Clustering for Algebraic K-Systems

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Abstract

We prove that for a von Neumann algebra that is an algebraic K system with respect to some automorphism, the invariant state is K–clustering and r–clustering. Further we study in examples how far the von Neumann algebra inherits the K property from the underlying C^* algebra.

1 Introduction

In a previous paper [NT1] we proved mixing properties for algebraic K systems but as we remarked in an erratum the assumptions under which the proof applies were not clearly stated. In this note we want to make more precise under which conditions the conclusion holds and give an appropriately modified proof.

An algebraic K system is a family of subalgebras $\mathcal{A}_n$, $n \in Z$, of an algebra $\mathcal{A}$ with the properties

(i) $\mathcal{A}_{n+1} \supset \mathcal{A}_n$

(ii) $\bigcup_n \mathcal{A}_n = \mathcal{A}$

(iii) $\bigcap_n \mathcal{A}_n = z\mathbf{1}$

and an automorphism σ of $\mathcal{A}$ with $\sigma(\mathcal{A}_n) = \mathcal{A}_{n+1}$ [NT1,NT2,E,KS,S].

Remarks (1)

1. The subalgebras $\mathcal{A}_n$ and $\mathcal{A}$ should be at least C^* algebras, however, unless stated otherwise we shall even require them to be von Neumann algebras. Only in this case does one have a chance to prove cluster properties in all σ–invariant states. C^* systems typically have several invariant states and these will not cluster if they are not extremal invariant.

2. Correspondingly $\bigcup_n$ means algebraic union together with strong closure in the representation with the invariant state ω. Thus for the axioms of a K system one actually does not need $\mathcal{A}$ but only an isomorphism $\sigma : \mathcal{A}_n \to \mathcal{A}_{n+1} \supset \mathcal{A}_n$. $\mathcal{A}$ can be thought of as defined by (ii) which is then a von Neumann algebra with automorphism σ. Thus a von Neumann K system is specified by the triple $(\mathcal{A}_n, \sigma, \omega)$.

3. $\bigcap_n$ is the set–theoretic intersection and (iii) is equivalent to saying that the isomorphism $\sigma : \mathcal{A}_n \to \mathcal{A}_{n+1}$ has no nontrivial σ–invariant subalgebra $\mathcal{B} \subset \mathcal{A}_n$. The "tail" $\bigcap_n \mathcal{A}_n$ is in fact an invariant subalgebra of all $\mathcal{A}_n$ and conversely an invariant subalgebra of any $\mathcal{A}_n$ would remain in the tail.

In the following we consider the standard setting of a representation Π_ω in a Hilbert space $\mathcal{H}_\omega$ given by the GNS construction with faithful invariant state $\omega = \omega \circ \sigma$, $\omega(a) = \langle\Omega|a|\Omega\rangle$ $\forall\, a \in \mathcal{A}$. We assume for simplicity that ω is faithful for $\mathcal{A} = \mathcal{A}''$ though for most of our results faithfulness on $\mathcal{A}_n$ $\forall n$ is needed. Since we use only this representation we take the same letter for elements of $\mathcal{A}$ and their representatives and call $\mathcal{A}'_n$ the commutant in this representation. The difficulty with the proof of mixing properties [1] has the following origin. It uses the orthogonal projection P_n defined by $P_n\mathcal{H}_\omega = \overline{\mathcal{A}_n|\Omega\rangle}$ and wants to conclude from (iii) that $\lim_{n\to-\infty} P_n = |\Omega\rangle\langle\Omega|$. This works for abelian algebras and in the tracial situation, where state preserving conditional expectations exist [T]. Only this situation was considered in the previous works on algebraic K systems [E], [KS]. It may, however, happen that $\mathcal{A}_n|\Omega\rangle$ is dense in $\mathcal{H}_\omega$ such that $P_n = \mathbf{1}$ $\forall n$. A theorem of Dixmier and Marechal [DM] states that this possibility is not so pathological for type III algebras and in fact it is physically realized by the wedge algebra in relativistic field theories. Thus we have to look for another quantity $\bar{P}$ with the property s–$\lim_{n\to-\infty} U^{-n}\bar{P}U^n = |\Omega\rangle\langle\Omega|$ where U implements σ. We shall now show that the modular operator of ω restricted to one of the $\mathcal{A}_n$ serves this purpose provided we use the Weidmann extension to define it on all of $\mathcal{H}_\omega$. The Weidmann extension is a generalization of the Friedrichs extension of a quadratic form to an operator and is explained below.

2 The Modular Operators

We define the modular operator Δ_0 of a sub–von Neumann algebra $\mathcal{A}_0 \subset \mathcal{A}$ as the Weidmann extension [W] of the positive quadratic form [BR]

$$\langle\Omega|a\Delta_0 b|\Omega\rangle = \langle\Omega|ba|\Omega\rangle \qquad \forall\, a, b \in \mathcal{A}_0 \tag{2.1}$$

and $|\Omega\rangle$ the separating and cyclic vector for Π_ω. The Weidmann extension is the Friedrichs extension on the closure of the form domain $\mathcal{D}(\Delta_0)$ and on $\mathcal{D}(\Delta_0)^\perp$ it is infinite in such a way that the resolvent $(1+\Delta_0)^{-1}$ is zero on $\mathcal{D}(\Delta_0)^\perp$. By the definition of Δ_0 in (2.1) $\mathcal{D}(\Delta_0)$ consists of those vectors $|\psi\rangle \in \mathcal{H}_\omega$ for which there is a sequence $a_k|\Omega\rangle \to |\psi\rangle$, $a_k \in \mathcal{A}_0$ such that $a_k^*|\Omega\rangle$ is also a Cauchy sequence. Since $a_k a'|\Omega\rangle = a' a_k|\Omega\rangle \to a'|\psi\rangle \;\forall\, a' \in \mathcal{A}_0'$ and $a'|\Omega\rangle = b'|\Omega\rangle \Rightarrow a'|\psi\rangle = b'|\psi\rangle$ we can define on $\mathcal{A}_0'|\Omega\rangle$ a linear operator A by the relation $Aa'|\Omega\rangle = a'|\psi\rangle$. Since $\mathcal{A}_0' \supset \mathcal{A}'$ we note that A is densely defined. Furthermore one easily sees $a_k^* a'|\Omega\rangle \to A^* a'|\Omega\rangle$ such that A^* is also densely defined and A therefore closable. Finally

$$Ab'a'|\Omega\rangle \leftarrow a_k b' a'|\Omega\rangle = b' a_k a'|\Omega\rangle \to b' A a'|\Omega\rangle$$

shows that $[A, \mathcal{A}_0'] = 0$ and therefore $\bar{A}$ is affiliated to $\mathcal{A}_0''$. Thus any $|\psi\rangle \in \mathcal{D}(\Delta_0)$ can be written as $A|\Omega\rangle$ with A possibly unbounded but affiliated to $\mathcal{A}_0''$.

Next we consider the effect of $\sigma : \mathcal{A}_n \to \mathcal{A}_{n+1}$. The operator Δ_n defined by

$$\langle\Omega|\sigma^n(a)\Delta_n\sigma^n(b)|\Omega\rangle = \langle\Omega|\sigma^n(b)\sigma^n(a)|\Omega\rangle = \langle\Omega|ba|\Omega\rangle$$

or $\Delta_n = U^{-n}\Delta_0 U^n$ with U the unitary implementor of σ coincides with Δ_m on $\mathcal{D}(\Delta_m)$ if $m < n$ but $\mathcal{D}(\Delta_m) \subset \mathcal{D}(\Delta_n)$. Thus in the sense of Weidmann we have $\Delta_m \geq \Delta_n$ for $m \leq n$ and we can appeal to his theory for the

Theorem (2.2) Let $(\mathcal{A}_n, \sigma, \omega)$ be a von Neumann K system then $(\Delta_n + 1)^{-1}$ converges strongly for $n \to -\infty$ to $\frac{1}{2}|\Omega\rangle\langle\Omega|$.

Proof: According to Weidmann's theory $(\Delta_n + 1)^{-1}$ converges strongly to zero on $(\bigcap_n \mathcal{D}(\Delta_n))^\perp$. Now $\psi \in \mathcal{D}(\Delta_n)$ can be written as $A|\Omega\rangle$ with $[A, \mathcal{A}_n'] = 0$. Thus $\psi \in \bigcap_n \mathcal{D}(\Delta_n)$ can be written as $A|\Omega\rangle$ with $[A, \mathcal{A}_n'] = 0 \;\forall\, n$ that is A is affiliated to $(\bigcup_n \mathcal{A}_n')'$. But $\mathcal{B}(\mathcal{H}) = (z\mathbf{1})' = (\bigcap_n \mathcal{A}_n)' = (\bigcup_n \mathcal{A}_n')''$ and thus $A = \lambda \in C$ and $\bigcap_n \mathcal{D}(\Delta_n) = \{\lambda|\Omega\rangle\}$. Therefore the limit of $(\Delta_n + 1)^{-1}$ is zero on $|\Omega\rangle^\perp$. Since $\Delta_n|\Omega\rangle = |\Omega\rangle \;\forall\, n$ we see that $(\Delta_n + 1)$ converges strongly to $1/2$ the projector onto $|\Omega\rangle$.

Corollary (2.3)

$$(\Delta_n^{1/2} + 1)^{-1} \to \frac{1}{2}|\Omega\rangle\langle\Omega|.$$

Proof: Strong convergence extends to continuous functions of Δ_n.

This result will be for our purpose as useful as $P_n \to |\Omega\rangle\langle\Omega|$.

3 Clustering

We shall now show that ω is clustering uniformly for all elements of any of the $\mathcal{A}_n$. $\forall\, b \in \mathcal{A}$, $a_n \in \mathcal{A}_n$, we have

$$\omega(b\sigma^k a_n) = \omega(b\sigma^k(\Delta_n^{1/2}+1)^{-1}(\Delta_n^{1/2}+1)a_n) = \langle\Omega|b(\Delta_{n+k}^{1/2}+1)^{-1}\sigma^k(\Delta_n^{1/2}+1)a_n|\Omega\rangle.$$

Now, on the one hand, we have

$$\text{s--}\lim_{k\to-\infty}(\Delta_{n+k}^{1/2}+1)^{1/2} = \frac{1}{2}|\Omega\rangle\langle\Omega|$$

and therefore

$$\langle\Omega|b(\Delta_{n+k}^{1/2}+1)^{-1} = \frac{1}{2}\langle\Omega|b|\Omega\rangle\langle\Omega| + \langle\psi|, \qquad \||\psi\rangle\| = \varepsilon(k,b)$$

with $\lim_{k\to-\infty}\varepsilon(k,b) = 0$. On the other hand, since $|\Omega\rangle$ is σ–invariant

$$\begin{aligned}\|\sigma^k(\Delta_n^{1/2}+1)a_n|\Omega\rangle\|^2 &= \|U^k(\Delta_n^{1/2}+1)a_nU^{-k}|\Omega\rangle\|^2 \\ &\le 2(\|a_n\|^2 + \langle\Omega|a_n^*\Delta_n a_n|\Omega\rangle) \le 2(\|a_n\|^2 + \langle\Omega|a_n a_n^*|\Omega\rangle) \le 4\|a_n\|^2.\end{aligned}$$

Thus

$$|\langle\Omega|b\sigma^k a_n|\Omega\rangle - \langle\Omega|b|\Omega\rangle\langle\Omega|a_n|\Omega\rangle| = |\langle\psi|\sigma^k(\Delta_n^{1/2}+1)a_n|\Omega\rangle| \le 2\varepsilon(k,b)\|a_n\|$$

and we get

Theorem (3.1) Let $(\mathcal{A}_n,\sigma,\omega)$ be a von Neumann K system. Then $\forall\, b \in \mathcal{A}$, $n \in Z$, $\varepsilon > 0$ $\exists\, N(b,n\varepsilon) \in Z^+$ such that

$$|\omega(b\sigma^k a) - \omega(b)\omega(a)| < \varepsilon\|a\| \qquad \forall\, a \in \mathcal{A}_n,\; k < -N.$$

(This clustering we call K–clustering.)

Remarks (3.2)

1. Though Theorem (3.1) holds for any n, uniformity in $\mathcal{A} = \bigcup_n \mathcal{A}_n$ is impossible. Take $a = \sigma^{-k}b^*$. Then Theorem (3.1) tells us that $\omega((b-\omega(b))(b-\omega(b)^*) \le \varepsilon$ for any $\varepsilon > 0$. Since ω is faithful this implies $b = \omega(b)$ or b is a multiple of unity. Thus K mixing means in some sense the greatest possible uniformity in mixing.

2. For $k > N$ uniformity cannot hold since for some $m > n$ $\exists\, \bar{b} \in \mathcal{A}_m$ such that $\|(\bar{b}-b)|\Omega\rangle\| < \varepsilon$. Now take $a = \sigma^{n-m-N}\bar{b}^* \in \mathcal{A}_n$ then $\omega(\bar{b}\sigma^{N+m-n}a) = \omega(\bar{b}\bar{b}^*) \ne \omega(\bar{b})\omega(\bar{b}^*)$ for $\bar{b} \ne \mathbf{1}\omega(\bar{b})$ since ω is faithful. Nevertheless without uniformity we have clustering in both directions:

Corollary (3.3) The state ω over $(\mathcal{A},\sigma)$ is mixing, i.e.

$$\lim_{k\to\pm\infty}\omega(a\sigma^k(b)c) = \omega(ac)\omega(b) \qquad \forall\, a,b,c \in \mathcal{A}.$$

For $k \to -\infty$ we have uniformity in b if $b \in \mathcal{A}_n$ for some n.

Proof: We may assume $\|a\|, \|b\|, \|c\| \leq 1$. First we observe, $\forall\, \varepsilon > 0\ \exists\, n$ and $b_n \in \mathcal{A}_n$, $\|b_n\| = 1$, such that $\|(b - b_n)|\Omega\rangle\| < \varepsilon$. Next we use

$$\omega(a\sigma^k(b)c) = \langle\Omega|c\Delta a\sigma^k b|\Omega\rangle = \langle\Omega|(c-\bar{c})\Delta^{1/2}\cdot\Delta^{1/2}a\sigma^k b|\Omega\rangle + \langle\Omega|\bar{c}\Delta a\sigma^k b|\Omega\rangle$$

where we can find $\bar{c}$, $\|\bar{c}\| = 1$ with $\|(c-\bar{c})\Omega\| < \delta$ and $\bar{c}$ analytic with respect to the modular automorphism such that $\Delta^{-1}\bar{c}\Delta \in \mathcal{A}$, $\|\Delta^{-1}\bar{c}\Delta\| < C(\delta)$. Finally we estimate

$$\langle\Omega|\bar{c}\Delta a\sigma^k b|\Omega\rangle = \langle\Omega|\bar{c}\Delta a\sigma^k(b-b_n)|\Omega\rangle + \langle\Omega|\bar{c}\Delta a\sigma^k b_n|\Omega\rangle.$$

The first term is dominated by $C(\delta)\cdot\varepsilon$, the last term tends to $\langle\Omega|a\bar{c}|\Omega\rangle\langle\Omega|b_n|\Omega\rangle$ for $k \to -\infty$ and does so uniformly if $b \in \mathcal{A}_n$. For $k \to +\infty$ observe $\omega(a\sigma^k(b)c) = \omega(\sigma^k(b)c\Delta a) = \omega(b\sigma^{-k}c\Delta a)$ and $\exists\, d_n \in \mathcal{A}_n$ with $\|(d_n - c\Delta a)|\Omega\rangle\| < \varepsilon$.

The result can be strengthened to r–clustering:

Theorem (3.4) A von Neumann K system is r–clustering, i.e.

$$\lim_{\inf|n_\ell - n_k|\to\infty} \omega(\sigma_{n_1}a_1\sigma_{n_2}a_2\ldots\sigma_{n_r}a_r) = \omega(a_1)\omega(a_2)\ldots\omega(a_r) \qquad \forall\, r \in N.$$

More precisely $\forall\,\varepsilon > 0$ and $a_1\ldots a_r \in \mathcal{A}\ \exists\, N$ such that if $|n_k - n_\ell| > N\ \forall\, k \neq \ell$ the two sides agree within ε.

Proof: We need two kinds of approximations for the elements a_i. By strictly local elements $\widehat{a}_i \in \mathcal{A}_n$ for some n and by elements analytic with respect to the modular automorphism $\widetilde{a}$ such that $\|\Delta\widetilde{a}\Delta^{-1}\| < \infty$. The second is needed for reordering the product and is trivial classically or in any tracial state. Some care is needed since the two approximations do not commute.

It is known that analytic elements are strongly dense, that is $\forall\, a_i,\ \varepsilon > 0,\ \exists\,\widetilde{a}_i$ such that $\|(a_i - \widetilde{a}_i)|\Omega\rangle\| < \varepsilon$, and $\bar{a}_i = \Delta\widetilde{a}_i\Delta^{-1}$ has norm $\leq c\,\forall\, i$. Furthermore, $\widetilde{a}_i|\Omega\rangle = a_i'|\Omega\rangle$ with $a_i' \in \mathcal{A}_i$, $\|a_i'\| = \|\Delta^{1/2}\widetilde{a}_i\Delta^{-1/2}\| \leq c'$. To show that we can restrict ourselves to analytic elements we first prove the

Lemma (3.4a) Given a set of operators $\{a_1,\ldots,a_r\}$. For given $\varepsilon > 0$ let $\widetilde{a}_i$ be the analytic element such that $\|(a_i - \widetilde{a}_i)|\Omega\rangle\| < \varepsilon$. Then $\exists\, C(r)$ such that

$$\left\|\left(\prod_{i=1}^r a_{i,n_i} - \prod_{i=1}^r \widetilde{a}_{i,n_i}\right)|\Omega\rangle\right\| < \varepsilon C(r) \text{ with } a_{i,n_i} = \sigma^{n_i}(a_i) \quad \forall\,\{n_i\} \in Z^r.$$

Proof: We proceed by induction in r, $C(1) = 1$. Now write

$$\prod_{i=1}^r a_{i,n_i} - \prod_{i=1}^r \widetilde{a}_{i,n_i} = \prod_{i+1}^{r-1} a_{i,n_i}(a_{r,n_r} - \widetilde{a}_{r,n_r}) + \left(\prod_{i=1}^{r-1} a_{i,n_i} - \prod_{i=1}^{r-1}\widetilde{a}_{i,n_i}\right)\widetilde{a}_{r,n_r}$$

On applying to $|\Omega\rangle$ we see $C(r+1) \leq \varepsilon + C'C(r)$ and since σ and Δ commute and $|\Omega\rangle$ is σ–invariant all approximations are uniform in $\{n_i\}$. Therefore $C(r) \leq rC'^r$.

Thus we see that it is enough to prove the theorem for analytic elements and henceforth we shall drop the $\tilde{}$ assuming a_i to be analytic. Next take j such that $\max_i n_i = n_j$, k such that $n_j - n_k = \min_{i\neq j}(n_j - n_i)$. Then

$$\begin{aligned} \omega\left(\prod_i a_{i,n_i}\right) &= \omega(a_{j,n_j} \dots a_{r,n_r} \bar{a}_{1,n_1} \dots \bar{a}_{j-1,n_{j-1}}) \\ &= \omega(a_j \sigma_{-n_j+n_k}(a_{j+1,n_{j+1}-n_k} \dots \bar{a}_{j-1,n_{j-1}-n_k})) \end{aligned} \tag{3.4b}$$

where all $n_\ell - n_k \leq 0$ since $\ell \neq j$. Now we have to make the local approximation and since $\mathcal{A}$ is the strong closure of $\bigcup_n \mathcal{A}_n$ $\forall \varepsilon$ $\exists n$ such that $\|(a_i - \widehat{a}_i)|\Omega\rangle\| < \varepsilon$ $\forall i$, $\widehat{a}_i \in \mathcal{A}_n$ (some a_i might be $\bar{a}_i$) and thus $\|(a_{i,n} - \widehat{a}_{i,n})|\Omega\rangle\| < \varepsilon$. Remember that the a_i are supposed to be analytic under the modular automorphism and so we can again appeal to the above lemma to conclude that

$$\left\| \left(\prod_{i=1}^r a_{i,\bar{n}_i} - \prod_{i=1}^r \widehat{a}_{i,\bar{n}_i} \right) |\Omega\rangle \right\| < \varepsilon C(r) \qquad \forall \bar{n}_i \in Z.$$

Inserting into (3.4b) we note that all $\widehat{a}_{\ell,n_\ell - n_k}$ are also in $\mathcal{A}_n$ since $n_\ell - n_k \leq 0$. Thus we can use the uniformity over $\mathcal{A}_n$ in (3.1) and $\|\widehat{a}\| < C$ from Kaplansky's theorem to conclude for $n_j > n_k + N$

$$|\omega(a_j \sigma_{-n_j+n_k}(a_{j+1,n_{j+1}-n_k} \dots \bar{a}_{j-1,n_{j-1}-n_k}) - \omega(a_j)\omega(a_{j+1,n_{j+1}} \dots \bar{a}_{j-1,n_{j-1}})| < \varepsilon C^{r-1}$$

since $n_\ell - n_k \leq 0$ $\forall \ell \neq j$. Then by $r-1$ inductive steps we arrive at (3.4).

Theorem (3.1) has a sort of converse:

Theorem (3.5) Let $\mathcal{A}$ be a C^* algebraic K system with respect to the automorphism σ and ω a σ–invariant separating state that is K–clustering with respect to the C^* system $\mathcal{A} = \bigcup \mathcal{A}_n$. Then it is also K–clustering for $\mathcal{A}'' = (\bigcup_n \mathcal{A}_n'')''$ and $\bigcap_n \mathcal{A}_n'' = \lambda \mathbf{1}$. Thus K–clustering ensures that a C^*–K system extends to a von Neumann K system and is also necessary (see Proposition (4.3)).

Proof: To every $a \in \mathcal{A}$, $\varepsilon > 0$ $\exists N$ such that

$$|\omega(a\sigma^n b) - \omega(a)\omega(b)| \leq \varepsilon \|b\| \qquad \forall n < 0,\ b \in \mathcal{A}_N.$$

Take $\bar{a} \in \mathcal{A}''$, $\bar{b} \in \mathcal{A}_N''$. Then they can be strongly approximated within ε by bounded operators a_k, b_k in $\mathcal{A}$ resp. in $\mathcal{A}_N$ such that $\forall n < 0$

$$\begin{aligned} |\omega(\bar{a}\sigma^n \bar{b}) - \omega(\bar{a})\omega(\bar{b})| \ &\leq \ |\omega(a_k \sigma^n b_k) - \omega(a_k)\omega(b_k)| + |\omega((\bar{a} - a_k)\sigma^n \bar{b}) \\ &\quad + \omega(a_k \sigma^n(\bar{b} - b_k)) + \omega(a_k)\omega(b_k) - \omega(\bar{a})\omega(\bar{b})| < 5\varepsilon \end{aligned}$$

if $\|\bar{a}\| = \|\bar{b}\| = 1$. Finally this implies

$$|\omega(bt) - \omega(t)\omega(b)| < \varepsilon \qquad \forall \varepsilon > 0,\ \forall t \in \text{Tail} \subset \mathcal{A}_N'' \ \forall N.$$

But $\omega(tb) = \omega(t)\omega(b)$ implies $t|\Omega\rangle = \omega(t)|\Omega\rangle$ and since $|\Omega\rangle$ was supposed to be separating (also for $\mathcal{A}''$) we conclude $t = c\mathbf{1}$. Therefore the tail is trivial.

Corollary (3.6)

(i) A von Neumann K system is weakly mixing (i.e. $\forall\, a,b \in \mathcal{A}\ \exists\, \delta > 0,\ T \in \mathbf{R}^+$ such that $\|ab_t\| > \delta\ \forall\, t > T$).

(ii) If it is strongly asymptotically abelian it is even strongly mixing (i.e. $\lim_{t\to\infty} \|ab_t\| = \|a\|\|b\|$).

Proof:

(i) $\|ab_t\|^2 \geq \omega(b_t^* a^* a b_t) = \omega(b^* a^*_{-t} a_{-t} b) \to \omega(b^* b)\omega(a^* a) > 0$ since ω is faithful.

(ii) If it is strongly asymptotic abelian it is even hyperclustering and (i) + hyperclustering implies (ii) [NT2].

4 Examples

Consider the C^* algebra $\mathcal{A}_c$ generated by the creation operators a_f^* and the annihilation operators a_g with $[a_f^*, a_g]_+ = (g|f),\ f,g \in L^2(\mathbf{R})$. As automorphism take the shift $\sigma^x a_f = a_{f_x}$, $f_x(y) = f(y-x)$ and for the state take any primary translation invariant state ω. Define the C^* algebras $\mathcal{A}_{(-\infty,x)}$ to be even algebras generated by $\{a_f | f \in L^2,\ \mathrm{supp}\ f \subset (-\infty, x\}$ and similarly $\mathcal{A}_{(x,\infty)}$. Then we take $\mathcal{A}_n = \mathcal{A}''_{(-\infty,n)}$ where the weak closure $\mathcal{A}''_{(-\infty,n)}$ is formed in the representation from ω. We claim $\{\mathcal{A}_n, \sigma, \omega\}$ is a von Neumann K system. The only nontrivial assertion is the triviality of the tail $\bigcap_n \mathcal{A}''_n = z\mathbf{1}$. For this purpose observe that $\mathcal{A}'_{(-\infty,x)} \supset \mathcal{A}_{(x,\infty)}\ \forall x$ and thus $\mathcal{A}'_{(-\infty,x)} \supset \mathcal{A}''_{(x,\infty)}$. Hence

$$\left(\bigcap_n \mathcal{A}''_n\right)' = \bigcup_n \mathcal{A}'_n = \bigcup_n (\mathcal{A}'_{(-\infty,n)} \cup \mathcal{A}''_{(n,\infty)}) = \bigcup_n \mathcal{A}'_{(-\infty,n)} \bigcup_m \mathcal{A}''_{(m,\infty)} = \mathcal{A}''' \cup \mathcal{A}'' = (\mathcal{A}'' \cap \mathcal{A}')'.$$

This is equivalent to $\bigcap_n \mathcal{A}''_n \subset \mathcal{A}'' \cap \mathcal{A}' = \{z\mathbf{1}\}$ since ω is primary which means that the center is trivial.

¿From (3.3) we conclude that $(\mathcal{A}, \sigma)$ is weakly mixing and its even part is strongly mixing since it is even norm asymptotic abelian. Strong mixing does not hold for mixtures of even and odd elements of $\mathcal{A}$. Consider $f,g \in L^2(\mathbf{R})$, supp $f,g \subset (0,1)$. Then $[a_f, \sigma^n a_g]_+ = 0\ \forall n \neq 0$ and $p_{f,g} = \frac{1}{2}(1 + a_{f,g} + a^*_{f,g})$ are two projections with $p_f \sigma^n(p_g) p_f = \frac{1}{2} p_f\ \forall n \neq 0$. Therefore

$$\lim_{n\to\infty} \|p_f \sigma^n(p_g) p_f\| = \frac{1}{2} \neq 1 = \|p_f\|^2 \|p_g\|^2. \tag{4.1}$$

We shall finally illustrate that a C^*K system need not generate a von Neumann K system even if it is based on the GNS construction with an extremal invariant state. As discretized variant of the previous example we consider an infinite spin chain. The C^* algebra $\mathcal{A} = \bigotimes_{i=-\infty}^{\infty} M_2^{(i)}$ is generated by the Pauli matrices $\sigma_\alpha^{(i)}$, $[\sigma_\alpha^{(n)}, \sigma_\beta^{(\ell)}] = i\delta_n^\ell \varepsilon_{\alpha\beta\gamma} \sigma_\gamma^{(k)}$ and $\mathcal{A}$ is the norm closure of polynomials in the σ's. The subalgebras $\mathcal{A}_n$ are $\bigotimes_{i=-\infty}^{n} M_2^{(i)}$ and the automorphism τ is $\tau(\sigma_\alpha^{(i)}) = \sigma_\alpha^{(i+1)}$. This satisfies the axioms (i–iii) on the C^* level only, that is the $\mathcal{A}_n$, $\mathcal{A}$ are not yet von Neumann algebras. A state ω is specified by the expectation value of a finite product of σ's. We consider the antiferromagnetic states

$$\begin{aligned} \omega_e(\sigma_{\alpha_1}^{(i_1)} \ldots \sigma_{\alpha_m}^{(i_m)}) &= (-)^{i_1+i_2+\ldots+i_m} n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_m}, \quad n \in \mathbf{R}^3,\ \|n\| \leq 1, \\ \omega_o(\sigma_{\alpha_1}^{(i_1)} \ldots \sigma_{\alpha_m}^{(i_m)}) &= (-)^{m+i_1+\ldots+i_m} n_{\alpha_1} n_{\alpha_2} \ldots n_{\alpha_m}. \end{aligned} \tag{4.2}$$

For $\|n\| = 1$ these states are pure, for $\|n\| < 1$ they are faithful. Neither is translation invariant but $\omega_e = \omega_o \circ \tau$, $\omega_o = \omega_e \circ \tau$, so they are invariant under τ^2. Therefore $\omega = \frac{1}{2}(\omega_e + \omega_o)$ is extremal τ–invariant for the following reason: τ is asymptotically abelian, therefore the decomposition into extremal invariant states is coarser than the factor decomposition which is given by $\omega = \frac{1}{2}(\omega_e + \omega_o)$. This does not allow for any coarser decomposition than the trivial one. Now we base our GNS construction on it. Thus $\mathcal{A}_n''$, $\mathcal{A}''$ denotes the strong closure of $\mathcal{A}_n$, $\mathcal{A}$ in this representation. One easily verifies the

Proposition (4.3) The mean magnetizations

$$\begin{aligned} \vec{m}_e &= \text{s–}\lim_{N\to\infty} \frac{1}{N+1} \sum_{i=-2N}^{-N} \vec{\sigma}^{(2i)}, \\ \vec{m}_o &= \text{s–}\lim_{N\to\infty} \frac{1}{N+1} \sum_{i=-2N}^{-N} \vec{\sigma}^{(2i+1)}, \qquad \vec{m}_o \neq \vec{m}_e, \end{aligned}$$

are contained in all $\mathcal{A}_n''$ and therefore $\bigcap_n \mathcal{A}_n'' \neq z\mathbf{1}$.

Proof: Since $\| \sum \vec{v}_i \cdot \vec{\sigma}^{(i)} \| = \sum_i \|\vec{v}_i\|$ we see that the limits do not exist as norm limits but the norms of $\frac{1}{N+1} \sum_{i=-2N}^{-N} \cdot$ are uniformly bounded in N and thus strong convergence on a total set is enough. It is immediately seen to converge strongly on vectors which are generated by a finite number of σ's on $|\Omega\rangle$. The representation Π_ω decomposes naturally into $\begin{pmatrix} \Pi_e & 0 \\ 0 & \Pi_o \end{pmatrix}$ and in this representation

$$\vec{m}_e = \begin{pmatrix} \vec{n} & 0 \\ 0 & -\vec{n} \end{pmatrix}, \qquad \vec{m}_o = \begin{pmatrix} -\vec{n} & 0 \\ 0 & \vec{n} \end{pmatrix},$$

and thus m_e and m_o are $\neq \lambda\mathbf{1}$. Notice that the center coincides with the tail and is not pointwise invariant under $\tau : \tau(m_e) = m_o$, $\tau(m_o) = m_e$.

Remarks

1. The state ω is not clustering since $\omega(\sigma_\alpha^{(i)}) = 0\ \forall i$ but $\omega(\sigma_\alpha^{(i)} \sigma_\beta^{(k)}) = (-)^{i+k} n_\alpha n_\beta\ \forall i \neq k$. Thus the $\mathcal{A}_n''$ cannot form a von Neumann K system.
2. In this case the subspaces $\mathcal{A}_n|\Omega\rangle$ are strictly decreasing but the proof of K–clustering of (3.1) using the projections P_n onto $\mathcal{A}_n|\Omega\rangle$ does not apply. The reason is that P_n projects onto $\overline{\mathcal{A}_n|\Omega\rangle}$ which contains $\mathcal{A}_n''|\Omega\rangle$ and the $\mathcal{A}_n''$ have a nontrivial tail.

3. Though norm asymptotic abelianness implies that a factor representation has to be based on an extremal invariant state we see that the converse is not true. $(\mathcal{A}, \tau)$ is norm asymptotic abelian and ω is extremal invariant but $\mathcal{A}''$ has a center which contains $\vec{m}_e$ and $\vec{m}_o$ and is thus nontrivial. Of course $(\mathcal{A}, \tau^2)$ is also asymptotically abelian but the center contains the τ^2–invariant elements m_e, m_o and thus ω is not τ^2 extremal invariant.

¿From these examples we see that a C^*K system will lead to von Neumann K systems only if the underlying state is extremal invariant, but this is not sufficient. Nevertheless the fewer invariant states exist the better are the chances to have a von Neumann K system. To restrict the number of invariant states we first mention

Theorem (4.4) Let σ be an automorphism of the CAR algebra $\mathcal{A}$ which is asymptotically antiabelian. (This means that for any two field operators a_f, a_g and $\varepsilon > 0$ exists T such that $\|[a_f, \sigma^t a_g]_+\| < \varepsilon$, $\|[a_f, \sigma^t a_g^*]_+\| < \varepsilon \; \forall t > T$.) Then any σ–invariant state ω is even.

Proof: Asymptotic antiabelianness carries over to all odd elements o and from $|\omega(a)|^2 = \omega(a^*)\omega(a) \leq \frac{1}{2}\omega([a^*, a]_+) \leq \frac{1}{2}\|[a^*, a]_+\|$ we infer

$$\begin{aligned} |\omega(o)|^2 &= \left| \sum_{k=1}^{N} \frac{1}{N} \omega(\sigma^{kT} o) \right|^2 \leq \frac{1}{2N^2} \sum_{i,k} \omega([\sigma^{iT} o^*, \sigma^{kT} o]_+) \\ &\leq \frac{1}{2N^2} \sum_{k=1}^{N} \omega(\sigma^{kT}[o^*, o]_+) + \varepsilon \frac{(N-1)}{2N} \leq \frac{\|o\|^2}{N} + \varepsilon \qquad \forall N, \varepsilon. \end{aligned}$$

Corollary (4.5) Let σ be a quasifree automorphism of $\mathcal{A}$ whose generator U has an absolutely continuous spectrum (in particular the shift) then any σ–invariant state is even.

Proof:

$$[a_f, \sigma^t a_g]_+ = [a_f, a_{U^t g}]_+ = 0$$

$$\|[a_f, \sigma^t a_g^*]_+\| = |\langle U^t g | f \rangle| \to 0 \text{ for } t \to \infty \text{ by Riemann–Lebesgue.}$$

It may even happen that commutativity and anticommutativity are so irregularly distributed that there is only one invariant state:

Example (4.6) (Price–Powers–shift) [Pr,Po]
Consider the (initial) C^* algebra $\mathcal{A}$ generated by hermitian elements e_i, $e_i^2 = 1$, $i \in Z$, with the commutation rules $e_i e_j = (-)^{g(i-j)} e_j e_i$ where $g : Z \to \{0, 1\}$, $g(0) = 0$. Thus $\mathcal{A}$ consists of linear combinations of "words" $w = e_{i_1} \ldots e_{i_k}$, $i_1 \leq i_2 \leq \ldots \leq i_k$, and norm limits thereof. $\sigma(e_i) = e_{i+1}$ generates an automorphism of $\mathcal{A}$ and the $\mathcal{A}_n$ generated by e_i, $-\infty < i \leq n$ make $(\mathcal{A}_n, \sigma)$ a C^*K system. Two words commute or anticommute and if g is so irregular that for each word $w \neq c\mathbf{1}$ there exists an infinite sequence $\{n_k\}$ such that $[\sigma^{n_i} w, \sigma^{n_k} w]_+ = 0 \; \forall i \neq k$ then we call $(\mathcal{A}, \sigma)$ a completely irregular Price–Powers–shift.

Theorem (4.7) For a completely irregular Price–Powers–shift only the tracial state ω is invariant. ($\omega(w) = 0 \; \forall w \neq c\mathbf{1}$.) This state is K–clustering so the system is a von Neumann K system.

Proof: Take

$$\omega(w) = \omega\left(\frac{1}{N}\sum_{k=1}^{N}\sigma^{n_k}w\right)$$

and proceed as for (4.4). Therefore $\omega(w) = 0\ \forall\, w \neq \lambda\mathbf{1}$ which characterizes the tracial state [Pr,Po,NT3].

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