_{N-1}(\zeta)\ . \tag{105}$$

if $N > 1$, and

$$\alpha(r) = a(r) + a(-r)$$

if $N = 1$. Now expand α in a power series at $r = 0$ and substitute this series in (104). In this way we obtain an asymptotic expansion

$$I(\kappa) \sim \sum_{j\geq 0} c_j\,\kappa^{-(N+j)/2}$$

with distribution coefficients c_j given by

$$c_j = c_j[a\varphi] = \frac{\alpha^{(j)}(0)}{j!}\int_0^\infty r^{j+N-1}\hat{\varphi}(r^2)\ dr.$$

$$= \frac{1}{2}\frac{\alpha^{(j)}(0)}{j!}\ \langle \rho_+^{\frac{N+j}{2}-1}, \hat{\varphi}(\rho)\rangle$$

If we finally use (98) again we arrive at the following result:

Theorem 3.11 *Suppose that Q is negative definite. Then*

$$I_{a,Q}(\kappa) \sim \sum_{j\geq 0} c_j\, \kappa^{-(N+j)/2}, \tag{106}$$

with

$$c_j = \frac{1}{2}\frac{\Gamma((j+N)/2)}{\Gamma(j+1)} e^{-i\pi(j+N)/4}\ \langle (t-i0)^{-(j+N)/2}, \partial_r^j \alpha(t,0)\rangle; \tag{107}$$

and $\alpha(t,\rho) = |\det\ Q|^{-1/2} \int_{S_{N-1}} a(t, |Q|^{-1/2}(r\zeta))\ d\sigma_{N-1}$ (if $N > 1$).

In particular, if $N = 2$, the expansion starts with a term of order $1/\kappa$, with coefficient

$$\frac{\pi}{2}\alpha(0,0)\ -\ \frac{i}{2}\int_{\mathbb{R}} \frac{\alpha(t,0)}{t}dt, \tag{108}$$

by the well-known identity $(t-i0)^{-1} = \mathrm{PV}(1/t) + i\pi\delta(t)$.

3.5 Proof of Theorem 1.1

We will now finish the proof of Theorem 1.1, which is almost immediate now. Part B and the first part of A follow from Proposition 2.5 and the stationary phase results above. To compute j_0, we have to apply theorem 3.7 with an α given by (80) - (82), where we have to take a equal to the amplitude $\check{\varphi}\beta$ of proposition 2.5 and where Q is also as in 2.5. Note that $|\det Q| = 2^{-N}$ and that ν will now be the number of *negative* eigenvalues of Q, because of the minus sign in the phase of (50). Thus α is of the form

$$\alpha(t,r,s)\ =\ r^p\, s^q\, F(t,r,s) \tag{109}$$

with $p = \nu - 1$ and $q = N - \nu - 1$, both even by assumption, and we are interested in the smallest value of j, j_0, for which

$$(\partial_r^2 - \partial_s^2)^j (r^p s^q F) \neq 0 \text{ at } r = s = 0. \tag{110}$$

By the binomial theorem, (110) equals a sum over l, from $l = 0$ to $l = j$ of

$$\frac{j!}{l!(j-l)!}\ \partial_s^{2(j-l)}\left[s^q\, \partial_r^{2l}\,(r^p\, F)\right]\ . \tag{111}$$

We are interested in the l's for which this is not zero at $(r,s) = (0,0)$. By Leibnitz's rule, (111) will be zero at $(0,0)$ if $p > 2l$ or if $q > 2(j-l)$. Hence (111) will be non-zero at the origin only if

$$p \leq 2l \leq 2j - q\ . \tag{112}$$

It is clear that the smallest value j_0 of j for which there exists an $l \leq j$ such that (112) holds is $j_0 = (p+q)/2 = N/2 - 1$, for which $l = p/2 = (\nu-1)/2$. This completes the proof of Theorem 1.1.