Normal forms and nonlinear symmetries

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Abstract.

We present some results concerning Poincaré-Dulac normal forms of dynamical systems which are symmetric under (possibly nonlinear) Lie point transformations. We show in particular that the vector fields defining the dynamical system and the Lie symmetry can be put into a "joint normal form".

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1. Introduction

The theory of Poincaré-Dulac normal forms [1-4] provides a classification of smooth dynamical systems (vector fields) in the neighbourhood of a fixed point (zero), up to $\mathcal{C}^\infty$ equivalence[1].

If the dynamical systems- or vector fields - are moreover known to be symmetric under a *linear* transformation, this fact is reflected into their normal form , and simplifies the normal form unfolding for equivariant setting [1,2,5,6].

In this note, we present some results concerning the case the dynamical system is symmetric under a general - i.e. possibly *nonlinear* - Lie-point transformation, i.e. a diffeomorphism. A more complete exposition of the results given here and extensions thereof, are presented elsewhere [7].

In section 2 below we fix notation and recall the Poincaré-Dulac procedure for transforming a dynamical system or a vector field into its normal form, and in section 3 we quickly reproduce geometrically some known results for the case of linear symmetries. In section 4 we deal with nonlinear diffeomorphisms, and obtain our main result as the final theorem 3.

2. Normal forms

By a (smooth) dynamical system (DS) we will mean a system of first order autonomous ODEs in[2] $\mathbf{R}^n$,

$$\dot{x} = f(x) \qquad ; \ x \in M \equiv \mathbf{R}^n \ , \ f : M \to TM \tag{1}$$

where f is a smooth ($\mathcal{C}^\infty$) function; equivalently, a DS is identified by a (smooth) vector field (VF) on $M \equiv \mathbf{R}^n$, i.e. for (1)

$$\varphi = f(x)\partial_x \equiv f^i(x)\frac{\partial}{\partial x^i} \tag{2}$$

We are in particular interested in the case (1) admits a fixed point x_0, that can be taken to be the origin of $\mathbf{R}^n$; i.e. we assume from now on

$$f(0) = 0 \tag{3}$$

We want then to study, by means of formal perturbative expansions, the flow of (1) in the vicinity of the fixed point.

We expand f in a series of homogeneous terms, dropping that of order zero due to (3)

$$f(x) = Ax + \sum_{k=2}^{\infty} F_k(x) \tag{4}$$

where F_k is homogeneous of order k, and A is an $n \times n$ (real) matrix[3].

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[1] Other kinds of normal forms do also exist, corresponding to different kind of equivalence, e.g. Shoshitashvili normal forms for $\mathcal{C}^0$ equivalence [1,2]. In this note, by normal forms we will always mean the Poincaré-Dulac ones.

[2] Since we are going to deal with a local problem, considering a general smooth manifold $M \subseteq \mathbf{R}^n$ would not add any generality to our considerations.

[3] By means of linear transformations, $x \to Tx$, $A \to TAT^{-1}$, we can classify the linear part of (1) in a standard way; if the fixed point is hyperbolic [4,8,9] this provides a topological ($\mathcal{C}^0$) classification of the flows, but even in this case does not give a $\mathcal{C}^\infty$ classification. In the sequel we will leave A unchanged.

The Poincaré-Dulac procedure shows that, by a sequel of nonlinear near- identity formal changes of coordinates, the DS (1) can be reduced to a $\mathcal{C}^\infty$ equivalent DS, its *normal form* (NF),

$$\dot{x} = g(x) = Ax + \sum_{k=2}^{\infty} G_k(x) \tag{5}$$

where the G_k are all *resonant* with A [1-4].

The changes of coordinates involved are of the form

$$x \to \widetilde{x} = x + h_k(x) \qquad k \geq 2 \tag{6}$$

and $h_k : \mathbf{R}^n \to \mathbf{R}^n$ is homogeneous of degree k. Under such a change of coordinates, the terms F_k of (4) will be changed to $\widetilde{F}_k$, and it is straightforward to see that $\widetilde{F}_m = F_m$ for $m < k$, and

$$\widetilde{F}_k = F_k - \mathcal{L}_A(h_k) \tag{7}$$

where $\mathcal{L}_A$ is the *homological operator* associated with A, given in terms of Poisson brackets by

$$\mathcal{L}_A(f) = \{Ax, f(x)\} \equiv (Ax \cdot \nabla)\, f(x) - (f(x) \cdot \nabla)\, Ax \tag{8}$$

Therefore, we can proceed sequentially and simplify F_k as much as possible by opportunely choosing h_k: the changes of coordinates (6) for $k' > k$ will not affect the terms F_k which have already been simplified. This gives a procedure to *normalize* f up to any desired order (we will take this to be formally infinite).

If π is the projection on the range[4] of $\mathcal{L}_A$, the "opportune" choice of h_k mentioned above corresponds to solving the *homological equation*

$$\mathcal{L}_A(h_k) = \pi(F_k) \tag{9}$$

after which we will be left with, cf. (7),

$$\widetilde{F}_k = (I - \pi)\, F_k \tag{10}$$

It should be stressed that the G_k *cannot* be obtained simply as $G_k = (I - \pi)F_k$, as the transformation (6) at order k does also change, in a very complicate way, the F_m with $m > k$.

Let us now make the simplifying assumption - which will be taken for granted in the following - that $A = (Df)(0)$ is a *normal* matrix, i.e.

$$[A, A^+] = 0 \tag{11}$$

This ensures [5] that $\mathcal{L}_A^+ = \mathcal{L}_{A^+}$ and, in particular, that $\mathrm{Ran}(\mathcal{L}_A)$ and $\mathrm{Ker}(\mathcal{L}_A)$ are complementary subspaces[5], so that (10) reads $\widetilde{F}_k \in \mathrm{Ker}(\mathcal{L}_A)$. In other words, the NF satisfies $g \in \mathrm{Ker}(\mathcal{L}_A)$ or, in geometrical terms,

$$\{Ax, g(x)\} = 0 \tag{12}$$

which expresses in particular the property of the $G_k(x)$ of being resonant with A.

3. Linear symmetries

[4] The considerations involving kernel and ranges, and projections to these, of homological operators can be made rigorous by considering spaces of homogeneous polynomial vector fields [7].

[5] Indeed, if (11) holds, then $\mathrm{Ker}(\mathcal{L}_A^+) = \mathrm{Ker}(\mathcal{L}_{A^+})$.

Let us now consider the case (1) admits a linear symmetry, i.e.

$$f(Sx) = Sf(x) \tag{13}$$

with S an $n \times n$ matrix. In geometrical terms, this means that there is a VF $\sigma = (Sx)\partial_x$ which commutes with the VF (2), i.e.

$$[\sigma, \varphi] \equiv \{Sx, f(x)\}\partial_x = 0 \tag{14}$$

The geometrical relation (14) is independent of the choice of coordinates, and must therefore also hold if $\varphi = f(x)\partial_x$ is expressed in NF, $\varphi = g(x)\partial_x$. Since σ contains only linear terms, its expression is not affected by the normalizing transformation (6). Therefore the NF will still admit the same symmetry, i.e. we are granted to have

$$g(Sx) = Sg(x) \tag{15}$$

Relation (14) can also be written as $\mathcal{L}_S(f) = 0$; therefore the above discussion[6] can be summarized by

Proposition. If $f(x)$ satisfies (3) and (13), it can be reduced to a NF $g(x)$ such that $g \in [\mathrm{Ker}(\mathcal{L}_A) \cap \mathrm{Ker}(\mathcal{L}_S)]$.

Notice also that (12) shows that the VF $\alpha = (Ax)\partial_x$ is a (linear) symmetry[7] of the NF.

4. Nonlinear symmetries

If (1) admits a general, i.e. possibly nonlinear, time-independent Lie-point symmetry σ, this fact is still expressed by the commutation relation (14), but now the representation of σ will contain nonlinear terms, i.e. we will have

$$\sigma = s(x)\partial_x \quad , \quad s(x) = Sx + \sum_{k=2}^{\infty} B_k(x) \tag{17}$$

where we assume S a normal matrix (cf. [7]); now the B_k will be changed by the normalizing transformation (6),(9) according to the equivalent of (7), i.e. $\widetilde{B}_m = B_m$ for $m < k$ and

$$\widetilde{B}_k = B_k - \mathcal{L}_S(h_k) \tag{18}$$

We will write (14) in the original coordinates by expanding $\varphi = f(x)\partial_x$ and $\sigma = s(x)\partial_x$ in homogeneous polynomial VF; it results

$$\{f(x), s(x)\} = \sum_{k=0}^{\infty} (C_k - u_k) \tag{19}$$

where $C_0 = \{f_0, s_0\}$, $u_0 = u_1 = 0$ and, with [8] $f_k = F_{k+1}$, $s_k = B_{k+1}$, $f_0 = Ax$, $s_0 = Sx$,

$$\begin{aligned} C_k \equiv & \{f_0, s_k\} + \{f_k, s_0\} \\ u_k \equiv & - \sum_{j=1}^{k-1} \{f_j, s_{k-j}\} \end{aligned} \tag{20}$$

It is easy to check that (6) at order k does leave u_m invariant for $m \le k$ and C_k invariant for $m < k$, acting on C_k by

$$\widetilde{C}_k = C_k + \{\{s_0, f_0\}, h_k\} \tag{21}$$

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[6] We have recast in geometrical terms known results, see [5] for the algebraic approach.
[7] If (11) is not satisfied, a weaker result holds, see [5,7].
[8] We resort to this notation in order to conform to the notation of [7].

Notice that if (14) is satisfied, this necessarily implies

$$\{f_0, s_0\} = 0 \longleftrightarrow [A, S] = 0 \tag{22}$$

so that C_k is also invariant, as it should.

We have then the

Theorem 1. If φ is expressed in NF as $\varphi = f(x)\partial_x$, with f satisfying (3) and (11), then any σ commuting with φ is expressed in the same coordinates as $\sigma = s(x)\partial_x$ with $s \in \mathrm{Ker}(\mathcal{L}_A)$, where $A = (Df)(0)$.

Proof: Indeed, (14) implies $C_k = u_k \ \forall k$; for $k = 0$ we get (22), and for $k = 1$

$$\{f_0, s_1\} + \{f_1, s_0\} = 0$$

Apply $\mathcal{L} \equiv \mathcal{L}_A$ to both sides, and recall $\mathcal{L} = \{f_0, .\}$. Using Jacobi identity and the fact that $\{f_0, f_k\} = 0$ since $f(x)$ is in NF, we get $\mathcal{L}(\mathcal{L}(s_1)) = 0$; if (11) is satisfied, this implies $s_1 \in \mathrm{Ker}(\mathcal{L})$. We can then proceed similarly and recursively for all k: indeed, $\{f_k, s_0\} \in \mathrm{Ker}(\mathcal{L})$, since both f_k and s_0 commute with f_0; if $s_j \in \mathrm{Ker}(\mathcal{L}) \ \forall j < k$, then $u_k \in \mathrm{Ker}(\mathcal{L})$; using again (11), $C_k = u_k$ is then solved only if

$$\mathcal{L}(s_k) = 0 \quad ; \quad \mathcal{L}_S(f_k) \equiv \mathcal{S}(f_k) = u_k \tag{23}$$

which proves the theorem. ••

Notice that in the above theorem there seems to be an asymmetry in the roles of φ and σ, as both f and s are in $\mathrm{Ker}(\mathcal{L})$, but we are not granted that neither f nor s belong to $\mathrm{Ker}(\mathcal{S})$. It is indeed possible to restore a symmetry of roles between φ and σ, as we do now prove.

Theorem 2. Let $\varphi = f(x)\partial_x$ be in NF, and let $\sigma = s(x)\partial_x$ commute with φ. Then it is possible to choose coordinates such that the representation of φ is not changed and that $s \in [\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})]$. Moreover, necessarily $f \in [\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})]$.

Proof: The homological equation (9) selects h_k only up to a $\delta h_k \in \mathrm{Ker}(\mathcal{L})$; by opportunely choosing this δh_k, we can eliminate terms of B_k in $\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ran}(\mathcal{S})$ (recall S is normal), so that due to theorem 1 we are left with terms $s \in [\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})]$ alone. To prove $f \in \mathrm{Ker}(\mathcal{S})$, notice that if $f_j \in \mathrm{Ker}(\mathcal{S}) \ \forall j < k$, then $\{f_j, s_m\} \in \mathrm{Ker}(\mathcal{S})$ by Jacobi identity and $s_m \in \mathrm{Ker}(\mathcal{S})$ (as we have just proved); but $f_0 \in \mathrm{Ker}(\mathcal{S})$, so indeed $u_k \in \mathrm{Ker}(\mathcal{S})$. Applying $\mathcal{S}$ to $(C_k - u_k) = 0$ we get therefore $\mathcal{S}(\mathcal{S}(f_k)) = 0$; since S is normal, we have necessarily $f_k \in \mathrm{Ker}(\mathcal{S})$. ••

The above theorem suggests to introduce the following

Definition. Let the VF $\varphi = f(x)\partial_x$ and $\sigma = s(x)\partial_x$ have a common fixed point, $f(x_0) = s(x_0) = 0$, and let their linearization at the fixed point be given by normal matrices $A = (Df)(x_0)$ and $S = (Ds)(x_0)$. The VF are in **joint normal form** if and only if both f and s are in $[\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})]$.

Our discussion and results will then be recast as the

Theorem 3. Let the vector fields φ and σ *i)* have a common fixed point x_0, *ii)* have normal linearizations at x_0, and *iii)* do commute. Then they can be put in joint normal form by means of Poincaré- Dulac transformations.

It should be stressed that the above theorem is also particularly useful in the unfolding of equivariant NFs for problem with assigned symmetry: in this case, indeed, the allowed terms for the DS in NF are not all those in $\mathrm{Ker}(\mathcal{L})$, as it would be expected on the basis of standard NF theory, but only those in $[\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})]$.

The problem of joint NF unfolding can be further simplified (i.e. the general form of joint NF further restricted) by an extension of the above theory, given in [7]. Roughly speaking, this consists in considering not only $\mathcal{L} = \{f_0, .\}$ (and the like for σ), but the sequence of operators $\mathcal{L}^{[k]} = \{\sum_{m=0}^k f_m, .\}$; this gives a filtration of the Lie algebra of vector fields in $\mathbf{R}^n$. Further details, extensions, examples and discussion can be found in [7], to which the reader is referred for these as well as for a more rigorous discussion of the results presented here.

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