Entropy Density for Relativistic Quantum Field Theory

Heide Narnhofer
Institut für Theoretische Physik
Universität Wien

Dedicated to Elliott H. Lieb
to his 60th birthday

Abstract

We show how the nuclearity condition of Buchholz and Wichmann allows to define in the ground state a local entropy with the desired properties despite the fact that local algebras are type III. Generalization to temperature states is also possible so that thermodynamic functions also exist in the context of relativistic quantum field theory.

1 Introduction

Since we know that we have to assign a temperature to the background of the universe [PW] and we let this temperature change in time in cosmological theories it is evident that we should not restrict ourselves to consider rigorously only the ground state of relativistic quantum fields.

Temperature states are either defined as Gibbs states. This only works for finite systems, if the Hamiltonian has a discrete and sufficiently increasing spectrum. In the thermodynamic limit we pass to the KMS condition as requirement for equilibrium states [Ku,MS,HHW]. That this requirement is the relevant one is justified by stability considerations [HKTP,PuW]. But the first path that led to this description was a variational problem: KMS states are tangent functionals to the free energy density [LR].

In [BJu] the KMS condition was taken to be the requirement to be satisfied by equilibrium states for relativistic theories. Buchholz and Junglas found an approximation procedure so that the weak limit state satisfies the KMS condition. But additional requirements – the nuclearity – are necessary. They essentially replace the condition for Gibbs states, that local Hamiltonians must have discrete spectrum. What is missing in this approach is a feeling, how phase transitions can occur. Here the description of an equilibrium state as tangent plane of a convex functional seems more promising. But to do so, we must find a reasonable definition of this functional as energy density minus entropy density.

In this paper we will concentrate on the problems and the possibility to define an entropy density and we will notice that as in the construction of KMS states the nuclearity condition is the key property that allows to estimate local entropy and entropy density.

2 Local Algebras and Their Entropy

If we want to get an entropy density corresponding to a state, we have to assign an entropy to a local algebra$\mathcal{A}_\Lambda$ with appropriate convexity properties on the state. The entropy density for a space translationally invariant state is given by $\lim S(\mathcal{A}_\Lambda,\omega)/|\Lambda|$. The minimal requirement on $S(\mathcal{A}_\Lambda,\omega)$ is that this procedure works for the ground state and there gives an entropy density 0.

The candidate for a local algebra $\mathcal{A}_\Lambda$ is the algebra over the double cone $\mathcal{O}$ with $\mathcal{O}\cap\mathbf{R}^3=\Lambda$. In the example of a free scalar field theory we have

$$\mathcal{A}_\Lambda=\{W(f)|f\in\mathcal{L}(\mathcal{O})\}'' \tag{2.1}$$

with

$$W(f)=\exp\{i[a^\dagger(f)+a(f)]\}$$
$$\mathcal{L}(\mathcal{O})=\frac{1}{\sqrt{\omega(p)}}\mathcal{D}(\Lambda)+i\sqrt{\omega(p)}\,\mathcal{D}(\Lambda)$$

with $\mathcal{D}(\Lambda)$ the space of real test functions with support in Λ, the basis of $\mathcal{O}$. A general quasifree state can be written as

$$\omega(W(f))=\exp[-\langle f|A|f\rangle] \tag{2.2}$$

A an operator on $\mathcal{L}(\mathbf{R}^3)$. If we calculate the entropy of the vacuum state restricted to $\mathcal{A}_\Lambda$, which is determined by P_0AP_0, P_0 the projection onto $\mathcal{L}(\mathcal{O})$, then it turns out to be infinite [A]. We can become more concrete: [BiW] showed that the algebra of the wedge has as modular automorphism group for the vacuum state – which must exist according to the Reeh-Schlieder theorem – the boost. Boosts are evidently weakly asymptotically abelian, therefore the spectrum of the modular operator Δ is $\mathbf{R}_+$ and the algebra of the wedge is of type III_1 [D,C1]. In [H,L] the authors managed to construct a map from the algebra over the wedge onto the algebra over a double cone, provided the system is conformally invariant. Thus also the algebra of double cones is for conformally invariant theories of type III_1. Finally [F] took into account that a reasonable quantum field theory should have a scaling limit and since in this scaling limit the edge of a double cone looks like the edge of a wedge he showed that the assumption on the scaling limit guarantees that the spectrum of the modular operator for the double cone coincides with that of the wedge so that the algebra of the double cone is again type III_1.

As a consequence we run into problems if we want to calculate the entropy of a state restricted to a local algebra, since no corresponding density matrix is available.

But we can use the alternative definition [NT,CNT] that was shown to coincide for type I algebras with the usual definition. It uses the relative entropy $S(\omega|\nu)$ which can be defined for arbitrary states ω, ν over C^*-algebras [Ko].

Definition: Let ω be a state over $\mathcal{A}$. Then

$$S_{\mathcal{A}}(\omega) = \sup_{\omega=\sum \omega_i, \widehat{\omega}_i=\omega_i/(\omega_i(1))} \sum \omega_i(1) S(\omega|\widehat{\omega}_i)_{\mathcal{A}} \tag{2.3}$$

where ω is decomposed into states $\widehat{\omega}_i$.

Lemma (2.4): For every hyperfinite type III_1 algebra $\mathcal{A}$ $S_{\mathcal{A}}(\omega) = \infty$ [OP, Lemma 6.9].

Proof: We consider the realization of the hyperfinite III_1 algebra [C1]

$$\mathcal{A} = \bigotimes_{i=1}^{\infty} M_i^3 \qquad M_i \ 3 \times 3 \text{ matrices}$$

with the state ω given as the limit $N \to \infty$ of local states corresponding to density matrices

$$\rho_N = \bigotimes_{i=1}^{N} \begin{pmatrix} e^{-\lambda_1} & & \\ & e^{-\lambda_2} & \\ & & e^{-\lambda_3} \end{pmatrix}$$

where λ_1, λ_2, λ_3 are incommensurable in the sense that

$$\text{spec } \ln \Delta = \overline{\{z_1(\lambda_1 - \lambda_2) + z_2(\lambda_2 - \lambda_3) + z_3(\lambda_3 - \lambda_1); z_i \in Z\}} = \mathbf{R}.$$

For this algebra and this state

$$\mathcal{M}_0 = \bigotimes_{i=0}^{\infty} \begin{pmatrix} a_{i1} & & \\ & a_{i2} & \\ & & a_{i3} \end{pmatrix}$$

form the centralizer. We can use elements of $\mathcal{M}_0$ to give a decomposition

$$S_{\mathcal{A}}(\omega) \geq S_{\mathcal{M}_0}(\omega) = \lim_{N\to\infty} N \cdot \sum_{i=1}^{3} \lambda_i e^{-\lambda_i} = \infty.$$

For general states φ we can use the homogeneity of type III_1, i.e. the following [CS,N]: Given two states ω, φ. Then there exist sequences of unitaries in $\mathcal{A}$, $\{U_n\}$ and $\{V_n\}$ such that

$$\begin{aligned} \omega(\cdot) &= n \lim \varphi(U_n \cdot U_n^*) \\ \varphi(\cdot) &= n \lim \omega(V_n \cdot V_n^*). \end{aligned}$$

We pick a decomposition of unity in $\mathcal{A}'$, $1 = \sum x_i$, $0 \leq x_i \leq 1$, that approaches the optimal decomposition for ω with $\omega_i(\cdot) = \langle \Omega | x_i \cdot | \Omega \rangle$. We denote $\varphi_i(\cdot) = \langle \Phi | x_i \cdot | \Phi \rangle$. Then

$$\begin{aligned} S(\omega|\omega_i)_{\mathcal{A}} &= S(\omega(V_n \cdot V_n^*)|\omega_i(V_n \cdot V_n^*))_{\mathcal{A}} \geq S(\varphi|\varphi_i)_{\mathcal{A}} \\ &= S(\varphi(U_n \cdot U_n^*)|\varphi_i(U_n \cdot U_n^*))_{\mathcal{A}} \geq S(\omega|\omega_i)_{\mathcal{A}} \end{aligned}$$

where we used the semicontinuity of the relative entropy following from [Ko], [OP].

These inequalities become equalities and imply

$$S(\omega) \geq S(\varphi) \geq S(\omega) = \infty$$

for all states φ over $\mathcal{A}$.

We have to look for an alternative definition for a local entropy and this is offered by [C2], [CNT], [NT]:

Definition (2.5):

$$H_{\mathcal{M}}(\omega, \mathcal{A}) = \sup_{\omega = \sum \omega_i} \sum \omega_i(1)\ S(\omega|\widehat{\omega}_i)_{\mathcal{A}}$$

where now ω_i has to be a state over $\mathcal{M} \supset \mathcal{A}$.

Corollary (2.6): If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, i.e. ω is pure over $\mathcal{M}$, then

$$H_{\mathcal{M}}(\omega, \mathcal{A}) = 0 \qquad \forall\, \mathcal{A} \subset \mathcal{M}.$$

This holds if ω is the ground state and therefore the definition satisfies our requirement.

It might be desirable to consider also $H_{\mathcal{A}_{\bar{\Lambda}}}(\omega, \mathcal{A}_\Lambda)$. This should give us information if we construct KMS states as limit of states, locally arranged on $\mathcal{A}_{\bar{\Lambda}}$. Especially we will study in the sequel the following quantity:

Let $\Lambda \subset \bar{\Lambda}$ correspond to two double cones. Consider

$$H_\omega(\Lambda, \bar{\Lambda}) = \sup_{\omega = \sum \omega_i, \omega, \omega_i \text{ states over } \mathcal{A}(\bar{\Lambda})} \sum \omega_i(1)\ S(\omega|\widehat{\omega}_i)_{\mathcal{A}_\Lambda}.$$

We will prove that under the nuclearity condition [BW] for the vacuum

$$\lim_{|\bar{\Lambda}| \to \infty} H_\omega(\Lambda, \bar{\Lambda}) = H_{\mathcal{M}}(\omega, \mathcal{A}_\Lambda) = 0$$

and we will consider whether

$$\lim_{\Lambda \to \infty, |\Lambda|/|\bar{\Lambda}| \to 1} \frac{H_\omega(\Lambda, \bar{\Lambda})}{|\Lambda|} = 0$$

so that the difference between $H(\Lambda, \bar{\Lambda})$ and $H_{\mathcal{M}}(\mathcal{A}_\Lambda)$ only is a surface effect and we will see that this is true for massive theories. For temperature states and massive theories we will get

$$\limsup_{\Lambda \to \infty, |\Lambda|/|\bar{\Lambda}| \to 1} \frac{H_\omega(\Lambda, \bar{\Lambda})}{|\Lambda|} < \infty.$$

3 Bounds on the Entropy in the Ground State

In order to work with $H_{\mathcal{B}}(\omega, \mathcal{A})$ we must find some control over this quantity. Lower bounds are available by trying decompositions. Simple upper bounds are consequences of the monotonicity

$$H_{\mathcal{B}}(\omega, \mathcal{A}) \leq H_{\underline{\mathcal{B}}}(\omega, \bar{\mathcal{A}}) \quad \text{for } \underline{\mathcal{B}} \subset \mathcal{B},\ \bar{\mathcal{A}} \supset \mathcal{A}. \tag{3.1}$$

Therefore

$$H_{\mathcal{B}}(\omega, \mathcal{A}) \leq \inf\{S_\omega(\mathcal{A}), S_\omega(\mathcal{B})\}.$$

In our case these estimates are useless, since both $\mathcal{B} = \mathcal{A}_{\bar{\Lambda}}$ and $\mathcal{A} = \mathcal{A}_\Lambda$ are type III_1. But from (3.1) we already get a hint how to find an upper bound:

Theorem (3.2): Let $\mathcal{A}_\Lambda \subset \mathcal{A}_{\bar{\Lambda}}$ be split in the sense [DL] that there exists a type I algebra $\mathcal{N}$ such that $\mathcal{A}_\Lambda \subset \mathcal{N} \subset \mathcal{A}_{\bar{\Lambda}}$. If $S_\omega(\mathcal{N}) < c$, then

$$H_{\mathcal{A}_{\bar{\Lambda}}}(\omega, \mathcal{A}_\Lambda) < c.$$

It is known [BD'AF] that the split property follows from the nuclearity condition, and this nuclearity condition will also supply us with explicit upper bounds on the entropy. It is an open problem whether the split property is also necessary for the entropy to become finite. We will give some observations that support this suspicion.

Lemma (3.3): Let $H_{\mathcal{B}}(\omega, \mathcal{A}) < \infty$. Then to every $\varepsilon > 0$ there exists an $\bar{\omega}$, normal with respect to ω, such that

$$H_{\mathcal{B}}(\bar{\omega}, \mathcal{A}) < \varepsilon$$

and

$$\sum |\bar{\omega}(x_i a_i) - \bar{\omega}(x_i)\bar{\omega}(a_i)| < \varepsilon$$

for all $0 \leq x_i \leq 1$, $a_i \in \mathcal{A}$, $\|a_i\| = 1$, $\sum x_i = 1$, $x_i \in \mathcal{B}'$.

Proof: There exists a decomposition $\{y_i, 0 \leq i \leq N\}$, $y_i \in \mathcal{B}'$, such that

$$H_{\mathcal{B}}(\omega, \mathcal{A}) \leq \sum_{i=1}^{N} \omega(y_i) S(\omega|\widehat{\omega}(y_i \cdot))_{\mathcal{A}} + \varepsilon.$$

A refinement can improve only up to ε, such that

$$\sum_{i,j} \omega(y_{ij}) S(\omega|\widehat{\omega}(y_{ij} \cdot)) \leq \sum_i \omega(y_i) S(\omega|\widehat{\omega}(y_i \cdot)) + \varepsilon.$$

Now

$$\sum_{i,j} \omega(y_{ij}) S(\omega|\widehat{\omega}(y_{ij} \cdot)) = \sum_i \omega(y_i) S(\omega|\widehat{\omega}(y_i \cdot)) + \sum_i \omega(y_i) \sum_j \frac{\omega(y_{ij})}{\omega(y_i)} S(\widehat{\omega}_i|\widehat{\omega}(y_{ij})).$$

Therefore

$$\min_i \sum_i \frac{\omega(y_{ij})}{\omega(y_i)} S(\widehat{\omega}_i|\widehat{\omega}_{ij})_{\mathcal{A}} < \varepsilon$$

and we can take the corresponding $\widehat{\omega}_i$ as $\bar{\omega}$ with

$$H_{\mathcal{B}}(\bar{\omega}, \mathcal{A}) < \varepsilon.$$

The estimate on $\bar{\omega}$ follows because for all possible decompositions [OP]

$$\varepsilon > \sum \bar{\omega}(x_i \cdot) S(\bar{\omega}|\widehat{\bar{\omega}}(x_i \cdot))_{\mathcal{A}} \geq \frac{1}{2} \sum \bar{\omega}(x_i) \|\bar{\omega}(\cdot) - \widehat{\bar{\omega}}(x_i \cdot)\|^2.$$

The condition on $\bar{\omega}$ should be compared with the following fact [DL]:

> $\mathcal{A}_\Lambda$, $\mathcal{A}_{\bar{\Lambda}}$ is split, if $\mathcal{A}_\Lambda \vee \mathcal{A}'_{\bar{\Lambda}}$ is isomorphic to $\mathcal{A}_\Lambda \otimes \mathcal{A}'_{\bar{\Lambda}}$, and this is the case, if there exists a state with $\|\bar{\omega} - \bar{\omega} \otimes \bar{\omega}\| < \varepsilon$ (see [DL], remarks after Def. 1.4 and Prop. 1.5 for the exact statement) (compare with Theorem 3.7).

Another observation that supports the importance of the split property is

Theorem (3.4):

$$H_{\mathcal{B}}(\omega, \mathcal{A}) \leq S(\omega \otimes \omega|\omega)_{\mathcal{A} \otimes \mathcal{B}'}.$$

Here $\mathcal{B}'$ is the commutant of $\mathcal{B}$ in the GNS representation corresponding to ω. $\mathcal{A} \otimes \mathcal{B}'$ is understood to be the algebra built by the elements $\sum_{n=1}^N a_n b_n$, $a_n \in \mathcal{A}$, $b_n \in \mathcal{B}'$, so far without topology. On this algebra the state ω is given by

$$\omega(\sum a_n b_n) = \sum \langle \Omega | a_n b_n | \Omega \rangle$$

whereas

$$\omega \otimes \omega(\sum a_n b_n) = \sum \omega(a_n)\omega(b_n).$$

The relative entropy is given by [Ko].

Proof: According to [ST] we can write

$$H_{\mathcal{B}}(\omega, \mathcal{A}) = \sup_{\lambda, \mathcal{C}} S(\omega \otimes \mu|\lambda)_{\mathcal{A} \otimes \mathcal{C}}$$

with $\mathcal{C}$ an abelian algebra and λ a state over $\mathcal{B} \otimes \mathcal{C}$ such that $\lambda|_{\mathcal{B}} = \omega$ and $\lambda|_{\mathcal{C}} = \mu$.

Due to the monotonicity property of the relative entropy it suffices to take the supremum over finite–dimensional abelian algebras $\mathcal{C}$. For such an algebra $\mathcal{C}$ the elements of $\mathcal{B} \otimes \mathcal{C}$ can be written as $\bigoplus b_i$, $b_i \in \mathcal{B}$, $1 \leq i \leq N$. Also $\lambda = \bigoplus \omega_i$ where $\omega_i = \omega(\widehat{b}_i \cdot)$ with $\widehat{b}_i \geq 0 \in \mathcal{B}'$ $\forall i$. Now we construct a completely positive map $\gamma : \mathcal{C} \to \mathcal{B}' \otimes \mathcal{C}$ by

$$\gamma m_i = \sum m_i \widehat{b}_i \otimes \mathbf{1}.$$

It is unity preserving

$$\gamma(1) = \sum \widehat{b}_i \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1}.$$

Further

$$\lambda(a \otimes m) = \sum \omega(a\widehat{b}_i) m_i = (\omega \otimes \mu)(a \otimes \gamma(m))$$

where ω is considered as state over $\mathcal{A} \otimes \mathcal{B}'$.

Now we use the monotonicity of the relative entropy under unital completely positive maps and obtain

$$\begin{aligned} S(\omega\otimes\mu|\lambda)_{\mathcal{A}\otimes\mathcal{C}} &= S((\omega\otimes\omega\otimes\mu)\circ(1\otimes\gamma)|(\omega\otimes\mu)\circ(1\otimes\gamma))_{\mathcal{A}\otimes\mathcal{C}} \\ &\leq S(\omega\otimes\omega\otimes\mu|\omega\otimes\mu)_{\mathcal{A}\otimes\mathcal{B}'\otimes\mathcal{C}} \\ &= S(\omega\otimes\omega|\omega)_{\mathcal{A}\otimes\mathcal{B}'}. \end{aligned}$$

Taking the supremum over all $\mathcal{C}$ gives (3.4).

Remark (3.5): Provided $\mathcal{B}$ is type I we get in addition an inequality chain

$$S_\omega(\mathcal{B})\leq S(\omega\otimes\omega|\omega)_{\mathcal{B}\otimes\mathcal{B}'}\leq 2S_\omega(\mathcal{B})$$

where the left side becomes an equality if $\mathcal{B}$ is abelian and the right side becomes an equality if $\mathcal{B}$ is a full matrix algebra.

Proof: Assume $\mathcal{B}$ is abelian and discrete (otherwise $S_\omega(\mathcal{B})=\infty$). Then ω corresponds to a density $\sum\lambda_i P_i\otimes Q_i$ for $\mathcal{B}\otimes\mathcal{B}'$, whereas $\omega\otimes\omega$ corresponds to $\sum\lambda_i\lambda_j P_i\otimes Q_j$. Therefore

$$\begin{aligned} S(\omega\otimes\omega|\omega) &= \sum\lambda_i\ln\lambda_i - \mathrm{Tr}\ \lambda_i P_i\otimes Q_i\ln\sum\lambda_i\lambda_j P_i\otimes Q_j \\ &= \sum\lambda_i\ln\lambda_i-\sum\lambda_i\ln\lambda_i^2=-\sum\lambda_i\ln\lambda_i=S_\omega(\mathcal{B}). \end{aligned}$$

If instead $\mathcal{B}$ is a full matrix algebra, then $\omega\otimes\omega$ again corresponds to $\sum\lambda_i\lambda_j P_i\otimes Q_j$, whereas ω corresponds to the projection

$$\left|\sum\sqrt{\lambda_i}\,\varphi_i\otimes\psi_i\right\rangle\left\langle\sum\sqrt{\lambda_j}\,\varphi_j\otimes\psi_j\right|$$

with the eigenvectors φ_i of P_i and ψ_k of Q_k. This leads to

$$\begin{aligned} S(\omega\otimes\omega|\omega) &= -\sum\ln(\lambda_i\lambda_j)\langle\sqrt{\lambda_k}\,\varphi_k\otimes\psi_k|P_i\otimes Q_j|\sqrt{\lambda_\ell}\,\varphi_\ell\otimes\psi_\ell\rangle \\ &= -\sum\lambda_i\ln\lambda_i^2=2S. \end{aligned}$$

The general inequality can be obtained by interpolation of the two extremal cases.

It remains an open problem if $\mathcal{H}_{\mathcal{B}}(\omega,\mathcal{A})$ can also be bounded from below by results on the relative entropy. If this were to hold, this would demonstrate that the split property is necessary to make $\mathcal{H}_{\mathcal{B}}(\omega,\mathcal{A})$ finite due to the following

Lemma (3.6): Let $S(\omega|\varphi)_{\mathcal{B}}<\infty$. Then

$$\Pi_\omega(\mathcal{B})''\simeq\Pi_\varphi(\mathcal{B})''.$$

Proof: We use Kosaki's [K] formula for the relative entropy

$$S(\omega|\varphi)_{\mathcal{B}}=\sup\int\left[\frac{\omega(1)}{1+t}-\omega(y^*(t)y(t))-\frac{1}{t}\varphi(x(t)x^*(t))\right]\frac{dt}{t}$$

where $x(t)=1-y(t)$.

To attain the supremum we must choose an $x(t)$ with st-$\lim_{t\to 0}\Pi_\varphi(x(t)) = 0$ sufficiently fast so that the singularity $1/t^2$ does not contribute. Now $[\omega(1) - \omega(y^*(t)y(t))]/t = (\omega(x(t)) + \omega(x^*(t)) - \omega(x^*(t)x(t)))/t$ remains. We can assume that $0 < x = x^* < 1$ so that we can estimate

$$\omega(1) - \omega(y^*(t)y(t)) \geq \omega(x(t))$$

If $\Pi_\omega \neq \Pi_\varphi$ we can find $x(t)$ with st-$\lim_{t\to 0}\Pi_\varphi(x(t)) = 0$ but with st-$\lim_{t\to 0}\Pi_\omega(x(t)) \neq 0$. So either $\lim \omega(x(t)) > 0$ or there exists some operator a such that $\lim \omega(a^* x(t) a) > 0$. But again st-$\lim \Pi_\varphi a^* x(t) a = 0$ and this operator can be used in the calculation of the relative entropy to make the integral infinite.

We close the section by showing how the split property and estimates on the relative entropy are related.

Theorem (3.7) [D,L], [H]: Let $\mathcal{R}$ and $\bar{\mathcal{R}}$ be factors acting on a Hilbert space $\mathcal{H}$ with a representation Π and $\mathcal{R} \subset \bar{\mathcal{R}}$. Let Ω be a vector cyclic and separating for $\mathcal{R}$, $\bar{\mathcal{R}}'$, $\bar{\mathcal{R}} \wedge \mathcal{R}'$. Then the following conditions are equivalent:

1. If φ_1 is a state over $\mathcal{R}$ and φ_2 is a state over $\bar{\mathcal{R}}'$, then there exists a state φ such that
$$\varphi(R_1 R_2') = \varphi_1(R_1)\varphi_2(R_2') \qquad \forall R_1 \in \mathcal{R}, \qquad R_2' \in \bar{\mathcal{R}}'.$$
2. There exists a vector η cyclic and separating for $\mathcal{R} \vee \bar{\mathcal{R}}'$ such that
$$\langle \eta | R_1 R_2' | \eta \rangle = \langle \Omega | R_1 | \Omega \rangle \langle \Omega | R_2' | \Omega \rangle.$$
3. There exists a unitary operator W from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$ such that
$$W \Pi(AB') W^* = \Pi(A) \otimes \Pi(B').$$
4. There exists an intermediate type I factor $\mathcal{N}$
$$\mathcal{R} \subset \mathcal{N} \subset \bar{\mathcal{R}}$$
namely,
$$\mathcal{N} = W^* \mathcal{B}(\mathcal{H}) \otimes 1\, W.$$
This property is called split property.

Lemma (3.8): If $S(\omega \otimes \omega | \omega)_{\mathcal{R} \vee \bar{\mathcal{R}}'} < \infty$ for some state ω over $\bar{\mathcal{R}}$, then $\mathcal{R} \subset \bar{\mathcal{R}}$ are split.

Proof: This is an immediate consequence of (3.6) and (3.7.2).

If therefore $\mathcal{A}_\Lambda \subset \mathcal{A}_{\bar{\Lambda}}$ are split, we can consider the type I sequence of algebras $\mathcal{N}_\Lambda$ and can calculate $\lim[S_{\mathcal{N}_\Lambda}(\omega)]/|\Lambda|$. But neither is $\mathcal{N}_\Lambda$ given very explicitly – though there exists a method to construct $\mathcal{N}_\Lambda = \mathcal{R} \vee J_{\bar{\mathcal{R}} \wedge \mathcal{R}'} \mathcal{R} J_{\bar{\mathcal{R}} \wedge \mathcal{R}'}$ – nor does a control over $S_{\mathcal{N}_\Lambda}(\omega)$ seem available. Instead we will stay with $H_{\mathcal{A}_{\bar{\Lambda}}}(\omega, A_\Lambda)$ and relate it to the nuclearity condition of [BW].

4 The Nuclearity Condition and Entropy Estimates in the Vacuum

In [BW] the nuclearity condition was considered to find a measure on the energy level density in finite regions. Take the map $\mathcal{A}(\Lambda) \to \mathcal{B}(\mathcal{H})$,

$$\Theta_{\beta,\Lambda}(A) = e^{-\beta h}\, A|\Omega\rangle, \qquad h \text{ the Hamiltonian.} \tag{4.1}$$

This map is nuclear, if there exists a sequence of linear functionals $\varphi_i \in \mathcal{A}(\Lambda)^*$ and vectors $\Phi_i \in \mathcal{H}$ such that if $\Theta_{\beta,\Lambda}$ can be written as

$$\Theta_{\beta,\Lambda}(A) = \sum_i \varphi_i(A)\Phi_i$$

then

$$\|\Theta_{\beta,\Lambda}\|_1 =: \inf \sum \|\varphi_i\|\|\Phi_i\| < \infty. \tag{4.2}$$

For free massive theories in three + one dimensions this nuclearity index was estimated in [BW] – without ambition to become optimal – and the bound was found

$$\|\Theta_{\beta,\Lambda}\|_1 \leq \exp\left[c\left(\frac{r}{\beta}\right)^3\right] \sum_{i=1}^{\infty} \ln(1 - e^{-\beta m_i/2}), \qquad r > \beta \tag{4.3}$$

where m_i are the masses of the theory and r the radius of Λ, with the restriction $r > m_i^{-1}$. Similar estimates are found in [BJ] for free massless field theories and for the free electromagnetic field

$$\|\Theta_{\beta,\Lambda}\|_1 \leq \begin{array}{ll} \exp\left[c\left(\frac{r}{\beta}\right)^3\right] & r \geq \beta \\ \exp\left[c'\left(\frac{r}{\beta}\right)\right] & r \leq \beta \end{array} \tag{4.4}$$

always for suitable constants c.

It seems reasonable that similar nuclearity bounds also hold for interacting theories.

This nuclearity bound was used to show [BD'AF] that local algebras $\mathcal{A}_\Lambda \subset \mathcal{A}_{\bar\Lambda}$ with some distance between Λ and $\bar\Lambda$ satisfy the split property. A slight variation of the arguments allow to find an upper bound on $\mathcal{H}_{\mathcal{A}_{\bar\Lambda}}(\omega, \mathcal{A}_\Lambda)$.

Lemma (4.5) [BD'AF]: Take $A \in \mathcal{A}_\Lambda$ and $B' \in \mathcal{A}'_{\bar\Lambda}$ with Λ, $\bar\Lambda$ balls of radius r resp. R and $R = r + \beta$. Then one can construct a continuous function f of the Hamiltonian h such that

$$\langle\Omega|AB'|\Omega\rangle = \langle\Omega|Af_\beta(h)B'|\Omega\rangle + \langle\Omega|B'f_\beta(h)A|\Omega\rangle$$

where f_β has almost exponential decrease in the sense that

$$\lim_{|E|\to\infty} \exp(|E|^\kappa)|f_\beta(E)| = 0 \qquad 0 \leq \kappa < 1$$

and $f_\beta(E) = f(E\beta)$, $f(0) = 1/2$.

Under the assumption of the nuclearity condition

$$\|\Theta_{\beta,\Lambda}\|_s \leq \exp\left[cr^3\beta^{-3}\ln(1 - e^{-\beta m})\right] = \exp g(r, \beta) \tag{4.6}$$

with

$$\|\Theta_{\beta,\Lambda}\|_s = \inf - \sum \|\varphi_i\| \ln \|\varphi_i\|, \qquad \|\Phi_i\| = 1,$$

we can use this function f to prove:

Theorem (4.6): Assume that the theory has a mass gap, i.e. 0 is an isolated point in the spectrum of h. Then

$$H_{\mathcal{A}_{\Lambda_R}}(\mathcal{A}_{\Lambda_r}) < \delta$$

provided $R = r + \beta$ and $\dfrac{r^3}{\beta^{3+\varepsilon}} < \bar{\delta}$ so that $\beta(r)$ can be chosen such that with $r \to \infty$ $\delta \to 0$ together with $\bar{\delta} \to 0$.

Proof: Following the construction of [BD'AF] we decompose the map $\Theta_f : \mathcal{A} \to \mathcal{H}$

$$\Theta_f A = f_\beta(h) A|\Omega\rangle$$

into

$$\Theta_k A = P_k f(h) e^{\gamma h} \Theta_\gamma$$

where P_k is the spectral projection of h for the interval $[k-1, k)$. We calculate the nuclearity bound for Θ_k to be

$$\Theta_1 A = \frac{1}{2}|\Omega\rangle\langle\Omega|A|\Omega\rangle$$

if the map gap is ≥ 1 (otherwise we have to scale the interval $[k-1, k)$). Therefore $\|\Theta_1\|_1 = 1$, $\|\Theta_1\|_s = 0$. Next with $f_k = \sup f(E\beta)$, $E \in [k-1, k)$ we have

$$\|\Theta_k\|_1 \leq f_k e^{\gamma k + g(r,\gamma)}$$

resp.

$$\|\Theta_k\|_s \leq (-f_k \ln f_k + f_k) e^{\gamma k + g(r,\gamma)}.$$

For every k we optimize γ by

$$\begin{aligned} k &= c r^3 \gamma^{-3} \left[-\frac{3}{\gamma} \ln(1 - e^{-\gamma m}) + \frac{e^{-\gamma m}}{1 - e^{-\gamma m}} \right] \\ &\simeq c r^3 \gamma^{-3} \left[-\frac{3 \ln \gamma m}{\gamma} + \frac{1}{\gamma m} \right] \qquad \text{for large } k \end{aligned}$$

therefore γ tends to 0 for large k. f_k and $f_k \ln f_k$ decrease almost exponentially. Therefore

$$\|\Theta_k\|_s < \exp\left[-k^{\nu/(\nu+1)} \widehat{c}_\beta \right] \qquad \text{for some } \nu < \infty.$$

This is summable in k. In addition the scaling property of f_β allows to estimate (for large β, that correspond to small k and determine the permitted radius R)

$$\widehat{c}_\beta \approx r^{3/(\nu+1)} \beta^{-(\nu-\varepsilon)/(\nu+1)}$$

so that if we adjust β appropriately to r

$$\lim_{r^3/(\beta^{\nu-\varepsilon/2}) = \text{const}, r \to \infty} \widehat{c}_\beta(r) = 0. \tag{4.7}$$

We use this result to write

$$\langle\Omega|AB'|\Omega\rangle = \langle\Omega|A|\Omega\rangle\langle\Omega|B'|\Omega\rangle + \sum \varphi_i(A)\psi_i(B')$$

with φ_i, ψ_i linear functionals over $\mathcal{A}_\Lambda$ resp. $\mathcal{A}_{\bar{\Lambda}}$ where $\|\varphi_i\| = 1$ and $-\sum \|\psi_i\| \ln \|\psi_i\| < \varepsilon$ provided according to the previous estimates β is large enough. The left hand side is positive and therefore also real. Every φ_i resp. ψ_i can be written as linear combination of at most four positive linear functionals $\varphi_{i\alpha}$, $\psi_{i\beta}$. For hermitean A, B' only the real combinations remain so that

$$\langle\Omega|AB'|\Omega\rangle = \langle\Omega|A|\Omega\rangle\langle\Omega|B|\Omega\rangle + \sum \varphi_i(A)\psi_i(B') - \sum \bar{\varphi}_i(A)\bar{\psi}_i(B')$$

where separately

$$-\sum \|\psi_i\| \ln \|\psi_i\| < \varepsilon, \qquad -\sum \|\bar{\psi}_i\| \ln \|\bar{\psi}_i\| < \varepsilon, \qquad \bar{\varphi}_k,\ \varphi_i \text{ normalized.}$$

Therefore

$$\omega + \sum_{k=1} \bar{\varphi}_k \otimes \bar{\psi}_k = \omega \otimes \omega + \sum_{i=1} \varphi_i \otimes \psi_i$$

as positive functionals over $\mathcal{A}_\Lambda \otimes \mathcal{A}'_{\bar{\Lambda}}$. Since $\mathcal{B}(\mathcal{H}) \supset \mathcal{A}_\Lambda \otimes \mathcal{A}'_{\bar{\Lambda}}$ we can extend $\omega + \sum \bar{\varphi}_k \otimes \bar{\psi}_k$ to a positive linear functional over $\mathcal{B}(\mathcal{H})$ and therefore also over $\mathcal{A}_{\bar{\Lambda}}$. After normalization we can write this functional $\bar{\omega} = (1-\varepsilon)\omega + \varepsilon\varphi = \sum_{i=0} \varphi_i \otimes \psi_i$ where we set $\psi_0 = (1-\varepsilon)\omega$, $\varphi_0 = \omega$. Next we use [CNT] that for any $\mathcal{A} \subset \mathcal{B}$

$$H_\mathcal{B}(\bar{\omega}, \mathcal{A}) \geq (1-\varepsilon) H_\mathcal{B}(\omega, \mathcal{A}) + \varepsilon H_\mathcal{B}(\varphi, \mathcal{A})$$

and therefore

$$H_\mathcal{B}(\omega, \mathcal{A}) \leq \frac{1}{1-\varepsilon} H_\mathcal{B}(\bar{\omega}, \mathcal{A}).$$

This will be applied for $\mathcal{A} = \mathcal{A}_\Lambda$, $\mathcal{B} = \mathcal{A}_{\bar{\Lambda}}$. We use $S(\sum \mu_i\varphi_i | \sum \mu_i\omega_i) \leq \sum \mu_i S(\varphi_i|\omega_i)$ and $S(\varphi|\lambda\omega) = \lambda \ln \lambda + \lambda S(\varphi|\omega)$ to estimate

$$\begin{aligned}
H_{\mathcal{A}_{\bar{\Lambda}}}(\bar{\omega}, \mathcal{A}_\Lambda) &= \sup_{\sum B_\ell = 1, B_\ell \in \mathcal{A}'_{\bar{\Lambda}}} \sum [-\bar{\omega}(B_\ell) \ln \bar{\omega}(B_\ell) + S(\bar{\omega}|\bar{\omega}(B_\ell \cdot))_{\mathcal{A}_\Lambda}] \\
&= \sup_L \sum_\ell \left[-\sum_k \psi_k(B_\ell) \ln \sum_k \psi_k(B_\ell) + S(\sum_k \varphi_k\psi_k(1) | \sum_k \varphi_k\psi_k(B_\ell))_{\mathcal{A}_\Lambda} \right] \\
&\leq \sup_L \left(S_L + \sum_{\ell,k} \psi_k(1) S\left(\varphi_k | \varphi_k \frac{\psi_k(B_\ell)}{\psi_k(1)}\right)_{\mathcal{A}_\Lambda} \right)
\end{aligned}$$

where we refined the decomposition of $\bar{\omega}$ given by B_ℓ further into $\sum_{k,\ell} \varphi_k \cdot \psi_k(B_\ell)$. Then we can continue

$$= \sup_L (S_L - S_{KL} + S_K) \leq 2S_K = -2\sum \|\psi_k\| \ln \|\psi_k\|$$

where S_L, S_{KL}, S_K are the entropies of the abelian models corresponding to the decompositions $\bar{\omega} = \sum_\ell \bar{\omega}(B_\ell \cdot)$, $\bar{\omega} = \sum_{k,\ell} \varphi_k\psi_k(B_\ell \cdot)$, $\bar{\omega} = \sum_k \varphi_k\psi_k$.

Remark: To obtain the desired result it was necessary that in (4.6) we need $\nu > n$ which asks for $m > 0$. This does not hold for massless theories. Also to control $\|\Theta\|_1$ the mass gap was necessary. Otherwise the estimate (4.7) guarantees

Theorem (4.8): For massive theories with corresponding nuclearity bound (4.3)

$$\lim_{|\Lambda|/|\bar{\Lambda}|\to 1,\Lambda\to\mathbf{R}^3} \frac{H_{\mathcal{A}_{\bar{\Lambda}}}(\omega,\mathcal{A}_\Lambda)}{|\Lambda|} = 0$$

with $\bar{\Lambda}$ larger than Λ by slightly more than a surface effect.

Theorem (4.9): For massless theories with nuclearity bound (4.6) with $m = n$ we have

$$\lim_{|\bar{\Lambda}|\to\infty} H_{\mathcal{A}_{\bar{\Lambda}}}(\omega,\mathcal{A}_\Lambda) = 0.$$

This follows from the same estimates with considering $\beta\to\infty$ for fixed Λ.

5 Entropy for Temperature States

Let us first assume that a KMS state exists. We want to evaluate estimates on the local entropy. Theorem (4.6) can be replaced by

Theorem (5.1): Let ω be a KMS state to the temperature β. Let Ω_β be the corresponding GNS vector. Assume that the following nuclearity condition holds: Let

$$e^{-\gamma h}A|\Omega\rangle = \tau_{i\gamma}A|\Omega\rangle = \sum\varphi_i(A)\Phi_i, \qquad \|\Phi_i\| = 1,$$

for all $A\in\mathcal{A}(\Lambda)$ with $\sum\|\varphi_i\|\ln\|\varphi_i\|\leq c_\gamma(\Lambda)$. If $\gamma<\beta/2$, then

$$H_{\mathcal{M}}(\omega,\mathcal{A}_\Lambda)\leq c_\gamma(\Lambda).$$

Remark: The above estimate seems to be optimal for $\gamma = \beta/4$, if we check it for lattice systems or the free massive case (compare the Appendix). Also for lattice systems we lose control on the nuclearity estimate for $\gamma > \beta/2$, because Ω_β does not necessarily remain in the domain of $\tau_{i\gamma}(A)$ (which can be unbounded). So we must be aware that $c_\gamma\to\infty$ for $\gamma\to\beta/2$ [BD'AL].

Proof: ω_β has to be decomposed by $x_k\in\mathcal{M}'$. Thus

$$\langle\Omega_\beta|x_kA|\Omega_\beta\rangle = \langle\Omega_\beta|y_k\tau_{i\beta/2}A|\Omega_\beta\rangle = \sum\varphi_i(A)\langle\Omega_\beta|\tau_{-i(\beta/2-\gamma)}y_k|\Phi_i\rangle$$

with $x_k\in\mathcal{M}'$, $y_k\in\mathcal{M}$, $\|y_k\|\leq 1$.

The rest is copying the previous proof taking into account that

$$|\langle\Omega_\beta|\tau_{i\gamma}y|\Phi_i\rangle|\leq\langle\Omega_\beta|ye^{-2\gamma h}y|\Omega_\beta\rangle^{1/2}\leq\langle\Omega_\beta|y^2|\Omega_\beta\rangle^{1/2}, \qquad 0\leq\gamma\leq\beta/2,$$

because $\langle\Omega_\beta|ye^{-2\gamma h}y|\Omega_\beta\rangle$ is a convex function on γ with $f(0) = f(\beta/2)$ and $f(\gamma)\leq(f(0)+f(\beta/2))/2$ for $0\leq\gamma\leq\beta/2$.

Again we can be interested in $H_{\mathcal{A}_{\bar{\Lambda}}}(\mathcal{A}_\Lambda)$ to control surface effects. We have to modify [BD'AF] slightly:

Lemma (5.2): Take $A \in \mathcal{A}_\Lambda$ and $B \in \mathcal{A}'_{\bar\Lambda}$ (in the GNS representation corresponding to ω_β). Then there exists $f_{\beta c}(|h|)$, h implementing the time evolution in the GNS representation, with almost exponential decrease such that

$$\langle \Omega_\beta | AB | \Omega_\beta \rangle = \langle \Omega_\beta | A f_{\beta c}(|h|) B | \Omega_\beta \rangle + \langle \Omega_\beta | B f_{\beta c}(|h|) A | \Omega_\beta \rangle + \sum \varphi_i(A) \psi_i(B)$$

with $\sum \|\varphi_i\|_s < \infty$, $\|\psi_i\| = 1$.

Proof: We can mimic the proof of [BD'AF] expressing f as a contour integral. Under the above assumption the function

$$g(t) = \frac{1}{2} \langle \Omega_\beta | A e^{iht} B | \Omega_\beta \rangle + \frac{1}{2} \langle \Omega_\beta | B e^{iht} A | \Omega_\beta \rangle$$

is analytic in the strip $I = \{ z = t + i\gamma, t \in \mathbf{R}, |\gamma| \leq \beta/2 \}$ and with a cut for $|t| > c_{\bar\Lambda}$. The boundary stems from $B \in \mathcal{M}'$, the cut stems from the fact that $[\tau_t A, B]$ no longer vanishes for $B \in \mathcal{A}'_{\bar\Lambda} \cap \mathcal{M}$, if $|t| > c_{\bar\Lambda}$, and $c_{\bar\Lambda}$ depends on the distance between Λ and $\bar\Lambda$ (compare [BD'AF] and [H]). Again we make the variable transformation $z = 2w/\beta(w^2+1)$ such that $z(0) = 0$. Therefore

$$g(0) = \frac{1}{2\pi i} \oint g(z(w)) \frac{dw}{w}$$

can be evaluated as integral over the boundary of the analytic region in w. The contribution along the cut in z remains the unit circle as for the vacuum. The contribution of the boundary in z can be shifted into the interior $|\gamma| < \beta/2$. In the w–plane it corresponds to a curve of finite length and we can evaluate it by using the nuclearity condition to

$$\oint \frac{dw}{w} \langle \Omega_\beta | A e^{iht - \gamma h} B | \Omega_\beta \rangle = \sum_i \oint \frac{dw}{w} \tilde\varphi_i(A) \langle \bar\psi_i e^{iht} B | \Omega_\beta \rangle = \sum_i \varphi_i(A) \psi_i(B).$$

$\sum \|\varphi_i\|_s < \infty$ follows because $\sum \|\tilde\varphi_i\|_s < \infty$ due to the nuclearity and the rest in the integral is uniformly bounded on the curve. This allows us to deduce

Theorem (5.3): Let the nuclearity condition for $e^{-\gamma h} A \Omega$ be satisfied with

$$\sum \|\varphi_i\| \ln \|\varphi_i\| \leq e^{-c_{\bar\Lambda,\Lambda} \gamma^{-n}} \qquad \forall \gamma \leq \beta/4$$

then

$$H_{\mathcal{A}_{\bar\Lambda}}(\mathcal{A}_\Lambda) < \infty.$$

Proof: This follows by the same arguments as in the vacuum. Notice that the necessary range of γ in (4.6) covers the permitted region in γ. Evidently now we have no mass gap, and also γ cannot vary arbitrarily to improve the bound. More control how $H_{\mathcal{A}_{\bar\Lambda}}(\mathcal{A}_\Lambda)$ tends to $H_{\mathcal{M}}(\mathcal{A}_\Lambda)$ asks for controlling the nuclearity condition, but even more how $f_{\beta c}$ tends to $f_\beta = \exp(-\frac{\beta}{2}|h|)$ if the cut moves to infinity with increasing distance between Λ and $|\Lambda|$.

Though we have no necessary conditions on the temperature state, which are satisfying in the sense that they can be proven for massive free fields, such that they guarantee the existence of the entropy density, it seems desirable to find conditions that have to be satisfied in the vacuum state and then guarantee the existence of temperature states with finite entropy density. We follow the proposal of [BJu] to construct KMS states.

Theorem (5.4) [BJu]: Let $\mathcal{A}_\Lambda \subset \mathcal{N}_{\bar\Lambda} \subset \mathcal{A}_{\bar\Lambda}$ be split with $\mathcal{N}_{\bar\Lambda}$ an appropriate type I algebra. Consider the state $\omega|_{\mathcal{N}_{\bar\Lambda}} \otimes \omega|_{\mathcal{N}'_{\bar\Lambda}}$ with GNS vector η. Define the projector $E_{\bar\Lambda}$ by

$$E_{\bar\Lambda}\mathcal{H} = \overline{\mathcal{N}_{\bar\Lambda}\eta}$$

and

$$\omega_{\beta,\bar\Lambda}(A) = \frac{\mathrm{Tr}\ E_{\bar\Lambda}e^{-\beta h}E_{\bar\Lambda}A}{\mathrm{Tr}\ E_{\bar\Lambda}e^{-\beta h}E_{\bar\Lambda}} \qquad \forall\, A \in \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}.$$

Then w-$\lim_{\bar\Lambda\to\mathbf{R}^3}\omega_{\beta,\bar\Lambda}$ satisfies the KMS condition.

We slightly change the definition and consider an imbedding

$$\mathcal{A}_\Lambda \subset \mathcal{N}_{\bar\Lambda} \subset \mathcal{A}_{\bar\Lambda} \subset \mathcal{N}_{\widehat\Lambda} \subset \mathcal{A}_{\widehat\Lambda}$$

with $\mathcal{A}_{\widehat\Lambda} \to \mathcal{M}$ and such that $\mathcal{N}_{\widehat\Lambda} = \mathcal{N}_{\bar\Lambda} \otimes \mathcal{N}_{\widetilde\Lambda}$. We define $\widetilde\eta$ to be the GNS vector corresponding to $\omega|_{\mathcal{N}_{\bar\Lambda}} \otimes \omega|_{\mathcal{N}_{\widetilde\Lambda}} \otimes \omega|_{\mathcal{N}'_{\widehat\Lambda}}$ and $E_{\widehat\Lambda}\mathcal{H} = \overline{\mathcal{N}_{\widehat\Lambda}\widetilde\eta}$. Again

$$\lim \omega_{\widetilde\Lambda}(A) = \lim \frac{\mathrm{Tr}\ E_{\widehat\Lambda}e^{-\beta h}E_{\widehat\Lambda}A}{\mathrm{Tr}\ E_{\widehat\Lambda}e^{-\beta h}E_{\widehat\Lambda}} \equiv \omega_\beta(A)$$

satisfies the KMS condition.

We want to use the semicontinuity [CNT]

$$\lim H_{\omega_n}(A) \geq H_{\lim \omega_n}(A)$$

and have therefore to estimate $H_{\omega_{\widetilde\Lambda}}(A)$ by a bound independent of $\widetilde\Lambda$.

The trace (for $A \in \mathcal{A}_\Lambda \subset \mathcal{N}_{\widetilde\Lambda}$) can be written as

$$\frac{\sum \langle \varphi_i \otimes \psi_j | e^{-\beta h} A | \varphi_i \otimes \psi_j \rangle}{\sum \langle \varphi_i \otimes \psi_j | e^{-\beta h} | \varphi_i \otimes \psi_j \rangle}$$

where

$$|\varphi_i \otimes \psi_j\rangle = |\varphi_i\rangle\langle\Omega| \otimes |\psi_j\rangle\langle\Omega|\ |\widetilde\eta\rangle = U_{i1} \otimes V_{j1}|\widetilde\eta\rangle$$

where $U_{i1} \in \mathcal{N}_{\bar\Lambda}$ and $V_{j1} \in \mathcal{N}_{\widetilde\Lambda}$. If we write (in the interpretation of (4.3) $\widetilde\eta = |\Omega\rangle \otimes |\Omega\rangle$, then we can use a nuclearity condition on $e^{-\gamma h}A|\Omega\rangle$ for $A \in \mathcal{N}_{\bar\Lambda}$, that is implemented by a nuclearity condition for $\mathcal{A}_{\bar\Lambda}$, to get

$$\begin{aligned}
&\langle \Omega \otimes \Omega | V_j^* \otimes U_i^* e^{-\beta h} V_j e^{\gamma h/2} e^{-\gamma h/2} A U_i | \Omega \otimes \Omega \rangle = \\
&= \ \langle \Omega \otimes \Omega | V_j^* \otimes U_i^* e^{-\beta h} V_j e^{\gamma h/2} | \Phi_k \otimes \Omega \rangle \varphi_k(AU_i).
\end{aligned}$$

We assume $|\varphi_k(AU_i)| \leq \|\varphi_k\|$ with $\sum \|\varphi_k\| \ln \|\varphi_k\| < \infty$. For the remaining term we estimate

$$\begin{aligned}
&|\langle \Omega \otimes \Omega | V_j^* \otimes U_i^* e^{-\beta h} V_j e^{\gamma h/2} e^{-\gamma h/2} | \Phi_k \otimes \Omega \rangle| \leq \\
&\leq \ |\langle \Omega \otimes \Omega | V_j^* \otimes U_i^* e^{-\beta h} U_i \otimes V_j | \Omega \otimes \Omega \rangle|^{1/2} \cdot \langle \Phi_k \otimes \Omega | e^{\gamma h/2} V_j^* e^{-\beta h} V_j e^{\gamma h/2} | \Phi_k \otimes \Omega \rangle|^{1/2}.
\end{aligned}$$

If we now assume that the $|\widetilde\Lambda|$ dependence in all terms is the same, i.e.

$$\sum_j \langle \Phi_k \otimes \Omega | e^{\gamma h/2} V_j^* e^{-\beta h} V_j e^{\gamma h/2} | \Phi_k \otimes \Omega \rangle \approx e^{c|\widetilde\Lambda| + c_k},$$

$$\sum_{i,k} \langle \varphi_i \otimes \psi_k | e^{-\beta h} | \varphi_i \otimes \psi_k \rangle \approx e^{c|\widetilde{\Lambda}|+\bar{c}}$$

then the estimate on the nuclearity bound for $\tau_{i\gamma} A|\widetilde{\eta}\rangle$ becomes independent of $|\widetilde{\Lambda}|$. This assumption does not seem implausible. $\tau_{i\gamma} A$ should essentially stay in a finite region and $\widetilde{\eta}$ should not be able to connect it with distant regions. Also $e^{-\gamma} V_j e^{\gamma}$ should stay inside of $\widetilde{\Lambda}$ up to boundary effects. But evidently the possibility of phase transitions and long range correlations can be of importance. So far it is improbable that one can deduce this $|\widetilde{\Lambda}|$ behaviour from the given nuclearity condition. Therefore we are unable to offer a definite answer whether the entropy density of the KMS states constructed by [BJu] remains finite.

Appendix

It remains to verify the necessary nuclearity condition for free field theories. They can be found in [BD'AL] for several norms. We want to sketch the proof and first concentrate on the ground state. Here we succeed for massive and massless theories. Since the considerations follow closely [BW] and [BJ].

We write $e^{-\beta h}A|\Omega\rangle = \sum\langle\Phi_n|A|\Omega\rangle \cdot e^{-\beta h}|\Phi_n\rangle$ with $|\Phi_n\rangle$ an orthonormal basis that can be chosen to be

$$\Phi_n = \prod_{n=\{n_1,\ldots,n_p\}} \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i}}\,|\Omega\rangle. \tag{A.1}$$

In [BW] it is shown: if for $G(f;\psi) = \langle e^{a^\dagger(f)}\Omega|\psi\rangle\ \forall f \in L^2(\mathbf{R}^3)$ and $\psi \in \mathcal{N} = \{e^{-h}A|\Omega\rangle, \|A\| = 1, A \in \mathcal{A}_\Lambda\}$ we can find a bound such that $|G(f;\psi)| \leq \exp\frac{1}{2}\|e^{-\gamma h}f - g\|^2\ \forall g \in K(O_r)'$, i.e. functions with support outside O_r. ¿From this it can be deduced that

$$|G(f;\psi)| \leq \exp\left[\frac{1}{2}\|Tf\|^2\right] \tag{A.2}$$

where T is a positive trace class operator with $0 \leq T < 1$ and eigenvalues t_k corresponding to eigenvectors e_k, when we will use in the construction of Φ_n $a_i^\dagger = a^\dagger(e_i)$ and can estimate

$$|\langle\psi|\Phi_n\rangle| \leq \prod_{k=1}^{p}(n_k+1)t_k^{n_k}, \qquad n = \{n_1,\ldots,n_p\}.$$

So far these results are found in [BW]. Denoting $\lambda_n = \sup_{\psi\in\mathcal{N}}|\langle\psi|\Phi_n\rangle|$ we estimate rather roughly

$$-\sum\lambda_n\ln\lambda_n \leq -c\det\frac{1}{(1-T)^3}\cdot\mathrm{Tr}\ T\ln T. \tag{A.3}$$

Again following [BW] and [BJ] we have to estimate $\det 1/(1-T)^3$ and $\mathrm{Tr}\ T\ln T$. They split

$$|G(f;e^{-\beta h}A\Omega)| \leq \left[\frac{1}{2}\|T_+\frac{1}{2}(1+J)f\|^2 + \frac{1}{2}\|T_-\frac{1}{2}(1-J)f\|^2\right]$$

with $(Jf)(p) = \overline{f(-p)}$ and $T_+ = E_r^+e^{-\beta\omega}$ and $T_- = E_r^-e^{-\beta\omega}$ where $E_r^\pm$ are the projections onto $\sqrt{\omega}^{\pm 1}\widetilde{f}_r$ (compare (2.1)). Then

$$T = \lim_{n\to\infty}[(T_+^*T_+)^n + (T_-^*T_-)^n]^{1/2n}.$$

For estimating $\mathrm{Tr}\ T\ln T$ a new difficulty arises: Though

$$\|(A+B)^\alpha\|_1 \leq \|A^\alpha\|_1 + \|B^\alpha\|_1, \qquad \alpha = \frac{1}{2n},$$

this does not hold for $T^\alpha\ln T$, because it fails to be an operator monotonic function for $0 < T \leq 1$. But using its integral representation one can see

$$\|(A+B)^\alpha\ln(A+B)\|_1 \leq \|A^\alpha\ln A\|_1 + \|B^\alpha\ln B\|_1, \qquad \alpha = \frac{1}{2n},$$

for $A, B < a$ and $\ln a < -1$. If therefore we can prove that $T_\pm^* T_\pm < 1$, then $(T_\pm^* T_\pm)^n < a$ for n sufficiently large so that we come into the range of the inequality. $T_+^* T_+ < 1$ is a consequence of the estimate [BJ]

$$\det(1 - |Ee^{-\gamma}|)^3 \leq \exp 3 \sum_{n=1}^{\infty} \frac{1}{n} \|Ee^{-n\gamma}\|_1 < \infty .$$

This also shows that $\det 1/(1-T)^3 < \infty$ or more explicitly [BW], [BJ]

$$\det \frac{1}{(1-T)^3} \leq \begin{array}{ll} \exp\left[c_1 \frac{r}{\beta} + c_2 \left(\frac{r}{\beta}\right)^3 \right] & m = 0 \\ \exp\left[c \left(\frac{r}{\beta}\right)^3 \sum \ln(1 - e^{-\beta m_i/2}) \right] & m_i \neq 0. \end{array} \tag{A.4}$$

Following [BW] we estimate $T \ln T$ where T stands for $(T_\pm^* T_\pm)^{1/2}$ by an appropriate decomposition $T = ABC$ and using

$$\text{Tr } T \ln T \leq \|C \ln T\| (\text{Tr } AA^\dagger)^{1/2} (\text{Tr } BB^\dagger)^{1/2}$$

and $e^{-\beta\omega}$ allows some splitting so that the estimates remain essentially unchanged.

It remains to evaluate what has to be changed in a temperature state. We restrict ourselves to the massive case. For the massless case problems must be anticipated, because, as is argued in [BJu], it can happen that the temperature state is not locally normal any more. We consider $\mathcal{M} \vee \mathcal{M}'$ and interpret ω_β as a vacuum state of a free massive field theory with Weyl algebra $\{\bar{W}(f_1, f_2)\}''$ of two classes of particles with an appropriate Bogoliubov transformation. Mimicking the above estimates we consider

$$|G(f_1, f_2; e^{-\gamma H} A)| = |\langle \Omega | e^{\bar{a}(f_1) + \bar{b}(f_2)} e^{-\gamma H} A \Omega \rangle|$$

where e^{iHt} implements the time evolution in the GNS representation and use

$$|G(f_1, f_2; e^{-\gamma H} A)| \leq \exp\left[\frac{1}{2} \|e^{-\gamma h} \vec{f} - \vec{g}\|^2 \right]$$

for all $\vec{g} \in K(O_r)'$. Notice that now $K(O_r)'$ also consists of elements corresponding to $\mathcal{M}'$. h implements the time evolution in the the one–particle space, notice that it is not positive definite anymore but we consider so far only $\vec{f}$ in the domain of $e^{-\gamma h}$. We decompose $e^{-\gamma h} \vec{f}$ into its components corresponding to $\mathcal{M}'$ which can be completely annihilated by $\vec{g}$ and those corresponding to $\mathcal{M}$. They are given in p–space by

$$e^{-\gamma\omega(p)} \sqrt{\frac{1}{1 - e^{-\beta\omega(p)}}} f_1(p) + e^{\gamma\omega(p)} \sqrt{\frac{1}{e^{\beta\omega(p)} - 1}} f_2(p).$$

If we choose $\gamma = \beta/4$ we get

$$e^{-\beta\omega(p)/4} \left[\sqrt{\frac{1}{1 - e^{-\beta\omega(p)}}} f_1(p) + \sqrt{\frac{1}{e^{\beta\omega(p)/2} - e^{-\beta\omega(p)/2}}} f_2 \right] = e^{-\beta\omega(p)/4} A \vec{f}.$$

Provided $m > 0$ the operator A is a bounded operator. Therefore we are back to the previous estimates with the replacement $|G(f; \psi)| \leq \exp[\frac{1}{2} \|TAf\|^2]$.

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