The Self-Avoiding-Walk and Percolation Critical Points in High Dimensions*

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Abstract

We prove existence of an asymptotic expansion in the inverse dimension, to all orders, for the connective constant for self-avoiding walks on $\mathbf{Z}^d$. For the critical point, defined to be the reciprocal of the connective constant, the coefficients of the expansion are computed through order d^{-6}, with a rigorous error bound of order d^{-7}. Our method for computing terms in the expansion also applies to percolation, and for nearest-neighbour independent Bernoulli bond percolation on $\mathbf{Z}^d$ gives the $1/d$-expansion for the critical point through order d^{-3}, with a rigorous error bound of order d^{-4}. The method uses the lace expansion.

1 Introduction

Expansions in the inverse dimension, or $1/d$-expansions, are ubiquitous in the physics literature and serve to give approximate information in situations where more precise results are unavailable. They provide useful comparisons for other approximate methods, as well as evidence which can be used in support of or against conjectures. In the physics literature, it is generally assumed that in such expansions the remainder should be of the order of the first omitted term, but this is almost never proved rigorously. However from a mathematical point of view it is desirable to have error bounds on such expansions. In this paper, we prove existence of an *asymptotic* $1/d$-expansion, to all orders, for the connective constant for self-avoiding walks, and compute several terms in this expansion in a systematic way. Unlike previous approaches to this problem [4, 24], there is little counting involved in our method for deriving the expansion. Indeed the derivation of the expansion presented here is reduced almost to mechanical computations.

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Our method for deriving the expansion applies also to percolation, and gives the expansion for the critical point for nearest-neighbour independent Bernoulli bond percolation on $\mathbf{Z}^d$ through order d^{-3}, with a rigorous error bound of order d^{-4}. The derivation for percolation is however more involved, and less mechanical, than for the self-avoiding walk. Unfortunately the proof of existence of an asymptotic expansion to all orders does not extend to percolation.

A review of the literature on $1/d$-expansions for the critical temperature of spin systems has been given by Fisher and Singh [5]. In particular, they describe an expansion for the critical temperature of the N-vector model [9], which corresponds formally in the limit $N \to 0$ to the connective constant for self-avoiding walks. It has not been proven that such expansions are asymptotic for general N. However for the spherical model, which corresponds to $N \to \infty$, the critical temperature is given by $2d[I_{1,0}(0)]^{-1}$, where

$$I_{1,0}(0) \;=\; \int_{[-\pi,\pi]^d} \frac{1}{1 - d^{-1}\sum_{j=1}^d \cos k_j} \frac{d^d k}{(2\pi)^d}. \tag{1.1}$$

The spherical-model critical temperature has an asymptotic expansion to all orders in powers of $1/d$; however this expansion is not convergent [9]. Although we are unable to prove it, we expect that the asymptotic $1/d$-expansion for the connective constant studied in this paper is also divergent.

Before stating our results more precisely, we first make some relevant definitions. An n-step self-avoiding walk on the d-dimensional hypercubic lattice $\mathbf{Z}^d$ is an ordered set $\omega = (\omega(0), \ldots, \omega(n))$ with each $\omega(i) \in \mathbf{Z}^d$, with $\omega(i+1)$ and $\omega(i)$ separated by Euclidean distance 1 for all i, and with $\omega(i) \neq \omega(j)$ for all $i \neq j$. Let $c_n(d)$ denote the number of n-step self-avoiding walks in $\mathbf{Z}^d$ with $\omega(0) = 0$. It was shown in [13] that the limit

$$\mu(d) = \lim_{n\to\infty} [c_n(d)]^{1/n} \tag{1.2}$$

exists; this limit is known as the connective constant. The critical point is then defined as $\beta_c(d) = [\mu(d)]^{-1}$; this is the radius of convergence of the generating function $\chi(\beta) = \sum_{n=0}^\infty c_n(d)\beta^n$ and is analogous in several respects to the critical percolation probability and to the inverse critical temperature of a spin system. For notational convenience, we shall use the abbreviation

$$s = \frac{1}{2d}. \tag{1.3}$$

The precise value of $\mu(d)$ is not known in any dimension (except the trivial case $\mu(1) = 1$), although numerical estimates and rigorous bounds have been obtained (see [1, 3, 18] and references therein). The behaviour of the connective constant as $d \to \infty$ was studied in [19], where it was shown that

$$\mu(d) = s^{-1} - 1 - s + O(s^2). \tag{1.4}$$

Here and throughout the paper, unless otherwise stated, $O(a)$ denotes a term which is bounded in absolute value by Ka, with a universal constant K. The lower bound was improved in [18] to

$$\mu(d) \geq s^{-1} - 1 - s - 3s^2 - 18s^3 - 122s^4 + O(s^5). \tag{1.5}$$

This should be compared with the series

$$\mu(d) = s^{-1} - 1 - s - 3s^2 - 16s^3 - 102s^4 + \cdots \tag{1.6}$$

which was given in [4, 24] with no rigorous bound on the remainder term.

Our first theorem is the following.

Theorem 1.1 *There is an asymptotic expansion in $s = 1/2d$, to all orders, for the connective constant $\mu(d)$. All coefficients in the expansion are integers.*

From (1.6) one might guess that all higher order coefficients will be *negative* integers (or equivalently that all terms in the expansion for $\beta_c(d)$ will be *positive*), but this is not apparent from our proof and may well be untrue. As pointed out in [9], there are sign changes in the expansion of the spherical-model inverse critical temperature $(2d)^{-1}I_{1,0}(0)$ at orders s^{12} and s^{20}.

The next theorem gives the natural error estimate for (1.6).

Theorem 1.2 *As $d \to \infty$,*

$$\mu(d) = s^{-1} - 1 - s - 3s^2 - 16s^3 - 102s^4 + O(s^5). \tag{1.7}$$

Equivalently,

$$\beta_c(d) = s + s^2 + 2s^3 + 6s^4 + 27s^5 + 157s^6 + O(s^7). \tag{1.8}$$

The proofs of both theorems are based on the lace expansion [2]. Theorem 1.1 rests also on a result of Kesten [19] concerning finite-memory walks. In principle our method could be used to derive the expansion to higher order than (1.7), but the necessary calculations become increasingly involved.

It has been pointed out already in [4] that the expansion (1.6) does remarkably well even in three dimensions: truncating (1.6) after its smallest term $-16s^3$ gives $\mu(3) \approx 4.6759$, compared with the estimate 4.6839... obtained from series extrapolation [12]. Unfortunately the rigorous numerical error bounds which can be obtained from (1.7) cannot compete with the very good numerical bounds on $\mu(d)$ of [1, 3, 18]. Nevertheless the $1/d$ expansion did play a role in obtaining the lower bounds on μ in [18], in that the derivation of the lower bounds was based on an attempt to capture as much of the $1/d$-expansion for μ as possible. A similar philosophy was at work in [15].

It is known that for $d \geq 5$, $c_n \sim A\mu^n$ and that $\langle \omega(n) \cdot \omega(n) \rangle \sim Dn$ (uniform measure on n-step self-avoiding walks) [16, 15], and our methods can also be used to compute the first few terms of the $1/d$-expansions for A and D, with error bounds. We have carried this out to an error of order $O(s^3)$, with the following result.

Theorem 1.3 *As $d \to \infty$,*

$$A = 1 + s + 4s^2 + O(s^3), \quad D = 1 + 2s + 8s^2 + O(s^3). \tag{1.9}$$

Because the proof is similar to that of Theorem 1.2, we omit the details [17]. The lace expansion provides explicit formulas for A and D which are used in the proof; see (3.3) and (3.5) of [16]. In [6, 24] and [23] respectively, $1/d$-expansions for A and D are carried out through order s^5, but with no rigorous error estimate.

Our method can also be used to derive some terms of the $1/d$-expansion for the critical point $p_c(d)$ of nearest-neighbour Bernoulli bond percolation on $\mathbf{Z}^d$ (see [11] for the definition of the model). However our reliance on Kesten's result for finite-memory walks prevents the extension of our proof of Theorem 1.1 to percolation. It is known

[20] that $p_c(2) = 1/2$, and it is trivial that $p_c(1) = 1$, but the precise value of p_c is not known for $d > 2$, and rigorous bounds are in short supply: for $d = 3$, p_c is believed to be about 0.2488 but the current best bounds are $0.2102 \leq p_c(3) \leq 0.4798$ (the upper bound is quoted on page 23 of [11], and the lower bound follows from the well-known bound $p_c(d) \geq \beta_c(d)$ together with the upper bound on μ of [1]). Concerning asymptotic results, it was shown in [14] that

$$p_c(d) = s + O(s^2), \tag{1.10}$$

improving results of Gordon [10] and Kesten [21] who obtained the leading order term in (1.10) but with weaker error bounds of respectively $O(s^{65/64})$ and $O(s|\log\log s|^2|\log s|^{-1})$. The method of [14] gives (1.10) also for nearest-neighbour independent site percolation. A lower bound on $p_c(d)$ is given by (1.8). The expansion

$$p_c = s + s^2 + \frac{7}{2}s^3 + 16s^4 + 103s^5 + \dots \tag{1.11}$$

was reported in [7], but with no rigorous bound on the remainder. Extending the method of proof of Theorem 1.2 to bond percolation, we have obtained the following result [17].

Theorem 1.4 *For nearest-neighbour independent Bernoulli bond percolation on* $\mathbf{Z}^d$*,*

$$p_c(d) = s + s^2 + \frac{7}{2}s^3 + O(s^4). \tag{1.12}$$

The computations involved in proving (1.12) are similar to, but more involved than, those involved in the proof of Theorem 1.2; for this reason we do not include the proof here. It should be possible to perform similar calculations for site percolation using these methods, and for oriented percolation using the methods of [25]. For site percolation the expansion

$$p_c^{\text{site}}(d) = s + \frac{5}{2}s^2 + \frac{31}{4}s^3 + \frac{75}{2}s^4 + \cdots \tag{1.13}$$

was reported in [8], but again with no bound on the remainder term. As indicated in [7, 8], the presumably asymptotic expansions (1.11) and (1.13) appear to give reasonable approximations even for $d = 3$.

The next section gives a brief review of some aspects of the lace expansion, which is our principal tool, with brief further commentary on the proof of Theorem 1.4. In Section 3 the proof of Theorem 1.1 is given. Section 4 contains the proof of Theorem 1.2, although because of the repetitive nature of the calculations, details are given only up to an error $O(s^6)$ for β_c (the calculation of an additional term is given in [17]).

2 The lace expansion

This section briefly reviews some previously known facts about the lace expansion.

2.1 Self-avoiding walks

Let $0 \leq \tau \leq \infty$. For $x \in \mathbf{Z}^d$, we write $|x| \equiv \sum_{j=1}^d |x_j|$. An n-step memory-τ walk is defined to be an ordered sequence $\omega = (\omega(0), \omega(1), \dots, \omega(n))$, with each $\omega(i) \in \mathbf{Z}^d$,

$|\omega(i+1)-\omega(i)|=1$, and $\omega(i)\neq\omega(j)$ for $|i-j|\leq\tau$. The last condition prohibits loops of size less than or equal to τ. Taking $\tau=0$ gives the simple random walk, and taking $\tau=\infty$ gives the self-avoiding walk. Finite memory will be used to prove Theorem 1.1, but is not needed for Theorems 1.2 and 1.3. Let $\Omega_n^{(\tau)}(x,y)$ denote the set of n-step memory-τ walks with $\omega(0)=x$ and $\omega(n)=y$. We write $\Omega^{(\tau)}(0,x)=\cup_{n\geq 0}\Omega_n^{(\tau)}(0,x)$, and $\Omega(0,x)=\Omega^{(\infty)}(0,x)$. Let $c_n^{(\tau)}(x,y)$ denote the cardinality of $\Omega_n^{(\tau)}(x,y)$ (with $c_0^{(\tau)}(x,y)=\delta_{x,y}$), and let $c_n^{(\tau)}=\sum_x c_n^{(\tau)}(0,x)$. A subadditivity argument implies existence of the limit $\mu_\tau=\lim_{n\to\infty}[c_n^{(\tau)}]^{1/n}$. Define $\beta_\tau=1/\mu_\tau$, so in particular $\beta_\infty=\beta_c$.

The proofs of Theorems 1.1 and 1.2 are based on an identity

$$1-2d\beta_\tau-\hat{\Pi}_{\beta_\tau}(0;\tau)=0, \tag{2.1}$$

from which it follows that

$$\beta_\tau=s[1-\hat{\Pi}_{\beta_\tau}(0;\tau)]; \tag{2.2}$$

this is proven in [26] for d sufficiently large and for all $0\leq\tau\leq\infty$ and in [16, 15] for $d\geq 5$ and $\tau=\infty$. The identity (2.2) will be used in an iterative fashion to derive an expansion for β_c in powers of s. The quantity $\hat{\Pi}_{\beta_\tau}(k;\tau)$ occurring in the above two equations is related to the *two-point function*. The two-point function is the generating function of the sequence $c_n^{(\tau)}(0,x)$:

$$G_\beta(0,x;\tau)=\sum_{n=0}^{\infty}c_n^{(\tau)}(0,x)\beta^n=\sum_{\omega\in\Omega^{(\tau)}(0,x)}\beta^{|\omega|}, \tag{2.3}$$

where we write $|\omega|=n$ for an n-step walk ω. Let $\hat{f}$ denote the Fourier transform of an absolutely summable function f on $\mathbf{Z}^d$:

$$\hat{f}(k)=\sum_{x\in Z^d}f(x)e^{-ik\cdot x},\quad k=(k_1,\ldots,k_d)\in[-\pi,\pi]^d. \tag{2.4}$$

The inverse transform is given by

$$f(x)=\int_{[-\pi,\pi]^d}\hat{f}(k)e^{ik\cdot x}\frac{d^dk}{(2\pi)^d}. \tag{2.5}$$

Then $\hat{\Pi}_\beta(k;\tau)$ is given implicitly by

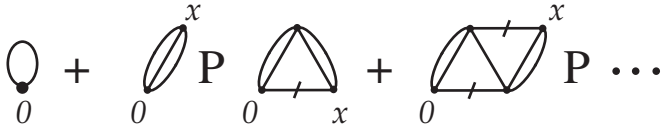
$$\hat{G}_\beta(k;\tau)=\frac{1}{1-2d\beta\hat{D}(k)-\hat{\Pi}_\beta(k;\tau)}, \tag{2.6}$$

with

$$\hat{D}(k)=\frac{1}{2d}\sum_{x\in Z^d:|x|=1}e^{ik\cdot x}=\frac{1}{d}\sum_{j=1}^{d}\cos k_j. \tag{2.7}$$

The function $\Pi_\beta(x;\tau)$ is given by a diagrammatic series

$$\begin{aligned}\Pi_\beta(x;\tau) &= \sum_{N=1}^{\infty}(-1)^N\Pi_\beta^{(N)}(x;\tau)\\ &= -\delta_{0,x}\end{aligned}$$



(2.8)

where each line denotes a summation over memory-τ walks, with some mutual avoidance constraints. Unlabeled vertices in diagrams are summed over $\mathbf{Z}^d$. The series (2.8) has been proved to converge absolutely for $d \geq 5$, $\tau = \infty$, and $0 \leq \beta \leq \beta_c$ [15], and for d sufficiently large, $0 \leq \tau \leq \infty$, and $0 \leq \beta \leq \beta_\tau$ [26]; in these situations it is also summable in x, so that the Fourier transform exists.

To define the diagrams more precisely, we consider first the case of $\tau = \infty$; in this case we drop labels τ. Writing e_i $(1 \leq i \leq d)$ for the ith canonical unit vector in $\mathbf{Z}^d$, the first two diagrams are given by

$$\Pi_\beta^{(1)}(x) = \delta_{0,x} \sum_{e:|e|=1} \beta \sum_{\omega \in \Omega(0,e)} \beta^{|\omega|} = \delta_{0,x}\, 2d\, \beta\, G_\beta(0, e_1) \tag{2.9}$$

and

$$\Pi_\beta^{(2)}(x) = (1 - \delta_{0,x}) \sum_{\omega_1,\omega_2,\omega_3 \in \Omega(0,x)} \beta^{|\omega_1|+|\omega_2|+|\omega_3|}\, \mathrm{I}[A], \tag{2.10}$$

where A is the event that no pair of $\omega_1, \omega_2, \omega_3$ shares a common site other than their endpoints 0 and x, and $\mathrm{I}[A]$ denotes the indicator function of the event A. Higher order diagrams are similarly defined by sums over self-avoiding walks, with some (but not complete) mutual avoidance between the walks. The precise definitions can be found in [2] or [22]. Lines in a diagram which are slashed correspond to walks which may consist of zero steps, while the unslashed lines correspond to nontrivial walks which must take at least one step.

For finite memory τ there are some modifications to the above description of the diagrams, and precise definitions can be found in [2, 22]. For our purposes in proving Theorem 1.1 the precise nature of the constraints involved in the diagram definitions is inessential, apart from the fact that for finite memory τ the N-loop diagram consists of at most $N\tau$ steps (in fact each loop consists of at most N steps), and that diagram configurations occur with the natural lattice symmetry.

2.2 Percolation

For percolation the lace expansion is related to that for the self-avoiding walk, but is more involved. The expansion and results of [14] lead to the following identity, analogous to (2.2), for the critical point:

$$p_c = \frac{s}{1 + \hat{g}(0)}. \tag{2.11}$$

Here $\hat{g}(0) = \sum_{n=1}^{\infty} (-1)^n \hat{g}_n(0)$ is defined in Proposition 2.3 of [14], and is given by a sum of terms which are conveniently represented by diagrams. The diagrams are themselves evaluated at p_c, so that (2.11) can be used iteratively. For percolation, the relevant diagrams are more complex than those for the self-avoiding walk. However one can proceed in a similar fashion to derive terms in an expansion for p_c. The detailed calculations involved in proving Theorem 1.4 are omitted [17]. As a byproduct of the calculations we obtain good estimates on some related quantities, e.g. for the two-point function $\tau_p(x, y)$ (the probability that x is connected to y) we have

$$\tau_{p_c}(0, x) = \begin{cases} s + 2s^2 + O(s^3) & (x = e_1) \\ s^2 + O(s^3) & (x = 2e_1) \\ 2s^2 + O(s^3) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2). \end{cases} \tag{2.12}$$

3 Proof of Theorem 1.1

In [19] it is proven that for each finite even integer $\tau \geq 0$,

$$0 \leq \mu_\tau(d) - \mu(d) \leq O(d^{-\tau/2}) \quad \text{as} \quad d \to \infty. \tag{3.1}$$

Thus to prove Theorem 1.1 it suffices to show that, for each finite even τ, $\mu_\tau(d)$ has an asymptotic expansion to all orders with integer coefficients, and for this is it sufficient to show that $\beta_\tau(d) = \mu_\tau(d)^{-1}$ has such an expansion with first term equal to s. The fact that the first term in the expansion of β_τ has coefficient unity follows e.g. from (1.4) and (3.1).

The proof proceeds by induction on the order of the expansion. That is, we fix an even nonnegative integer τ and assume that there are integers $\alpha_{n,\tau}$ and a positive integer M such that

$$\beta_\tau = \sum_{n=1}^{M-1} \alpha_{n,\tau} s^n + O(s^M). \tag{3.2}$$

We prove below that (3.2) then holds also when M is replaced by $M+1$. This is sufficient, once we observe that (3.2) with $M = 1$ follows from the elementary bounds $d \leq \mu_\tau \leq 2d$.

By (2.2) and (2.8), β_τ satisfies

$$\beta_\tau = s \left[1 - \sum_{N=1}^{\infty} (-1)^N \hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau) \right]. \tag{3.3}$$

The estimates of [26] or [22] can be used to show that there is a constant K independent of τ and (large) d such that

$$\hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau) \leq (Ks)^N; \tag{3.4}$$

in fact the estimates of [26, 22] are somewhat weaker, but can be easily improved to (3.4) by carefully dealing with contributions to diagrams arising from some combination of slashed lines all corresponding to zero-step walks. This gives

$$\beta_\tau = s \left[1 - \sum_{N=1}^{M-1} (-1)^N \hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau) \right] + O(s^{M+1}). \tag{3.5}$$

By definition,

$$\hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau) = \sum_{\ell=1}^{L(N,\tau)} C_{N,d,\tau}(\ell) {\beta_\tau}^\ell, \tag{3.6}$$

where $C_{N,d,\tau}(\ell)$ is the number of ℓ-step configurations of a memory-τ N-loop diagram. Because each loop must have length less than or equal to τ, $L(N,\tau) \leq N\tau$.

By definition, $C_{N,d,\tau}(\ell)$ is a finite nonnegative integer. Since each diagram with ℓ steps corresponds to a sum over random walk configurations having ℓ steps, with the last step constrained to close a loop, $C_{N,d,\tau}(\ell) \leq (2d)^{\ell-1}$. In addition, $C_{N,d,\tau}(\ell)$ is a polynomial in $2d$, with integer coefficients which are independent of d; this can be seen as follows. We begin by defining a mapping φ from the set of all walk configurations contributing to $C_{N,d,\tau}(\ell)$ to itself. Given such a walk ω, $\varphi(\omega)$ is the walk having the same "shape" as ω, obtained by aligning the first step of ω with the e_1 direction, the first step which leaves the x_1 axis with the e_2 direction, the first step which leaves the x_1-x_2

plane with the e_3 direction, and so on. Each resulting walk is inside a subspace of $\mathbf{Z}^d$ of some dimension $\delta(\omega) \le N\tau$. By symmetry, the inverse image under φ of a walk ω in its range will have cardinality equal to the product $(2d)(2d-2)\cdots(2d-2\delta(\omega))$, and hence is a polynomial in $2d$, with integer coefficients and no constant term. Since the size of the range of φ is independent of d, summing these polynomials over the elements of the range of φ then gives a polynomial in $2d$, again with integer coefficients. It follows from the above properties of $C_{N,d,\tau}(\ell)$ that we can write

$$C_{N,d,\tau}(\ell) = \sum_{q=1}^{\ell-1} a_{N,q,\ell,\tau}(2d)^q \tag{3.7}$$

with integer coefficients $a_{N,q,\ell,\tau}$ which are independent of d.

By (3.2),

$$\beta_\tau^\ell = \left[\sum_{n=1}^{M-1} \alpha_{n,\tau} s^n + O(s^M)\right]^\ell = s^\ell \left[\sum_{n=0}^{M-2} \gamma_{n,\ell,\tau} s^n + O(s^{M-1})\right] \tag{3.8}$$

with integer coefficients $\gamma_{n,\ell,\tau}$. With (3.7) and (3.6), this gives

$$\hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau) = \sum_{\ell=1}^{L(N,\tau)} \sum_{q=1}^{\ell-1} a_{N,q,\ell,\tau} s^{\ell-q} \left[\sum_{n=0}^{M-2} \gamma_{n,\ell,\tau} s^n + O(s^{M-1})\right]. \tag{3.9}$$

The only term in the above product which can give rise to an non-integral power or coefficient is the $O(s^{M-1})$ term. However this term is multiplied by $s^{\ell-q} \le O(s)$, and hence gives rise to a contribution which is $O(s^M)$. Therefore

$$\hat{\Pi}_{\beta_\tau}^{(N)}(0;\tau) = \sum_{n=1}^{M-1} b_{N,n,\tau} s^n + O(s^M), \tag{3.10}$$

with integer $b_{N,n,\tau}$. Substitution of (3.10) into (3.5) then gives

$$\beta_\tau = \sum_{n=1}^{M} b'_{n,\tau} s^n + O(s^{M+1}), \tag{3.11}$$

with integer $b'_{n,\tau}$ (which must of course agree with $\alpha_{n,\tau}$ for $n \le M-1$). This gives (3.2) with M replaced by $M+1$, and the proof is complete.

4 Proof of Theorem 1.2

In this section we consider only the case of $\tau = \infty$ and drop τ from the notation. From (2.2), to prove (1.8) it is sufficient to obtain the expansion for $\hat{\Pi}_{\beta_c}(0)$ to an error of order $O(s^6)$. In particular, it suffices to prove that

$$\hat{\Pi}_{\beta_c}^{(1)}(0) = s + 3s^2 + 11s^3 + 52s^4 + 304s^5 + O(s^6) \tag{4.1}$$

$$\hat{\Pi}_{\beta_c}^{(2)}(0) = s^2 + 6s^3 + 33s^4 + 200s^5 + O(s^6) \tag{4.2}$$

$$\hat{\Pi}_{\beta_c}^{(3)}(0) = s^3 + 9s^4 + 64s^5 + O(s^6) \tag{4.3}$$

$$\hat{\Pi}_{\beta_c}^{(4)}(0) = s^4 + 12s^5 + O(s^6) \tag{4.4}$$

$$\hat{\Pi}_{\beta_c}^{(5)}(0) = s^5 + O(s^6) \tag{4.5}$$

$$\sum_{N=6}^{\infty} \hat{\Pi}_{\beta_c}^{(N)}(0) = O(s^6). \tag{4.6}$$

Because of the repetitive nature and increasing complexity of the calculations required, here we explicitly prove (4.1)–(4.6) only to error estimates of order $O(s^5)$. The necessary calculations to go to the next order are similar but more tedious [17], and are omitted.

We will prove (4.1)–(4.6) to an error of order $O(s^5)$ by using two basic estimates derived from [22, 26], together with some elementary but detailed estimates on diagrams as used in [15]. We will make use of the integrals

$$I_{n,m}(x) \;=\; \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^m e^{ik\cdot x}}{[1-\hat{D}(k)]^n} \frac{d^dk}{(2\pi)^d} \tag{4.7}$$

and of their $1/d$-expansions; these expansions are given in Section A.2.

Using (2.6) and (2.1), we obtain

$$\hat{G}_{\beta_c}(k) = \frac{1}{1-2d\beta_c\hat{D}(k)-\hat{\Pi}_{\beta_c}(k)} = \frac{1}{2d\beta_c[1-\hat{D}(k)]+\hat{\Pi}_{\beta_c}(0)-\hat{\Pi}_{\beta_c}(k)}. \tag{4.8}$$

The quantity $\hat{\Pi}_{\beta_c}(0)-\hat{\Pi}_{\beta_c}(k)$ should be regarded as a small perturbation of the Gaussian term $1-\hat{D}(k)$.

For the remainder of this section we shall drop all subscripts β_c to simplify the notation, since henceforth all β-dependent quantities are evaluated at $\beta=\beta_c$.

4.1 The basic strategy

The identity (4.8) relates the two-point function on the left side and various diagrams on the right side, with the diagrams themselves capable of being bounded in terms of the two-point function. Estimates will be derived to increasingly high accuracy by iteration. In principle this can be carried out to arbitrarily high order, although the amount of labour involved increases with the order. An intermediate quantity in the iteration is

$$B(x) = \sum_{y\neq 0,x} G(0,y)G(y,x). \tag{4.9}$$

In more detail, the basic strategy is to perform the following two procedures A and B repeatedly, alternating between the two, until the desired order of the $1/d$-expansion is obtained. We will perform three iterations below to obtain (4.1)–(4.6) to within an error $O(s^5)$. One additional iteration gives a further order of accuracy, yielding (4.1)–(4.6).

A Given estimates on $G(0,x)$ and $B(x)$, derive estimates on the diagrams in $\hat{\Pi}(k)$. From these obtain an estimate on β_c.

B Given estimates on $\hat{\Pi}(k)$ and β_c, derive estimates on $G(0,x)$ and $B(x)$.

4.2 First iteration: A

We begin with the following estimates, which can be proved using the methods of [26] or [22]. For some constant K independent of d,

$$\left|\hat{\Pi}(0)-\hat{\Pi}(k)\right| \leq O(s)[1-\hat{D}(k)], \tag{4.10}$$

$$\sum_x \Pi^{(N)}(x) \le (Ks)^N, \quad (N = 1, 2, 3, \ldots). \tag{4.11}$$

To an error of order $O(s^5)$, (4.5) and (4.6) follow immediately from (4.11). With (2.2) and (2.8), (4.11) gives a first-order bound on β_c:

$$2d\beta_c = 1 - \sum_{N=1}^{\infty} (-1)^N \hat{\Pi}^{(N)}(0) = 1 + O(s). \tag{4.12}$$

4.3 First iteration: B

Here we prove the following bounds:

$$\beta_c = s + s^2 + O(s^3). \tag{4.13}$$

$$G(0,x) = \begin{cases} s + 2s^2 + O(s^3) & (|x| = 1) \\ s^2 + O(s^3) & (x = 2e_1) \\ 2s^2 + O(s^3) & (x = e_1 + e_2) \\ O(s^3) & (|x| > 2) \end{cases} \tag{4.14}$$

$$B(x) = \begin{cases} s + 6s^2 + O(s^3) & (x = 0) \\ O(s^2) & (x \neq 0) \end{cases} \tag{4.15}$$

$$\sum_{|x|>2} G(0,x)^2 = O(s^3). \tag{4.16}$$

Defining $\alpha = 2d\beta_c$ and $\Delta(k) = \hat{\Pi}(0) - \hat{\Pi}(k)$, by (4.8) we have

$$\begin{aligned} G(0,x) &= \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik\cdot x}}{2d\beta_c[1 - \hat{D}(k)] + \Delta(k)} \\ &= \int \frac{d^dk}{(2\pi)^d} \left[\frac{e^{ik\cdot x}}{\alpha[1 - \hat{D}(k)]} - \frac{e^{ik\cdot x}\Delta(k)}{\{\alpha[1 - \hat{D}(k)]\}\{\alpha[1 - \hat{D}(k)] + \Delta(k)\}} \right]. \end{aligned} \tag{4.17}$$

The first term is $\alpha^{-1} I_{1,0}(x)$. The second term is estimated using the following lemma, whose proof is deferred to Section A.1. The lemma is stated in a more general form than is needed here, for later use.

Lemma 4.1 *Let f be a real function on $\mathbf{Z}^d$ which respects lattice symmetry, satisfies*

$$|\hat{f}(0) - \hat{f}(k)| \le \frac{1}{2}[1 - \hat{D}(k)], \quad \textit{for} \quad k \in [-\pi, \pi]^d, \tag{4.18}$$

and for which there exists a (small) positive ϵ such that

$$\sum_{x \neq 0} |f(x)| \le \epsilon. \tag{4.19}$$

Then for any integer $n \ge 1$ and for $d \ge 2n + 1$, $\alpha \ge 1$,

$$\int \frac{d^dk}{(2\pi)^d} \frac{[\hat{f}(0) - \hat{f}(k)]e^{ik\cdot x}}{\{\alpha[1 - \hat{D}(k)]\}^{n-1}\{\alpha[1 - \hat{D}(k)] + \hat{f}(0) - \hat{f}(k)\}} = O\left(s\epsilon + \epsilon^2\right) + \frac{\delta_{0,x}}{\alpha^n} \sum_{y \neq 0} f(y). \tag{4.20}$$

Here $O(A)$ denotes a term bounded in absolute value by KA, where K is a positive constant independent of ϵ, d, α (but dependent on n), and as usual $s = \frac{1}{2d}$.

We now apply Lemma 4.1, with $f(x) = \Pi(x)$ and $\alpha = 2d\beta_c \geq 1$, to the second term of (4.17). By (4.11) and the fact that $\Pi^{(1)}(x) = 0$ for $x \neq 0$, we have $\sum_{x\neq 0} |\Pi(x)| \leq O(s^2)$, and hence can take $\epsilon = O(s^2)$. Recalling the definition of $I_{n,m}(x)$ in (4.7), the result is

$$G(0,x) = \frac{1}{2d\beta_c} I_{1,0}(x) + O(s^3) - \frac{\delta_{x,0}}{(2d\beta_c)^2} \sum_{y\neq 0} \Pi(y). \tag{4.21}$$

Taking $x = e_1$ in (4.21), and using (4.11) and (A.9), gives the improved bound (4.13) on β_c, since by (2.2) and (2.9),

$$2d\beta_c = 1 + 2d\beta_c G(0,e_1) - \sum_{N\geq 2}^{\infty} (-1)^N \hat{\Pi}^{(N)}(0). \tag{4.22}$$

In addition, (4.14) follows from the improved estimate (4.13) on β_c, together with (A.9)–(A.11) and (A.16).

Proceeding now to $B(x)$, we begin with the related quantity $B''(x) \equiv \sum_y G(0,y)G(y,x) = B(x) + 2G(0,x) - \delta_{x,0}$. Writing $\alpha = 2d\beta_c$ and $\Delta(k) = \hat{\Pi}(0) - \hat{\Pi}(k)$, and using (4.17) and the fact that the Fourier transform of a convolution is the product of Fourier transforms, we have

$$\begin{aligned} B''(x) &= \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \left[\frac{1}{\alpha[1-\hat{D}(k)] + \Delta(k)} \right]^2 \\ &= \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} \left[\frac{1}{\alpha[1-\hat{D}(k)]} - \frac{\Delta(k)}{\{\alpha[1-\hat{D}(k)]\}\{\alpha[1-\hat{D}(k)] + \Delta(k)\}} \right]^2 . \end{aligned} \tag{4.23}$$

After expanding the square, the first term gives $\alpha^{-2} I_{2,0}(x)$. The cross term can be bounded using Lemma 4.1, and the final term simply by taking absolute values. The result is

$$B''(x) = \frac{1}{(2d\beta_c)^2} I_{2,0}(x) + O(s^3) - \delta_{x,0} \frac{2}{(2d\beta_c)^3} \sum_{y\neq 0} \Pi(y). \tag{4.24}$$

With (4.21) , this leads to

$$\begin{aligned} B(x) &= \frac{1}{(2d\beta_c)^2} \left[I_{2,0}(x) - 2(2d\beta_c) I_{1,0}(x) + (2d\beta_c)^2 \delta_{x,0} + O(s^3) \right] \\ &= \frac{1}{(2d\beta_c)^2} \left[I_{2,0}(x) - \delta_{x,0} - 2(2d\beta_c) \left\{ I_{1,0}(x) - \delta_{x,0} \right\} + (2d\beta_c - 1)^2 \delta_{x,0} + O(s^3) \right] . \end{aligned} \tag{4.25}$$

[Here the terms involving $\sum_{y\neq 0} \Pi(y)$ in (4.21) and (4.24) have almost canceled, with difference $O(s^3)$.] Appealing to (4.13) and (A.8)–(A.16), we obtain the desired estimates (4.15).

Finally, we note that by definition

$$\sum_{|x|>2} G(0,x)^2 = B(0) - 2dG(0,e_1)^2 - 2d(d-1)G(0,e_1+e_2)^2 - 2dG(0,2e_1)^2. \tag{4.26}$$

Using (4.15) and (4.14) gives (4.16).

4.4 Second iteration: A

We use the results of Step B of the first iteration to prove the following:

$$\hat{\Pi}^{(n)}(0) = \begin{cases} s^2 + 6s^3 + O(s^4) & (n=2) \\ s^3 + 9s^4 + O(s^5) & (n=3) \\ s^4 + O(s^5) & (n=4) \end{cases} \tag{4.27}$$

$$\hat{\Pi}^{(2)}(0) = 2d\,\Pi^{(2)}(e_1) + O(s^5), \tag{4.28}$$

and

$$2d\beta_c = 1 + s + 2s^2 + O(s^3). \tag{4.29}$$

In particular, this gives (4.3) and (4.4) to within an error $O(s^5)$, and we are progressing towards (4.2). We also prove

$$\sum_{n=2}^{\infty} (-1)^n \Pi^{(n)}(x) = (s^3 + 6s^4)\mathrm{I}[|x| = 1] + h(x), \quad (x \neq 0) \tag{4.30}$$

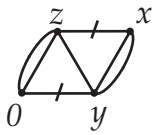
with

$$\sum_{x \neq 0} |h(x)| = O(s^4). \tag{4.31}$$

4.4.1 Four-loop diagram

The four-loop diagram can be represented as follows, with the two slashed lines possibly of length zero:

$$\Pi^{(4)}(x) = \sum_{y,z: y \neq x,\, y \neq z,\, z \neq 0} \tag{4.32}$$



There is some mutual avoidance between the various lines of the diagram, but this will not be relevant for our bounds.

We will estimate this diagram by applying the simple inequality

$$\sum_x f(x)g(x) \leq \left[\sup_x f(x)\right] \left[\sum_x g(x)\right] \tag{4.33}$$

for $f(x), g(x) \geq 0$. We define $B'(x) = B(x) + G(0,x) = \sum_{y \neq 0} G(0,y)G(y,x)$ for $x \neq 0$, and also

$$A_1(z) = \sum_{y: y \neq z} G(z,y)^2 G(y,0), \quad A_2(z) = \sum_{x \neq 0} G(0,z)G(z,x)G(0,x)^2. \tag{4.34}$$

Overcounting by neglecting mutual avoidances between the lines in the diagram (4.32) (with the role of 0 and y interchanged), and then applying (4.33), gives

$$\sum_x \Pi^{(4)}(x) \leq \sum_{z \neq 0} A_1(z)A_2(z) \leq \left[\sup_{z \neq 0} A_1(z)\right] \left[\sum_{z \neq 0} A_2(z)\right]. \tag{4.35}$$

Further application of (4.33) to the two factors on the right side of the above equation, together with (4.14), gives

$$\sum_x \Pi^{(4)}(x) \le \left[\sup_{x\neq 0} G(0,x)\right] \left[\sup_{x\neq 0} B'(x)\right]^2 B(0) = s^4 + O(s^5). \tag{4.36}$$

For a lower bound, we consider only the simplest contribution to $\Pi^{(4)}(e_1)$, in which $x = z = e_1$, $y = 0$, due to the 5-step walk which steps repeatedly back and forth from the origin to e_1, and obtain

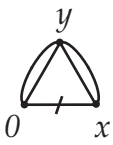
$$\Pi^{(4)}(e_1) \ge {\beta_c}^5 = s^5 + O(s^6).$$

Combined with the upper bound (4.36), this gives the last bound of (4.27):

$$\sum_x \Pi^{(4)}(x) = s^4 + O(s^5). \tag{4.37}$$

4.4.2 Three-loop diagram

The three-loop diagram is given by

$$\Pi^{(3)}(x) = \sum_{y: y\neq 0, x}$$ 

. (4.38)

In this diagram the walks corresponding to each line are mutually avoiding apart from their common endpoints, except for the fact that the two lines joining 0 and y can intersect one of the lines joining y to x (say the one on the right). For the sum over x and y, we consider separately the two cases where $x = 0$ and $|y| = |x - y| = 1$.
Case (1): When $x = 0$, we have from (4.14) and (4.16) that

$$\Pi^{(3)}(0) \le \sum_{y\neq 0} G(0,y)^4 \le 2dG(0,e_1)^4 + \left[\sup_{|y|>1} G(0,y)\right]^2 B(0) = s^3 + 8s^4 + O(s^5). \tag{4.39}$$

For a lower bound we count the 4-step and 6-step contributions only, obtaining

$$\Pi^{(3)}(0) \ge 2d[{\beta_c}^4 + 4(2d-2){\beta_c}^6] = s^3 + 8s^4 + O(s^5). \tag{4.40}$$

Case (2): When $|y| = |x - y| = 1$, we have by symmetry $2d(2d-2)$ contributions with $y = e_1$ and $x = e_1 + e_2$, and $2d$ contributions with $y = e_1$ and $x = 2e_1$. Therefore, using (4.14), this contribution to $\sum_x \Pi^{(3)}(x)$ is bounded above by

$$2d(2d-2)G(0,e_1)^4\{G(0,e_1+e_2) - {\beta_c}^2\} + 2dG(0,e_1)^4 G(0,2e_1) = s^4 + O(s^5), \tag{4.41}$$

and is bounded below by

$$2d(2d-2){\beta_c}^6 = s^4 + O(s^5). \tag{4.42}$$

In (4.41), $G(0,e_1+e_2) - {\beta_c}^2$ appears rather than $G(0,e_1+e_2)$, due to the mutual-avoidance constraint inherent in the diagram, which in particular prohibits the walk joining 0 to $e_1 + e_2$ from passing through $y = e_1$.

Case (3): The other contributions to the diagram, arising from $x \neq 0$ and $|y| \geq 1, |x-y| > 1$ or $|y| > 1, |x-y| \geq 1$, can be bounded above as in Section 4.4.1 by

$$2 \left[\sup_{|y|>1} G(0,y) \right] \left[\sup_{u \neq 0} B(u) \right] B(0) \leq O(s^5). \tag{4.43}$$

using (4.14) and (4.15). To summarize, we have proved the bound on the three-loop diagram given in (4.27), and also

$$\sum_{x \neq 0} \Pi^{(3)}(x) = s^4 + O(s^5). \tag{4.44}$$

4.4.3 Two-loop diagram

The two-loop diagram is defined in (2.10). We have

$$\hat{\Pi}^{(2)}(0) = 2d\Pi^{(2)}(e_1) + 2d\Pi^{(2)}(2e_1) + 2d(d-1)\Pi^{(2)}(e_1+e_2) + \sum_{|x|>2} \Pi^{(2)}(x). \tag{4.45}$$

By (4.14) and (4.16),

$$\sum_{|x|>2} \Pi^{(2)}(x) \leq \sum_{|x|>2} G(0,x)^3 \leq \left[\sup_{|x|>2} G(0,x) \right] \sum_{|x|>2} G(0,x)^2 \leq O(s^6). \tag{4.46}$$

Since the three diagram lines correspond to mutually-avoiding walks, we have from (4.14) and (4.13) that

$$\Pi^{(2)}(2e_1) \leq G(0,2e_1)^3 - {\beta_c}^6 = O(s^7), \tag{4.47}$$

$$\Pi^{(2)}(e_1+e_2) \leq G(0,e_1+e_2)^3 - 8{\beta_c}^6 = O(s^7). \tag{4.48}$$

This gives (4.28):

$$\hat{\Pi}^{(2)}(0) = 2d\Pi^{(2)}(e_1) + O(s^5). \tag{4.49}$$

Also,

$$\Pi^{(2)}(e_1) \leq G(0,e_1)^3 = s^3 + 6s^4 + O(s^5) \tag{4.50}$$

and

$$\Pi^{(2)}(e_1) \geq {\beta_c}^3 + 3[G(0,e_1) - \beta_c]{\beta_c}^2 = s^3 + 6s^4 + O(s^5), \tag{4.51}$$

where we have used (4.14) and (4.13). We thus have (4.27) for $n = 2$, and

$$\Pi^{(2)}(x) = (s^3 + 6s^4)\mathrm{I}[|x| = 1] + O(s^5). \tag{4.52}$$

Using (4.44), (4.37) and (4.11), this completes the proof of (4.27)–(4.31), apart from (4.29) which we prove next.

4.4.4 Bound on β_c

By (4.27) and (4.11),

$$\sum_{n=2}^{\infty} (-1)^n \hat{\Pi}^{(n)}(0) = s^2 + O(s^3), \tag{4.53}$$

and by (4.13) and (4.14),

$$\hat{\Pi}^{(1)}(0) = 2d\beta_c G(0,e_1) = s + 3s^2 + O(s^3). \tag{4.54}$$

Hence (4.29) follows from the first equality of (4.12).

4.5 Second Iteration: B

For our present needs it suffices to estimate the one-loop diagram $2d\beta_c G(0,e_1)$ in more detail. By (4.10), (4.30) and (4.31), we can write

$$2d\beta_c[1-\hat{D}(k)]+\hat{\Pi}(0)-\hat{\Pi}(k)=(2d\beta_c+s^2+6s^3)[1-\hat{D}(k)]+\hat{h}(0)-\hat{h}(k), \quad (4.55)$$

with $\sum_{y\neq 0}|h(y)|=O(s^4)$ and $|\hat{h}(0)-\hat{h}(k)|\leq O(s)[1-\hat{D}(k)]$. Defining $f(x)=h(x)$, $\alpha=2d\beta_c+s^2+6s^3$ and $\Delta(k)=\hat{h}(0)-\hat{h}(k)$, we apply Lemma 4.1 to the second term of

$$\begin{aligned} G(0,x) &= \int \frac{d^dk}{(2\pi)^d}e^{ik\cdot x}\frac{1}{\alpha[1-\hat{D}(k)]+\Delta(k)} \\ &= \int \frac{d^dk}{(2\pi)^d}\left[\frac{e^{ik\cdot x}}{\alpha[1-\hat{D}(k)]}-\frac{\Delta(k)e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}}\right]. \end{aligned} \quad (4.56)$$

This gives

$$G(0,x)=\alpha^{-1}I_{1,0}(x)+O(s^5), \quad (x\neq 0). \quad (4.57)$$

Combined with (4.29) and (A.9), this gives

$$G(0,e_1)=s+2s^2+7s^3+O(s^4). \quad (4.58)$$

Moreover keeping $2d\beta_c$ in α explicitly, we have

$$2d\beta_c G(0,e_1)=I_{1,0}(e_1)-(s^2+6s^3)G(0,e_1)+O(s^5)=s+3s^2+11s^3+52s^4+O(s^5), \quad (4.59)$$

which proves (4.1) to error $O(s^5)$.

4.6 Third Iteration: A

To complete the proof of Theorem 1.2 to $O(s^5)$, we need to prove (4.2) for $\hat{\Pi}^{(2)}(0)$ to this order. By (4.28), it suffices to show that

$$\Pi^{(2)}(e_1)=s^3+6s^4+33s^5+O(s^6). \quad (4.60)$$

For an upper bound, using (4.58) we have

$$\Pi^{(2)}(e_1)\leq G(0,e_1)^3=s^3[1+6s+33s^2]+O(s^6). \quad (4.61)$$

For a lower bound, we just consider low order terms and use (4.29) and (4.58) to obtain

$$\Pi^{(2)}(e_1)\geq {\beta_c}^3+3\{G(0,e_1)-\beta_c\}{\beta_c}^2+3(2d-2)(2d-3){\beta_c}^7=s^3[1+6s+33s^2]+O(s^6). \quad (4.62)$$

This completes the proof of Theorem 1.2, to error $O(s^5)$.

A Appendix

A.1 Proof of Lemma 4.1

Throughout the proof, we write $\Delta(k)\equiv\hat{f}(0)-\hat{f}(k)$. Note that the assumption (4.19) on f implies that

$$|\Delta(k)|=|\hat{f}(0)-\hat{f}(k)|\leq 2\sum_{y\neq 0}|f(y)|\leq 2\epsilon. \quad (A.1)$$

We begin by reducing the proof to the case where $\hat{f} \equiv 0$ in the denominator of the integral (4.20). For this it suffices to show that the difference between the two integrals, one having $\hat{f} \equiv 0$ in the denominator and the other $\hat{f} \neq 0$ in the denominator, is $O(\epsilon^2)$. But this difference is

$$\left| \int \frac{d^d k}{(2\pi)^d} \frac{\Delta(k) e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n-1}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}} - \int \frac{d^d k}{(2\pi)^d} \frac{\Delta(k) e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n}} \right|$$

$$\begin{aligned} &= \left| \int \frac{d^d k}{(2\pi)^d} \frac{[\Delta(k)]^2 e^{ik\cdot x}}{\{\alpha[1-\hat{D}(k)]\}^{n}\{\alpha[1-\hat{D}(k)]+\Delta(k)\}} \right| \\ &\leq 4\epsilon^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{\{[1-\hat{D}(k)]\}^{n}\{\frac{1}{2}[1-\hat{D}(k)]\}} = 8\epsilon^2 I_{n+1,0}(0) = O(\epsilon^2), \end{aligned} \tag{A.2}$$

using $\alpha \geq 1$, (4.18) and (A.1), and in the last step (A.17).

To prove (4.20) for the case $\Delta \equiv 0$ in the denominator, we for the moment omit α^{-n} and write

$$\int \frac{d^d k}{(2\pi)^d} \frac{\Delta(k) e^{ik\cdot x}}{[1-\hat{D}(k)]^n} = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \Delta(k) \left[1 + \hat{D}(k) + \frac{\hat{D}(k)^2}{1-\hat{D}(k)} \right]^n \tag{A.3}$$

and expand the nth power. Explicitly extracting the zeroth and first powers of $\hat{D}(k)$ in the last factor, (A.3) is equal to

$$\int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \Delta(k) \left[1 + n\hat{D}(k) + \frac{P(\hat{D}(k))\hat{D}(k)^2}{[1-\hat{D}(k)]^n} \right] \tag{A.4}$$

for some polynomial P depending only on n. Setting $P_{max} = \max_{-1\leq x\leq 1} |P(x)|$, and using (A.1) and (A.17), we have

$$\left| \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \Delta(k) \frac{P(\hat{D}(k))\hat{D}(k)^2}{[1-\hat{D}(k)]^n} \right| \leq 2\epsilon P_{max} \int \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^n} = O(s\epsilon). \tag{A.5}$$

In addition, explicit calculation gives

$$\int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \Delta(k) \left[1 + n\hat{D}(k) \right] = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} [\hat{f}(0) - \hat{f}(k)] \left[1 + \frac{n}{2d} \sum_{|u|=1} e^{-ik\cdot u} \right]$$

$$= \mathrm{I}[x=0] \sum_{y\neq 0} f(y) - \mathrm{I}[x\neq 0] f(x) + \frac{n}{2d} \mathrm{I}[|x|=1] \sum_{y\neq 0} f(y) - \frac{n}{2d} \sum_{|e|=1} \mathrm{I}[x\neq e] f(x-e). \tag{A.6}$$

In the last equality we made use of the identity

$$\hat{f}(0) - \hat{f}(k) = \sum_{y\neq 0} \left[1 - e^{-ik\cdot y} \right] f(y).$$

The lemma then follows, using the facts that $\alpha \geq 1$, $f(x) \leq \frac{1}{2d} \sum_{y\neq 0} f(y) \leq \epsilon s$ for $x \neq 0$, and $|\sum_{|e|=1} \mathrm{I}[x\neq e] f(x-e)| \leq \sum_{y\neq 0} |f(y)| \leq \epsilon$.

A.2 $1/d$ expansions for $I_{m,n}(x)$

Expansions for the integrals

$$I_{n,m}(x) \;=\; \int_{[-\pi,\pi]^d} \frac{\hat{D}(k)^m e^{ik\cdot x}}{[1-\hat{D}(k)]^n} \frac{d^d k}{(2\pi)^d} \tag{A.7}$$

in powers of $1/d$ can be derived using the method described in the Appendix to [18], yielding

$$\begin{aligned}
I_{1,0}(0) &= 1 + s + 3s^2 + 12s^3 + 60s^4 + 355s^5 + O(s^6) && \text{(A.8)}\\
I_{1,0}(e_1) &= s + 3s^2 + 12s^3 + 60s^4 + 355s^5 + O(s^6) && \text{(A.9)}\\
I_{1,0}(2e_1) &= s^2 + 6s^3 + 37s^4 + 255s^5 + O(s^6) && \text{(A.10)}\\
I_{1,0}(e_1+e_2) &= 2s^2 + 12s^3 + 66s^4 + 390s^5 + O(s^6) && \text{(A.11)}\\
I_{2,0}(0) &= 1 + 3s + 15s^2 + 90s^3 + 630s^4 + 5005s^5 + O(s^6) && \text{(A.12)}\\
I_{2,0}(e_1) &= 2s + 12s^2 + 78s^3 + 570s^4 + 4650s^5 + O(s^6) && \text{(A.13)}\\
I_{2,0}(2e_1) &= 3s^2 + 30s^3 + 275s^4 - 735s^5 + O(s^6) && \text{(A.14)}\\
I_{2,0}(e_1+e_2) &= 6s^2 + 60s^3 + 510s^4 - 2310s^5 + O(s^6) && \text{(A.15)}\\
\sup_{|x|>2} I_{n,0}(x) &\le O(s^3) \quad (n=1,2) && \text{(A.16)}\\
\sup_{x\in Z^d} I_{n,2m}(x) &\le O(s^m) \quad (m,n = 0,1,2,\ldots). && \text{(A.17)}
\end{aligned}$$

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