Symmetry Invariance and Center Manifolds

for Dynamical Systems

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Abstract

In this paper we analyze the role of general (possibly nonlinear) time-independent Lie point symmetries in the study of finite-dimensional autonomous dynamical systems, and their relationship with the presence of manifolds invariant under the dynamical flow. We first show that stable and unstable manifolds are left invariant by all Lie point symmetries admitted by the dynamical system. An identical result cannot hold for the center manifolds, because they are in general not uniquely defined. This nonuniqueness, and the possibility that Lie point symmetries map a center manifold into a different one, lead to some interesting features which we will discuss in detail. We can conclude that - once the reduction of the dynamics to the center manifold has been performed - the reduced problem automatically inherites a Lie point symmetry from the original problem: this permits to extend properties, well known in standard equivariant bifurcation theory, to the case of general Lie point symmetries; in particular, we can extend classical results, obtained by means of Lyapunov-Schmidt projection, to the case of bifurcation equations obtained by means of reduction to the center manifold. We also discuss the reduction of the dynamical system into normal form (in the sense of Poincaré-Birkhoff-Dulac) and respectively into the "Shoshitaishvili form" (in both cases one center manifold is given by a "flat" manifold), and the relationship existing between nonuniqueness of center manifolds, perturbative expansions, and analyticity requirements. Finally, we present some examples which cover several aspects of the preceding discussion.

1. Introduction

In this paper we analyze the role of general Lie symmetries [1-3] (precisely, time-independent Lie point (LP) symmetries) in the study of finite-dimensional autonomous dynamical systems (DS), and their relationship with the presence of manifolds invariant under the dynamical flow. Our interest will be mainly devoted on the properties of *center manifolds* (CM) [4-8]: it is well known in fact that the reduction of the dynamics to the CM is a fundamental step in the analysis of the DS.

Our approach permits to extend properties, well known in standard equivariant bifurcation theory [9-13], to the case of general LP symmetries; in particular, we can extend classical results obtained by means of Lyapunov-Schmidt projection to the case of bifurcation equations obtained by means of CM reduction (cf. [14-15]).

In Sect.2 we give some basic results concerning the symmetry properties of flow-invariant manifolds: we show that stable and unstable manifolds are left invariant by all LP symmetries of the DS. An identical result cannot hold for CMs, because they are in general not uniquely defined. This nonuniqueness of the CM, and the possibility that LP symmetries map a CM into a different one, lead to some interesting features which will be discussed in Sections from 4 to 8. However, we can adfirm (Sect.7) that, in short, any LP symmetry leaves invariant one CM, and, conversely, that any CM is left invariant by some LP symmetry. This is enough to conclude that - once the reduction of the DS to the CM has been performed - the reduced problem automatically inherites a LP symmetry from the original problem (Sect. 3 and 7). In Sect. 4 and 6 we discuss the reduction of the DS into normal form (in the sense of Poincaré-Birkhoff-Dulac [16,17,18], and respectively into the "Shoshitaishvili form" [19]: in both cases one CM is given by a "flat" manifold of the form $y = 0$. These methods give us the occasion to add some comment on the relationship existing between nonuniqueness of CM, perturbative expansions, and analyticity requirements (Sect. 5 and 6). Sect.8 is devoted to some short additional remark concerning specifically the case of bifurcation problems. In Sect.9, finally, we present some examples, which cover several aspects of the preceding discussion.

2. Some basic preliminaries

We consider $n-$dimensional autonomous DS

$$\dot{u} = F(u) \tag{2.1}$$

defined for u in a smooth manifold $M \subseteq R^n$, where $F(u)$ is assumed to be smooth (e.g. analytical); F may also depend on some real parameter λ, in order that bifurcation problems can be considered within this scheme (see Sect. 8 for a specific discussion of this point). Let $\mathcal{G}$ be the algebra of time-independent smooth vector fields on M

$$\eta = \sum_{i=1}^{n} \vartheta_i(u) \frac{\partial}{\partial u_i} \equiv \vartheta \partial_u \tag{2.2}$$

which are the generators of LP symmetries for the DS (2.1). As well known [2,4,15], $\eta \in \mathcal{G}$ if and only if

$$[\eta, \eta_F] = 0 \tag{2.3}$$

where

$$\eta_F = F \partial_u \tag{2.4}$$

is the generator of the dynamical flow (which is a "trivial" LP symmetry generator, of course).

We then have the following basic lemmas.

Lemma 2.1. *Any manifold $W \subseteq M$ which is invariant under the dynamical flow is transformed by the LP symmetries $\eta \in \mathcal{G}$ into a flow-invariant manifold.*

Proof. Let $\mathcal{W}(u) = 0$ be the equation describing the CM W, then W is flow-invariant iff $\eta_F \cdot \mathcal{W} = 0$. From (2.3), one has immediately

$$\eta_F \cdot (\eta \cdot \mathcal{W}) = 0 \tag{2.5}$$

which implies the flow-invariance of the manifold W_ε given by the transformation $(\exp \varepsilon\, \eta) \cdot \mathcal{W}$ (where $\varepsilon \in R$ parametrizes the finite group action). •

It is understood that, here and in the following, our results are true "locally": this is an obvious restriction when dealing with nonlinear DS; on the other hand, this holds for symmetry transformations too, which are in fact defined locally by means of their Lie algebra [2].

¿From now on, we will assume that there is a stationary point $u_0 \in M$ for the DS, and that all LP symmetries satisfy the condition

$$\vartheta_i(u_0) = 0 \qquad \forall\ i = 1,\ \dots, n \tag{2.6}$$

in order that the stationary point is left fixed by the LP symmetries. Notice that this condition is automatically satisfied if u_0 is an isolated stationary point. It is not restrictive to put $u_0 = 0$ and then write the DS (2.1) in the form

$$\dot{u} = F(u) = Lu + h(u) \tag{2.7}$$

where $L = \partial_u F(0)$ is the linear part of F.

Let now $\{\sigma_i\}$ be the set of the eigenvalues of L, and let us denote by V_0 the $k-$dimensional subspace spanned by the (generalized) eigenvectors corresponding to any subset of these eigenvalues (if one of these is complex, we include in V_0 also the eigenvectors with the complex conjugated eigenvalue), and by V_1 the subspace of the (generalized) eigenvectors corresponding to all other eigenvalues of L. Starting from Sect. 3, we will choose V_0 as the subspace of the (generalized) eigenvectors with eigenvalues σ_i^0 such that $\mathrm{Re}\sigma_i^0 = 0$, so that V_0 will be the center eigenspace E_c for the DS; but for the moment the only hypothesis we need is that all eigenvectors in V_0 have different eigenvalues from those in V_1. Performing a linear change of coordinates, and denoting by x the vectors in V_0 and by y those in V_1, the matrix L can be put in block-diagonal form and the problem (2.7) becomes

$$\begin{aligned} \dot{x} &= X(x,y) = L_0 x + X_h(x,y) \\ \dot{y} &= Y(x,y) = L_1 y + Y_h(x,y) \end{aligned} \tag{2.8}$$

Writing a LP symmetry for this problem in the form

$$\eta = \vartheta(u)\partial_u = \sum_{r=1}^{k} \varphi_r(x,y)\partial_{x_r} + \sum_{p=k+1}^{n} \psi_p(x,y)\partial_{y_p} \equiv \varphi\partial_x + \psi\partial_y \tag{2.9}$$

we have the following results.

Lemma 2.2. *With the above notations* (2.8,9) *for the DS and the LP symmetry, one has, for all indices* r, s, p, q

$$\frac{\partial \psi_p(0,0)}{\partial x_r} = \frac{\partial \varphi_s(0,0)}{\partial y_q} = 0 \qquad (r,s = 1,\ldots,k\ ;\ p,q = k+1,\ldots,n) \tag{2.10}$$

Proof. Writing explicitly condition (2.2) with the above notations, one has in particular (sum over repeated indices)

$$\varphi_r \frac{\partial Y_p}{\partial x_r} + \psi_q \frac{\partial Y_p}{\partial y_q} = X_r \frac{\partial \psi_p}{\partial x_r} + Y_q \frac{\partial \psi_p}{\partial y_q}$$

Differentiating with respect to x_s, and putting $x = y = 0$, one finds, using also (2.6),

$$(L_1)_{pq} \frac{\partial \psi_q(0,0)}{\partial x_s} = \frac{\partial \psi_p(0,0)}{\partial x_r} (L_0)_{rs}$$

or $L_1 D = D L_0$ where D is the constant matrix $D_{pr} = \partial \psi_p(0,0)/\partial x_r$, which implies $D = 0$. An identical argument shows $\partial \varphi_s(0,0)/\partial y_q = 0$. •

Lemma 2.3. *Let $y = m(x)$ be the equation describing any $(n-k)-$dimensional smooth manifold M tangent at the origin to the subspace $y = 0$:*

$$m(0) = 0 \qquad \partial_x m(0) = 0$$

Then all transformations $\eta = \varphi \partial_x + \psi \partial_y$ which satisfy the conditions (2.6) *and* (2.10) *change this manifold into a manifold which is still tangent at the origin to $y = 0$.*

Proof. The manifold changed under the action of the transformations η can be represented by means of an expansion in powers of the parameter ε, where $\mathcal{M} \equiv y - m(x) = \mathcal{M}(x,y;0)$:

$$\begin{aligned}\mathcal{M}(x,y;\varepsilon) =& \Big(1 + \varepsilon\eta + \frac{\varepsilon^2}{2}\eta^2 + \ldots\Big)\mathcal{M}(x,y;0) = \\ &= y - m(x) + \varepsilon(\psi - \partial_x w \cdot \varphi) + \frac{\varepsilon^2}{2}(\varphi\partial_x + \psi\partial_y)(\psi - \partial_x w \cdot \varphi) + \ldots = 0\end{aligned}$$

It is easy to check at the first order in ε that $y = 0$ is tangent at the origin, and then extend the result to all orders. •

It can be interesting to notice that (in contrast with Lemma 2.1) the result of Lemma 2.3 is essentially "geometrical" (not "dynamical"), in fact we do not need here the hypothesis that M is an invariant manifold under a dynamical flow, nor that η is a LP symmetry.

An obvious consequence obtained combining all the above conclusions (in short, LP symmetries preserve flow-invariance of manifolds and their tangency property) is the following.

Proposition 2.4. *All LP symmetries $\eta \in \mathcal{G}$ leave invariant (i.e. map into themselves, not necessarily pointwise) both the stable and unstable manifolds W of the DS, i.e.*

$$\eta : W \to TW \tag{2.11}$$

Also, LP symmetries transform any CM into a CM (possibly the same).

The symmetry invariance of stable and unstable manifolds can actually be inferred directly from Lemma 2.1: in fact, they are flow-invariant and uniquely defined, being characterized by the time-exponential behaviour of the trajectories in them, so that LP symmetries, being time-independent, cannot map them into different invariant manifolds. The situation is not so simple (and in fact a weaker result is true) in the case of CMs, which in fact are in general not unique, and where the dynamics is more complicated.

It should be remarked that the last statement of the above Proposition means, in the language of fluid dynamics (see, e.g. [20,21]), that a LP transformation cannot connect *slow* and *fast* modes.

The nonuniqueness of the CM, and the related possibility that a LP symmetry map a CM into a different CM, lead to some nontrivial features within the present context, which will be discussed in Sections 4-7.

To conclude this section, it can be interesting to check, from the point of view of the symmetry transformations - at least in a specially simple situation - one of the peculiar properties of CMs. It is known in fact that, if W is a CM, there is an open subset $U \subset M$, with $U \cap W \neq \emptyset$, such that if a solution $f(t) \in U$, $\forall t \in R$, then $f(t)$ belongs to a CM [5-8]. Assume here for simplicity that the CM is unique, $\bar{U}$ is compact and that a solution $f(t)$ actually belongs $\forall\ t \in R$, to $W \cap U$: applying the transformation generated by a LP symmetry $\eta \in \mathcal{G}$, one obtains a $f_\varepsilon(t) \equiv \big((\exp \varepsilon\ \eta) \cdot f\big)(t)$ which is again a solution. But, $\forall \delta > 0$, $\exists \varepsilon$ such that $\sup_{t\in R} |f_\varepsilon(t) - f(t)| < \delta$, due to the smoothness of η; then if δ is small enough, $f_\varepsilon(t) \in U$ and then also $f_\varepsilon(t) \in W$ for the mentioned property of the CM, which confirms the previous result that W is mapped into itself by the symmetries $\eta \in \mathcal{G}$.

For some related considerations, and the possible extension of these properties to the infinite-dimensional case, see [22,23,24].

3. Reduction of the DS to a CM

We now examine more closely the dynamics. Let us recall our notations: we put (here and in the following - unless otherwise stated)

$$u \equiv (x, y) \in R^n \tag{3.1}$$

where $x \in R^k$ span the center eigenspace E_c and $y \in R^{n-k}$ the orthogonal subspace, then the DS can be written

$$\begin{aligned} \dot{x} =& X(x, y) = L_0 x + X_h(x, y) \\ \dot{y} =& Y(x, y) = L_1 y + Y_h(x, y) \end{aligned} \tag{3.2}$$

where the eigenvalues of L_0 have vanishing real part, and those of L_1 have nonvanishing real part.

Let us choose now a CM W, described by the $(n-k)$ equations

$$y = w(x) \tag{3.3}$$

and put

$$z = y - w(x) \qquad\qquad z \in R^{n-m} \tag{3.4}$$

Then the original DS becomes

$$\begin{aligned} \dot{x} =& \widetilde{X}(x, z) = L_0 x + \widetilde{X}_h(x, z) \\ \dot{z} =& Z(x, z) = L_1 z + Z_h(x, z) \end{aligned} \tag{3.5}$$

where $\widetilde{X}(x, z) = X(x, y(x, z))$ etc., and $Z_h(x, 0) = 0$, as a consequence of the flow-invariance of the CM $z = 0$. Notice that if the CM (3.3) is C^k, then the DS (3.5) will be only C^k although the original DS was C^∞ (or even analytic).

Let

$$\eta = \vartheta(u)\partial_u = \varphi(x, y)\partial_x + \psi(x, y)\partial_y \tag{3.6}$$

generate a LP symmetry of the original DS (3.2): then the system (3.5) admits the LP symmetry generated by

$$\widetilde{\eta} = \widetilde{\varphi}(x, z)\partial_x + (\widetilde{\psi} - \partial_x w \cdot \widetilde{\varphi})\partial_z \equiv \Phi(x, z)\partial_x + \Psi(x, z)\partial_z \tag{3.7}$$

Then we can say:

Proposition 3.1. *The CM* (3.3) *is left invariant under the LP symmetry* (3.6) *if and only if*

$$\psi(x,w(x)) = \big(\partial_x w(x)\big)\cdot\varphi(x,w(x)) \qquad i.e. \qquad \Psi(x,0)=0 \tag{3.8}$$

An immediate but important consequence of the above arguments concerns the symmetries inherited by the dynamics restricted to the CM. Precisely, we have the following result.

Proposition 3.2. *If the LP symmetry $\eta \in \mathcal{G}$* (3.6) *admitted by the DS* (3.2) *leaves invariant a given CM $y = w(x)$, then the DS restricted to this CM, i.e.*

$$\dot{x} = \widetilde{X}(x,0) = X(x,w(x)) \tag{3.9}$$

possesses the LP symmetry generated by

$$\widehat{\eta} = \Phi(x,0)\partial_x = \varphi(x,w(x))\partial_x \tag{3.10}$$

If in particular the CM is unique, this is true for any LP symmetry admitted by the DS.

This is an useful result: indeed, if the CM is unique, this proposition ensures that the DS once reduced to the CM, automatically inherites the LP symmetries of the original problem. Then, this result can be viewed as the counterpart of the Sattinger theorem [10]: this theorem in fact, stated in the context of equivariant bifurcation theory (and also extended to the case of LP symmetries [14,25]), relates the symmetries of the original problem to those of the reduced problem obtained by means of Lyapunov-Schmidt projection.

Some further discussion is needed in the case the CM is not unique. We shall show that this nonuniqueness is related to the presence of non-analytic terms, and that in some sense this nonuniqueness can be "controlled" by some analyticity arguments; in addition, we shall give some argument to show that, generically, any LP symmetry leaves invariant one CM, and conversely any CM is left invariant by some LP symmetry. Then, we can conclude that the above Proposition is also true even if the CM is not unique.

4. Normal forms

In the previous Section we have introduced the new variable z (3.4) in such a way that the CM can be expressed in the form $z = 0$. An alternative way to obtain a DS in a form where "automatically" one CM is given by the "flat" manifold $y = 0$, is provided by the Poincaré-Birkhoff-Dulac procedure of reducing the DS to normal form [16,17,18]. We can give in fact a direct proof (cf. also [18]) of the following result.

Proposition 4.1. *If the DS is written in the Poincaré-Birkhoff-Dulac normal form*

$$\dot{u} = Lu + h(u) \tag{4.1}$$

where all nonlinear terms $h(u)$ are resonant with the linear part L, then one CM is given by

$$y = 0 \ .$$

The same is true even if the nonlinear terms $h(u)$ are resonant with L^+.

Proof. The resonance condition (with L) can be written [16,17,18]

$$D_{L^+}h \equiv L^+_{ij} h_j - (L^+u)_k \partial_k h_i = 0 \tag{4.2}$$

(where D_{L^+} is the "homological operator" associated to L^+ [16]), which implies the characteristic equations

$$\frac{du_1}{(L^+u)_1} = \ldots = \frac{du_n}{(L^+u)_n} = \frac{dh_1}{(L^+u)_1} = \ldots = \frac{dh_n}{(L^+u)_n}$$

The most general solution can be written in the form [26]

$$h = Ku \qquad \text{with} \qquad KL^+ = L^+K$$

and where all the entries K_{ij} of the matrix K are functions of the constants of motions $k_\alpha(u)$ of the *linear* system

$$\dot{u} = L^+u \tag{4.3}$$

Writing $u \equiv (x, y)$ and $L^+ = \begin{pmatrix} L_0^+ & 0 \\ 0 & L_1^+ \end{pmatrix}$ the above results imply $K = \begin{pmatrix} K_0 & 0 \\ 0 & K_1 \end{pmatrix}$ and that the DS has the form

$$\begin{aligned} \dot{x} &= L_0 x + K_0(k_\alpha(x, y))x \\ \dot{y} &= L_1 y + K_1(k_\alpha(x, y))y \end{aligned} \tag{4.4}$$

The conclusion then easily follows observing that (4.4) shows that $y = 0$ is a flow-invariant manifold. In the case the nonlinear terms $h(u)$ are resonant with L^+, i.e. if $D_L h = 0$, the result (4.4) is again valid with the difference that now K is required to commute with L and $\kappa_\alpha(x, y)$ are the constants of motion of the linear system $\dot{u} = Lu$. •

Notice that it is easy to obtain all constants of motions $k_\alpha(x, y)$ of the *linear* system (4.3), and then conclude also the following.

Proposition 4.2. *The most general DS written in normal form (with $h(u)$ resonant with L) is given by eq. (4.4). If in particular the real part of* all *the eigenvalues σ_i of L_1 is negative (or positive), then the only smooth constants of motions $k_\alpha(x, y)$ may actually* depend only on the variables *x. The same is true if there is no "resonance relation" $m_1\sigma_1 + \ldots + m_n\sigma_n = 0$, where m_i are integer numbers ≥ 0 (not all 0), between the eigenvalues of L_1.*

This result generalizes Theorem 3 of ref. [18]. In the hypotheses of Proposition 4.2 above, the time evolution of the variables x turns out to be independent of the variables y:

$$\dot{x} = L_0 x + K_0(\kappa_\alpha(x))x \tag{4.4'}$$

(cf. also eq. $(5)_1$ of [18], and [27,28]). The same result is also true if the nonlinear terms are resonant with L^+: the above proof of Proposition 4.1 can in fact be repeated for this case.

For some other considerations on LP symmetries and normal form reduction, see [29,30].

5. CMs, perturbative expansions, and analyticity

The nonuniqueness of CMs is essentially related to non-perturbative effects [7,8,17,31], i.e. to terms beyond all orders in perturbative expansion; this also corresponds to the fact that the Poincaré-Birkhoff-Dulac normal form is determined up to any given order, but maintains an ambiguity related again to terms of arbitrarily high order in the perturbative expansion. It is maybe worth recalling shortly these aspects of CM and normal form construction, and their mutual relations, as the nonuniqueness is particularly relevant to the problems at hand in this paper.

The Poincaré-Birkhoff-Dulac procedure allows for determining the normal form of a DS around a fixed point (equivalently, giving local classification of DS up to differential equivalence) up to any desired order $k \leq \infty$. It is well clear from the perturbative character of this procedure that the normal form construction cannot distinguish among DS which differ for terms beyond all orders in perturbative expansion (such terms will be called BAO for short); e.g., no perturbative analysis around the fixed point $x = 0$ can distinguish among the two DS in R^1

$$\dot{x} = 0 \quad \text{and} \quad \dot{x} = e^{-1/x^2} \tag{5.1}$$

Notice, though, that if we consider only *analytic* DS, the problem with BAO terms does not appear, as these are by definition zero. Notice, anyway, that already in the C^∞ case, as the previous trivial example clearly shows, BAO terms play a role.

Similarly, we can determine a perturbative expansion of the CM order by order, up to any order $k \leq \infty$; such an approach determines an equivalence class of center manifolds, which is called the *k-jet* of the CM [17].

If we are able to uniquely determine the k-jet of the CM, we can only assert that the CM is unique up to terms of order $k+1$ (of BAO terms if $k = \infty$). It is also worth recalling that while the stable and unstable manifolds are as regular as the DS, the center manifold is less regular; in particular, it is of class C^r for a C^{r+1} DS, and it can be only finitely differentiable for a C^∞ DS [17].

A trivial and well known example of nonuniqueness of the CM (due in fact to BAO terms) is provided by the CM for the DS

$$\begin{aligned} \dot{x} &= -\, x^3 \\ \dot{y} &= -\, y \end{aligned} \tag{5.2}$$

in which case any curve

$$y = a e^{-1/2x^2} \tag{5.3}$$

where a is any constant (possibly different for $x < 0$ and $x > 0$) gives a CM; notice that the ∞-jet $y = 0$ determines the unique *analytic* CM.

The unique CM obtained by imposing the BAO terms to vanish could be called the "Poincaré-Birkhoff-Dulac CM", in analogy with the case of normal form, where in fact the Poincaré-Birkhoff-Dulac normal form is obtained by (implicitely) putting BAO terms to zero.

It should be mentioned that considering BAO terms effects would lead straight - and inevitably - into very difficult problems, connected with "resurgent functions" [17,32], well beyond the goal and scope of the present paper.

6. Shoshitaishvili theorem and CMs

Some other information (essentially of "topological" nature) about the role of LP symmetries can be more easily obtained using the Shoshitaishvili [19] theorem, which ensures the local topological, i.e. C^0, equivalence of any DS to a system of this simple form

$$\begin{aligned} \dot{x} &= A(x) \\ \dot{y} &= By \end{aligned} \tag{6.1}$$

where B is a constant matrix whose eigenvalues have nonzero real parts. A CM for this DS is clearly $y = 0$; assume that another CM is given by $y = w(x)$, then, putting as in Sect.3, $z = y - w(x)$, we obtain

$$\dot{z} = By - \partial_x w \cdot A(x)$$

but the flow-invariance of this CM implies

$$\dot{y} = Bw = \partial_x w \cdot A(x)$$

therefore

$$\dot{z} = By - Bw = Bz$$

which shows the remarkable result that in this form the time-evolution of the $y-$variable is the same as that of the $z-$variable.

We want now to evaluate explicitely, in the specially simple form (6.1), how a LP symmetry can transform a CM into a different CM. Recalling from Sect. 3 that the component $\Psi(x,0)$ of the LP symmetry is the term "pushing away" from the CM, let us write down the equations determining this function $\Psi(x,0)$. These are obtained writing explicitly the symmetry-invariance condition (2.3) applied to to the DS (3.5) and to the LP symmetry $\widetilde{\eta}$ (3.7), and then restricting it to the CM $z = 0$. We obtain

$$\big(\partial_z Z(x,0)\big) \cdot \Psi(x,0) = \big(\partial_x \Psi(x,0)\big) \cdot \widetilde{X}(x,0) \tag{6.2}$$

Here (cf. (3.5)) we have $y \equiv z,\ Z \equiv By,\ \widetilde{X} \equiv A(x)$. Let us choose now a CM, e.g. $y = 0$, and try to solve eq. (6.2), at least for some particular but significant cases.

If e.g. $m = 2,\ n = 3\ B = -1$, and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then we obtain from (6.2)

$$\Psi = \text{const} \times \exp\big(\arctan \frac{x_2}{x_1}\big)$$

which implies const $= \Psi = 0$ due to the regularity assumptions on η. A similar situation occurs if e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, in these cases, the CM is invariant under the LP symmetry. If instead $n = 2,\ m = 1,\ B = -1,\ A = -x^r\ (r > 1)$, eq. (3.9) is solved by

$$\Psi = \text{const} \times \exp\big(- 1/(r-1)x^{r-1}\big) \tag{6.3}$$

which shows that, if r is odd, Ψ may well be $\neq 0$ and the LP symmetries may connect the CM $y = 0$ to other CM $y = \text{const} \times \exp\big(- 1/(r-1)x^{r-1}\big)$, but, as expected, this CM is not analytic.

7. The relationship between CMs and LP symmetries

We want now to study the invariance properties of the CM under the LP symmetries; precisely, we propose some general argument illustrating the close relationship existing between CM's and LP symmetries.

In this Sect., we may assume that the DS is written in the Shoshitaishwili form (6.1); in addition, essentially for notational convenience, we assume $n-m=1$ and $B=-1$.

First, we want to verify that, in general, given any CM of a DS, we can find a nontrivial LP symmetry $\eta \in \mathcal{G}$ leaving invariant this CM: indeed, if $y=w(x)$ is the given CM, a LP symmetry η $(\neq \eta_F)$ leaving it invariant is

$$\eta = A(x)\partial_x - \mathcal{F}(k)\ y\partial_y \tag{7.1}$$

where $k=k(x,y)$ is any constant of motion of the DS, and $\mathcal{F}=\mathcal{F}(k)$ is any regular nonconstant function subject to the only condition that $\mathcal{F}(c)=1$, where $c=k(x,w(x))$ is the value assumed by $k(x,y)$ along the given CM. Here, we are clearly assuming that "generically" such a function $\mathcal{F}$ exists.

Conversely, let η be a LP symmetry, which can be written in the most general form

$$\begin{aligned}\eta &= \sum_\alpha k_\alpha(x,y)\xi_\alpha + k_0(x,y)y\partial_y \\ &\equiv \sum_{\alpha'} k_{\alpha'}(x,y)\xi_{\alpha'} + k_1(x,y)A(x)\partial_x + k_0(x,y)y\partial_y \equiv \widehat{\xi} + \eta_1\end{aligned} \tag{7.2}$$

where k_α, k_0, k_1 are constant of motions of the DS, ξ_α are in particular also LP symmetries of the subsystem $\dot{x}=A(x)$, and in the second sum we have extracted the term with the symmetry $A(x)\partial_x$, putting finally

$$\eta_1 = k_1(x,y)A(x)\partial_x + k_0(x,y)y\partial_y \tag{7.3}$$

We want to show how to construct a CM $y=w(x)$ which is invariant under this η (7.2): to this aim, we try first of all to satisfy the condition

$$\eta_1 \cdot \big(y - w(x)\big) = 0 \tag{7.4}$$

i.e.

$$k_1(x,w)A(x)\partial_x w = k_0(x,w)w$$

Now, under the only condition that $\lim_{y\to 0} yk_0(x,y) = 0$ (notice that the regularity of η ensures only that $yk_0(x,y)$ is regular), one has that

$$y = w \equiv 0 \tag{7.5}$$

is a flow-invariant solution of (7.4), and also of $\eta\cdot y = 0$, as required (being clearly $\widehat{\xi}\cdot y = 0$). Actually, even the "exceptional" case of divergent $k_0(x,y)$ for $y \to 0$ could be considered, and at least under mild conditions, one can obtain a flow-invariant manifold $y = w(x)$ satisfying both (7.4) and $\widehat{\xi}\cdot w = 0$ (and therefore invariant under η as required). Instead of giving tedious details, let us illustrate this point by means of a typical example.

Let $u \equiv (x_1, x_2, y) \in R^3$, and consider, with $r^2 = x_1^2 + x_2^2$

$$\begin{aligned} \dot{x}_1 &= -\, x_1 r^2 \\ \dot{x}_2 &= -\, x_2 r^2 \\ \dot{y} &= -\, y \end{aligned} \tag{7.6}$$

A LP symmetry admitted by this DS is the rotation generated by

$$\xi = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \tag{7.7}$$

Another LP symmetry is

$$\eta = r^2 \Big(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \Big) - \mathrm{e}^{-1/2r^2} \frac{\partial}{\partial y} \tag{7.8}$$

Notice that this symmetry (7.8) does *not* satisfy the "generic" condition above that $yk_0(x,y) = \mathrm{e}^{-1/2r^2}$ vanishes for $y \to 0$; anyway, consider the family of flow-invariant manifolds (where $\mathcal{F}$ is an arbitrary function)

$$y = \mathcal{F}\Big(\frac{x_2}{x_1}\Big)\mathrm{e}^{-1/2r^2} \tag{7.9}$$

It is sufficient to choose e.g. $\mathcal{F} =$ const, in order to obtain a CM which is left invariant by *both* LP symmetries ξ and η above. Notice that, instead, the CM $y = 0$ is *not* invariant under the symmetry η (7.8).

In conclusion, the above discussion can be summarized in the following statement.

Proposition 7.1 *Given any CM of a DS, there is, generically, some nontrivial LP symmetry $\eta \in \mathcal{G}$ leaving invariant this CM, and conversely any LP symmetry of the DS leaves invariant some CM.*

8. Bifurcation setting

We give here some additional remarks concerning specifically the case the DS (2.1) is a bifurcation problem. Writing the problem in the form

$$\dot{u} = F(\lambda, u) \tag{8.1}$$

where λ is a real "control parameter", we assume as usual $F(\lambda, 0) = 0$ and that for some critical value $\lambda = \lambda_0$, some of the eigenvalues $\sigma = \sigma(\lambda)$ of $F_u(\lambda, 0) \equiv L(\lambda)$ cross the imaginary axis with positive speed

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{Re}\ \sigma(\lambda_0) > 0$$

and all other eigenvalues lie in the left half complex plane. The definition of the "critical subspace" V_0 is then the same as in Sect.3, and the discussion on the symmetry properties goes in the same way (apart from the fact that the vector field (2.2) defining the LP symmetry may also depend parametrically on λ: i.e. we have $\vartheta = \vartheta(\lambda, u)$).

It is clear that, in the bifurcation setting, all previous results hold, strictly speaking, only at $\lambda = \lambda_0$, and not in a neighbourhood of λ_0; it is anyway not difficult to extend the results to be useful in bifurcation theory. This can be done either by standard suspension argument (see e.g. [7]), either by using the persistence of normally hyperbolic manifolds (see [5]) applied to the center unstable manifold (these are essentially different names for the same approach).

It can be noted, on the other hand, that for $\lambda > \lambda_0$ there is a *unique* unstable manifold (and a unique stable manifold, as well); this provides a natural criterion for selecting a unique CM W_c at $\lambda = \lambda_0$, just by requiring continuity: i.e., if $W_u(\lambda)$ is the unique unstable manifold for $\lambda > \lambda_0$, we just require $\lim_{\lambda \to \lambda_0^+} W_u(\lambda) = W_c$.

Nevertheless, it may be interesting to see, in an explicit example, how the infinite number of CMs emerge when $\lambda \to 0^+$. In fact, considering, with $u \equiv (x, y) \in R^2$

$$\dot{x} = \lambda x - x^3$$

$$\dot{y} = -y$$

the unique stable manifold $y = 0$ for fixed $\lambda > 0$ lives in the strip $|y| < \sqrt{\lambda}$, whereas, at the exterior of this strip, there is a family of (non-analytic !) flow-invariant manifolds given by

$$y = \text{const} \times \Big(\frac{\sqrt{x^2-\lambda}}{|x|}\Big)^{1/\lambda}$$

tangent to the $x-$axis at $x = \pm\sqrt{\lambda}$. It is clear that when $\lambda \to 0^+$, these manifolds become the family of CM const $\times \exp(-1/2x^2)$. This picture can be easily completed by considering the case $\lambda < 0$ and $\lambda \to 0^-$: the stable manifold is now the whole R^2, and the flow-invariant manifolds are, for fixed $\lambda = -\mu^2 < 0$,

$$y = \text{const} \times \Big(\frac{|x|}{\sqrt{x^2+\mu^2}}\Big)^{1/\mu^2}$$

It should also be recalled that, as already hinted in section 5, a unique CM can also be selected on the basis of regularity arguments, i.e. by selecting the unique maximally regular (analytic, if the DS is analytic) CM; notice that since the unstable manifold is as regular as the DS itself [17], this is an essentially equivalent criterion.

Still a different way of looking at the same problem is provided by the change of variables outlined in sect.3: since the stable and unstable manifolds are as regular as the DS, it is possible - for $\lambda > \lambda_0$ - to perform a λ-dependent change of variables such that the unstable manifold $W_u(\lambda)$ is given by $z = 0$, and this with no harm for the (local) regularity of the DS. In this way, we can choose the CM to be also given by $z = 0$, corresponding to the limit for $\lambda \to \lambda_0^+$ of $W_u(\lambda)$, i.e. of the associated change of variables. Notice that for $\lambda \to \lambda_0$ the change of variables could become less regular, which correspond to the weaker differentiability of the CM (see again the remarks in sect.3 and, more in general, [17]). The Shoshitashvili theorem [19] ensures anyway the C^0 regularity of this procedure, and therefore that in this way we do indeed select a unique CM.

As already stated, the above results show that theorems and proofs given in [14] immediately apply also to bifurcation equations obtained by CM reduction: starting from the LP symmetry of the original equation, a LP symmetry is inherited in this way by the reduced bifurcation equation.

9. Examples

Consider first the following DS, with $u \equiv (x_1, x_2, y) \in R^3$, according to the notations of Sect. 3,

$$\begin{aligned} \dot{x}_1 =& x_1^2 f_1(\kappa_1,\kappa_2) \\ \dot{x}_2 =& x_1(x_1^2-y) f_2(\kappa_1,\kappa_2) + f_3(\kappa_1,\kappa_2) + x_1^2(x_1^2-y) f_1(\kappa_1,\kappa_2) \\ \dot{y} =& (y-x_1^2) f_2(\kappa_1,\kappa_2) + 2x_1^3 f_1(\kappa_1,\kappa_2) \end{aligned} \tag{9.1}$$

where f_i are arbitrary smooth functions of the quantities

$$\kappa_1 = x_1^2 - y \ ; \qquad \kappa_2 = x_2 - x_1^3 + x_1 y \tag{9.2}$$

with the conditions

$$\lim_{u\to 0} f_2(\kappa_1,\kappa_2) = \ell \neq 0; \qquad \lim_{u\to 0} f_3(\kappa_1,\kappa_2) = \lim_{u\to 0} \partial_u f_3 = 0 \ . \tag{9.3}$$

It is easy to verify that a LP symmetry admitted by (9.1) is

$$\eta = x_1^2 \partial_{x_1} + x_1^2(x_1^2-y)\partial_{x_2} + 2x_1^3 \partial_y \tag{9.4}$$

and the (unique) CM for this DS is

$$y = x_1^2 \tag{9.5}$$

Notice that, as this and the foregoing examples clearly indicate, the invariance of the CM under a LP symmetry may be also useful for explicitly finding the CM itself: indeed, the expression defining the CM must be a function only of the quantities $\kappa_\alpha(u)$ which are left invariant by the LP symmetry η, i.e.

$$\eta \cdot \kappa_\alpha = 0 \tag{9.6}$$

In this example, these quantities are precisely given by (9.2). According to the discussion in Sect. 3, the restriction of the DS (9.1) to the CM is

$$\begin{aligned} \dot{x}_1 =& x_1^2 f_1(0, x_1) \\ \dot{x}_2 =& f_3(0, x_2) \end{aligned} \tag{9.7}$$

which inherites from (9.1,4) the LP symmetry

$$\widehat{\eta} = x_1^2 \partial_{x_1} \tag{9.8}$$

admitted in fact by (9.7).

To provide a large class of examples (see [15]), it is convenient now to change notations. Let us put (instead of (3.1))

$$u \equiv (x, y, z_1, \ldots, z_{n-2}) \ \in R^n \tag{9.9}$$

Let

$$v_a = v_a(y, z_a) \qquad\qquad (a = 1, \ldots, k = n-2) \tag{9.10}$$

be arbitrary smooth functions, satisfying the conditions

$$\frac{\partial v_a}{\partial z_a} \neq 0 \qquad \text{(no sum over } a) \tag{9.11}$$

and put

$$\chi_a = \chi_a(y, z_a) = \frac{\partial_y v_a}{\partial_{z_a} v_a} \qquad \text{(no sum over } a) \tag{9.12}$$

Consider systems of the following form

$$\begin{aligned} \dot{x} &= x\ f - y\ g \\ \dot{y} &= y\ f + x\ g \\ \dot{z}_1 &= \ - y\ f\ \chi_1 - x\ g\ \chi_1 + \frac{h_1}{\partial_{z_1} v_1} \\ &\vdots \\ \dot{z}_k &= \ - y\ f\ \chi_k - x\ g\ \chi_k + \frac{h_k}{\partial_{z_k} v_k} \end{aligned} \tag{9.13}$$

where $f, g, h_1, \ldots, h_k$ are n arbitrary smooth functions of the quantities

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad v_1, \ldots, v_k \tag{9.14}$$

(and possibly also of one or more real parameters λ, to include the case of bifurcation problems). Then, using (2.3), it is easy to verify that the above DS (9.13) admits the LP symmetry generated by

$$\eta = y\partial_x - x\partial_y + x\chi_1\partial_{z_1} + \ldots + x\chi_k\partial_{z_k} \tag{9.15}$$

In particular (with $n = 3$, the extension to $n > 3$ is straightforward; we write here z, v, etc. instead of z_1, v_1, etc.), if $v = v(y, z)$ and $h = h(r^2, v)$ in (9.13) satisfy

$$\lim_{y,z\to 0} \partial_y v = 0 \qquad \text{and} \qquad \lim_{y,z\to 0} \frac{h(r^2, v)}{z\ \partial_z v} = \ell < 0 \tag{9.16}$$

then the origin $u_0 = 0$ is an equilibrium point for the problem (9.13), with an attractive 2-dimensional center manifold W tangent at $u_0 = 0$ to the center eigenspace E_c ($z = 0$). If the functions $f(\lambda, r^2, v)$, $g(\lambda, r^2, v)$ (with $\lambda \in R$) satisfy also

$$f(0,0,0) = 0, \quad \partial_\lambda f(0,0,0) \neq 0, \quad g(0,0,0) \neq 0 \tag{9.17}$$

then the standard conditions for the appearance of Hopf bifurcation are met.

As remarked above, the invariance under the symmetry (9.15) implies that the CM must be a function of the quantities $\kappa_\alpha(u)$ defined in (9.6), which are in this case r^2 and v_a.

E.g., let us choose in (9.10,13), with $n = 3$,

$$v = z\ \mathrm{e}^{-y} \quad \text{and} \quad f = h = -v + r^2 \tag{9.18}$$

leaving $g = g(r^2, v)$ arbitrary; this gives $\chi = -z$, $h/\partial_z v = -z + \mathrm{e}^y r^2$, and the nonlinear LP symmetry

$$\eta = y\partial_x - x\partial_y - xz\partial_z\ . \tag{9.19}$$

It is easy to explicitly obtain the CM, which is given by the surface

$$z = w(x, y) = \mathrm{e}^y r^2 \tag{9.20}$$

and to verify also that the LP symmetry inherited, according to Sect.3, by the problem reduced to the CM is the rotation in the (x, y) plane, generated by

$$\widehat{\eta} = y\partial_x - x\partial_y \tag{9.21}$$

It can be interesting to perform explicitly the calculations indicated in Sect.3 to transform this problem to its CM: introducing the variable

$$\zeta = z - \mathrm{e}^y r^2$$

(just as in (3.4), apart from the notations), we obtain with the above choice (9.18) the problem

$$\begin{aligned} \dot{x} &= -\ x\zeta\mathrm{e}^{-y} - yg \\ \dot{y} &= -\ y\zeta\mathrm{e}^{-y} + xg \\ \dot{\zeta} &= \zeta(-1 + xg + 2r^2 - y\zeta\mathrm{e}^{-y}) \end{aligned} \tag{9.22}$$

where $g = g(r^2, r^2 + \zeta\mathrm{e}^{-y})$. We then see from (9.22) that $\zeta = 0$ is in fact an invariant attractive solution (the CM, as expected !), and also that *only on the* CM $\zeta = 0$ the system exhibits the rotation symmetry (9.21).

Let us point put that, in this example, the LP symmetry inherited by the reduced problem is a *linear* symmetry (9.21), even if the original symmetry (9.19) is *not* a linear one. This may be a remarkable evenience: in fact, if the original problem has *no* linear symmetries, then standard (linear) equivariant theory cannot predict any symmetry constraints on the form of the reduced equations. This may be important e.g. when dealing with Hopf bifurcation theory, where symmetry properties play a relevant role, as shown by the foregoing example.

Let us choose in (9.10,13), again with $n = 3$, $v = z - y^2$, $h = -v$, $f = \lambda - r^2$, vith arbitrary $g(\lambda, r^2, v)$ $\big($satisfying in particular all conditions (9.16,17)$\big)$: the resulting bifurcation problem

$$\begin{aligned} \dot{x} &= x\ (\lambda - r^2) - y\ g(\lambda, r^2, v) \\ \dot{y} &= y\ (\lambda - r^2) + x\ g(\lambda, r^2, v) \\ \dot{z} &= -\ z + y^2\big(1 + 2(\lambda - r^2)\big) + 2xy\ g(\lambda, r^2, v) \end{aligned} \tag{9.23}$$

admits the LP symmetry generated by

$$\eta = y\partial_x - x\partial_y - 2xy\partial_z \tag{9.24}$$

It is easy to verify that at the critical point $\lambda_0 = 0$, the CM is given by $z = y^2$ (this manifold is a flow-invariant manifold also for $\lambda \neq 0$), and the reduced bifurcation equations exhibit the (linear) rotation symmetry (9.21). Now, a SO_2-symmetric Hopf periodic solution to (9.23) bifurcates at $\lambda > \lambda_0 = 0$, along the lines

$$r^2 = x^2 + y^2 = \lambda, \qquad z = y^2$$

and with frequency $\omega(\lambda) = g(\lambda, \lambda, 0)$.

References

[1] L.V. Ovsjannikov, "Group properties of differential equations" (Novosibirsk, 1962); "Group analysis of differential equations" (Academic Press, New York, 1982)

[2] P.J. Olver, "Applications of Lie groups to differential equations" (Springer, Berlin, 1986)

[3] G.W. Bluman, S. Kumei, "Symmetries and differential equations" (Springer, Berlin, 1989)

[4] J. Guckenheimer and P. Holmes, "Nonlinear oscillations, dynamical systems, and bifurcation of vector fields" (Springer, New York, 1983)

[5] D. Ruelle, "Elements of differentiable dynamics and bifurcation theory" (Academic Press, London 1989)

[6] A. Vanderbauwhede, "Center manifolds, normal forms, and elementary bifurcations", Dynamics Reported **2** (1989), 89

[7] J.D. Crawford, Rev. Mod. Phys. **63** (1991), 991

[8] G. Iooss and M. Adelmeyer, "Topics in bifurcation theory and applications" Winter School on Dynamical Systems and Frustrated Systems, CIMPA, UNESCO, Chile (1991, 1992) (World Scientific, Singapore, 1993)

[9] D. Ruelle, Arch. Rat. Mech. Anal. **51** (1973), 136

[10] D.H. Sattinger, "Group theoretic methods in bifurcation theory" (Springer, Berlin, 1979); "Branching in the presence of symmetry" (SIAM, Philadelphia, 1983)

[11] A. Vanderbauwhede, "Local bifurcation and symmetry" (Pitman, Boston, 1982)

[12] M. Golubitsky, I. Stewart and D. Schaeffer, "Singularity and groups in bifurcation theory - vol. II" (Springer, New York, 1988)

[13] J.D. Crawford and E. Knobloch, Ann. Rev. Fluid Mech. **23** (1991), 341

[14] G. Cicogna and G. Gaeta, Ann. Inst. H. Poincaré **56** (1992), 375

[15] G. Cicogna and G. Gaeta, Phys. Letters A **172** (1993), 361

[16] V.I. Arnold, "Geometrical methods in the theory of ordinary differential equations" (Springer, New York 1988)

[17] V.I. Arnold and Yu.S. Il'yashenko, "Ordinary differential equations"; in *Encyclopaedia of Mathematical Sciences - vol. I, Dynamical Systems I*, (D.V. Anosov and V.I. Arnold eds.), p. 1-148 Springer, Berlin, 1988

[18] C. Elphic, E. Tirapegui, M.E. Brachet, P. Coullet, and G. Iooss, Physica D **29** (1987), 95

[19] A.N.Shoshitaishvili, Funct. Anal. Appl. **6** (1972), 169; Tr. Semin. I.G. Petrovskogo **1** (1975), 279

[20] P. Bergé, Y. Pomeau, Ch. Vidal, "L'ordre dans le chaos" (Hermann, Paris 1984)

[21] P. Manneville, "Dissipative structures and weak turbulence" (Academic Press, Boston 1990)

[22] A. Vanderbauwhede and G. Iooss, "Centre manifold theory in infinite dimensions", Dynamics Reported (new series) **1** (1992)

[23] Th. Gallay, Comm. Math. Phys. **152** (1993), 249

[24] J. Carr, R.G. Muncaster, J. Diff. Eq. **50** (1983), 260 and 280

[25] G. Gaeta, Nonlinear Analysis **17** (1991), 825

[26] R. Courant, D. Hilbert. "Methods of Mathematical Physics" (Interscience Publ., New York 1962)

[27] F. Takens, Topology **10** (1971), 133

[28] Yu.S. Il'yashenko and S.Yu. Yakovenko, Russ. Math. Surv. **46** (1991), 3

[29] G. Cicogna and G. Gaeta, J.Phys. A **23** (1990), L799, and **25** (1992), 1535

[30] G. Cicogna and G. Gaeta, "Poincaré normal forms and Lie point symmetries", preprint 1993, to be published in J. Phys. A

[31] T.K. Leen, Phys. Letters A **174** (1993), 89

[32] J. Ecalle, Ann. Inst. Fourier **42** (1992), 73