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# Dynamical Entropy for the Quantized CAT Map

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**Abstract**

We estimate the dynamical entropy for the quantized CAT map. It turns out that the result depends on the rotation parameter and positivity of the dynamical entropy demands asymptotic abelianness. Further the estimates offer a counterexample to the additivity of the dynamical entropy for tensor products.

# 1 Introduction

When in [CS] the concept of dynamical entropy was generalized to nonabelian von Neumann algebras of type II the purpose was to find a quantitative characterization to distinguish some automorphisms. Since then the definition was generalized to work also for arbitrary nonabelian von Neumann algebras [C,CNT] but it remained unclear what kind of automorphisms are distinguished by the fact that they have positive dynamical entropy.

The first result in this direction can be found in [S,T]: due to the fact that the definition was translated into a more transparent form, i.e. the quantal dynamical case has to be compared with classical dynamical models, they could prove that the spectrum of the unitary operator implementing the automorphism must contain an absolutely continuous part. This is in agreement and also a consequence of the classical result [CFS,W]. But also in the classical theory spectral analysis of the unitary operator does not suffice to characterize apart from mixing the ergodic behaviour of an automorphism. The situation is even worse in the quantal situation because we have to distinguish between weak mixing and hyperclustering [NTW]. In [S] the entropy of the shift of the $\mathrm{II}_1$ factor of the free group was shown to be zero. Here the weak mixing properties are optimal, but hyperclustering is violated, which raises the suspicion that hyperclustering is an essential ingredient. Support for this belief can be found in [N1] where it was shown that under the assumption that there exists an optimal classical model this optimal model is already an abelian subalgebra of the noncommutative dynamical system.

In this paper we want to continue in this context the research on the noncommutative version of the Arnold CAT map. In [BNS] we gave the definition as automorphism on the rotational algebra [CR] and showed that the hyperclustering behaviour depends sensitively on the rotation parameter. In [N2] we proved that for those parameters where the system is hyperclustering the dynamical entropy is strictly positive. Now we will improve the result to:

If the rotation parameter $\Theta$ is rational, then $h(\sigma) = \ln\lambda$. If the rotation parameter $\Theta$ is irrational and the system is hyperclustering (for a subalgebra) then $0 < h(\sigma) \leq \ln\lambda$. Otherwise $h(\sigma) = 0$.

One might consider this result as unsatisfactory because the rotation algebra is taken as the prototype of a noncommutative algebra and all relevant properties should remain unaffected by the noncommutativity. Therefore one might ask if one should not change the definition of the dynamical entropy. On the other hand, one of the important consequences in the classical theory is the fact that observables become completely independent in the course of the evolution. Since complete independence [Su] means tensor product type structure it needs commutativity resp. hyperclustering. If we therefore look for a measure that tells us, if and how quickly observables become independent as it can be done with the help of the dynamical entropy [NT] (compare also [BN]) then our dynamical entropy is the right candidate.

There is another peculiarity of the definition: For a long time it was an open problem [SV] whether the dynamical entropy is additive for tensor products. In our proof we give a counterexample, namely $h_{\mathcal{M}_\Theta}(\sigma) = 0$ for some (almost all) $\Theta$ whereas $h_{\mathcal{M}\otimes\mathcal{M}}(\sigma\otimes\sigma) \geq \ln\lambda$ for all irrational $\Theta$.

## 2 The CAT Map on the Rotational Algebra

We consider the algebra $\mathcal{M}_\Theta$ built by the Weyl operators $W(n_1,n_2)$ with multiplication law

$$W_\Theta(n_1,n_2)W_\Theta(m_1,m_2)=e^{i\Theta\pi(n_1m_2-n_2m_1)}W_\Theta(n_1+m_1,n_2+m_2),\quad n_i,m_i\in\mathbf{Z},\ \Theta\in[0,1).$$

Over $\mathcal{M}_\Theta$ we have the tracial state defined by

$$\omega(\sum c_{\vec{n}}W_\Theta(\vec{n}))=c_0.$$

The Arnold CAT map is an automorphism

$$\sigma W(\vec{n})=W(T\vec{n})$$

with $T=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d\in\mathbf{Z}$ and $ad-bc=1$ so that the map respects the multiplication law. As shown in [BNS,N2] the state $\omega$ corresponds to a GNS representation where the automorphism is implemented by a unitary operator independent from the rotation parameter $\Theta$ (see also the Appendix). Therefore $\sigma$ as automorphism of $\mathcal{M}_\Theta$ inherits the weak mixing properties from $\mathcal{M}_\Theta$, namely

$$\begin{aligned}\lim_{n\to\infty}\omega(A\sigma^nB) &= \lim_{n\to\infty}\langle\Omega|\pi_\Theta(A)U^n\pi_\Theta(B)|\Omega\rangle\\ &= \lim\langle\Omega|A_0U^nB_0|\Omega\rangle=\omega(A)\omega(B)\end{aligned}$$

with $A=\sum f(\vec{n})W_\Theta(\vec{n})$ and $A_0=\sum f(\vec{n})W_0(\vec{n})$.

In [BNS] we discussed: If $\Theta=p^2/q^2$, then $\mathcal{M}_\Theta$ is isomorphic to $M^q\otimes\mathcal{M}_0$, $M^q$ a $q\times q$ matrix. $\sigma(1\otimes\mathcal{M}_0)\subset1\otimes\mathcal{M}_0$. $\mathcal{M}_0$ corresponds to the abelian algebra of functions on the two torus. On $M^q$ $\sigma$ produces a periodic phase factor.

If $\Theta$ is irrational, we have to compare it with the eigenvalues of the matrix $T$, $\lambda$ and $1/\lambda$. If $\Theta=s\lambda^2$, $s\in\mathbf{Z}$ [N2] the automorphism is strongly asymptotically abelian and the dynamical entropy positive. In fact, we have an entropic $K$-system as defined in [NT]. For $\Theta=\frac{p}{q}\lambda^2$, we have the subalgebra $\mathcal{M}_{\Theta q^2}=\{W(nq,mq)\}''$ on which $\sigma$ acts strongly asymptotically abelian.

For all other $\Theta$

$$\overline{\lim_{k\to\infty}}\|[W(\vec{n}),W(T^km)]\|>0$$

and except for a countable set of $\Theta$ [BNS]

$$\overline{\lim_{k\to\infty}}\|[W(\vec{n}),W(T^k\vec{m})]\|=2.$$

## 3 The Abelian Model and the Dynamical Entropy

We base our considerations on the definition in [ST]:

**Definition:** Let $(\mathcal{M},\omega,\sigma)$ by a dynamical system, i.e. $\mathcal{M}$ a von Neumann algebra, $\sigma$ an automorphism and $\omega=\omega\circ\sigma$ an invariant state. A stationary coupling with a classical dynamical system $(\mathcal{B},\mu,\alpha)$ is a state $\lambda$ over the $C^*$-algebra $\mathcal{M}\otimes\mathcal{B}$ such that $\lambda\circ\sigma\otimes\alpha=\lambda$ and $\lambda|_{\mathcal{M}}=\omega$ and $\lambda|_{\mathcal{B}}=\mu$. We call $(\mathcal{B},\lambda|_{\mathcal{B}},\alpha)$ an abelian model.

**Definition:**

$$H_\lambda(\mathcal{A}|\mathcal{C}) = \sup_{\lambda=\sum\lambda_i} \sum S(\lambda|\lambda_i)_\mathcal{A} - S(\lambda|\lambda_i)_\mathcal{C}.$$

**Definition:** Let $(\mathcal{M}\otimes\mathcal{B},\lambda,\sigma\otimes\alpha)$ be a stationary coupling of an abelian model. Let $P$ be a finite partition in $\mathcal{B}$ and $P_-$ its past. Define

$$h(P,\lambda) = H_\mu(P|P_-) - H_\lambda(P|\mathcal{M}\otimes P_-).$$

Then the dynamical entropy $h_\omega(\sigma)$ of $(\mathcal{M},\omega,\sigma)$ is given by

$$h_\omega(\sigma) = \sup h(P,\lambda)$$

where the supremum is taken over all stationary couplings and all partitions $P$. If we want to indicate the algebra, we will write $h_\omega(\sigma,\mathcal{N})$.

As consequence of the following lemma the dynamical entropy for $\mathcal{M}_\Theta$ and $\mathcal{M}_{\bar\Theta}$ is the same, if $\Theta/\bar\Theta$ is rational.

**Lemma:** Let $\mathcal{M}\supset\mathcal{N}$ and $\sigma\mathcal{N}=\mathcal{N}$. Then

$$h_\omega(\sigma,\mathcal{N}) \le h_\omega(\sigma,\mathcal{M}) \le h_\omega(\sigma,\mathcal{N}) + H(\mathcal{M}|\mathcal{N}) \le h_\omega(\sigma,\mathcal{N}) + [\mathcal{M}:\mathcal{N}].$$

Here $[\mathcal{M}:\mathcal{N}]$ is the index of the imbedding $\mathcal{N}\subset\mathcal{M}$ [J]. The inequalities are a consequence of

$$H(P|\mathcal{N}) \overset{\mathcal{N}\subset\mathcal{M}}{\ge} H(P|\mathcal{M}) \ge H(P|\mathcal{N}) - H(\mathcal{M}|\mathcal{N})$$

and the last inequality can be found in [PP]. This can be strengthened to

**Lemma:** $h_\omega(\sigma,\mathcal{N}) = h_\omega(\sigma,\mathcal{M})$ if $[\mathcal{M}:\mathcal{N}]<\infty$.

**Proof:** We use the above inequality for $\sigma^k$, then use $h_\omega(\sigma^k,\mathcal{N}) = kh_\omega(\sigma,\mathcal{N})$ and then take the limit $k\to\infty$.

Now it only remains to show

**Lemma:** $[\mathcal{M}_\Theta : \mathcal{M}_{k\Theta}] = k$ for $k$ a positive integer ($k\Theta$ understood mod 1).

**Proof:** We imbed $\mathcal{M}_{k\Theta}$ naturally into $\mathcal{M}_\Theta$ via the isomorphism $c$:

$$cW_{k\Theta}(n_1,n_2) = W_\Theta(kn_1,n_2).$$

There is a $\omega$ preserving conditional expectation $\varepsilon : \mathcal{M}_\Theta \to c\mathcal{M}_{k\Theta}$, namely

$$\varepsilon W_\Theta(n_1,n_2) = \delta_{n_1/k,g} W_\Theta(N_1,n_2).$$

If $\Theta$ is irrational where the imbedding $c\mathcal{M}_{k\Theta}\to\mathcal{M}_\Theta$ is irreducible ($c\mathcal{M}'_{k\Theta}\cap\mathcal{M}_\Theta = \lambda 1$) because

$$[W(m_1,m_2),W(kn_1,n_2)] = \sin\Theta\pi(kn_1m_2 - n_2m_1)W(m_1+kn_1,m_2+n_2)$$

and for $\Theta$ irrational $\Theta(kn_1m_2 - n_2m_1) = g \; \forall n_1, n_2$ only if $m_1 = m_2 = 0$, then this is the only conditional expectation. We choose the GNS representation on $\mathcal{H}(T^2)$ with (see Appendix)

$$\begin{aligned} W_\Theta(n_1, 0) &= \exp[2\pi i n_1(x + \Theta p_y)] \\ W_\Theta(0, n_2) &= \exp[2\pi i n_2 y] \end{aligned}$$

$\omega$ is obtained by $\Omega \equiv 1$.

The vector $\psi = \chi(0 \leq x < 1/k)$ where $\chi$ is the characteristic function is separating for $\mathcal{M}_{k\Theta}$ and satisfies $\overline{\mathcal{M}_{k\Theta}\psi} = \overline{\mathcal{M}_\Theta\psi}$. The projection operator on $\overline{\mathcal{M}_{k\Theta}\psi}$ is given by the multiplication operator $\chi(0 \leq 1 \leq k)$ and its expectation value $\langle\Omega|P|\Omega\rangle = 1/k$ gives the index [J]

$$[\mathcal{M}_\Theta : \mathcal{M}_{k\Theta}] = k = \frac{1}{\langle\Omega|P|\Omega\rangle}.$$

Similarly, with the imbedding of $W(kn_1, kn_2)$ we get

$$[\mathcal{M}_\Theta : \mathcal{M}_{k^2\Theta}] = [\mathcal{M}_\Theta : \mathcal{M}_{k\Theta}] \cdot [\mathcal{M}_{k\Theta} : \mathcal{M}_{k^2\Theta}] = k^2.$$

Therefore we can conclude from the classical result

**Theorem:** If $\Theta$ is rational, then

$$h_{\omega,\mathcal{M}_\Theta}(\sigma) = h_{\omega,\mathcal{M}_P}(\sigma) = h_{\omega,\mathcal{M}_0}(\sigma) = \ln \lambda$$

where $\lambda$ is the Lyapunov exponent [Pe], given as the eigenvalue $> 1$ of the matrix $T$.

## 4 Upper Limit for Irrational $\Theta$

**Theorem:**

$$h_{\omega,\mathcal{M}_\Theta}(\sigma) \leq \ln \lambda \qquad \forall \Theta \in [0, 1).$$

**Proof:** We consider the tensor product $\mathcal{M}_\Theta \otimes \mathcal{M}_{-\Theta}$ with the state $\omega \otimes \omega$ and the automorphism $\sigma \otimes \sigma$. In this case we can find an upper limit, i.e. control the increase in time of a properly chosen algebra and then use

**Lemma:**

$$h_{\omega\otimes\varphi,\mathcal{M}\otimes\mathcal{N}}(\sigma \otimes \tau) \geq h_{\omega,\mathcal{M}}(\sigma) + h_{\varphi,\mathcal{N}}(\tau).$$

Evidently models for $\mathcal{M}$ and $\mathcal{N}$ give a model for $\mathcal{M} \otimes \mathcal{N}$ giving the inequality [SV]. It was conjectured that equality should hold as it does for abelian algebras, but we will give a counterexample to the equality in the next chapter.

**Lemma:**

$$h_{\omega,\mathcal{M}_\Theta}(\sigma) = \mathcal{M}_{\omega,\mathcal{M}_{-\Theta}}(\sigma).$$

Notice that $\pi(\mathcal{M}_{-\Theta}) = \pi(\mathcal{M}_\Theta)'$ in the GNS construction corresponding to $\omega$: Choose

$$\begin{aligned} \pi(W_\Theta(n_1, 0)) &= \exp[2\pi i n_1(x + \frac{\Theta}{2} p_y)] \\ \pi(W_\Theta(0, n_2)) &= \exp[2\pi i n_2(y - \frac{\Theta}{2} p_x)] \quad \text{in } L^2(T^2). \end{aligned}$$

Then $\exp[2\pi i n_1(x-\frac{\Theta}{2}p_y)]$, $\exp[2\pi i n_2(y+\frac{\Theta}{2}p_x)]$ commute with $\pi(\mathcal{M}_\Theta)$ and form $\pi(\mathcal{M}_{-\Theta})$. Further

$$J\pi W_\Theta(\vec{n})|\Omega\rangle = \pi W_{-\Theta}(-\vec{n})|\Omega\rangle, \qquad J\pi(\mathcal{M}_\Theta)J = \pi(\mathcal{M}_{-\Theta}) = \pi(\mathcal{M}_\Theta)'.$$

Take $(\mathcal{B},\lambda,\alpha)$ as abelian model for $(\mathcal{M}_\Theta,\sigma,\omega)$. A finite subset of $\mathcal{B}$ is represented by its index set $I$ corresponding to $\mu(P_I)=\omega(x_I)$, $x_I\in\mathcal{M}'$. The same algebra $\mathcal{B}$ serves for $\mathcal{M}_{-\Theta}$ with $\mu(P_I)=\langle\Omega|Jx_IJ|\Omega\rangle$, $Jx_IJ\in\mathcal{M}_\Theta$ and

$$S(\omega|\langle\Omega|Jx_IJ\cdot|\Omega\rangle)_{J\mathcal{A}J\subset\mathcal{M}'_\Theta} = S(\omega|\langle\Omega|x_I\cdot|\Omega\rangle\rangle)_{\mathcal{A}\subset\mathcal{M}_\Theta}$$

and therefore gives the same dynamical entropy.

We turn to estimate the dynamical entropy for $\mathcal{M}=\mathcal{M}_\Theta\otimes\mathcal{M}_{-\Theta}$, $\bar{\omega}=\omega\otimes\omega$. This algebra is again type $\mathrm{II}_1$. It contains a maximally abelian subalgebra built by the operators $W_\Theta(\vec{n})\otimes W_{-\Theta}(\vec{n})$ which in a natural way is isomorphic to the classical algebra over $T^2$ and we may therefore denote it by $\mathcal{M}_0$. That it is maximally abelian can be seen from

$$[W_\Theta(\vec{n})\otimes W_{-\Theta}(\vec{n}),\sum f_{k\ell}W_\Theta(\vec{k})\otimes W_{-\Theta}(\vec{\ell})] = \sum f_{k\ell}\sin\Theta\pi\sigma(\vec{n},\vec{k}-\vec{\ell})W(\vec{n}+\vec{k})\otimes W(\vec{n}+\vec{\ell}) = 0$$

$$\Longleftrightarrow f_{k\ell}=\delta_{\vec{k},\vec{\ell}}f_k.$$

This algebra is regular [D] in the sense that its normalizer $\mathcal{N}(\mathcal{M}_0)=\{u\in\mathcal{M}_\Theta\otimes\mathcal{M}_{-\Theta},\ u$ unitary, $u\mathcal{M}_0u^*=\mathcal{M}_0\}$ creates already $\mathcal{M}_\Theta\otimes\mathcal{M}_{-\Theta}=\{\mathcal{N}(\mathcal{M}_0)\cup\mathcal{M}_0\}''$. In fact,

$$\mathcal{M}_\Theta\otimes\mathcal{M}_{-\Theta}=\{\mathcal{M}_0,W_\Theta(\vec{n})\otimes 1\}''=\{\mathcal{M}_0,W_\Theta(n,0)\otimes W_{-\Theta}(0,k)\}''.$$

As a regular maximal abelian subalgebra it is a Cartan subalgebra of $\mathcal{M}_\Theta$ in the sense that $\mathcal{M}_0=\bigotimes_{i=1}^\infty D_{n_i}$ with $D_{n_i}$ the diagonal subalgebra in $M_{n_i}$, the full matrix algebra of dimension $n_i$, and $\mathcal{M}_\Theta=\bigotimes_{i=1}^\infty M_{n_i}$ [D1,D2]. Further we observe that $\sigma\mathcal{M}_0\subset\mathcal{M}_0$. Therefore

**Lemma:**

$$h_{\mathcal{M}_\Theta\otimes\mathcal{M}_{-\Theta}}(\sigma\otimes\sigma)\geq h_{\mathcal{M}_0}(\sigma)=\ln\lambda.$$

**Proof:** We take $(\mathcal{M}_0,\sigma,\lambda=\bar{\omega})$ as abelian model.

In order to find an upper limit we turn to the initial condition of the dynamical entropy to be found in [C,CNT]:

**Definition:**

$$\begin{aligned} h_\mathcal{M}(\sigma) &= \sup h_\mathcal{M}(\sigma,\mathcal{N}) \\ &= \sup_{\mathcal{N}\subset\mathcal{M}}\lim\frac{1}{n}H(\mathcal{N},\sigma\mathcal{N},\ldots,\sigma^{n-1}\mathcal{N}) \\ &= \lim_{k\to\infty}h_\mathcal{M}(\sigma,\mathcal{N}_k), \qquad \mathcal{N}_k\nearrow\mathcal{M} \end{aligned}$$

where $\mathcal{N}$ runs over finite-dimensional subalgebras of $\mathcal{M}$ and (see for details [C,CNT]

$$H(\mathcal{N},\ldots,\sigma^{n-1}\mathcal{N})=\sup_{\sum x_I=1,0\leq x_I\leq 1\in\mathcal{M}'}\sum\{\eta(\omega(x_I)+S(\omega|\omega_{ik}^{(k)})_{\sigma^k\mathcal{N}}\}.$$

That the two definitions give the same dynamical entropy is shown in [ST]. We concentrate on the relevant properties of $H(\mathcal{N}_1,\ldots,\mathcal{N}_k)$ proven in [CNT]:

$$H(\mathcal{N}_1,\ldots,\mathcal{N}_k) \leq H(\bar{\mathcal{N}}) \quad \text{for } \mathcal{N}_i \subset \bar{\mathcal{N}} \quad \forall i,$$

$$H(\mathcal{N}) \leq H(\bar{\mathcal{N}}) + \varepsilon \quad \text{for } \mathcal{N} \overset{\delta}{\subset} \bar{\mathcal{N}} \quad \text{with } \varepsilon = \varepsilon(\delta, \dim \mathcal{N})$$

and

$$\lim_{\delta\to 0} \varepsilon(\delta, \dim \mathcal{N}) = 0.$$

Here $\mathcal{N} \overset{\delta}{\subset} \bar{\mathcal{N}}$ if $\forall x \in \mathcal{N}$, $\|x\| \neq 1$, $\exists\, y \in \bar{\mathcal{N}}$, $\|y\| = 1$ with $\omega((x-y)^*(x-y)) < \delta$.

If $\omega$ is the tracial state, then $H(\mathcal{N}) = H(\mathcal{N}_0)$ for $\mathcal{N}_0$ maximally abeliaan in $\mathcal{N}$.

Evidently we will use the regularity of $\mathcal{M}_0$ in order to construct an $\mathcal{N}_1$ such that we can control the increae property of $\bar{\mathcal{N}}$.

Without loss of generality we concentrate on $\mathcal{N}_1 \overset{\delta}{\supset} W(1,0) \otimes 1$. Considering $\sum f(\vec{n}) W(\vec{n}) \otimes W(\vec{n})$ the Fourier transforms of $f$ are functions on the torus $T^2$, $\tilde{f}(x,y)$, by which we also denote the above operator. Then

$$W(1,0) \otimes 1 \cdot \tilde{f}(x,y) \cdot W(-1,0) \otimes 1 = \tilde{f}(x, y+\Theta).$$

$\Theta$ can be approximated by rationals $p/q$ such that $(\Theta q - p) < \varepsilon$. If we therefore take step functions

$$f_{q\ell}(x,y) = \chi\left(\frac{\ell}{q} \leq y < \frac{\ell+1}{q}\right), \qquad 0 \leq \ell < q,$$

then

$$\omega(|W(r,0) \otimes 1 \cdot f_{q\ell} \cdot W(-r,0) \otimes 1 - f_{q,\ell+rp}|^2) = \int dxdy |f_{q\ell}(x, y+r\Theta) - f_{q,\ell+rp}(x,y)|^2 < r\varepsilon.$$

We know that there exist unitary operators $V$ in $\mathcal{M}$ [D2,N3,CFW] such that

$$V^r f_{q\ell} V^{r*} = f_{q,\ell+rp}, \qquad V^q = 1.$$

If $V$, $\bar{V}$ satisfy the condition, then $V\bar{V}^* \in \{f_{q\ell}\}'$.

Now we approximate $W(1,0) \otimes 1$ by step functions $U_q = \sum_{s=1}^q z_s P_s$ where the $P_s$ are orthogonal projections such that $U_q^q = 1$ and $\|W(1,0) \otimes 1 - U_q\| \leq 1/q$. For these $U_q$ we get

$$\omega(|U_q^r f_{q\ell} U_q^{r*} - f_{q,\ell+rp}|^2) < 3r\varepsilon.$$

Now we can choose $V$ satisfying the above condition such that

$$\|(U_q - V)\Omega\| \leq 6r\varepsilon$$

or

$$\|(W(1,0) \otimes 1 - V)|\Omega\| \leq 7r\varepsilon.$$

$V$ and $f_{q\ell}$ together build an algebra $\mathcal{N}_q$ that is isomorphic to $M^q$ and contains $\mathcal{N}_q^0 = \{f_{q\ell}\}''$ as maximally abelian subalgebra. Further $W(1,0) \otimes 1 \overset{\delta}{\subset} \mathcal{N}_q$ with $\delta \to 0$ for increasing $q$.

Similarly we can proceed for $1 \otimes W(0,1)$ and get an algebra ${}_q\mathcal{N} = \{\widehat{V}, g_{q\ell}\}$ where

$$g_{q\ell}(x,y) = \chi\left(\frac{\ell}{q} \leq x < \frac{\ell+1}{q}\right), \qquad [g_{q\ell}, f_{qr}] = [\widehat{V}, f_{qr}] = [V, g_{q\ell}] = [V, \widehat{V}] = 0.$$

The last equality can be satisfied for $V \overset{\delta}{\sim} W(1,0) \otimes 1$, $\widehat{V} \overset{\delta}{\sim} 1 \otimes W(0,1)$ because

$$[W(1,0) \otimes 1, 1 \otimes W(0,1)] = 0.$$

Therefore

$$\{{}_q\mathcal{N} \cup \mathcal{N}_q\}'' = {}_q\mathcal{N}_q \approx M^{q^2}.$$

Finally we have to argue that

$$(\sigma \otimes \sigma)^n \; {}_q\mathcal{N}_q \overset{\delta}{\subset} {}_{\bar{q}}\mathcal{N}_{\bar{q}}, \qquad \bar{q} = q(\lambda^n + \lambda^{-n}) \in \mathbf{Z}.$$

First we notice that $\sigma^n \; {}_q\mathcal{N}_q^0 \subset {}_{\bar{q}}\mathcal{N}_{\bar{q}}^0$. Next we see that

$$\sigma^n W(1,0) \otimes 1 \equiv W(a_n, b_n) \otimes 1 = e^{i\alpha_n} W(a_n, 0) \otimes 1 \cdot 1 \otimes W(0, -b_n) \cdot W(0, b_n) \otimes W(0, b_n)$$

where

$$W(a_n, 0) \otimes 1 \overset{\delta}{\subset} \mathcal{N}_{\bar{q}},$$
$$1 \otimes W(0, -b_n) \overset{\delta}{\subset} {}_{\bar{q}}\mathcal{N},$$
$$W(0, b_n) \otimes W(0, b_n) \subset {}_{\bar{q}}\mathcal{N}_{\bar{q}}^0.$$

This gives

$$\lim \frac{1}{n} H({}_q\mathcal{N}_q, \ldots (\sigma \otimes \sigma)^n \; {}_q\mathcal{N}_q) \leq \lim \frac{1}{n} H({}_{\bar{q}}\mathcal{N}_{\bar{q}}) = \lim \frac{1}{n} H({}_{\bar{q}}\mathcal{N}_{\bar{q}}^0) = \lim \frac{1}{n} \ln \bar{q}^2 = 2 \ln \lambda.$$

**Remark:** It should be noted that $\sigma^n \; {}_q\mathcal{N}_q^0 \subset {}_{\bar{q}}\mathcal{N}_{\bar{q}}^0$ is a very poor estimate that in the classical situation gives only an upper limit definitely different from the correct result. i.e. $\ln \lambda$. The refinement of ${}_q\mathcal{N}_q^0$ should be done in agreement with the map $\sigma$, i.e. approximating the direction of the dilation of $T$. But the construction of the ${}_p\mathcal{N}_{p'}$ out of ${}_p\mathcal{N}_{p'}^0$ then involves $W(a_n, 0) \otimes W(0, b_n)$ in order to keep commutativity, whereas $(\sigma \otimes \sigma)^n \cdot W(1,0) = W(a_n, -b_n) \otimes 1$. The additional term $W(0, b_n) \otimes W(0, b_n)$ asks again for a refinement of ${}_p\mathcal{N}_p^0$ that again leads to ${}_{\bar{q}}\mathcal{N}_{\bar{q}}^0$. Using directly $W(a_n, b_n) \otimes 1$ for a refinement of ${}_q\mathcal{N}_q$ would ask for a more careful analysis. In fact, we conjecture that for $\Theta$ irrational $h_{\mathcal{M}_\Theta \otimes \mathcal{M}_{-\Theta}} = h_{\mathcal{M}_0}$ because this is the result of explicit calculations of the generalized 2-shift [NT], which has a rather similar structure.

# 5 Optimal Abelian Models

As discussed in Sect. 3 we have to find an abelian algebra $\mathcal{B}$ and a coupling between $\mathcal{M}$ and $\mathcal{B}$ represented by a state $\lambda$ over the algebra $\mathcal{M} \otimes \mathcal{B}$. We have to expect that the corresponding representation has a nontrivial kernel. But since $\lambda|_{\mathcal{M}} = \omega$ the representation cannot contain inequivalent representations of $\mathcal{M}$ so that we can write

$$\pi_\lambda(m \otimes b) = \int_\oplus m \cdot b(x) d\mu(x)$$

with $\mu$ the measure over some measure space. $\sigma \otimes \alpha$ acts as automorphism $\Theta$ and the only possibility satisfying $\Theta(m \otimes 1) = \sigma m \otimes 1$ is

$$\Theta \int_{\oplus} m(x) d\mu(x) = \int \sigma m(Tx) d\mu(x)$$

The state is given by

$$\lambda(\int_{\oplus} m(x) d\mu(x) = \int \omega_x(m(x)) d\mu(x).$$

For a typical element $m \cdot b(x)$ we can write

$$\lambda(m \otimes b) = \int \omega_x(m) b(x) d\mu(x) = \int \omega(\widehat{a}(x) m) b(x) d\mu(x) = \int \omega(\widehat{b}(x) m) d\mu(x)$$

with $\widehat{a}(x) \in \mathcal{M}'$ $\mu$ a.e. and $\widehat{b}(x) \in \mathcal{M}' \otimes \mathcal{B}$, depending linearly on $b$.

$$[\widehat{a}(x) b_1(x), \widehat{a}(x) b_2(x)] = 0, \qquad b_1(x),\ b_2(x) \text{ functions}$$

so that also

$$[\widehat{b}_1(x), \widehat{b}_2(x)] = 0 \qquad \mu \text{ a.e.}$$

This means especially for shifted elements – that stay in $\mathcal{B}$ –

$$[\Theta^n \widehat{b}(\cdot), \widehat{b}(\cdot)] = 0$$

or

$$[\sigma^n \widehat{b}(T^n x), \widehat{b}(x)] = 0, \qquad \mu \text{ a.e.}$$

We may assume that $T$ acts ergodically on the measure space. Otherwise, the abelian model can be decomposed into submodels, and the resulting dynamical entropy becomes the mean of the ergodic contributions as can be seen as follows:

If $\chi_\alpha(x)$ the characteristic functions on the invariant subspaces do not belong to the abelian algebra $\mathcal{B}$ we can add them and improve in this way the model according to

$$\begin{aligned} H(P|P_-) - H(P|\mathcal{M} \otimes P_-) &= \lim \frac{1}{n} \left\{ H\left( \bigvee_{k=1}^{n} \alpha^k P \right) - H\left( \bigvee_{k=1}^{n} \alpha^k P | \mathcal{M} \right) \right\} = \\ &= \lim \frac{1}{n} S(\omega \otimes \mu | \lambda)_{\bigvee_{k=1}^{n} \alpha^k P \otimes \mathcal{M}} \leq \lim \frac{1}{n} S(\omega \otimes \mu | \lambda)_{\bigvee_{k=1}^{n} \alpha^k Q \otimes \mathcal{M}} \end{aligned}$$

where we used the monotonicity of the relative entropy and $Q$ is the refinement of $P$ by adding $\chi_\alpha(x)$. If we write $Q = \{P_1, \ldots, P_r\}$ then

$$H(Q|Q_-) = \sum \mu(\chi_\alpha) H(P_\alpha | P_\alpha^-)$$

and

$$H(Q|\mathcal{M} \otimes Q_-) = \sum \mu(\chi_\alpha) H(P_\alpha | \mathcal{M} \otimes P_\alpha^-).$$

If the abelian model gives a positive dynamical entropy then this is already true for a subalgebra $\mathcal{B}_0 \subset \mathcal{B}$ built by some projection $b(x)$. This is first true for $H(P|P_-)$:

$$\begin{aligned} H(P|P_-) > 0 \quad &\iff \quad \exists \text{ projection } p_0 \in P \text{ such that} \\ p_0 \notin P_- \quad &\iff \quad H(\{p_0\}|P_-) > 0 \\ &\Rightarrow \quad H(\{p_0\}|\{p_0\}_-) > H(\{p_0\}|P_-) > 0. \end{aligned}$$

Next we know from [ST] that we can also demand

$$H(P|P_-) - H(P|\mathcal{M}) > 0 \Rightarrow H(P|\mathcal{M}) < H(P) \text{ together with } H(P|P_-) > 0.$$

For the last term we can consider only projections with $H(\{p_0\}|\{p\}_-) > 0$. Decreasing $P$ so that we restrict ourselves only on those $p_0$ only improves the contribution of $H(P|\mathcal{M})$. Next

$$H_\lambda(P|\mathcal{M}) < H(P) \iff \exists\, p_0 \in P,\ m \in \mathcal{M}$$

such that $\lambda(p_0 m) \neq \lambda(p_0)\lambda(m) = \mu(p_0)\omega(m)$ or $\lambda \neq \mu \otimes \omega$.

We look for the requirements on $b(x)$ to give positive dynamical entropy.

First

$$H_\mu(\{b(\cdot)\}|\{b(\cdot)\}^-) > 0,$$

$$\mu\{\Theta^n \widehat{b}(\cdot)\} = \int d\mu(x)\omega(\sigma^n \widehat{b}(T^n x)).$$

If we use the ergodic theorem that

$$\int d\mu(x) f(x) = \eta_k f(T^k x) \qquad \mu \text{ a.e.}$$

we get

$$\mu\{\Theta^n \widehat{b}(\cdot)\} = \omega(\sigma^n \eta_k \widehat{b}_k)$$

with $\widehat{b}_k(x) = \widehat{b}(T^k x)$ and we may suppress the dependence on $x$.

Thus we have to find a sequence of positive operators $\widehat{b}_k \in \mathcal{M}'$ with nontrivial invariant mean and

$$[\sigma^n \widehat{b}_n, \sigma^\ell \widehat{b}_\ell] = 0 \qquad \forall\, n, \ell$$

such that $\eta \widehat{b}_n \neq \lambda \mathbf{1}$. Otherwise $h(\{\widehat{b}_n\}, \{\widehat{b}_n\}^-) = 0$. This forbids the evident choice $\widehat{b}_n = \sigma^{-n}\widehat{b}_0$ for a mixing automorphism. If, on the other hand, there exists an abelian subalgebra $\mathcal{M}_0$ with $\sigma\mathcal{M}_0 \subset \mathcal{M}_0$ then we can take $\widehat{b}_n = \widehat{b} \subset \mathcal{M}_0$. If only $\sigma^k \mathcal{M}_0 \subset \mathcal{M}_0$ then we can take

$$\widehat{b}_n = \sigma^{-k(n/k) - k[n/k]} \widehat{b}_0$$

with $[\alpha]$ the largest integer $\leq \alpha$ and $(\alpha) = \alpha - [\alpha]$.

We turn to the irrational algebra and check if we can succeed to find some $\widehat{b}_0$ for the CAT map.

We start with some $\widehat{b}_0 = \sum c_n W(n)$. We can approximate $\widehat{b}_0$ by $\sum_{|n|<N_0} c_n W(n)$. Next we notice that the coupling of $\mathcal{B}$ to $\mathcal{M}$ has to be nontrivial. This demands

$$S(\omega|\omega(\eta \widehat{b}_n \cdot))_{\mathcal{M}} > 0,$$

$$S(\omega|\omega(\eta \widehat{b}_n \cdot)_{\mathcal{M}} \leq \lim S(\omega|\omega(\eta \widehat{b}_n \cdot))_{\mathcal{M}_k} \tag{$*$}$$

with $\mathcal{M}_k$ an increasing sequence of finite dimensional subalgebras. Therefore there must exist some $W(n_0)$ such that

$$\omega(\eta \widehat{b}_n \cdot W(n_0)) \neq \omega(\eta \widehat{b}_n))\omega(W(n_0)) = 0.$$

We consider the eigenvectors $u_\pm$ to the matrix $T$, remember that they do not belong to $\mathbf{Z}^2$. Then $n_0 = \alpha u_+ + \beta u_-$.

$$[\sigma^n \widehat{b}_n, \sigma^\ell \widehat{b}_\ell] = 0$$

means with $\widehat{b}_n = \sum c_{nk} W(k)$

$$\sum c_{nk} c_{\ell r} \sin \Theta \sigma(k, T^{\ell - n} r) W(k + T^{\ell - n} r) = 0.$$

Since $\|\widehat{b}_n\| < \infty$ we can assume that $c_{nk}$ only runs over a finite range of $k$. Every $\vec{k}$ corresponds to $(\alpha_k, \beta_k)$ such that $T^\ell \vec{k}$ corresponds to $(\lambda^\ell \alpha_k, \lambda^{-\ell} \beta_k)$. Replacing $\sum c_{nk} w(k)$ by finite sums $(|c_{nk}| > \varepsilon)$ corresponds to demand

$$\| \sum_{kr} c_{nk} c_{\ell r} \sin \Theta \sigma(k, T^{\ell - n} r) W(k + T^{\ell - n} r) \| < \bar{\varepsilon} \qquad \forall n, \ell.$$

Now either $\sigma(k, T^{\ell - n} r)$ is sufficiently close to an integer or $W(k + T^{\ell - n} r) = W(\bar{k} + T^{\ell - n} \bar{r})$ for several pairs of $(k, r)$ and their contributions add up to a negligible value.

Let us concentrate on the first possibility: In order to contribute in (*) we can demand $|\beta_k| < \widehat{\beta}$. Since $u_\pm$ do not belong to $\mathbf{Z}^2$, we can demand in addition $|\alpha_k| > \check{\alpha}$ to stay within $|\vec{n}| < N_0$, $\vec{n} \neq (0,0)$. This gives

$$\sigma(k, T^{\ell - n} r) = \lambda^{\ell - n} \beta_k \alpha_r + \lambda^{-\ell + n} \alpha_k \beta_r.$$

The second term is negligible for $\ell \gg n$, the first term is dominating and exponentially close to an integer [BNS,N]. Multiplied with $\Theta \neq s\lambda^2$ it will differ from an integer significantly (up to $1/2$) for values $\ell - n$ with positive density among the numbers, and those values $\ell - n$, for which $s\lambda^2$ is close to an integer up to $\varepsilon$, do not form a subgroup (this subgroup would correspond to $T^s$ with eigenvalues $\lambda^s$ and $\Theta = s\lambda^2 (\mathrm{mod}\ 1) \equiv k\lambda^s (\mathrm{mod}\ 1)$). Therefore we can either put $c_{nk} = 0$ for almost all $n$, this will violate (*), or we take $c_{nk} \neq 0$ for $n$ such that $\eta_n |c_{nk}| > 0$ and will violate

$$c_{nk} c_{\ell r} \sin \Theta \sigma(k, T^{\ell - n} r) W(k + T^{\ell - n} r) = 0$$

for infinitely many $\ell$ fixed $n$. Therefore we must find (consider $n$ fixed) other values $\bar{k}$, $\bar{r}$ such that

$$W(k + T^{\ell - n} r) = W(\bar{k} + T^{\ell - n} \bar{r})$$

or

$$\bar{k}_n - k = T^{\ell - n} (\bar{r}_\ell - r).$$

This equality must hold for infinitely many $(n, \ell)$. Expressed again by the values $\alpha$, $\beta$, we have

$$\alpha_n + \beta_n + \alpha_0 + \beta_0 = \lambda^{\ell - n} \alpha_\ell + \lambda^{-\ell + n} \beta_\ell + \lambda^{\ell - n} \bar{\alpha} + \lambda^{-\ell - n} \bar{\beta}.$$

Ignoring $\alpha_0, \bar{\alpha}, \beta_0, \bar{\beta}$ this implies $\alpha_\ell \sim \lambda^{-\ell}$, combined with $\bar{\alpha}$ we cannot arrange $\alpha_\ell$ in such a way that the equality holds for infinitely many $(n, \ell)$. This shows that we are unable to find an abelian model with positive dynamical entropy and nontrivial coupling. Thus

**Theorem:** $h_{\mathcal{M}_\Theta}(\sigma) = 0$ for $\Theta \neq s\lambda^2$ and irrational.

We have shown in Section 4 that

$$h_{\mathcal{M}_\Theta \otimes \mathcal{M}_{-\Theta}}(\sigma \otimes \sigma) \geq \ln \lambda \qquad \forall \Theta \text{ irrational}.$$

Now $h_{\mathcal{M}_\Theta}(\sigma) = 0$. Therefore

**Corollary:** The dynamical entropy is not additive for tensor products.

**Remark:** In [NT] we studied the mixing properties of the generalized 2-shift built by unitaries $e_i$, $e_i^2 = 1$ and $e_i e_j = e^{ig(i-j)} e_j e_i$, $\sigma e_i = e_{i+1}$. Mixing properties depend again on $g(|i-j|)$. If $g \equiv 0$ then we deal with an abelian algebra and $h(\sigma) = \ln 2$. If $g \neq 0$ but sufficiently regular $h(\sigma) = \frac{1}{2} \ln 2$. Again we can mimic the proof in Section 4 to construct the maximal abelian subalgebra $e_I \otimes f_I$ of $\mathcal{M}_g \otimes \mathcal{M}_g$ to show in general $h(\sigma \otimes \sigma) \geq \ln 2$. Mimicking the proof in this chapter for sufficiently irregular $g$ we get $h(\sigma) = 0$ and thus another counterexample for additivity.

## Appendix: Representation of the Weyl Algebra

We represent $\omega(W_t(\vec{n}))$ on $L^2(T^2)$. The GNS vector $\Omega$ corresponds to the constant function on $T$? Then we can choose

$$W_\Theta(n_1, 0) = e^{2\pi i n_1 (x + \alpha p_y)}, \qquad W_\Theta(0, n_2) = e^{2\pi i n_2 (y + \beta p_x)},$$

$$p_x = \frac{1}{i}\frac{\partial}{\partial x}, \qquad p_y = \frac{1}{i}\frac{\partial}{\partial y} \quad \text{with periodic boundary conditions.}$$

In order to satisfy the commutation relations

$$W_\Theta(n_1,0) W_\Theta(0,n_2) = e^{2\pi i n_1 n_2 (\alpha - \beta)} W_\Theta(0,n_2) W_\Theta(n_1,0) = e^{2\pi i \Theta n_1 n_2} W_\Theta(0,n_2) W_\Theta(n_1,0)$$

we have to take $\alpha = \Theta + \beta$. If we call the corresponding represntation $\pi_\alpha$ ($\Theta$ fixed) then

$$\pi_\alpha(\cdot) = e^{2\pi i \alpha p_x p_y} \pi_0(\cdot) e^{-2\pi i \alpha p_x p_y}.$$

The choice $\alpha = \Theta/2$ has the advantage that

$$W_\Theta(n_1, n_2)|\Omega\rangle = e^{2\pi i (n_1 x + n_2 y)}$$

so that

$$W_\Theta(\sigma n)|\Omega\rangle = W_0(\sigma n)|\Omega\rangle = U_\Theta(\sigma) W_\Theta(n)|\Omega\rangle = U_0(\sigma) W_0(n)|\Omega\rangle$$

or

$$U_\Theta(\sigma) = U_0(\sigma).$$

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