A Note on Essential Duality

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ABSTRACT

By considering some simple models it is shown that the essential duality condition for local nets of von Neumann algebras associated with Wightman fields need not be fulfilled if Lorentz covariance is dropped. These models illustrate a point made by Borchers in the proof of his two dimensional CPT theorem for local nets: The Lorentz covariant net constructed from the wedge algebras of a given two dimensional net may not be unique. It is also shown that in higher dimensions the Lorentz boosts constructed by means of the modular groups of wedge algebras may act nonlocally in the directions parallel to the edge of the wedge.

1. Introduction

One of the most remarkable results in algebraic quantum field theory in recent years is Borchers' derivation of the CPT theorem for local nets in two dimensional space-time [B]. The main ingredient in this proof is a certain converse of the Bisognano-Wichmann analysis of modular structures in quantum field theory [BW1,2], (see also [BY]): Whereas Bisognano and Wichmann assume covariance with respect to Lorentz transformations and show that the Lorentz boosts coincide with the modular group of the algebra corresponding to a wedge-domain, Borchers proves that in two dimensions the modular group of a wedge algebra can be used to define a representation af the Poincaré group, if the net is covariant with respect to a representation of the translation group satisfying the spectrum condition. Moreover, the initial net can be embedded into a net that is covariant with respect to this representation af the Poincaré group, has CPT symmetry and satisfies Haag-duality.

As pointed out in [B], the Poincaré covariant net obtained by this method may not be unique unless the initial net satisfies wedge duality [BW1,2]. This condition of wedge duality is also important in another context: It implies essential duality [Ro], which is one of the chief assummptions on which the analysis of superselection rules is based [H]. By the results of Bisognano and Wichmann wedge duality always holds for local nets generated by Wightman fields transforming with respect to finite dimensional representations of the Lorentz group, but examples of local nets violating this condition have apparently not been known so far.

In the present note we discuss a class of very simple fields in two dimensions, whose duality properties can be completely analyzed. Among these fields are both such that violate essential duality (and hence, *a fortiori* wedge duality), as well as fields that satisfy wedge duality without being Lorentz covariant. The former provide examples where the net has infinitely many different extensions satisfying duality. Higher dimensional fields of a similar type demonstrate clearly that some additional assumptions are needed to extend Borchers' result to space-time dimensions larger than 2, because the action of the modular group of a wedge algebra in directions parallel to the edge of the wedge is not local in these examples.

2. Generalized free fields on a light ray

Let (x^0, x^1) denote the usual time and space coordinates in two dimensional Minkowski space and $x_\pm = x^0 \pm x^1$ the corresponding light cone coordinates. Suppose Φ is a hermitian Wightman field that transforms covariantly under space-time translations, but not necessarily under Lorentz transformations, and depends only on one light cone coordinate, say x_+. Locality implies that the commutator $[\Phi(x_+), \Phi(y_+)]$ has support only for $x_+ = y_+$. Moreover, from the spectrum condition it follows that the generator for translations of Φ in the x_+-direction, $P^0 - P^1$, is positive semidefinite. This implies readily that the Fourier transform of the two point function, $\mathcal{W}_2$, defined by $\langle \Omega, \Phi(x_+)\Phi(y_+)\Omega\rangle = (1/2\pi)\int \exp[ip(x_+ - y_+)]\tilde{\mathcal{W}}_2(p)dp$, with Ω the vacuum vector, has the form

$$\tilde{\mathcal{W}}_2(p) = \theta(p)pQ(p^2) + c\delta(p), \tag{2.1}$$

where $Q(p^2)$ is a positive, even polynomial in $p \in \mathbf{R}$, $\theta(s) = 1$ for $s \geq 0$ and zero else, and $c = \langle \Omega, \Phi(x_+)\Omega\rangle^2 \geq 0$ is a constant. Subtracting $c^{1/2}$ from Φ if necessary, we may drop the $\delta(p)$-term. For simplicity of notation we also from now on write x, y instead of x_+, y_+.

The models we consider are generalized free fields with the two point function (2.1) (without the δ-term). They are characterized by the commutation relations

$$[\Phi(x), \Phi(y)] = DQ(D^2)\delta(x-y)\mathbf{1}, \tag{2.2}$$

where we have for convenience denoted id/dx by D. The simplest case is the chiral $U(1)$-current Φ_0, given by

$$[\Phi_0(x), \Phi_0(y)] = D\delta(x-y)\mathbf{1}. \tag{2.3}$$

The other models can be obtained from Φ_0 by differentiation. To see this, one notes that since $Q(p^2)$ is positive and an even function of p, one can write

$$Q(p^2) = L(p)L(-p), \tag{2.4}$$

where $L(p)$ is a polynomial satisfying

$$L(-p) = L(p)^*. \tag{2.5}$$

The polynomial $L(p)$ is uniquely fixed (up to a sign) for a given $Q(p^2)$ by the requirement that its zeros lie in the closed upper half plane. The most general form of $L(p)$ is then

$$L(p) = R(p)P(p), \tag{2.6}$$

where $R(p)$ has zeros only on the real axis and can be written as

$$R(p) = (ip)^n \prod_{j=1}^{k} (p^2 - r_j^2), \tag{2.7}$$

with $r_j > 0$, while $P(p)$ is a polynomial whose zeros have a strictly positive imaginary part. Note that $P(p)$ satisfies the reality condition (2.5) and $R(-p) = (-1)^n R(p)$. From (2.4) and (2.5) it is clear that the field Φ defined by

$$\Phi(x) = L(D)\Phi_0(x) \tag{2.8}$$

satisfies the commutation relation (2.2). It also follows that the smeared field operators $\Phi(f)$ are well defined and essentially self adjoint on their natural domain for every tempered real distribution f such that $L(p)\tilde{f}(p)$ is square integrable with respect to the measure $|p|dp$. Such f are necessarily measurable functions, but they may have singularities and grow at infinity if $L(p)$ has zeros on the real axis.

Since on a light ray Lorentz boosts are the same as dilatations, the field Φ is Lorentz covariant if and only if $Q(p^2) = p^{2n}$ for some natural number n, i.e.

$$\Phi(x) = (d/dx)^n \Phi_0(x). \tag{2.9}$$

In fact, if $U_0(\lambda)$ is the unitary operator implementing dilatation by $\lambda > 0$ for Φ_0, i.e., $U_0(\lambda)\Phi_0(x)U_0(\lambda)^{-1} = \lambda\Phi_0(\lambda x)$, then

$$U_0(\lambda)\Phi(x)U_0(\lambda)^{-1} = \lambda^{n+1}\Phi(\lambda x) \tag{2.10}$$

for Φ as in (2.9).

The two point function for the field (2.9) is

$$\mathcal{W}_2(x-y) = (2\pi)^{-1}(x - y + i0)^{-2(n+1)}. \tag{2.11}$$

A Möbius transformation, $x \mapsto x' = \frac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbf{R}$ and $ad - bc > 0$, changes $\mathcal{W}_2$ according to the formula

$$\mathcal{W}_2(x' - y')dx'dy' = \left[\frac{(cx+d)(cy+d)}{ad-bc}\right]^{2n} \mathcal{W}_2(x-y)dxdy. \tag{2.12}$$

It follows that there is a unitary representation of the Möbius group implementing the transformations $\Phi(f) \mapsto \Phi(f')$ with

$$f'(x') = \frac{(ad-bc)^n}{(cx+d)^{2n}} f(x) = \frac{(cx'-a)^{2n}}{(ad-bc)^n} f(-(dx'-b)/(cx'-a)). \tag{2.13}$$

As a consequence the field (2.9) can be extended to a conformally covariant field on the compactified light ray [BSch].

Returning to the general case (2.8) we define the unitary Weyl operators as usual by

$$W(f) = e^{i\Phi(f)}, \tag{2.14}$$

where f is a real function such that $L(p)\tilde{f}(p)$ is square integrable with respect to the measure $|p|dp$. The Weyl relations are

$$W(f)W(g) = e^{-K(f,g)/2} W(f+g) \tag{2.15}$$

with

$$K(f,g) = \langle \Omega, [\Phi(f), \Phi(g)]\Omega \rangle = \int_{-\infty}^{\infty} p\, Q(p^2)\tilde{f}(-p)\tilde{g}(p)dp. \tag{2.16}$$

It follows that $W(f)$ commutes with $W(g)$ if and only if $K(f,g) = 0$, in particular if f and g have disjoint supports.

Using the concrete realization of Φ as a derivative of Φ_0 we can write

$$W(f) = W_0(L(D)f) \tag{2.17}$$

with $W_0(g) = \exp(i\Phi_0(g))$, and

$$K(f,g) = K_0(L(D)f, L(D)g) = K_0(Q(D^2)f, g) = K_0(f, Q(D^2)g) \tag{2.18}$$

where K_0 is the vaccum expectation value of the commutator for Φ_0. We note also that the Weyl operators are weakly continuous in f with respect to the norm defined by the two point function, *viz*,

$$\|\tilde{f}\|^2 = \int_0^\infty p|L(p)\tilde{f}(p)|^2 dp. \tag{2.19}$$

The Weyl operators generate on Fock space a local net of von Neumann algebras $\mathcal{M}(I) = \{W(f) \mid \text{supp } f \subset I\}''$ associated with bounded intervals I of the light ray $\mathbf{R}$. Alternatively, one may think of these algebras as a net over two-dimensional Minkowski space. If K is an open double cone on Minkowski space we define $\mathcal{A}(K) = \mathcal{M}(I)$, where I is the projection of K onto the light ray $x_- = 0$. Locality of the net $\mathcal{M}(\cdot)$ for disjoint intervals is equivalent to the locality of $\mathcal{A}(\cdot)$ for space like separated double cones.

Duality conditions for the net $\mathcal{A}(\cdot)$ also translate immediately into corresponding conditions for $\mathcal{M}(\cdot)$. *Duality* for double cones, i.e., the condition $\mathcal{A}(K)' = \mathcal{A}(K')$ for all doble cones K, is equivalent to $\mathcal{M}(I)' = \mathcal{M}(I')$ for all intervals I. Here K' denotes as usual the space like complement of the closure of K, whereas I' denotes the set theoretical complement of the closure of I in $\mathbf{R}$. The algebras corresponding to unbounded subsets of $\mathbf{R}^2$ or $\mathbf{R}$ are by definition generated by the subalgebras corresponding to double cones or intervals, repectively, contained in the unbounded sets. *Essential duality* for $\mathcal{A}(\cdot)$ means that the net $K \mapsto \mathcal{A}(K')'$ satisfies locality; this is equivalent to $I \mapsto \mathcal{M}(I')'$ being local. Finally, *wedge duality*, which implies essential duality [BW1,2] (see also [BY]), is the condition $\mathcal{A}(W)' = \mathcal{A}(W')$ for all space like wedges W. This is equivalent to $\mathcal{M}(\mathbf{R}^+)' = \mathcal{M}(\mathbf{R}^-)$, where $\mathbf{R}^\pm$ denotes the positive or negative half axis.

3. The Algebras $\mathcal{M}(I)$, $\mathcal{M}(I')$ and $\mathcal{M}(\mathbf{R}^\pm)$

The net $\mathcal{M}(\cdot)$ generated by $\Phi = L(D)\Phi_0$ may be regarded as a subnet of the net $\mathcal{M}_0(\cdot)$ generated by the chiral $U(1)$-current Φ_0. One may ask to what extent the algebras $\mathcal{M}(I)$, $\mathcal{M}(I')$ and $\mathcal{M}(\mathbf{R}^\pm)$ depend on the polynomial $L(p)$, in particular when they are equal to the corresponding algebras generated by Φ_0. The answer given below can be summarized as follows: If I is a bounded interval, then polynomials differing by more than a constant factor lead to different $\mathcal{M}(I)$, while $\mathcal{M}(I')$ depends only on the roots of $L(p)$ in the open upper half plane, and the same holds for $\mathcal{M}(\mathbf{R}^\pm)$.

To discuss this in more detail let $\mathcal{M}_1(\cdot)$ and $\mathcal{M}_2(\cdot)$ denote two such nets corresponding to different polynomials $L_1(p)$ and $L_2(p)$. It is clear that if $L_1(p)$ is a factor of $L_2(p)$, then $\mathcal{M}_2(\cdot)$ is a subnet of $\mathcal{M}_1(\cdot)$. We now show that conversely, if $L_1(p)$ is not a factor of $L_2(p)$, then $\mathcal{M}_2(I)$ is not a subalgebra of $\mathcal{M}_1(I)$ for a bounded interval I. In particular, $\mathcal{M}_1(I) \neq \mathcal{M}_2(I)$, unless L_1 and L_2 are proportional to each other.

To see this, note that if L_1 is not a factor of L_2, then one can find a real test function g such that $DL_1(-D)L_2(D)g(x) = 0$ on I, while $DL_2(-D)L_2(D)g$ is nonzero on some open subinterval of I. (Recall that L_1 and L_2 satisfy the reality condition (2.5).) By the first property it is clear that $W_2(g) = W_0(L_2(D)g)$ commutes with all $W_1(f) = W_0(L_1(D)f)$ with $\mathrm{supp} f \subset I$, so $W_2(g) \in \mathcal{M}_1(I)'$. On the other hand, if $DL_2(-D)L_2(D)g$ is nonzero on a subinterval of I then one has for some f with support in I that

$$K_0(L_2(D)f, L_2(D)g) = \int f(x)DL_2(-D)L_2(D)g(x)dx \neq 0, \tag{3.1}$$

and thus $W_2(g) \notin \mathcal{M}_2(I)'$. Hence $\mathcal{M}_1(I)' \not\subset \mathcal{M}_2(I)'$.

By an analogous argument, with I replaced by I' or $\mathbf{R}^\pm$, one sees that $\mathcal{M}_2(I') \not\subset \mathcal{M}_1(I')$ and $\mathcal{M}_2(\mathbf{R}^\pm) \not\subset \mathcal{M}_1(\mathbf{R}^\pm)$ if the polynomial $P_1(p)$, obtained by cancelling the real zeros of $L_1(p)$, is not a factor of the corresponding polynomial $P_2(p)$ for $L_2(p)$. For instance, suppose $P_1(\alpha) = 0$ while $P_2(\alpha) \neq 0$, for some α in the open upper half plane. The test function $g(x) = h(x)\exp(i\alpha x)$, where h is a C^∞-function with $h(x) = 1$ on $\mathbf{R}^+$, satisfies the equation $DL_1(-D)L_2(D)g = 0$ on $\mathbf{R}^+$, but $DL_2(-D)L_2(D)g$ is nonzero on some interval in $\mathbf{R}^+$. As above, we conclude from this that $W_2(g) \in \mathcal{M}_1(\mathbf{R}^+)'$ and $W_2(g) \notin \mathcal{M}_2(\mathbf{R}^+)'$.

If L_1 and L_2 have the same zeros in the open upper half plane, we assert that $\mathcal{M}_1(\mathbf{R}^\pm) = \mathcal{M}_2(\mathbf{R}^\pm)$, and hence also $\mathcal{M}_1(I') = \mathcal{M}_2(I')$ for bounded intervals I. This may be seen as follows. Suppose $L_i(p) = R_i(p)P(p)$, where R_i has only real zeros, $i = 1, 2$, while the zeros of $P(p)$ have strictly positive imaginary parts. If f is a test function with support in $\mathbf{R}^\pm$ and $\varepsilon > 0$ we define g_ε by $\tilde{g}_\varepsilon(p) = R_1(p)\tilde{f}(p)/R_2(p \pm i\varepsilon)$. Since $\tilde{g}_\varepsilon$ has no poles in the upper (lower) half plane, g_ε has again support in $\mathbf{R}^\pm$. Since $L_2(p)\tilde{g}_\varepsilon(p)$ obviously converges to $L_1(p)f(p)$ in the one particle space of Φ_0, it follows that $W_2(g_\varepsilon) = W_0(L_2(D)g_\varepsilon)$ converges weakly to $W_1(f) = W_0(L_1(D)f)$ as $\varepsilon \to 0$. Hence $\mathcal{M}_2(I') \subset \mathcal{M}_1(I')$. Interchanging the role of 1 and 2 one then obtains $\mathcal{M}_1(\mathbf{R}^\pm) = \mathcal{M}_2(\mathbf{R}^\pm)$.

4. Violation of Duality and Essential Duality

It is well known that the chiral $U(1)$-current Φ_0 satisfies duality, i.e.,

$$\mathcal{M}_0(I)' = \mathcal{M}_0(I') \tag{4.1}$$

for all bounded intervals I. This follows from the analysis in [HL], see also [BSch]. If the polynomial L is nonconstant with only real zeros, then $\mathcal{M}(I) \neq \mathcal{M}_0(I)$ and $\mathcal{M}(I') = \mathcal{M}_0(I')$, by the results above. Thus duality is violated for such fields. They still satisfy wedge duality, however, because $\mathcal{M}_0$ does, and $\mathcal{M}(\mathbf{R}^\pm) = \mathcal{M}_0(\mathbf{R}^\pm)$. Note also that the conformally covariant fields (2.9) can be extended to the compactified light ray, and duality holds for this extension by the general analysis in [BSch] (see also [BGL] and [GF]).

To complete the picture we shall now show directly (i.e., without using the duality properties of $\mathcal{M}_0(\cdot)$) that duality is always violated for the fields (2.8) with a nonconstant $L(p)$, and that essential duality is violated if $L(p)$ has zeros with a nonvanishing imaginary part.

Let us consider duality first. Let $I \subset \mathbf{R}$ be a bounded interval. If $L(p)$ and hence $Q(p)$ is not a constant, we can obviously find a test function g such that $DQ(D^2)g(x) = 0$ for $x \in I$, while $DL(D)g(x) \neq 0$ for all x in some open subinterval of I. The first property implies that $W(g) \in \mathcal{M}(I)'$ because of (2.2). On the other hand, by the latter property of g one can find a test function h with support in I such that

$$\int \tilde{h}(-p)pL(p)\tilde{g}(p)dp = \int h(x)DL(D)g(x)dx \neq 0. \tag{4.2}$$

Let f be the tempered distribution whose Fourier transform is $\tilde{f}(p) = \lim_{\varepsilon\to 0^+} \tilde{h}(p)/L(p+i\varepsilon)$. Since $\int |p||L(p)\tilde{f}(p)|^2 dp < \infty$, $W(f)$ and $K(f,g)$ are well defined. By (2.16), (2.4) and (4.2) we have

$$K(f,g) = \int pL(-p)\tilde{f}(-p)L(p)\tilde{g}(p)dp = \int \tilde{h}(-p)pL(p)\tilde{g}(p)dp \neq 0. \tag{4.3}$$

Hence $W(f)$ and $W(g)$ do not commute. Since $W(g) \in \mathcal{M}(I)'$, it is enough to check that $W(f) \in \mathcal{M}(I')'$ in order verify our assertion that duality is violated. But if $u(x)$ is a test function with support in I', then

$$K(f,u) = \int pL(p)L(-p)\tilde{f}(-p)\tilde{u}(p)dp = \int pL(p)\tilde{h}(-p)\tilde{u}(p)dp = K(-DL(-D)h, u) = 0 \tag{4.4}$$

because $DL(-D)h$ has support in I. Hence $W(f) \in \mathcal{M}(I')'$, so duality does not hold.

To check for essential duality let h be a nonzero test function with support in a bounded interval I and define f in the same way as above, i.e., $\tilde{f}(p) = \lim_{\varepsilon \to 0^+} \tilde{h}(p)/L(p+i\varepsilon)$. As before, $W(f) \in \mathcal{M}(I')'$, and hence also $W(f_a) \in \mathcal{M}(I_a')'$, where f_a is defined by $f_a(x) = f(x-a)$ and $I_a = I + a$. Define g by $\tilde{g}(p) = \lim_{\varepsilon \to 0^+} \tilde{h}(p)/L(-p-i\varepsilon)$. Since $DL(D)h$ has support in I, a computation analogous to (4.4) shows that $W(g) \in \mathcal{M}(I')'$. Since $I_a \cap I = \emptyset$ for large enough $|a|$, essential duality implies that the function $F(a) = K(f_a, g)$ vanishes for large enough $|a|$. Hence the Fourier transform of F must in that case be entire analytic. From (2.16), (2.6) and (2.7) it follows that this Fourier transform is given by

$$\tilde{F}(p) = \frac{pL(p)L(-p)\tilde{h}(p)\tilde{h}(-p)}{L(p)^2} = (-1)^n \frac{P(-p)}{P(p)} \tilde{h}(p)\tilde{h}(-p). \tag{4.5}$$

If $P(p)$ is not constant, then $P(p)$ has zeros in the upper half plane while the zeros of $P(-p)$ lie in the lower half plane. The function $\tilde{F}$ is then not entire analytic and essential duality is violated.

5. Modular Structures of the Wedge Algebras

We shall now compute the modular groups defined by the vacuum state for the wedge algebras $\mathcal{M}(\mathbf{R}^+)$ and $\mathcal{M}(\mathbf{R}^-)$. In order to compare the modular groups for different Φ's it is convenient to realize all fields on the same Hilbert space as above, namely the Hilbert space $\mathcal{H}_0$ of Φ_0. This is the Fock space over the one particle space $\mathcal{H}_{01} = L^2(\mathbf{R}, \theta(p)pdp)$. As before we write $L(p) = R(p)P(p)$, where $R(p)$ has only real zeros, $P(p)$ has only zeros with a strictly positive imaginary part, $P(-p) = P(p)^*$, and

$R(-p) = (-1)^n R(p)$ for real p. For each $\lambda > 0$ we define two unitary operators, $U_+(\lambda)$ and $U_-(\lambda)$ by

$$U_+(\lambda)\psi(p) = \lambda \frac{P(p)}{P(-p)} \frac{P(-\lambda p)}{P(\lambda p)} \psi(\lambda p) \tag{5.1}$$

$$U_-(\lambda)\psi(p) = \lambda \frac{P(-p)}{P(p)} \frac{P(\lambda p)}{P(-\lambda p)} \psi(\lambda p) \tag{5.2}$$

for $\psi \in \mathcal{H}_{01}$ and canonical extension to the Fock space $\mathcal{H}_0$. That $U_\pm(\lambda)$ is unitary follows from the reality condition $P(-p) = P(p)^*$ for $p \in \mathbf{R}$, which means that $P(\lambda p)/P(-\lambda p)$ is just a phase factors for real p. It is clear that $U_\pm(\cdot)$ is a representation of the multiplicative group $\mathbf{R}^+$. Moroever, $U_+ = U_-$ if and only if $P(p)$ is a constant, in which case $U_\pm(\lambda^{-1})$ is the representation $U_0(\lambda)$ of the Lorentz boosts for Φ_0, cf. (2.10).

If ψ is of the form $\psi(p) = L(p)\varphi(p)$, then $U_+(\lambda)\psi(p) = L(p)V_+(\lambda)\varphi(p)$ with

$$V_+(\lambda)\varphi(p) = \lambda \frac{L(-\lambda p)}{L(-p)}\varphi(\lambda p). \tag{5.3}$$

Here we have used that $R(p)/R(-p) = \pm 1$ for real p. In the same way $U_-(\lambda)\psi(p) = L(p)V_-(\lambda)\varphi(p)$ with

$$V_-(\lambda)\varphi(p) = \lambda \frac{L(\lambda p)}{L(p)}\varphi(\lambda p). \tag{5.4}$$

$V_\pm$ is of course just the unitarily equivalent realization of $U_\pm$ in the Fock space constructed over the one particle space of Φ, *viz.* $\mathcal{H}_1 = L^2(\theta(p)pQ(p^2)dp)$.

Since $L(\lambda p)/L(-\lambda p)$ is analytic in p in the upper half plane for $\lambda > 0$ it is clear that if $\tilde{f}$ is the Fourier transform of a test function with support in the right wedge $\mathbf{R}^+$, then the same holds for $V_+(\lambda)\tilde{f}$. The analogous statement for the left wedge $\mathbf{R}^-$ and $V_-(\lambda)$ is also clear. It follows that one can define one paprameter groups $\sigma_t^\pm$ of automorphisms of $\mathcal{M}(\mathbf{R}^\pm)$ (realized as algebras on $\mathcal{H}_0$) by

$$\sigma_t^+(W(f)) = U_+(e^{-2\pi t})W(f)U_+(e^{2\pi t}) \tag{5.5}$$

for supp $f \subset \mathbf{R}^+$, and

$$\sigma_t^-(W(f)) = U_-(e^{2\pi t})W(f)U_-(e^{-2\pi t}) \tag{5.6}$$

for supp $f \subset \mathbf{R}^-$.

In order to show that the groups defined by (5.5) and (5.6) are indeed the modular groups associated with the vacuum state on $\mathcal{M}(\mathbf{R}^+)$ and $\mathcal{M}(\mathbf{R}^-)$ respectively, it is sufficient to verify the KMS condition. We discuss this for the algebra $\mathcal{M}(\mathbf{R}^+)$, the other is treated in the same way. One must show that for test functions f, g with support in $\mathbf{R}^+$ the function $F(t) := \langle \Omega, \sigma_t^+(W(f))W(g)\Omega\rangle$ has an analytic continuation from the real axis into the half strip $\{t + is \mid 0 < s < 1\}$ with

$$\lim_{s \uparrow 1} F(is) = \langle \Omega, W(g)W(f)\Omega\rangle. \tag{5.7}$$

¿From the Weyl relation (2.15) and the equation $\langle\Omega, W(h)\Omega\rangle = \exp(-\|\tilde{h}\|^2/2)$, with $\|\cdot\|$ as in (2.19), it follows that (5.7) is equivalent to a corresponding relation for the two point function, where we find it convenient to use the variable $\lambda = e^{2\pi t}$ instead of t: The function $G(\lambda) := \langle\Omega, \Phi(f)U_+(\lambda)\Phi(g)\Omega\rangle$ has an analytic continuation in λ from the positive real axis up into the cut plane $\mathbf{C}\backslash\mathbf{R}^+$, and

$$\lim_{\theta\uparrow 2\pi} G(e^{i\theta}) = \langle\Omega, \Phi(g)\Phi(f)\Omega\rangle. \tag{5.8}$$

We shall now verify (5.8). Since $\Phi(f) = \Phi_0(L(D)f)$ we have by (5.3) for $\lambda > 0$:

$$\begin{aligned} G(\lambda) = \langle\Omega, \Phi(f)U_+(\lambda)\Phi(g)\Omega\rangle &= \int_0^\infty \tilde{f}(-p)\lambda\frac{L(-\lambda p)}{L(-p)}\tilde{g}(\lambda p)pQ(p^2)dp \\ &= \lambda\int_0^\infty \tilde{f}(-p)\tilde{g}(\lambda p)pL(p)L(-\lambda p)dp. \end{aligned} \tag{5.9}$$

Since g is a test function with support in $\mathbf{R}^+$, $\tilde{g}(\lambda p)$ is analytic in λ in the upper half plane and rapidly decreasing in $p \in \mathbf{R}^+$, together with $d\tilde{g}(\lambda p)/d\lambda$, for fixed λ. Hence we may continue G analytically, obtaining

$$G(\lambda e^{i\pi}) = -\lambda\int_0^\infty \tilde{f}(-p)\tilde{g}(-\lambda p)pL(p)L(\lambda p)dp. \tag{5.10}$$

Now the integrand is analytic in the lower half plane in p with rapid decrease at infinity, so we may rotate the integration contour clockwise by π, obtaining

$$G(\lambda e^{i\pi}) = -\lambda\int_0^\infty \tilde{f}(p)\tilde{g}(\lambda p)pL(-p)L(-\lambda p)dp. \tag{5.11}$$

Finally, we again continue analytically in λ and obtain

$$\lim_{\theta\uparrow 2\pi} G(\lambda e^{i2\theta}) = \lambda\int_0^\infty \tilde{f}(p)\tilde{g}(-\lambda p)pL(-p)L(\lambda p)dp. \tag{5.12}$$

For $\lambda = 1$ the right side is just

$$\int_0^\infty \tilde{g}(-p)\tilde{f}(p)pQ(p^2)dp = \langle\Omega, \Phi(g)\Phi(f)\Omega\rangle. \tag{5.13}$$

Having identified $\sigma_t^\pm$ as the modular groups of $\mathcal{M}(\mathbf{R}^\pm)$ we can now write the polar decomposition of the Tomita operators $S_\pm$ that map $A\Omega$ into $A^*\Omega$ for $A \in \mathcal{M}(\mathbf{R}^\pm)$: We have $S_\pm = J_\pm\Delta_\pm^{1/2}$ with $\Delta_\pm^{1/2} = U_\pm(e^{i\pi})$, in particular

$$\Delta_+^{1/2}\psi(p) = -\frac{P(p)^2}{P(-p)^2}\psi(-p) \tag{5.14}$$

for $\psi \in \mathcal{H}_{01}$ of the form $\psi(p) = L(p)\tilde{f}(p)$, where f has support in $\mathbf{R}^+$, so $\tilde{f}$ is analytic in the upper half plane, and

$$J_+\psi(p) = -\frac{P(p)^2}{P(-p)^2}\psi(p)^* \tag{5.15}$$

for all $\psi \in \mathcal{H}_{01}$. This defines the operators on the one particle space; the extension to the whole Fock space is by sums of tensor products. The corresponding formulae for $\mathcal{M}(\mathbf{R}^-)$ are

$$\Delta_-^{1/2}\psi(p) = -\frac{P(-p)^2}{P(p)^2}\psi(-p) \tag{5.16}$$

and

$$J_-\psi(p) = -\frac{P(-p)^2}{P(p)^2}\psi(p)^* \tag{5.17}$$

We see that $J_+ = J_-$ if and only if $P(p)^2 = P(-p)^2$, i.e, if and only if $P(p)$ is constant, because $P(p)$ and $P(-p)$ have their zeros in different half planes. This shows anew that $\mathcal{M}(\mathbf{R}^+)' = \mathcal{M}(\mathbf{R}^-)$ if and only if $L(p)$ has only real zeros.

Finally we remark that by (5.15) the algebras $\mathcal{M}_1(\mathbf{R}^+)$ and $\mathcal{M}_2(\mathbf{R}^+)$ generated by two different fields Φ_1 and Φ_2 have the same J-operators if and only if the corresponding polynomials P_1, P_2 differ at most by a constant factor. Since Ω is separating for $\mathcal{M}_1(\mathbf{R}^+) \vee \mathcal{M}_2(\mathbf{R}^+)$ (the commutant contains $\mathcal{M}_0(\mathbf{R}^-)$), Tomita's Theorem implies that the algebras $\mathcal{M}_1(\mathbf{R}^+)$ and $\mathcal{M}_2(\mathbf{R}^+)$ are equal if and only if their J-operators are equal. Hence $\mathcal{M}_1(\mathbf{R}^+) = \mathcal{M}_2(\mathbf{R}^+)$ if and only if P_1, P_2 are proportional, as already noted in Section 3.

6. Connection with the Two Dimensional CPT Theorem

In [B] it is proved that any two dimensional local net with translational covariance and spectrum condition can be embedded into a Poincaré covariant net that is CPT invariant and satisfies duality. As pointed out in [B], it is not to be expected in general that the extended net is unique, in fact the construction in [B] gives rise to infinite families of such nets. If the original net satisfies wedge duality, all these nets coincide, but in general they will all be different. We shall now show that this is indeed the case for the field (2.8) if $L(p)$ has a root away from the real axis.

The construction of [B] applied to a net $\mathcal{M}(\cdot)$ on a light ray defines to begin with two local nets $\mathcal{N}_+(\cdot)$ and $\mathcal{N}_-(\cdot)$ as follows: If $I_{a,b}$ denotes the interval $]a,b[$ and $\mathbf{R}_a^+ =]a,\infty[$, $\mathbf{R}_a^- =]-\infty,a[$, then one defines

$$\mathcal{N}_+(I_{a,b}) = \mathcal{M}(\mathbf{R}_a^+) \cap \mathcal{M}(\mathbf{R}_b^+)' \tag{6.1}$$

$$\mathcal{N}_-(I_{a,b}) = \mathcal{M}(\mathbf{R}_b^-) \cap \mathcal{M}(\mathbf{R}_a^-)' \tag{6.2}$$

Both nets contain the original net $\mathcal{M}(\cdot)$ as a subnet and they are Poincaré covariant with CPT symmetry. The Lorentz boosts for $\mathcal{N}_\pm(\cdot)$ are given by the modular group of $\mathcal{M}_\pm(\mathbf{R}^\pm) = \mathcal{N}_\pm(\mathbf{R}^\pm)$, and the CPT operator is the J-operator of $\mathcal{M}_\pm(\mathbf{R}^\pm)$.

The nets $\mathcal{N}_\pm(\cdot)$ generated by the field (2.8) can be decribed explicitly in terms of the net $\mathcal{M}_0(\cdot)$ of the $U(1)$-current Φ_0. For this purpose we define two unitary operators $Y_\pm$ on the Fock space $\mathcal{H}_0$ by

$$Y_+\psi(p) = \frac{P(p)}{P(-p)}\psi(p) \tag{6.2}$$

$$Y_-\psi(p) = \frac{P(-p)}{P(p)}\psi(p) \tag{6.3}$$

for $\psi \in \mathcal{H}_{01}$ and canonical extension to $\mathcal{H}_0$. We assert that

$$\mathcal{N}_+(I) = Y_+\mathcal{M}_0(I)Y_+^{-1} \qquad \text{and} \qquad \mathcal{N}_-(I) = Y_-\mathcal{M}_0(I)Y_-^{-1} \tag{6.4}$$

for all I. To show this for $\mathcal{N}_+(\cdot)$ (the other case is analogous) we note first that since $\mathcal{M}_0(\cdot)$ satisfies duality we have $\mathcal{M}_0(I_{a,b}) = \mathcal{M}_0(\mathbf{R}_a^+) \cap \mathcal{M}_0(\mathbf{R}_b^+)'$. This implies the corresponding relation for the net $Y_+\mathcal{M}_0(\cdot)Y_+^{-1}$. Moreover, we have $\mathcal{N}_+(\mathbf{R}^+) = \mathcal{M}(\mathbf{R}^+)$ and hence also $\mathcal{N}_+(\mathbf{R}_a) = \mathcal{M}(\mathbf{R}_a^+)$ for all a by translational covariance. It therefore suffices to check that $Y_+\mathcal{M}_0(\mathbf{R}^+)Y_+^{-1} = \mathcal{M}_+(\mathbf{R}^+)$. But this is an easy consequence of the fact that Y_+ establishes a one to one correspondence between test functions $\tilde{f}(p)$ that are analytic in the upper half plane and functions of the form $L(p)\tilde{g}(p)$ with $\tilde{g}(p)$ analytic in the upper half plane: Since $L(p) = R(p)P(p)$ with $R(-p) = \pm R(p)$, we have $Y_+\tilde{f}(p) = L(p)\tilde{g}(p)$ with $\tilde{g}(p) = \pm\tilde{f}(p)/L(-p)$.

The fact that $\mathcal{N}_+(I) = Y_+\mathcal{M}_0(I)Y_+^{-1}$ is reflected in the corresponding relations for the Lorentz boosts and the CPT operators. The Lorentz boosts for $\mathcal{M}_0(\cdot)$ are given by (2.10) and hence for $\mathcal{N}_+(\cdot)$ by

$$Y_+U_0(\lambda)Y_+^{-1} = U_+(\lambda) \tag{6.5}$$

where $U_+(\lambda)$ is the modular group for $\mathcal{N}_+(\mathbf{R}^+) = \mathcal{M}(\mathbf{R}^+)$, cf. (5.1). The CPT operator for $\mathcal{M}_0(\cdot)$ is the J operator for $\mathcal{M}_0(\mathbf{R}^\pm)$, given by

$$J_0\psi(p) = -\psi(p)^* \tag{6.5}$$

for $\psi \in \mathcal{H}_{01}$, and the CPT operator for $\mathcal{N}_+(\cdot)$ is thus

$$Y_+J_0Y_+^{-1} = Y_+^2J_0 = J_+, \tag{6.6}$$

cf. (5.15). The corresponding operators for $\mathcal{N}_-(\cdot)$ are given by (5.2) and (5.17). It is clear that the nets $\mathcal{N}_+(\cdot)$ and $\mathcal{N}_-(\cdot)$ are equal if and only if $\mathcal{M}(\cdot)$ satisfies wedge duality, or equivalently, if and only if $J_+ = J_-$. As noted above, this holds if and only if $P(p)$ is a constant, i.e. $L(p)$ has only real roots.

As discussed in [B] one may for each of the nets $\mathcal{N}_\pm(\cdot)$ produce an infinite family of local nets

$$\mathcal{N}_\pm^k(I) = (J_-J_+)^k\mathcal{N}_\pm(I)(J_-J_+)^{-k}, \quad k \in \mathbf{Z}. \tag{6.7}$$

All these nets are again Poincaré covariant, satisfy CPT invariance and duality and extend the original net $\mathcal{M}(\cdot)$. In the present case we have

$$J_-J_+ = Y_+^4 = Y_-^{-4} \tag{6.8}$$

and thus

$$\mathcal{N}_\pm^k(I) = Y_+^{4k\pm1}\mathcal{M}_0(I)Y_+^{-(4k\pm1)} \tag{6.9}$$

because $\mathcal{N}_\pm(I) = Y_\pm\mathcal{M}(I)_0Y_\pm^{-1}$ and $Y_- = Y_+^{-1}$. The CPT operator for $\mathcal{N}_\pm^k(I)$ is

$$Y_+^{4k\pm1}J_0Y_+^{-(4k\pm1)} = Y_+^{2(4k\pm1)}J_0. \tag{6.10}$$

It is clear that unless $P(p)$ is a constant these operators are all different, and the same is therefore true for the nets $\mathcal{N}_\pm^k(\cdot)$.

The models considered above have the special feature that the nets $\mathcal{N}_+(\cdot)$ and $\mathcal{N}_-(\cdot)$ are unitarily equivalent to each other, although they are represented differently on the Hilbert space of the net $\mathcal{M}(\cdot)$. In fact, since $\mathcal{N}_\pm(I) = Y_\pm \mathcal{M}_0(I) Y_\pm^{-1}$ and $Y_+ = Y_-^{-1}$, we have $\mathcal{N}_+(I) = Y_+^2 \mathcal{N}_-(I) Y_+^{-2}$. More generally, the unitary equivalence of $\mathcal{N}_\pm(\cdot)$ holds for all nets such that $\mathcal{M}(\mathbf{R}^-)$ is unitarily equivalent to $\mathcal{M}(\mathbf{R}^+)'$, provided the equivalence is implemented by an operator commuting with translations. It is not to be expected that this holds in general.

A last point worth mentioning is that the nets $\mathcal{N}_\pm^k(\cdot)$ do not exhaust the list of extensions of $\mathcal{M}(\cdot)$ satisfying duality in the examples considered. In fact, $\mathcal{M}(\cdot)$ is a subnet of $\mathcal{M}_0(\cdot)$, which is not equal to any of $\mathcal{N}_\pm^k(\cdot)$ if $P(p)$ is not constant, although these nets are all unitarily equivalent to $\mathcal{M}_0(\cdot)$.

7. Higher Dimensional Examples

In this last section we consider fields in n-dimensional Minkowski space, $n > 2$. The most general two-point function consistent with positivity, translational covariance, spectrum condition and locality has in Fourier space the form

$$\mathcal{W}_2(p) = \sum_{i=1}^{N} M_i(p) d\mu_i(p) \tag{7.1}$$

where $d\mu_i$ is a positive Lorentz-invariant measure with support in the forward light cone and M_i is a polynomial that is positive on the support of $d\mu_i$, $i = 1, \ldots, N$. Guided by the low-dimensional examples considered above we shall compute the modular groups of the wedge algebras for generalized free fields on $\mathbf{R}^n$ in the special case that the sum in (7.1) contains only one term, i.e.,

$$\mathcal{W}_2(p) = M(p) d\mu(p), \tag{7.2}$$

and the polynomial M allows a factorization,

$$M(p) = F(p) F(-p) \tag{7.3}$$

where $F(p)$ is a function (in general not a polynomial) with certain analyticity properties to be specified below.

To describe the properties of F we use the light cone coordinates $x_\pm = x^0 \pm x^1$ for $x = (x^0, \ldots, x^{(n-1)}) \in \mathbf{R}^n$ and denote $(x^2, \ldots, x^n)$ by $\hat{x}$. The Minkowski scalar product is

$$\langle x, y\rangle = \tfrac{1}{2}(x_+y_- + x_-y_+) - \hat{x}\cdot\hat{y}. \tag{7.4}$$

The right wedge, W_R, is characterized by $x_+ > 0$, $x_- < 0$; hence the Fourier transform, $\tilde{f}(p) = \int \exp(-i\langle p, x\rangle) f(x) d^n x$ of a test function f with support in W_R has for fixed $\hat{p} \in \mathbf{R}^{n-2}$ an analytic continuation in p_+ and p_- into the half planes $\operatorname{Im} p_+ > 0$, $\operatorname{Im} p_- < 0$. The property required for F is that $F(\pm p)$ is analytic and $F(-p)$ *without zeros* in this domain, with $F(-p) = F(p)^*$ for $p \in \mathbf{R}^n$. There is no lack of polynomials M allowing such a factorization; one example (suggested by H.J. Borchers) is

$$M(p) = (p^1)^2 + \cdots + (p^n)^2 + m^2 \tag{7.5}$$

with

$$F(p) = \sqrt{\hat{p}\cdot\hat{p} + m^2} + ip^1 = \sqrt{\hat{p}\cdot\hat{p} + m^2} + \tfrac{i}{2}(p_+ - p_-). \tag{7.6}$$

If $d\mu(p) = \theta(p^0)\delta(\langle p, p\rangle - m^2)$ we can replace the polynomial (7.5) by $(p^0)^2$, hence the corresponding generalized free field is nothing but the time derivative $(d/dx^0)\Phi_m(x)$, where Φ_m is the free field of mass m.

In analogy with (5.3) we now define for $\lambda > 0$ the unitary operators $V_R(\lambda)$ on the Fock space $\mathcal{H}$ over the one-particle space $\mathcal{H}_1 = L^2(\mathbf{R}^n, M(p)d\mu(p))$ by

$$V_R(\lambda)\varphi(p) = \frac{F(-\lambda p_+, -\lambda^{-1}p_-, -\hat{p})}{F(-p_+, -p_-, -\hat{p})}\varphi(\lambda p_+, \lambda^{-1}p_-, \hat{p}) \tag{7.7}$$

for $\varphi \in \mathcal{H}_1$ and canonical extension to $\mathcal{H}$. By means of $V_R(\lambda)$ we then define a one parameter group of automorphisms of the von Neumann algebra $\mathcal{M}(W_R)$ on $\mathcal{H}$ generated by the Weyl operators $W(f)$ with $\operatorname{supp} f \subset W_R$:

$$\sigma_t^R(W(f)) = V_R(e^{-2\pi t})W(f)V_R(e^{2\pi t}). \tag{7.8}$$

Note that we are working on the Fock space constructed over $L^2(\mathbf{R}^n, M(p)d\mu(p))$ and not $L^2(\mathbf{R}^n, d\mu(p))$, hence we do not make use of the analogues of the operators U_+ defined in (5.1).

By essentially the same computation that verified (5.8) one shows that (7.8) satisfies the KMS condition and is therefore the modular group defined by the vacuum state on $\mathcal{M}(W_R)$. The corresponding modular operator is given by

$$\Delta_R^{1/2}\varphi(p) = \frac{F(p_+, p_-, -\hat{p})}{F(-p_+, -p_-, -\hat{p})}\varphi(-p_+, -p_-, \hat{p}) \tag{7.9}$$

where φ is analytic in p_+ in the upper half plane and in p_- in the lower half plane. The modular conjugation is

$$J_R\varphi(p) = \frac{F(p_+, p_-, -\hat{p})}{F(-p_+, -p_-, -\hat{p})}\varphi(p_+, p_-, -\hat{p})^*. \tag{7.10}$$

Note that (7.9) and (7.10) are written for φ in the space $L_2(M(p)d\mu(p), \mathbf{R}^n)$ that depends on the field, while ψ in (5.14) and (5.15) belongs to one particle space $L_2(\theta(p)pdp, \mathbf{R})$ of the chiral $U(1)$-current. This is the reason why there is a square in the phase factor in (5.14)-(5.15), while (7.9)-(7.10) has only the first power of F.

For the left wedge $W_L = \{x \mid x_+ < 0, x_- > 0\}$ the corresponding operators are

$$V_L(\lambda)\varphi(p) = \frac{F(\lambda p_+, \lambda^{-1}p_-, \hat{p})}{F(p_+, p_-, \hat{p})}\varphi(\lambda p_+, \lambda^{-1}p_-, \hat{p}), \tag{7.11}$$

$$\Delta_L^{1/2}\varphi(p) = \frac{F(-p_+, -p_-, \hat{p})}{F(p_+, p_-, \hat{p})}\varphi(-p_+, -p_-, \hat{p}) \tag{7.12}$$

and

$$J_L\varphi(p) = \frac{F(-p_+, -p_-, \hat{p})}{F(p_+, p_-, \hat{p})}\varphi(p_+, p_-, -\hat{p})^*. \tag{7.13}$$

By comparing (7.10) and (7.13) we see that the field satisfies the wedge duality condition $\mathcal{M}(W_R)' = \mathcal{M}(W_L)$ if and only if $F(p) = F(-p)$ on the support of $d\mu$. This condition is, e.g., violated in the example (7.6), and the arguments of Sect. 4 are easily generalized to show that neither does essential duality hold in this case.

The example (7.6) demonstrates also that the modular group of $\mathcal{M}(W_R)$ may act nonlocally in the $\hat{x}$-directions. In fact, let f be a test function with compact support in W_R. Under the transformation (7.7) the Fourier transform $\tilde{f}$ is mapped into

$$\tilde{f}_\lambda(p) = \frac{\sqrt{\hat{p}\cdot\hat{p} + m^2} - \frac{i}{2}(\lambda p_+ - \lambda^{-1}p_-)}{\sqrt{\hat{p}\cdot\hat{p} + m^2} - \frac{i}{2}(p_+ - p_-)}\tilde{f}(\lambda p_+, \lambda^{-1}p_-, \hat{p}). \tag{7.10}$$

This is no longer the Fourier transform of a function of compact support in the $\hat{x}$-directions, because it is not analytic in $\hat{p}$. From this lack of analyticity it is not difficult to deduce that $W(f_\lambda)$ does not belong to any wedge algebra generated by the field unless the wedge is a translate of W_R or W_L, but we refrain from presenting a formal proof of this. The operator $W(f_\lambda)$ is still localized in the x^0, x^1-directions in the sense that it is contained in $\mathcal{M}(W_R+a)\cap\mathcal{M}(W_R+b)'$ for some $a, b \in W_R$, in accordance with the theorem of Borchers [B].

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