A Geometric Framework for time-dependent Billiards

Jair Koiller*
Laboratório Nacional de Computação Científica
and Instituto de Matemática, UFRJ, Brasil.

Roberto Markarian†
Inst. de Matemática y Estadística "Prof.Ing. Rafael Laguardia"
Fac. de Ingeniería, Uruguay.

Sylvie Oliffson Kamphorst
Sônia Pinto de Carvalho
Departamento de Matemática, ICEx, UFMG, Brasil.

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Abstract

Billiards with moving boundaries can be viewed as the limiting case of rigid ones, constructed on an augmented configuration space. The metric is defined by a new kinetic energy, singular in the limit. We describe the symplectic structure of the time-dependent billiard map and specify suitable coordinates for the convex case of dimension 2.

1 Introduction

Billiard flows are defined by the free motion of a point mass in a connected domain with piecewise smooth boundary on a Riemannian manifold. The motion is along the geodesics inside the domain and the particle is reflected elastically when it reaches the smooth parts of the boundary. *Billiards* aroused mathematical interest since G.D.Birkhoff used them to illustrate many geometrical ideas for Dynamical Systems [5]. The studies of billiards gained further impulse when Krylov and Sinai related them with the ergodic hypothesis of Boltzman-Gibbs [10]. For surveys and a complete introduction to the subject, with emphasis on ergodic properties, see [6],[14].

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Billiards also provide a natural arena for comparisons between classical and quantum mechanics. Number theorical results and conjectures (including about the Riemann zeta-function) came out of semiclassical studies [9],[4].

Billiards whose boundaries may *deform* are also important in Physics. For instance, "Bohr's liquid drop model" introduced in the thirties [16], assumes that the nucleous is populated by particles, moving inside it like billiard balls. Collisions among these balls are neglected, and exchange of energy occurs only between the particles and the deforming nucleous boundary. This model is still in vogue and many questions remain open [1]. The 1-dimensional case already is quite complicated, even if the motion of one wall is prescribed and the other is fixed; this "ping-pong" ball problem is known as "Fermi accelerator", a basic model in Plasma physics [8],[13]. By registering the velocity change and time at collisions, one gets an area preserving map in (energy $\times$ time) space. Numerical studies, with different periodic wall functions, indicated the existence of invariant curves crossing the time direction for a whole period, thus limiting the maximum energy gain. Rigorous mathematical proofs of the existence of such invariant curves were given, in some cases, via KAM theory (see, [7], 3.IV, and [11] §4). Using similar methods, M. Levi has given examples of "pulsating soft circular billiards" whose energies stay bounded for all time ([12] §1). We believe it is natural to consider "higher dimensional Fermi accelerators", that is, billiards whose boundaries move, in particular, periodically.

The techniques we use in this paper are elementary. In section 2 we show that a billiard with moving boundary can be viewed as a rigid billiard in an augmented phase space. The metric, however, is *singular*. Section 3 presents suitable coordinates, although not canonical, for the Poincaré section of the 2-dimensional moving billiard mapping. These coordinates consist on time and energy, as in the 1-dimensional Fermi accelerator, and a pair (φ, p_φ) where φ parametrizes the frozen boundaries. In section 4 we obtain the derivative of the billiard mapping. Section 5 contains a simple application, which hopefully could surprise some readers.

2 Time-dependent billiards as rigid billiards in augmented space

The problem of the time-dependent billiard consists on the study of the billiard map with a moving boundary. In this section we show that deforming billiards can be viewed as the limiting case of *rigid* billiards in suitably space, with a special *singular* metric. For simplicity we shall restrict ourselves to two dimensional time-dependent billiard in euclidean space, but everything carries over for higher dimensions and general riemannian manifolds.

The time dependence of the potential energy V will be indicated by a third "spatial" coordinate z. The special metric is defined by a generalized kinetic energy which depends on a small parameter γ. (See [2] §19.D)

More precisely, consider a particle in $\mathbf{R}^3 = (x, y, z)$ moving under the potential energy $V(x,y,z)$ and with kinetic energy defined by

$$T_\gamma(x',y',z') = \frac{1}{2}(x'^2 + y'^2 + \frac{1}{\gamma^2}z'^2) \tag{1}$$

where γ is a small parameter, $'$ stands for d/dt and t is a new independent variable. The equations of motion of this autonomous lagrangian system are, evidently,

$$\begin{aligned} (x'',y'') &= -\mathrm{grad}_{(x,y)} V(x,y,z) \\ z'' &= -\gamma^2 \partial V/\partial z \ . \end{aligned} \tag{2}$$

Now consider in $\mathbf{R}^3$ a tube $f(x,y,z) \leq 1$, where f is a sufficiently smooth function, with $\partial f/\partial z$ bounded. In our setting, we define

$$V(x,y,z) = \lim_{N\to\infty} f^N(x,y,z) = \begin{cases} 0 & \text{inside the tube} \\ \infty & \text{outside the tube} \end{cases} \tag{3}$$

(for large but finite N we have a "soft billiard").

Therefore we can make computations like in a usual rigid billiard, using the special metric. (See [6], ch.6).

Let $\vec{v}_0$ and $\vec{v}_1$ be the velocity vectors immediately before and after the collision of the particle with a point $\vec{q}_1$ of the boundary $f(x,y,z) = 1$. Then

$$\begin{aligned} \vec{v}_1 &= \vec{v}_0 - 2 < \vec{v}_0, \vec{n} >_\gamma \vec{n} \\ \vec{n} &= \frac{\nabla_\gamma f(\vec{q}_1)}{\| \nabla_\gamma f(\vec{q}_1) \|_\gamma} \end{aligned} \tag{4}$$

where $<,>_\gamma$ is the riemannian metric given by T_γ and ∇_γ its associated gradient operator.

Since

$$\begin{aligned} \nabla_\gamma f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \gamma^2 \frac{\partial f}{\partial z}\right) \\ \| \nabla_\gamma f \|_\gamma^2 &= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \gamma^2 \left(\frac{\partial f}{\partial z}\right)^2 \end{aligned}$$

It follows that, for $\vec{v}_0 = (a_0, b_0, c_0)$,

$$\vec{v}_1 = (a_1,b_1,c_1) = (a_0,b_0,c_0) - 2\frac{\left(a_0\frac{\partial f}{\partial x} + b_0\frac{\partial f}{\partial y} + c_0\frac{\partial f}{\partial z}\right)}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \gamma^2\left(\frac{\partial f}{\partial z}\right)^2}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \gamma^2\frac{\partial f}{\partial z}\right) \ . \tag{5}$$

Proposition 1 *Take $c_0 = 1$. Then $\lim_{\gamma\to 0} \vec{v}_1$ exists and is equal to*

$$\vec{v}_1 = (a_1,b_1,1) = (a_0,b_0,1) - 2\frac{\left(a_0\frac{\partial f}{\partial x} + b_0\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right)}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 0\right) \ . \tag{6}$$

Proof: Take $c_0 = 1$ in (5) and then take the limit when γ goes to 0.

Observe that proposition 1 tells us that the section $\{z' = 1\}$ is invariant by the flow defined by (2), when $\gamma \to 0$.

Consider now a time-dependent billiard whose boundary at time t is given by $C(t) = \{f(x, y, t) = 1\}$. Suppose that the particle leaves the boundary $C_0 = C(t_0)$ at the point $q_0 \in C_0$ with velocity $\vec{v}_0 = (a_0, b_0)$. It reaches the boundary again, in some future time t_1, at a point $q_1 \in C_1 = C(t_1)$, with an exit velocity $\vec{v}_1 = (a_1, b_1)$. If we look the moving boundary as a surface in the (x, y, t) space the trajectory of the particle at each instant will have velocity $\vec{v} = (\dot{x}, \dot{y}, 1)$. So, the billiard map shall change an initial condition $\vec{q}_0 = (q_0, t_0)$ with $\vec{v}_0 = (a_0, b_0, 1)$ into $\vec{q}_1 = (q_1, t_1)$ with $\vec{v}_1 = (a_1, b_1, 1)$.

We will show now that proposition 1 also implies that the time-dependent billiard can be thought as the limit case, $\gamma \to 0$, of the movement given by (2) and (3).

To check that this is the physically correct formula, denote by $\vec{\eta}$ the instantaneous external normal vector in xy-plane, with the usual metric,

$$\vec{\eta}(x, y; t) = \frac{(\partial f/\partial x, \partial f/\partial y)}{[(\partial f/\partial x)^2 + (\partial f/\partial y)^2]^{1/2}} \tag{7}$$

Caveat: do not confound $\vec{\eta}$ with $\vec{n}$. The latter is 3-dimensional, see (5).

Equation (6) is equivalent to

$$(a_1, b_1)_{mb} = (a_1, b_1)_{rb} + 2\,\frac{-\partial f/\partial t}{[(\partial f/\partial x)^2 + (\partial f/\partial y)^2]^{1/2}}\,\vec{\eta}$$

where "mb" means "for our billiard with moving boundary" and "rb" means "for the instantaneous billiard with rigid boundary". In other words, while $(a_1, b_1)_{rb}$ is given by the usual specular reflection,

$$(a_1, b_1)_{rb} = (a_0, b_0) - 2\,\frac{(a_0 \partial f/\partial x + b_0 \partial f/\partial y)}{(\partial f/\partial x)^2 + (\partial f/\partial y)^2}\,(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

there is an extra momentum $2u\vec{\eta}$, with

$$u = \frac{-\partial f/\partial t}{[(\partial f/\partial x)^2 + (\partial f/\partial y)^2]^{1/2}}$$

It is easy to see that u is the velocity of the boundary in the direction of the normal $\vec{\eta}$. There is a simple physical explanation for the factor 2 in formulas (5) and (6). Usual specular reflection law holds if one goes to a reference frame moving with constant velocity $-u\vec{\eta}$, because in this frame the boundary velocity at the point of collision is *zero*. The normal component of the incoming vector in the new frame is equal to the normal component in the old frame minus $u\vec{\eta}$. Thus, under reflection, the normal component of the velocity of the particle gets an extra $u\vec{\eta}$ term. Going back to the old frame adds one more $u\vec{\eta}$.

We recall that conservation of energy holds for any rigid billiard; in particular, using the γ-metric,

$$\| \vec{v}_1 \|_\gamma^2 = \| \vec{v}_0 \|_\gamma^2$$

or equivalently,

$$a_0^2 + b_0^2 + (\frac{1}{\gamma^2})c_0^2 = a_1^2 + b_1^2 + (\frac{1}{\gamma^2})c_1^2.$$

We are interested in the x, y part of the energy,

$$2\Delta E = (a_1^2 + b_1^2) - (a_0^2 + b_0^2) = (\frac{1}{\gamma^2})c_0^2 - (\frac{1}{\gamma^2})c_1^2$$

where from (5) we have

$$c_1 = c_0 - 2\gamma^2(\frac{\partial f}{\partial z})(\frac{\partial f/\partial x\, a_0 + \partial f/\partial y\, b_0 + \partial f/\partial z\, c_0}{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + \gamma^2(\partial f/\partial z)^2})$$

Therefore,

$$2\Delta E_\gamma = \frac{c_0^2}{\gamma^2} - \frac{1}{\gamma^2}\left[c_0 - 2\gamma^2(\frac{\partial f}{\partial z})(\frac{\partial f/\partial x a_0 + \partial f/\partial y b_0 + \partial f/\partial z c_0}{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + \gamma^2(\partial f/\partial z)^2})\right]^2$$

Taking the limit as $\gamma \to 0$ we get

Proposition 2 *The energy change at each collision is given by*

$$\Delta E = 2(\frac{\partial f}{\partial z})(\frac{\partial f/\partial x\, a_0 + \partial f/\partial y\, b_0 + \partial f/\partial z}{(\partial f/\partial x)^2 + (\partial f/\partial y)^2})$$

Remark 1: Although the metric is singular in $\gamma = 0$, if $\partial V/\partial z$ (or equivalently $\partial f/\partial z$) remains bounded, then $z''(t) \to 0$ as $\gamma \to 0$ (see (2)). So, if we choose initial conditions such that $z'|_{t=0} = 1$ and $z|_{t=0} = 0$, then $\lim_{\gamma\to 0} z(t) = t$, or $z(t) = t + \mathcal{O}(\gamma^2)$. Indeed, a philosophical principle, well known among physicists, states that *time is just a spatial coordinate whose inertia tends to infinity (so its motion is uniform).* We refer to Montgomery [15] for related equations and bibliography on singular metrics, in the context of Calculus of Variations.

Remark 2: Under the change of scale $z = \gamma\zeta$, the Lagrangian system (1),(3) becomes $T = (1/2)\,(x'^2 + y'^2 + \zeta'^2)$, $V = V_\gamma = V(x, y, \gamma\zeta)$, and the singular behavior of the metric as $\gamma \to 0$ is removed. However, this change is not very useful if the natural initial conditions indicated above are considered; the velocity component in the ζ direction is very large, $d\zeta/dt = 1/\gamma + \mathcal{O}(\gamma)$ and it will remain as such, at least for $t = \mathcal{O}(1)$.

Remark 3: At first sight, the small parameter γ seems to be artificial. However, taking $\gamma = i/c$, (3) becomes the Lorentz's metric. Relativistic billiards may turn out to be good models for highly accelerated particles confined on magnetic bottles (this, incidentally, was Fermi's initial motivation).

3 Poincaré's integral invariant: coordinates for a section

Under Legendre's transformation $p_x = x'$, $p_y = y'$, $p_z = (1/\gamma^2)z'$, our autonomous Lagrangian of mechanical type (1),(2) turns into the Hamiltonian

$$H = \frac{1}{2}[p_x^2 + p_y^2 + \gamma^2 p_z^2] + V(x, y, z). \tag{8}$$

Observe that since $z' = 1 + \mathcal{O}(\gamma^2)$, $p_z = \mathcal{O}(1/\gamma^2)$ and $\gamma^2 p_z^2 = \mathcal{O}(1/\gamma^2)$. Hence the term $\gamma^2 p_z^2$ cannot be dropped when γ goes to zero. One way to circumnvent this difficulty is to consider the Hamiltonian $H = (1/2)[p_x^2 + p_y^2] + V(x,y,z)$ together with the *constraint* $z' = 1$ in the context of Dirac's theory. For our purposes it is simpler to rewrite (8) as

$$\begin{aligned} H &= E + \frac{1}{2\gamma^2} z'^2, \\ E &= \frac{1}{2}\,[p_x^2 + p_y^2] + V(x,y,z), \end{aligned}$$

with E being the (relevant) finite part of the energy when γ is infinitesimal. Differentiating, we get

$$dH = dE + z'(1/\gamma^2)dz' = dE + z'dp_z\ , \qquad z' = 1 + O(\gamma^2),$$

If we confine ourselves to a fixed total energy manifold, then $dH = 0$ and

$$dp_z = -dE/z' = -dE + O(\gamma^2). \tag{9}$$

We want to construct suitable coordinates for the moving billiard mapping, describing consecutive phase space points immediately after collision, and calculate the associated Poincaré's integral invariant (see [2] §38). This mapping is *four* dimensional: fixing the total energy H in (8), one variable is cut down, so the flow is restricted to a 5-dimensional manifold; the successive collisions represent a 4-dimensional submanifold transversal to the flow.

We change the cartesian coordinates x, y, z to generalized coordinates ζ, φ, ξ as follows: let $x_0 = \alpha(z,\varphi)$, $y_0 = \beta(z,\varphi)$ represent parametrically the moving boundary $f(x,y,z) = 1$. We dont need to assume that the z-curves are orthogonal to the φ-curves. Let ξ describe outward motion along normals (with frozen z). The desired coordinate transformation $X : (\zeta,\varphi,\xi) \longrightarrow (x,y,z)$ is

$$\begin{aligned} x = \quad x(z,\varphi,\xi) &= \alpha(z,\varphi) + \xi\,\frac{\partial\beta/\partial\varphi}{[(\partial\alpha/\partial\varphi)^2 + (\partial\beta/\partial\varphi)^2]^{1/2}} \\ y = \quad y(z,\varphi,\xi) &= \beta(z,\varphi) - \xi\,\frac{\partial\alpha/\partial\varphi}{[(\partial\alpha/\partial\varphi)^2 + (\partial\beta/\partial\varphi)^2]^{1/2}} \\ z = \quad \gamma\zeta & \end{aligned}$$

Note that $\xi = 0$ means that we are on the moving boundary. The kinetic energy T will be given by

$$T = (1/2)(\zeta',\varphi',\xi')G(\zeta',\varphi',\xi')^{tr}$$

with $G = A^{tr}A$, where A^{tr} is the matrix whose rows are $X_\zeta, X_\varphi, X_\xi$. Moreover, Legendre's transformation yields

$$(p_\zeta, p_\varphi, p_\xi) = G(\zeta',\varphi',\xi')^{tr} = A^{tr}AA^{-1}(x',y',z')^{tr} = A^{tr}(x',y',z')^{tr}$$

Thus for instance, p_φ is equal to the scalar product of X_φ with (x',y',z'). In our case, since $z_\varphi \equiv 0$, we have

$$p_\varphi = x'x_\varphi + y'y_\varphi \tag{10}$$

Poincaré's 1-form writes, via pullback to the new coordinates, as

$$\lambda = x'(x_\varphi d\varphi + x_\xi d\xi + x_z dz) + y'(y_\varphi d\varphi + y_\xi d\xi + y_z dz) + p_z dz$$

Our 4-dimensional section is obtained by fixing the total energy in (8) and taking $\xi = 0$. So λ simplifies into

$$\lambda = (x'x_\varphi + y'y_\varphi)d\varphi + (x'x_z + y'y_z + p_z)dz$$

or, in view of (10),

$$\lambda = p_\varphi d\varphi + (F + p_z)dz$$

where we have defined $F = x'x_z + y'y_z$.

Finally, from (9) and letting $\gamma \to 0$, we get:

Proposition 3 *The conserved 2-form for the Poincaré section is*

$$\omega = dp_\varphi d\varphi - dtd(E - F) \ . \tag{11}$$

Although the pairs (φ, p_φ), $(E - F, t)$ are canonically conjugate, physically it would be more natural to map over $((\varphi, p_\varphi), (E, t))$. *Helas*, the quantity F is an unavoidable nuisance, inherited from the time-dependence.

Perhaps it is instructive to look into the 1-dimensional case, where the motion is confined to the x–axis and there are elastic collisions with moving walls $x = a(t)$ and $x = b(t)$. Let's look at a collision with the left wall. The first term in (11) does not appear, and the second gives

$$\omega = -dt\, d\left[\frac{1}{2}(x')^2 - x'a_t\right] = dt(-x'dx' + a_t dx' + x'a_{tt}dt) = (a_t - x')dtdx' . \tag{12}$$

As a motivation for the next section, we present now the derivative of the mapping which gives successive collisions with the two moving walls. This is obtained by an elementary calculation. Suppose the particle leaves the left wall at time t_0 with velocity x'_0 , and that it reaches the right wall at a later time $t_{1/2}$, given by the implicit equation

$$b(t_{1/2}) = a(t_0) + x'_0(t_{1/2} - t_0) \ . \tag{13}$$

After the collision we have

$$x'_{1/2} = -x'_0 + 2\frac{db}{dt}(t_{1/2}) \ . \tag{14}$$

Analogously, we get

$$\begin{aligned} a(t_1) \quad &= b(t_{1/2}) + x'_{1/2}(t_1 - t_{1/2}) \ , \\ x'_1 \quad &= -x'_{1/2} + 2\frac{da}{dt}(t_1) \ . \end{aligned}$$

Differentiating (13),(14) we obtain

$$\begin{aligned} b'(t_{1/2})dt_{1/2} \quad &= a'(t_0)dt_0 + x'_0(dt_{1/2} - dt_0) + dx'_0(t_{1/2} - t_0) \\ dx'_{1/2} \quad &= -dx'_0 + 2b''(t_{1/2})dt_{1/2} \end{aligned}$$

so that, after a simple algebra,

$$(dt_{1/2}, dx'_{1/2})^{tr} = B(dt_0, dx'_0)^{tr}$$

where

$$B = \begin{pmatrix} (a'_0 - x'_0)/(b'_{1/2} - x'_0) & (t_{1/2} - t_0)/(b'_{1/2} - x'_0) \\ 2b''_{1/2}(a'_0 - x'_0)/(b'_{1/2} - x'_0) & -1 + 2b''_{1/2}(t_{1/2} - t_0)/(b'_{1/2} - x'_0) \end{pmatrix} \tag{15}$$

Similarly we can write $(dt_1, dx'_1)^{tr} = A(dt_{1/2}, dx'_{1/2})^{tr}$ with

$$A = \begin{pmatrix} (b'_{1/2} - x'_{1/2})/(a'_1 - x'_{1/2}) & (t_1 - t_{1/2})/(a'_1 - x'_{1/2}) \\ 2a''_1(b'_{1/2} - x'_{1/2})/(a'_1 - x'_{1/2}) & -1 + 2a''_1(t_1 - t_{1/2})/(a'_1 - x_{'1/2}) \end{pmatrix}$$

Now we can present a direct verification of the invariance of the measure on the plane (x, t) given by (12). Since

$$(b'_{1/2} - x'_0) = -(x'_{1/2} - b'_{1/2})$$

then

$$detB = -(a'_0 - x'_0)/(b'_{1/2} - x'_0) = (a'_0 - x'_0)/(x'_{1/2} - b'_{1/2}),$$
$$(a'_0 - x'_0)dt_0 dx'_0 = (x'_{1/2} - b'_{1/2})dt_{1/2}dx'_{1/2}$$

as desired.

4 Derivative of the moving billiard mapping. Case n=2

Inspired by the last calculation, we now compute the derivative of the time-dependent billiard mapping in two dimensions. The relevant variables are depicted in Fig.1. We assume for convenience that the curves $C(t)$ are convex; following Berry [3] we use the angle φ with a fixed direction as parameter, $\varphi \in [0, 2\pi)$. If $C(t)$ ceases to be convex, one may pass to arc length $s \in [0, l)$.

Suppose the particle leaves the boundary $C_0 = C(t_0)$ from point q_0 with velocity $\vec{v}_0$, making an angle α_0 with the instantaneous tangent vector $\vec{\tau}_0 = (\cos\varphi_0, \sin\varphi_0)$. It reaches the boundary again in some future time t_1, at point $q_1 \in C_1 = C(t_1)$, where the exit velocity $\vec{v}_1$ makes angle α_1 with the vector $\vec{\tau}_1 = (\cos\varphi_1, \sin\varphi_1)$. We denote by α_1^* the *entry* angle at q_1 . In general $\alpha_1^* \neq \alpha_1$. Elementary geometry yields (observe that no harm is done from C_0 and C_1 being different curves):

$$\alpha_0 + \alpha_1^* = \varphi_1 - \varphi_0 \qquad (\text{hence } d\alpha_1^* = -d\alpha_0 - d\varphi_0 + d\varphi_1)$$

Given $q_0 \in C(t_0)$, $q_1 \in C(t_1)$, clearly

$$\| \vec{v}_0 \| = (2E_0)^{1/2} = \frac{\| q_1 - q_0 \|}{t_{01}} \quad , \quad t_{01} = t_1 - t_0 \ .$$

Figure 1:

If the boundary is $C(t) : f(x, y, t) = 1$, then given $q_0 \in C(t_0)$ and $\vec{v}_0$, in order to obtain t_{01} (and hence $q_1 = q_0 + t_{01}\vec{v}_0$) we must solve the implicit equation (for t_{01})

$$f(q_0 + t_{01}\vec{v}_0, t_0 + t_{01}) = 1,$$

Lemma 4 *Let u_1 be the velocity of the boundary at (q_1, t_1) in the direction of the exterior normal and R the radius of curvature of the instantaneus boundary at the impact point. Then*

$$\begin{aligned} p_\varphi &= (2E)^{1/2} R \cos\alpha \\ \cos\alpha_1 &= (E_0/E_1)^{1/2} \cos\alpha_1^* \\ E_1 &= E_0 + 2u_1[u_1 - (2E_0)^{1/2} \sin\alpha_1^*] \ . \end{aligned} \tag{16}$$

Proof: From (10) we get $p_\varphi = (x_\varphi, y_\varphi) \cdot (x', y') = (\partial s/\partial\varphi)\vec{\tau} \cdot \vec{v}$. The boundary is parametrized by $(x, y) = (x(\varphi, t), y(\varphi, t))$, $R = \partial s/\partial\varphi$ is the radius of curvature and $\vec{\tau}$ is the instantaneous tangent vector. There is no ambiguity by taking the velocity $\vec{v} = (x', y')$ either immediately before or after impact, because the tangential component does not change. In particular $(E_0)^{1/2} \cos\alpha_1^* = (E_1)^{1/2} \cos\alpha_1$. Proposition 2 gives $E_1 = E_0 + 2(f_t/\|\nabla f\|)(f_t/\|\nabla f\| + \vec{v}_0 \cdot \nabla f/\|\nabla f\|)$. Recall also (see (7)) that $\vec{\eta} = \nabla f/\|\nabla f\|$ is the exterior normal and $u = -f_t/\|\nabla f\|$. Finally, observe from Fig.1 that $\sin\alpha_1^* = \vec{\eta}_1 \cdot \vec{v}_0/\|\vec{v}_0\|$. Formulas (16) follow.

Now we describe the derivative of the billiard mapping

$$(\varphi_0, \alpha_0, E_0, t_0) \to (\varphi_1, \alpha_1, E_1, t_1)$$

where, in order to compare results with rigid billiards, we have replaced the variable p_φ by α. A tedious calculation is outlined on the Appendix:

Proposition 5 *("bank of differentials" for the 4×4 jacobian).*

i) $$dt_1 = C_{41} d\varphi_0 + C_{42} d\alpha_0 + C_{43} dE_0 + C_{44} dt_0$$

$$\begin{aligned}
C_{41}[-\sin\alpha_1^*(2E_0)^{1/2} + u_1] &= -t_{01}\cos\alpha_1^*(2E_0)^{1/2} - R_0\sin(\varphi_0 - \varphi_1)\\
C_{42}[-\sin\alpha_1^*(2E_0)^{1/2} + u_1] &= -t_{01}(2E_0)^{1/2}\cos\alpha_1^*\\
C_{43}[-\sin\alpha_1^*(2E_0)^{1/2} + u_1] &= t_{01}\sin\alpha_1^*/(2E_0)^{1/2}\\
C_{44}[-\sin\alpha_1^*(2E_0)^{1/2} + u_1] &= -\sin\alpha_1^*(2E_0)^{1/2} + u_0\cos(\varphi_0 - \varphi_1)
\end{aligned}$$

ii) $$d\varphi_1 = C_{11} d\varphi_0 + C_{12} d\alpha_0 + C_{13} dE_0 + C_{14} dt_0$$

$$\begin{aligned}
R_1 C_{11} &= R_0\cos(\varphi_0 - \varphi_1) + t_{01}\sin\alpha_1^*(2E_0)^{1/2} + (2E_0)^{1/2}\cos\alpha_1^* C_{41}\\
R_1 C_{12} &= t_{01}\sin\alpha_1^*(2E_0)^{1/2} + (2E_0)^{1/2}\cos\alpha_1^* C_{42}\\
R_1 C_{13} &= t_{01}\cos\alpha_1^*/(2E_0)^{1/2} + (2E_0)^{1/2}\cos\alpha_1^* C_{43}\\
R_1 C_{14} &= u_0\sin(\varphi_0 - \varphi_1) + (2E_0)^{1/2}\cos\alpha_1^*(-1 + C_{44})
\end{aligned}$$

iii) $$dE_1 = C_{31} d\varphi_0 + C_{32} d\alpha_0 + C_{33} dE_0 + C_{34} dt_0$$

$$\begin{aligned}
C_{31} = &\quad 2C_{11}\partial u_1/\partial\varphi_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] +\\
&+ 2C_{41}\partial u_1/\partial t_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] +\\
&+ 2u_1\cos\alpha_1^*(1 - C_{11})(2E_0)^{1/2}\\
C_{32} = &\quad 2C_{12}\partial u_1/\partial\varphi_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] +\\
&+ 2C_{42}\partial u_1/\partial t_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] +\\
&+ 2u_1\cos\alpha_1^*(1 - C_{12})(2E_0)^{1/2}\\
C_{33} = &\quad 1 + 2C_{13}\partial u_1/\partial\varphi_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] +\\
&+ 2C_{43}\partial u_1/\partial t_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] -\\
&- 2u_1[\cos\alpha_1^* C_{13}(2E_0)^{1/2} + \sin\alpha_1^*/(2E_0)^{1/2}]\\
C_{34} = &\quad 2C_{14}\partial u_1/\partial\varphi_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] +\\
&+ 2C_{44}\partial u_1/\partial t_1[2u_1 - \sin\alpha_1^*(2E_0)^{1/2}] -\\
&- 2u_1\cos\alpha_1^* C_{14}(2E_0)^{1/2}
\end{aligned}$$

iv) $$d\alpha_1 = C_{21} d\varphi_0 + C_{22} d\alpha_0 + C_{23} dE_0 + C_{24} dt_0$$

$$\begin{aligned}
-\sin\alpha_1 C_{21} &= (E_0/E_1)^{1/2}[\sin\alpha_1^*(1 - C_{11}) - C_{31}\cos\alpha_1^*/(2E_1)]\\
-\sin\alpha_1 C_{22} &= (E_0/E_1)^{1/2}[\sin\alpha_1^*(1 - C_{12}) - C_{32}\cos\alpha_1^*/(2E_1)]\\
-\sin\alpha_1 C_{23} &= -(E_0/E_1)^{1/2}\sin\alpha_1^* C_{13} + \cos\alpha_1^*/(2(E_0E_1)^{1/2}) -\\
&\quad -C_{33}(E_0/E_1)^{1/2}\cos\alpha_1^*/(2E_1)\\
-\sin\alpha_1 C_{24} &= (E_0/E_1)^{1/2}[-\sin\alpha_1^* C_{14} - C_{34}\cos\alpha_1^*/(2E_1)]
\end{aligned}$$

(Observe the "telescopic" structure of these formulas)

5 Example: loss of stability of the energy component

In this section we consider the simplest possible situation, where at the points of contact

$$\alpha_0 = \alpha_1 = \pi/2 \quad , \quad \varphi_1 - \varphi_0 = \pi \quad , \quad u_1 = u_0 = 0 \ .$$

i.e, we have a 2-bounce periodic orbit whose energy is unchanged at the collisions. First we look at the "geometrical block" A associated to the corresponding rigid billiard mapping $(\varphi_0, \alpha_0) \to (\varphi_1, \alpha_1)$. Proposition 5 yields (compare with Berry [3])

$$A = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= -R_0/R_1 + t_{01}(2E_0)^{1/2}/R_1 \\ C_{12} &= t_{01}(2E_0)^{1/2}/R_1 \\ C_{21} &= -1 + C_{11} \\ C_{22} &= -1 + C_{12} \end{aligned}$$

We take for simplicity $R_0 = R_1$ and choose units so that $2E_0 = 1$ and $t_{01} = 1$. The distance between the contact points is $d = t_{01}(2E_0)^{1/2} = 1$ so

$$A = \begin{pmatrix} 2a+1 & 2a+2 \\ 2a & 2a+1 \end{pmatrix} \quad \text{,where} \quad a = (1/2R) - 1 > -1$$

The stability index is $\chi_A = tr(A)/2 = 2a+1 = (1/R) - 1$, so the orbit is unstable for $R < 1/2$, that is, if the osculating circles do not touch (these facts are well known: a two periodic orbit is linearly stable if and only if $t_{01} < R_0 + R_1$ and $(t_{01} - R_0)(t_{01} - R_1) > 0$; see [3] §3 and [17]). Next we look at the "Fermi block" B, associated, by restriction, to the 1-dimensional Fermi accelerator $(E_0, t_0) \to (E_1, t_1)$. From proposition 5 we have (equivalently, it follows from (15))

$$B = \begin{pmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{pmatrix}$$

$$\begin{aligned} C_{33} &= 1 - 2(\partial u_1/\partial t_1)t_{01}/(2E_0)^{1/2} \\ C_{34} &= -2(\partial u_1/\partial t_1)(2E_0)^{1/2} \\ C_{43} &= t_{01}/2E_0 \\ C_{44} &= 1 \end{aligned}$$

so that, since $t_{01} = 1$, $2E_0 = 1$,

$$B = \begin{pmatrix} -2b+1 & -2b \\ 1 & 1 \end{pmatrix} \quad \text{,where} \quad b = \partial u_1/\partial t_1$$

The stability index is $\chi_B = tr(B)/2 = -b + 1$: the orbit is stable for $0 < b < 2$.

To illustrate the loss of stability due to the underlying 2-dimensional geometry, we take the following values for the parameters, yielding the saddle-center case:

1. $b = 1$, so the eigenvalues of B are $i, -i$ (most stable case for B) .
2. $a = 1/8$, so for A we have $\lambda_1 = 2, v_1 = (3,1); \lambda_2 = 1/2, v_2 = (-3,1)$.

Due to the symmetry $R_0 = R_1$, the derivative of the second bounce mapping corresponds to a matrix C^2, where C is

$$C = \begin{pmatrix} \multicolumn{2}{c}{A} & 0 & 0 \\ & & 0 & 0 \\ C_{31} & C_{32} & \multicolumn{2}{c}{B} \\ 0 & 0 & & \end{pmatrix}$$

$$C_{31} = -(5/2)\partial u_1/\partial \varphi_1,$$
$$C_{32} = -(9/2)\partial u_1/\partial \varphi_1.$$

The full eigenvector corresponding to $\lambda_1 = 2$ is

$$v = (3, 1, -(12/5)\partial u_1/\partial \varphi_1, (4/5)\partial u_1/\partial \varphi_1).$$

The presence of the coupling terms C_{31}, C_{32} implies linear loss of stability for the energy. This effect would not be seen if we looked only to the 1-dimensional Fermi accelerator obtained by restricting the dynamics to the line $q_0 q_1$.

Appendix: The jacobian matrix: proof of proposition 5

i) Calculation of dt_1.

Linearizing $f(q_0 + dq_0, t_0 + dt_0) = 1, f(q_1 + dq_1, t_1 + dt_1) = 1$ we get

$$\nabla f_{(q_0,t_0)} \cdot dq_0 + (\partial f/\partial t)_{(q_0,t_0)} dt_0 = \nabla f_{(q_1,t_1)} \cdot dq_1 + (\partial f/\partial t)_{(q_1,t_1)} dt_1 \tag{17}$$

Clearly, $q_0 + t_{01}\vec{v}_0 = q_1$ implies to first order

$$dq_0 + t_{01} d\vec{v}_0 + (dt_1 - dt_0)\vec{v}_0 = dq_1. \tag{18}$$

Multiplying 18 by $\vec{\eta}_1 = \nabla f(q_1, t_1)/ \parallel \nabla f(q_1, t_1) \parallel$ we obtain

$$dt_1(u_1 - \vec{v}_0 \cdot \vec{\eta}_1) = -dt_0 \vec{v}_0 \cdot \vec{\eta}_1 + t_{01} d\vec{v}_0 \cdot \vec{\eta}_1 + dq_0 \cdot \vec{\eta}_1 \tag{19}$$

We must express the scalar products in (19) in terms of the geometrical variables of Fig.1. Clearly $dq_0 = d\varphi_0 R_0 \tau_0 + d\xi_0 \vec{\eta}_0$ where, because of (17), $d\xi_0 = u_0 dt_0$ so that

$$dq_0 = d\varphi_0 R_0 \vec{\tau}_0 + u_0 dt_0 \vec{\eta}_0 \qquad (dq_1 = d\varphi_1 R_1 \vec{\tau}_1 + u_1 dt_1 \vec{\eta}_1) \tag{20}$$

¿From $\vec{v}_0 = (2E_0)^{1/2}[\cos(\alpha_0)\vec{\tau}_0 - \sin(\alpha_0)\vec{\eta}_0]$ we get

$$\begin{aligned} d\vec{v}_0 = & \quad [dE_0/(2E_0)^{1/2}][\cos(\alpha_0)\vec{\tau}_0 - \sin(\alpha_0)\vec{\eta}_0] + \\ & + \quad (2E_0)^{1/2}[-\sin(\alpha_0)\vec{\tau}_0 + \cos(\alpha_0)\vec{\eta}_0] d\alpha_0 + \\ & + \quad (2E_0)^{1/2}[\cos(\alpha_0) d\vec{\tau}_0 + \sin(\alpha_0) d\vec{\eta}_0] \end{aligned}$$

where $\vec{\tau}_0 = (\cos\varphi_0, \sin\varphi_0)$, $\vec{\eta}_0 = (\sin\varphi_0, -\cos\varphi_0)$. Finally, after a few manipulations we get (observe Fig.1)

$$\begin{aligned} \vec{v}_0 \cdot \vec{\eta}_1 &= \sin\alpha_1^*(2E_0)^{1/2} \\ dq_0 \cdot \vec{\eta}_1 &= -d\varphi_0 R_0 \sin(\varphi_0 - \varphi_1) + dt_0 u_0 \cos(\varphi_0 - \varphi_1) \\ d\vec{v}_0 \cdot \vec{\eta}_1 &= [dE_0/(2E_0)^{1/2}] \sin\alpha_1^* - (2E_0)^{1/2} \cos\alpha_1^* [d\alpha_0 + d\varphi_0] \ . \end{aligned} \tag{21}$$

Substituting (21) into (19) we obtain the desired formula for dt_1.

ii) Calculation of $d\varphi_1$.

From (18) and (20) we have

$$R_1 d\varphi_1 = dq_1 \cdot \vec{\tau}_1 = [dq_0 + t_{01} d\vec{v}_0 + (dt_1 - dt_0)\vec{v}_0] \cdot \vec{\tau}_1$$

and analogously to (21)

$$\begin{aligned} \vec{v}_0 \cdot \vec{\tau}_1 &= \cos\alpha_1^*(2E_0)^{1/2} \\ dq_0 \cdot \vec{\tau}_1 &= d\varphi_0 R_0 \cos(\varphi_0 - \varphi_1) + dt_0 u_0 \sin(\varphi_0 - \varphi_1) \\ d\vec{v}_0 \cdot \vec{\tau}_1 &= [dE_0/(2E_0)^{1/2}] \cos\alpha_1^* + (2E_0)^{1/2} \sin\alpha_1^* [d\alpha_0 + d\varphi_0] \end{aligned}$$

from which the formula for $d\varphi_1$ follows.

iii) Calculation of dE_1 and $d\alpha_1$.

Recall that (lemma 4)

$$\begin{aligned} &E_1 = E_0 + 2u_1^2 - 2u_1 \sin\alpha_1^*(2E_0)^{1/2}, \\ &\cos\alpha_1 = \cos\alpha_1^*(E_0/E_1)^{1/2}. \end{aligned}$$

The rest of proposition 5 is obtained by differentiating these expressions, using that $d\alpha_1^* = -d\alpha_0 - d\varphi_0 + d\varphi_1$, and inserting the previously computed differentials for dt_1 and $d\varphi_1$.

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References

[1] G.Abal, R.Donangelo, C.O. Dorso: One-body dissipation at intermediate nuclear connection regimes, Physical Review C **46:1**, 380-381 (1992)
see also J.Warnbach: Nuclear sound, Contemp.Phys. **32**, 291-304 (1991).

[2] V.I.Arnold: *Mathematical Methods of Classical Mechanics*, Springer-Verlag, Berlin, Heidelberg, New York (1978).

[3] M.V.Berry: Regularity and chaos in classical mechanics, illustrated by three deformations of a circular billiard, Eur.J.Phys **2**, 91-102 (1981).

[4] M.V.Berry: Some quantum to classic asymptotics, in *Les Houches Lecture series 52* (ed.M.J.Giannoni, A.Voros, J.Zinn Justin) p.251-303, Amsterdam, North-Holland (1991).

[5] G.D.Birkhoff: *Dynamical Systems*, A.M.S. Colloquium Publications, Providence, RI, (1966). (Original ed. 1927).

[6] I.P.Cornfeld, S.V. Fomin, Ya.G. Sinai: *Ergodic Theory*, Berlin, Heidelberg, New York, Springer-Verlag (1982).

[7] R.Douady: Applications du théorème des tores invariants, Thèse de 3ème Cycle, Univ. Paris VII (1982).

[8] E.Fermi: On the origin of the cosmic radiantion, Phys.Rev. **75**, 1169-1174 (1949).
See also S.Ulam, in *Proc. 4th Berkeley Symp. on Math.Stat. and Prob.*, vol 3, p. 315, University of California Press (1961).

[9] M.C.Gutzwiller: *Chaos in classical and quantum mechanics*, Berlin, Heidelberg, New York, Springer-Verlag (1990).

[10] N.S.Krylov: *Works on the foundations of Statistical Physics*, Princeton, PUP (1979).
Ya.G.Sinai: On the foundations of the ergodic hypothesis for a dynamical system of Statistical Mechanics, Sov.Math.Doklady **4**, 1818-1822 (1963).

[11] S.Laederich, M. Levi: Invariant curves and time dependent potentials, Erg.th.& Dyn.Syst., **11**, 365-378 (1991).

[12] M.Levi: Quasiperiodic motions in superquadratic time-periodic potentials, Commun.Math.Phys, **143** 43-83 (1991).

[13] A.J.Lichtenberg, M.A. Liebermann: *Regular and Stochastic Motion*, Springer-Verlag Appl.Math.Sci. **38**, Berlin, Heidelberg, New York, Springer-Verlag (1983).

[14] R.Markarian: Introduction to the ergodic theory of plane billiards, in *Dynamical Systems, Santiago de Chile, 1990* Harlow, Longman (1992) (to appear).

[15] R.Montgomery: Isoholonomic problems and some applications, Commun.Math.Phys. **128**, 565-592 (1990).

[16] A.Pais: Niels Bohr's Times, in *Physics, Philosophy and Policy*, Clarendon (Oxford University Press), New York (1991).
See also the review by F.Wilczek , Science, **255**, 345-347 (1992).

[17] M.Wojtkowski: Principles for the design of billiards with nonvanishing Lyapounov exponents, Comm.Math.Phys. **105**, 391-414 (1986).
A.Hayli, Th.Dumont: Experiences numériques sur les billiards C^1 formés de quatre arcs de cercles, Celestial Mechanics **38**, 23-66 (1986).

Jair Koiller
Laboratório Nacional de Computação Científica
R. Lauro Muller 455
22290 Rio de Janeiro, Brasil.
email:userjako@lncc.bitnet

Roberto Markarian
IMERL, Facultad de Ingeniería
C.C. $N^{\underline{o}}$ 30
Montevideo, Uruguay.
email:roma@iie.edu.uy

Sylvie Oliffson Kamphorst
Departamento de Matemática, ICEx, UFMG
C.P. 702
30161 Belo Horizonte, Brasil.
email:syok@brufmg.bitnet

Sônia Pinto de Carvalho
Departamento de Matemática, ICEx, UFMG
C.P. 702
30161 Belo Horizonte, Brasil.
email:icedo@brufmg.bitnet