Deformations of Calogero-Moser Systems*

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Abstract. Recent results are surveyed pertaining to the complete integrability of some novel n-particle models in dimension one. These models generalize the Calogero-Moser systems related to classical root systems.

1. Introduction

The Hamiltonian of the celebrated Calogero-Moser (CM) system [1] is given by

$$H_{cm} = 1/2 \sum_{1\leq j\leq n} \theta_j^2 + g^2 \sum_{1\leq j<k\leq n} \wp(x_j - x_k), \tag{1}$$

where $\wp(z)$ denotes the Weierstraß $\wp$-function [2] or a degeneration thereof ($1/z^2$, $1/\mathrm{sh}^2(z)$ or $1/\sin^2(z)$). The integrability of H_{cm} (1) was proved with the aid of a Lax matrix [1].

Some years ago a relativistic generalization of H_{cm} was introduced [3, 4, 5]. The Hamiltonian of the relativistic system (RCM) reads

$$H_{rcm} = \sum_{1\leq j\leq n} \mathrm{ch}(\beta\theta_j) \prod_{k\neq j} \left[1 + \beta^2 g^2 \wp(x_j - x_k)\right]^{1/2}. \tag{2}$$

One can look upon the RCM system as a one-parameter deformation of the CM model, with $\beta \sim 1/c$ (the inverse of the speed of light) acting as deformation parameter. For $\beta \to 0$, which corresponds to the nonrelativistic limit, $\beta^{-2}(H_{rcm}(\beta) - n)$ converges to H_{cm}. The relativistic system is also integrable; explicit formulas have been found for a complete set of integrals in involution:

$$H_{l,rcm} = \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=l}} e^{-\beta \sum_{j\in J} \theta_j} \prod_{\substack{j\in J\\ k\in J^c}} \left[1 + \beta^2 g^2 \wp(x_j - x_k)\right]^{1/2}, \qquad l = 1,\ldots,n. \tag{3}$$

From a Lie-theoretic perspective the above n-particle models are connected with the root system A_{n-1}. Here, we will take a look at similar deformations of the CM systems related to classical root systems other than A_{n-1} (i.e. B_n, C_n, D_n and BC_n). A more detailed discussion of the material covered below (including proofs) can be found in the papers [6, 7].

2. Trigonometric Potentials

In the case of trigonometric potentials our system is characterized by the Hamiltonian

$$H = \sum_{1\leq j\leq n} \left(\mathrm{ch}(\beta\theta_j)\, V_j^{1/2} V_{-j}^{1/2} \;-\; (V_j + V_{-j})/2\right) \tag{4}$$

with

$$V_{\varepsilon j} \;=\; w(\varepsilon x_j) \prod_{k\neq j} v(\varepsilon x_j + x_k) v(\varepsilon x_j - x_k), \qquad \varepsilon = \pm 1, \tag{5}$$

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$$v(z) = \frac{\sin\alpha(\mu+z)}{\sin(\alpha z)}, \tag{6}$$

$$w(z) = \frac{\sin\alpha(\mu_0+z)}{\sin(\alpha z)}\frac{\cos\alpha(\mu_1+z)}{\cos(\alpha z)}\frac{\sin\alpha(\mu_0'+z)}{\sin(\alpha z)}\frac{\cos\alpha(\mu_1'+z)}{\cos(\alpha z)}. \tag{7}$$

One can look upon the functions v and w as potentials: v governs the interaction between the particles and w models an external field. The parameters μ, μ_r and μ_r' $(r=0,1)$ act as coupling constants; after setting them equal to zero the particles become free $(v, w = 1)$.

Just as for the RCM system, explicit formulas have been found that constitute a complete set of integrals in involution for the Hamiltonian H (4)-(7)

$$H_l = \sum_{\substack{J\subset\{1,\ldots,n\},\,|J|\leq l\\ \varepsilon_j=\pm1,\,j\in J}} \mathrm{ch}(\beta\theta_{\varepsilon J})\, V_{\varepsilon J;\,J^c}^{1/2}\, V_{-\varepsilon J;\,J^c}^{1/2}\, U_{J^c,\,l-|J|}, \qquad l=1,\ldots,n, \tag{8}$$

with

$$\theta_{\varepsilon J} = \sum_{j\in J}\varepsilon_j\,\theta_j, \tag{9}$$

$$V_{\varepsilon J;\,K} = \prod_{j\in J} w(\varepsilon_j x_j) \prod_{\substack{j,j'\in J\\ j<j'}} v^2(\varepsilon_j x_j+\varepsilon_{j'}x_{j'}) \prod_{\substack{j\in J\\ k\in K}} v(\varepsilon_j x_j + x_k)v(\varepsilon_j x_j - x_k), \tag{10}$$

$$U_{I,p} = \sum_{\substack{1\leq q\leq p\\ \varepsilon_i=\pm1,\,i\in I}} (-1)^q \sum_{\substack{\emptyset\subsetneq I_1\subsetneq\cdots\subsetneq I_q\subset I\\ |I_q|=p}} \prod_{1\leq q'\leq q} V_{\varepsilon(I_{q'}\setminus I_{q'-1});\,I\setminus I_{q'}} \tag{11}$$

$(U_{I,0}=1,\ I_0=\emptyset)$. Notice that H_1 coincides with the Hamiltonian H (4) up to a factor two.

Theorem 1 (Liouville integrability): *The functions $H_1,\ldots,H_n$ are in involution (with respect to the standard Poisson bracket induced by the symplectic form $\omega=\sum_j dx_j\wedge d\theta_j$).*

After reparametrization according to

$$\mu = i\beta g, \qquad \mu_r = i\beta g_r, \qquad \mu_r' = i\beta g_r', \tag{12}$$

the asymptotics for $\beta\to 0$ leads to the CM system associated with the root system BC_n:

$$H_1(\beta) = H_{1,0}\,\beta^2 \ +\ o(\beta^2), \tag{13}$$

with

$$H_{1,0} = \sum_{1\leq j\leq n}\theta_j^2 + g^2\alpha^2 \sum_{1\leq j\neq k\leq n}\left(\frac{1}{\sin^2\alpha(x_j+x_k)}+\frac{1}{\sin^2\alpha(x_j-x_k)}\right)$$
$$+\alpha^2\sum_{1\leq j\leq n}\left(\frac{(g_0+g_0')^2}{\sin^2(\alpha x_j)}+\frac{(g_1+g_1')^2}{\cos^2(\alpha x_j)}\right) \ +\ const. \tag{14}$$

More generally, one has

$$H_l(\beta) = H_{l,0}\,\beta^{2l} \ +\ o(\beta^{2l}) \tag{15}$$

with

$$H_{l,0} = \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=l}} \prod_{j\in J}\theta_j^2 \ + l.o. \tag{16}$$

($l.o.$ stands for terms of lower order in the momenta θ_j). An immediate consequence of Theorem 1 and the above expansion formulas is the integrability of the CM Hamiltonian $H_{1,0}$ (14).

Theorem 2 (transition to the BC_n-type CM system): *The limits*

$$H_{l,0} = \lim_{\beta\to 0} \beta^{-2l} H_l(\beta), \qquad l = 1, \ldots, n, \tag{17}$$

exist and the resulting functions $H_{1,0}, \ldots, H_{n,0}$ are in involution.

3. Elliptic Potentials

As elliptic counterpart of H (4)-(7), I propose the following Hamiltonian:

$$H = \sum_{1\leq j\leq n} \mathrm{ch}(\beta\theta_j)\, V_j^{1/2} V_{-j}^{1/2} \quad + \; U \tag{18}$$

where $V_{\varepsilon j}$ is again of the form (5), but now with v and w given by

$$v(z) = \frac{\sigma(\mu+z)}{\sigma(z)}, \qquad w(z) = \prod_{0\leq r\leq 3} \frac{\sigma_r(\mu_r+z)\;\sigma_r(\mu_r'+z)}{\sigma_r(z)\;\sigma_r(z)}. \tag{19}$$

The function U is defined by

$$U \;=\; \sum_{0\leq r\leq 3} c_r \prod_{1\leq j\leq n} v(\omega_r + x_j) v(-\omega_r - x_j), \tag{20}$$

$$c_r \;=\; \sigma(\mu)^{-2} \prod_{0\leq s\leq 3} \sigma_s(\mu_{\pi_r(s)})\sigma_s(\mu'_{\pi_r(s)}), \tag{21}$$

where the following permutations have been introduced: $\pi_0 = id$, $\pi_1 = (01)(23)$, $\pi_2 = (02)(13)$ and $\pi_3 = (03)(12)$. In the above expressions $\sigma(z)$ denotes the Weierstraß σ-function with quasi-periods $2\omega_r$, $r = 1, 2, 3$, and the σ_r are the associated functions [2]

$$\sigma_r(z) = \exp(-\eta_r z)\sigma(\omega_r + z)/\sigma(\omega_r), \qquad r = 1, 2, 3. \tag{22}$$

(By convention $\omega_0 \equiv 0$ and $\sigma_0(z) \equiv \sigma(z)$).

Although the integrability of H (18)-(21) has not been demonstrated yet, some partial results have been obtained [7]: *i.* I found an additional Hamiltonian, which commutes with H (18)-(21) if the coupling constants of the external field satisfy the condition $\sum_{0\leq r\leq 3}(\mu_r + \mu_r') = 0$; *ii.* for special values of the coupling constants H (18)-(21) can be seen as a reduction of the RCM Hamiltonian (2); the integrability of the model then follows from [4]; *iii.* if μ equals a half-period ω_r, then straightforward generalization of the expressions in Section 2 results in an ansatz for the higher integrals; their commutativity has been verified for $n \leq 4$.

After setting parameters as in Eq. (12), one has

$$H(\beta) = const \; + \; H_0\, \beta^2/2 \quad + o(\beta^2) \tag{23}$$

with

$$H_0 = \sum_{1\leq j\leq n} \theta_j^2 + \; g^2 \sum_{1\leq j\neq k\leq n} (\wp(x_j+x_k) + \wp(x_j - x_k)) \; + \sum_{\substack{1\leq j\leq n\\ 0\leq r\leq 3}} (g_r + g_r')^2\, \wp(\omega_r + x_j). \tag{24}$$

A Lax pair for the flow generated by H_0 (24) has been presented by Inozemtsev [8].

Remarks: i. For special values of the parameters g_r, g_r', the Hamiltonians $H_{1,0}$ (14) and H_0 (24) reduce to CM Hamiltonians that are associated with the root systems B_n, C_n and D_n.

ii. Quantization of the Hamiltonians for $\beta \neq 0$ gives rise to difference operators instead of the usual partial differential operators. The reason is the occurrence of exponentials of the form $\exp(\pm \beta i \partial_{x_j})$ in the quantized Hamiltonian. In the case of trigonometric potentials the eigenfunctions of the quantum system turn out to be the product of a factorized ground-state wave function and recently discovered multivariable generalizations of the Askey-Wilson polynomials [6, 7].

iii. Further limits of the Hamiltonian with elliptic potentials in Eqs. (18)-(21) lead to novel n-particle models with nearest neighbor interaction [9]. These models generalize the nonperiodic relativistic Toda chain [10], and form a deformation of known Toda chains with very general boundary conditions [11].

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