# Beta function and Schwinger functions for a many fermions system in one dimension. Anomaly of the Fermi surface.

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**Abstract**: *We present a rigorous discussion of the analyticity properties of the beta function and of the effective potential for the theory of the ground state of a one dimensional system of many spinless fermions. We show that their analyticity domain as a function of the running couplings is a polydisk with positive radius bounded below, uniformly in all the cut offs (infrared and ultraviolet) necessary to give a meaning to the formal Schwinger functions. We also prove the vanishing of the scale independent part of the beta function showing that this implies the analyticity of the effective potential and of the Schwinger functions in terms of the bare coupling. Finally we show that the pair Schwinger function has an anomalous long distance behaviour.*

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*§1 Introduction.*

In this paper we study a system of interacting one dimensional fermions. The Hamiltonian for $n$ spinless particles in a periodic box of length $L$ will be:

$$H=\sum_{i=1}^{n}(\frac{-\Delta_{\vec{x}_i}}{2m}-\mu)+2\lambda\sum_{i<j}\bar{v}(\vec{x}_i-\vec{x}_j) \tag{1.1}$$

where $m>0$ is the particles mass, $\mu$ is the chemical potential, $2\lambda\bar{v}(\vec{r})$ is the pair potential, which we suppose bounded, smooth, even in $\vec{r}$ and with finite range $p_0^{-1}$.

Physically one defines the Fermi momentum $p_F$ so that the ground state energy of $H$ has the minimum at $n=2p_F L/2\pi$ when $\mu=p_F^2/2m$, while the mass of the particles is defined by computing the minimum energy increase obtained by adding one particle to the ground state.

Usually one requires that $p_F$ has a given value and that the minimum energy increase has the form:

$$e(\vec{k}_0)=(\vec{k}_0^2-p_F^2)/2m \tag{1.2}$$

where $\vec{k}_0$ is the smallest $\vec{k}$ of the form $2\pi sL^{-1}$, $s$ integer, larger than $p_F$; this, however, cannot be imposed on (1.1) as there are not free parameters.

Hence we shall study:

$$H=\sum_{i=1}^{n}(\frac{-\Delta_{\vec{x}_i}}{2m}-\mu)+\alpha\sum_{i=1}^{n}(\frac{-\Delta_{\vec{x}_i}}{2m}-\mu)+\nu n+2\lambda\sum_{i<j}\bar{v}(\vec{x}_i-\vec{x}_j)\equiv T+\alpha T+\nu N+\bar{V} \tag{1.3}$$

and tune $\alpha,\nu$ so that the ground state of (1.3) has the above two properties. We require, given $p_F, m>0$, that $p_F$ and $L$ are so related that $2\pi L^{-1}(n_F+1/2)=p_F$ with $n_F$ integer; this implies, in particular, that no particle can have momentum $\pm p_F$ and that $\vec{k}_0=p_F+\frac{\pi}{L}$.

It is very useful to write the Hamiltonian $H$ in second quantization, *i.e.* in terms of creation and annihilation operators $a^+_{\vec{k}}, a^-_{\vec{k}}$. Defining:

$$\psi^{\pm}_{\vec{x}}=L^{-1/2}\sum_{\vec{k}}e^{\pm i\vec{k}\cdot\vec{x}}a^{\pm}_{\vec{k}} \tag{1.4}$$

we have:

$$\begin{aligned}
T&=\sum_{\vec{k}}(\frac{\vec{k}^2}{2m}-\mu)a^+_{\vec{k}}a^-_{\vec{k}}=\int_L d\vec{x}\,(\frac{1}{2m}\vec{\partial}\psi^+_{\vec{x}}\vec{\partial}\psi^-_{\vec{x}}-\mu\psi^+_{\vec{x}}\psi^-_{\vec{x}})\\
N&=\sum_{\vec{k}}a^+_{\vec{k}}a^-_{\vec{k}}=\int_L d\vec{x}\,\psi^+_{\vec{x}}\psi^-_{\vec{x}} \qquad (1.5)\\
\bar{V}&=\lambda\int_{L\times L}d\vec{x}d\vec{y}\,\bar{v}(\vec{x}-\vec{y})\;\psi^+_{\vec{x}}\psi^+_{\vec{y}}\psi^-_{\vec{y}}\psi^-_{\vec{x}}
\end{aligned}$$

Let us denote $E^{(n)}$ the ground state energy of the system with $n$ particles and let us define $N = 2n_F + 1$.

By first order perturbation theory it is easy to see that:

$$E^{(N+1)} - E^{(N)} = (1+\alpha)e(p_F + \frac{\pi}{L}) + \nu + 2\lambda\frac{2\pi}{L}\sum_{\vec{k}:e(\vec{k})<0}[\hat{v}(0) - \hat{v}(p_F + \frac{\pi}{L} - \vec{k})] \quad (1.6)$$

$$E^{(N-1)} - E^{(N)} = -(1+\alpha)e(p_F - \frac{\pi}{L}) - \nu - 2\lambda\frac{2\pi}{L}\sum_{\vec{k}:e(\vec{k})<0}[\hat{v}(0) - \hat{v}(p_F - \frac{\pi}{L} - \vec{k})] \quad (1.7)$$

where $2\pi\hat{v}(\vec{k}) = \int d\vec{x}e^{-i\vec{k}\vec{x}}\bar{v}(\vec{x})$.

The conditions that the system has $p_F$ as Fermi momentum and $m$ as mass in presence of interaction can be translated in the conditions:

$$E^{(N\pm1)} - E^{(N)} = \pm e(p_F \pm \frac{\pi}{L}) \quad (1.8)$$

which imply, by using $\hat{v}(p_F \pm \frac{\pi}{L} - \vec{k}) = \hat{v}(p_F - \vec{k}) \pm \frac{\pi}{L}\hat{v}'(p_F - \vec{k}) + O(L^{-2})$, that:

$$\begin{aligned} &\nu + 2\lambda\int_{e(\vec{k})<0} d\vec{k}[\hat{v}(0) - \hat{v}(p_F - \vec{k})] + O(L^{-1}) = 0 \\ &\alpha\frac{p_F}{m} - 2\lambda\int_{e(\vec{k})<0} d\vec{k}\hat{v}'(p_F - \vec{k}) + O(L^{-1}) = 0 \end{aligned} \quad (1.9)$$

Recursively one can determine the higher order corrections to $\alpha, \nu$. So far with formal perturbation theory.

If one, however, attempts at estimating the remainders one meets serious difficulties. Unless one is willing to take $\lambda$ so small to be of order much smaller than $L^{-1}$, say of $O(\eta L^{-1})$ for some small $\eta$. In the latter case one can easily check that there are no convergence problems for the perturbation expansions (and in fact the first order is dominant), as it is physically obvious. For $\eta$ small enough and $L$ fixed the perturbation theory converges, the ground state is unique and separated by a gap of order $L^{-1}p_F/m$ from the first excited state.

One possible approach to the theory of low temperature Fermi gases, that we shall follow, is to study the above perturbation expansions as well as the expansions of the other interesting quantities (like the system reduced density matrices or the Schwinger functions, see §2), and to show that they can be resummed so that, after resummation, they admit analytic continuation in $\lambda$ up to $\lambda$'s of size of order 1 uniformly in $L$ and $\beta$.

If this goal is achieved, it is clear that we have constructed objects of interest for low temperature physics: they can be interpreted as Gibbs states of the system provided they verify the necessary positivity properties. The latter are, essentially, automatically verified as we know that for $L, \beta > 0$ fixed none of the correlations functions has a singularity for $\lambda, \alpha, \nu$ real (small or large).

In this paper we study a resummation algorithm, generated by the application of the renormalization group methods to the study of the above series. We show that the resummation can be described in terms of stability properties of a well defined dynamical system.

We call *beta function* the functions defining the dynamical system iteration map $B_h$: the latter operates on a three dimensional set of parameters called the running couplings denoted by $\underline{r}$. Each triple $\underline{r}_0$ of initial data generates, for $h = 0, -1, -2, \ldots$ a trajectory $\underline{r}_{h-1} = \underline{r}_h + B_h(\underline{r}_h, \underline{r}_{h+1}, \ldots, \underline{r}_0)$ which, under the condition that $|\underline{r}_h|$ remains small, provides a set of parameters in terms of which the relevant dynamical quantities (Schwinger functions) can be expanded in a *convergent power series.*

The reason we call the above a *resummation* is because the expansion constructed is not in power series of $\underline{r}_0$: if we express $\underline{r}_h$ in powers of $\underline{r}_0$ it may well be that the convergence radius of the expansion shrinks to zero as $h \to -\infty$.

Our main results are:

1) *the existence and boundedness and analyticity of the functions* $B_h(\underline{r}_h, \ldots, \underline{r}_0)$ *as functions of their arguments (regarded as independent arguments), if they are small enough:* $|\underline{r}_h| \le \varepsilon$, *for all* $h \le 0$.

2) *We also show that* $B_h(\underline{r}, \underline{r}, \ldots, \underline{r}) \equiv \beta_h(\underline{r})$ *is the sum of two parts* $\beta_h(\underline{r}) = \beta(\underline{r}) + \hat{\beta}_h(\underline{r})$ *with* $\hat{\beta}_h(\underline{r}) \to 0$, *for* $h \to -\infty$ *and for* $|\underline{r}| \le \varepsilon$, *exponentially fast, and with* $\beta(\underline{r})$ *("scaling part of the beta function") which we show (in § 7) to be zero.*

3) *We deduce from 1),2) an expansion in powers of* $\{\underline{r}_h\}_{h \le 0}$, *convergent if* $|\underline{r}_h| \le \varepsilon$ *for all* $h \le 0$, *of the pair (and higher) Schwinger functions. The expansion implies, if* $|\underline{r}_h| \le \varepsilon$ *for all* $h \le 0$, *that the pair Schwinger function approaches* $0$ *as its argument* $\xi \to \infty$ *faster than the free Schwinger function does, and we compute exactly how fast (i.e. we compute the anomaly exponent).*

Some support to the validity of the vanishing of $\beta(\vec{r})$ in 2) above was given in [BG], [BGM], by reducing it to the proof of a similar conjecture for the Luttinger model. In [BGM] the proof of the conjecture was reduced to a property of the Schwinger functions which is implied by the results in 3) above, plus the independence of the exact solubility of the Luttinger model from the cut offs necessary to define it. Thus we showed that the exact solubility of the latter model would allow us to establish a rigorous proof of the conjecture if we knew suitable uniformity properties on the Luttinger model running constants defined in a way entirely analogous to the one followed for our problem, see [BGM].

The above scheme of proof is discussed in §7 and, using the new results derived in the previous sections (§3÷6), it is completed.

The discussion of 1) requires the solution of two distinct problems. The first is an ultraviolet problem, which should be considered trivial as it is technically similar (but easier) to the theory of the ultraviolet stability of the Gross Neveu model in field theory, whose solution is well known [GK]: we perform it in detail (as our formalism differs from that of

[GK]), but we find no unexpected difficulty (§3). The second problem is an infrared problem: this presents new difficulties as it requires the discussion of a *anomalous dimension* (physically this means that the perturbed system has correlations which decay at $\infty$ faster than the free ones). To our knowledge this is the first example of a rigorous theory of the beta function of an anomalous renormalization group flow and, technically, represents the major part of this work.

The results of §7 also imply the existence of a one parameter family of non trivial fixed points of our renormalization group transformation: this can be regarded as the origin of the anomalous dimension: however we only allude (§5) to such a corollary as it not essential for our work.

In the next section we set up the formalism in a self consistent way trying to discuss the rigour issues growing out of the functional integral representation of the Schwinger functions that we plan to use in the rest of the paper.

It is useful to state our main result in a form independent on the subsidiary concepts (like running coupling, beta function, *etc.*) and based solely on the hamiltonian (1.3) and on the standard notion of pair Schwinger function, $S(x)$, of the model (introduced formally in the next section); it can be summarized in the following theorem:

**Theorem:**

*Given a pair potential $\lambda\bar{v}(\vec{x}-\vec{y})$, with $\bar{v}$ smooth and with short range $p_0^{-1}$, one can find analytic functions $\alpha(\lambda), \nu(\lambda)$, holomorphic near $\lambda = 0$ and of order $\lambda$, such that the one dimensional spinless Fermi gas with hamiltonian:*

$$\sum_{i=1}^{N}\Big(\frac{-\Delta_{\vec{x}_i}}{2m(\lambda)} - \frac{p_F^2}{2m(\lambda)} + \nu(\lambda)\Big) + 2\lambda\sum_{i<j}\bar{v}(\vec{x}_i - \vec{x}_j) \tag{1.10}$$

*with $m(\lambda) = [1+\alpha(\lambda)]^{-1}m$, $m > 0$, admits a zero temperature Gibbs state (defined as the $T \to 0$ limit of a $T > 0$ Gibbs state) with a Euclidean pair Schwinger function $S(x-y)$ verifying, for $|x-y|p_0$ large, the relation:*

$$S(x-y) = (1 + A_0(\lambda))\frac{S_0(x-y)}{(p_0|x-y|)^{2\eta(\lambda)}} + A_1(\lambda)\frac{1}{(p_0|x-y|)^{1+2\eta(\lambda)}} \tag{1.11}$$

*with $\eta(\lambda)$, $A_i(\lambda)$ analytic near $\lambda = 0$, $\eta(\lambda) = O(\lambda^2)$, $A_i(\lambda) = O(\lambda)$, $A_0$ independent of $x, y$ and with $S_0$ being the pair Schwinger function for the free gas with Fermi momentum $p_F$ and mass $m$.*

Note that $S_0(x-y)$ tends to zero with oscillations on scale $p_F^{-1}$ and speed $|x-y|^{-1}$, so that the first term in (1.11) dominates over the second "when non zero".

The theorem was proposed by Tomonaga who developed theoretical arguments for its validity, [T]; on the basis of Tomonaga's work Luttinger proposed a model which, if Tomonaga's ideas were correct, should behave in the same way as the system (1.1) that we are considering, [L]. The model differs from (1.1) in two respects: first there are two spinless particles and second the kinetic energy is linear in the momentum. Luttinger also gave arguments to suggest that the model might be exactly soluble. The model was solved exactly, later, by Mattis and Lieb, proving that indeed it did behave as expected on the basis of its heuristic equivalence to the Tomonaga's theory of the model (1.1), [ML].

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## §2 *Functional integral representation of fermionic correlation functions*

The Schwinger functions of a Hamiltonian $H$, like (1.3) are defined by:

$$S(\vec{x}_1,t_1,\sigma_1,\ldots,\vec{x}_s,t_s,\sigma_s)=\frac{Tr(e^{-(\beta-t_1)H}\psi^{\sigma_1}_{\vec{x}_1}\ldots e^{-(t_{s-1}-t_s)H}\psi^{\sigma_s}_{\vec{x}_s}e^{-t_sH})}{Tr\,e^{-\beta H}} \tag{2.1}$$

for $\beta>t_1>t_2>\ldots>t_s>0$, $\psi^{\sigma}_{\vec{x}},\sigma=\pm$, being field operators on the Fock space of a fermion system confined in a box of size $L$, with periodic boundary conditions, and at temperature $\beta^{-1}>0$.

At fixed $\beta,L$ the (2.1) are, by inspection, real analytic in $\lambda,\alpha,\nu$: their holomorphy domain has complex size which, for the time being, is totally out of control and it may shrink to 0 as $\beta\to\infty$ or $L\to\infty$.

If we are willing to take $\lambda,\nu,\alpha$ of $O(\eta L^{-1})$ with $\eta$ small, it is not difficult to see that we have in fact uniformity in $\beta$ as $\beta\to\infty$. The basic reason is that, if $\lambda,\nu,\alpha$ are so small, we see by perturbation theory that the lowest eigenvalue of $H$ is separated by a gap from the next. Hence the limit as $\beta\to\infty$ is simply expressed in terms of the expectation value in the ground state $|0\rangle_{\lambda,\nu,\alpha}$ (which is also analytic in such small $\lambda,\nu,\alpha$), as:

$$S(\vec{x}_1,t_1,\sigma_1,\ldots,\vec{x}_s,t_s,\sigma_s)={}_{\lambda,\nu,\alpha}\langle 0|\psi^{\sigma_1}_{\vec{x}_1}\ldots e^{-(t_{s-1}-t_s)H}\psi^{\sigma_s}_{\vec{x}_s}|0\rangle_{\lambda,\nu,\alpha} \tag{2.2}$$

This is manifestly analytic in $\lambda,\nu,\alpha$. Knowing the above analyticity property we can find the expansion coefficients in powers of $\lambda,\nu,\alpha$. The classical calculation is as follows.

We define the imaginary time fields (see (1.4)) as:

$$\psi^{\pm}_{\vec{x},t} = L^{-1/2} \sum_{\vec{k}} e^{\pm i\vec{k}\vec{x} \pm e(k)t} a^{\pm}_{\vec{k}} \equiv e^{tT} \psi^{\pm}_{\vec{x}} e^{-tT} \tag{2.3}$$

Then by using the representation (where $V \equiv \bar{V} + \nu N + \alpha T$, see (1.3)):

$$e^{-tH} = \lim_{n\to\infty} \big(e^{-tT/n}(1 - \frac{tV}{n})\big)^n \tag{2.4}$$

we find that the numerator of (2.1) becomes:

$$\sum \pm \int \mathrm{Tr}\{e^{-\beta T} V(t'_1) \ldots V(t'_{p_1-1}) \psi^{\sigma_1}_{\vec{x}_1, t'_{p_1}} \ldots \psi^{\sigma_s}_{\vec{x}_s, t'_{p_1+\ldots+p_s}} \ldots V(t'_{p_1+\ldots+p_{s+1}})\} \, d\underline{t}' \tag{2.5}$$

where $V(t) = e^{tT} V e^{-tT}$ and the sum runs over integers $p_1, p_2, \ldots$ while the integral is over all the $t'_j$ variables with $j \neq p_1, p_1+p_2, \ldots, p_1+p_2+\ldots+p_s$ and $t'_{p_1}, t'_{p_1+p_2}, \ldots, t'_{p_1+p_2+\ldots+p_s}$ are fixed to be $t_1 > t_2 > \ldots > t_s \geq 0$, respectively; finally the $t'$ variables are constrained to decrease in their index $j$, and the sign $\pm$ is $+$ if the number of $V$ factors is even and $-$ otherwise.

Since the product of $V$'s is an integral of a sum of products of $\psi^{\pm}_{\vec{x},t}$ operators and since the $T$ is a quadratic hamiltonian in the $\psi^{\pm}$ operators, the Wick's theorem holds for evaluating $\mathrm{Tr}(\exp -\beta T(\cdot))/\mathrm{Tr}(\exp -\beta T)$ (see, for example, [NO]) and therefore it will be possible to express the various terms in (2.5) as suitable integrals of sums of products of expressions like:

$$\begin{aligned} g_+(\vec{\xi}, \tau) =& \mathrm{Tr}\, e^{-\beta T} \psi^-_{\vec{x},t} \psi^+_{\vec{x}',t'} / \mathrm{Tr}\, e^{-\beta T} \\ g_-(\vec{\xi}, \tau) =& \mathrm{Tr}\, e^{-\beta T} \psi^+_{\vec{x},t} \psi^-_{\vec{x}',t'} / \mathrm{Tr}\, e^{-\beta T} \end{aligned} \tag{2.6}$$

if $\vec{\xi} = \vec{x} - \vec{x}', \;\; \tau = t - t' > 0$, which we combine to form a single function:

$$g(\vec{\xi}, \tau) = \begin{cases} g_+(\vec{\xi}, \tau) & \text{if } \tau > 0 \\ -g_-(-\vec{\xi}, -\tau) & \text{if } \tau \leq 0 \end{cases} \tag{2.7}$$

Then it is easy to see, from Wick's theorem, that the generic term in (2.5) can be expressed graphically as follows.

One lays down graph elements like:

Fig. 1

symbolizing respectively:

$$\begin{aligned}
&-\lambda\bar{v}(\vec{x}_1-\vec{x}_2)\psi^+_{\vec{x}_1,t}\psi^+_{\vec{x}_2,t}\psi^-_{\vec{x}_2,t}\psi^-_{\vec{x}_1,t}\\
&-(\nu-\mu\alpha)\psi^+_{\vec{x},t}\psi^-_{\vec{x},t}\\
&(\alpha/2m)\psi^+_{\vec{x},t}(-\Delta)\psi^-_{\vec{x},t}\\
&\psi^+_{\vec{x},t}\qquad\text{and}\qquad\psi^-_{\vec{x},t}
\end{aligned}\tag{2.8}$$

One should then draw $n+s$ such elements so that the first $n$ have a shape of one of the first three forms with labels $(\vec{y}_i,t_i)$ attached arbitrarily to the vertices ("free labels") and the last $s$ have a shape of the last two forms (representing respectively $\psi^-_{\vec{x},t}$ or $\psi^+_{\vec{x},t}$) and carry "external labels" $(\vec{x}_1,t_1),\ldots,(\vec{x}_s,t_s)$.

Then one considers all *Feynman graphs*, that is all possible ways of joining together lines in pairs so that no unpaired line is left over and so that only lines with consistent orientations are allowed to form a pair.

To each graph we assign a sign $\sigma=\pm$ obtained by considering the permutation necessary to bring next to each other the pairs of operators which, in the given graph, are paired (one says also *contracted*), with the $\psi^-$ to the left of the associated $\psi^+$, and then setting $\sigma=(-1)^\pi$ if $\pi$ is the permutation parity.

To each graph we assign a *value* which is the integral over the free vertices of the product of the sign factor times the product of factors $g(\vec{\xi},\tau)$ (or of some of its derivatives) for every line with an arrow pointing from $(\vec{x}_1,t_1)$ to $(\vec{x}_2,t_2)$ with $\vec{\xi}=(\vec{x}_2-\vec{x}_1),\ \tau=t_2-t_1$, times a factor $-\lambda\bar{v}(\vec{x}_1-\vec{x}_2)$ for every wiggly line joining $(\vec{x}_1,t)$ to $(\vec{x}_2,t)$, times a factor $-(\nu-\mu\alpha)$ or $-\alpha/2m$ for every vertex of the type with only two lines.

The *propagator* function $g$ is given by (2.7) and can be represented as:

$$g(\vec{\xi},\tau)=L^{-1}\sum_{\vec{k}}e^{-i\vec{k}\vec{\xi}}\{\frac{e^{-\tau e(\vec{k})}}{1+e^{-\beta e(\vec{k})}}\chi(\tau>0)-\frac{e^{-(\beta+\tau)e(\vec{k})}}{1+e^{-\beta e(\vec{k})}}\chi(\tau\le 0)\}\tag{2.9}$$

where $\chi(\text{“}condition''\text{)}=1$ if "*condition*$''$ is verified and $\chi=0$ otherwise.

This can be written:

$$g=\lim_{K\to\infty}\frac{1}{\beta L}\sum_{\substack{k_0,\vec{k}\\ e^{-ik_0\beta}=-1,e^{i\vec{k}L}=1}}\frac{e^{-i(k_0(\tau+0^-)+\vec{k}\vec{\xi})}}{-ik_0+e(\vec{k})}\Delta(\frac{\sqrt{k_0^2+\vec{k}^2}}{K})\tag{2.10}$$

with the sum running over the $k_0,\vec{k}$ verifying $e^{-ik_0\beta}=-1,\ e^{-i\vec{k}L}=+1$; and $\Delta$ is a cutoff function like one of the following:

$$\Delta_s(x)=\chi(x<1),\qquad\Delta_\alpha(x)=(1+\frac{x^2}{\alpha})^{-\alpha},\qquad\Delta_\infty(x)=e^{-x^2}\tag{2.11}$$

The (2.10) can be proved, in the case of the first regularization $\Delta = \Delta_s$ ("sharp momentum regularization"), by remarking that, if $\tau > 0$ and $\vec{k}$ is fixed in the r.h.s. of (2.10), the sum over $k_0$ has a limit, for $K \to \infty$, equal to:

$$\frac{1}{2\pi} \oint \frac{e^{-iz\tau}}{(-iz + e(\vec{k}))(1 + e^{-iz\beta})} dz \tag{2.12}$$

with the contour running parallel to the real axis (nearer than $|e(\vec{k})| \geq e_{min} \sim \frac{p_F}{m}\frac{\pi}{L}$) and going from $-\infty$ to $+\infty$ if Im$z < 0$ and from $+\infty$ to $-\infty$ if Im$z > 0$. Using that $\beta > \tau > 0$ we easily see that (2.10) implies (2.7). If $\tau < 0$ (but $\beta > |\tau|$ so that $\beta + \tau > 0$) we see from (2.10) that the sum has value $-1$ times the value when $\tau$ is replaced by $\beta + \tau$ (because $e^{i\beta k_0} \equiv -1$). Hence for such values of $\tau$ the value of $g$ is given by $-1$ times the value of (2.12) with $\tau$ replaced by $\beta + \tau$, and (2.9) follows also for $\tau < 0$.

The cutoff $\Delta_\alpha$ can be treated in the same way (if $\alpha$=positive integer as we suppose): one finds instead of (2.10) a complex integral that can be, essentially, explicitly evaluated and one can therefore estimate easily the difference between (2.10) and (2.9) as $K \to \infty$

The gaussian cut off $\Delta_\infty(x)$ cannot be treated by using complex integrals because $\Delta_\infty$ has bad behaviour at $\pm i\infty$. But $\Delta_\infty(x) - \Delta_\alpha(x) = O(x^4)$ as $x \to 0$ and this, together with the fact that we know that (2.10) holds with the regularization $\Delta_\alpha$, easily implies the validity of (2.10) with the gaussian cut off as well.

Therefore we can compute the coefficients of the perturbation theory for the Schwinger functions by the above graphical algorithms and by using propagators with one of the above cut offs and then removing it.

The above discussion suggests the following definition:

**Definition**: *suppose that, for $\lambda$ in a small neighbourhood $D$ of the origin in the complex plane, and for $\alpha, \nu$ suitably chosen as analytic functions of $\lambda$ in $D$, the perturbation series for the Schwinger functions can be shown to admit an analytic continuation to the domain $D$, extending on the real axis to $\lambda$'s of $O(1)$, i.e. $\beta, L$-independent, and suppose that the limits as $\beta \to \infty$, $L \to \infty$ of the Schwinger functions exist in $D$. Then we say that the limit as $L \to \infty$ of the Schwinger functions defines a Gibbs state for our system with Fermi momentum $p_F$ and particles mass $m$. The limit of the Schwinger functions as $\beta \to \infty$ will be called a ground state with Fermi momentum $p_F$ and particles mass $m$.*

Note that such a definition would certainly not be adequate for $d = 3$ (because changing the sign of $\lambda$ destroys the stability of the Hamiltonian, see [R], [Th], and the system collapses) and probably not even for $d$=2 (although in this case the sign of $\lambda$ does not affect stability, if $\lambda$ is small enough). Hence, for $d > 1$, we would replace the requirement that $D$ is a neighbourhood of the origin by the requirement that it is a domain in the right half plane.

This shows that one can conceive a purely perturbative approach to the low temperature Fermi systems. One starts with some expressions of the perturbation expansion for the

Schwinger functions depending on various parameters to be eventually sent to $\infty$ (e.g. $\beta$ or $L$ or others that will be introduced later). At fixed values of the parameters the expansions should be obviously convergent for small $\lambda, \alpha, \nu$. Then one proves uniform analyticity in a region $D$ of complex $\lambda$, where $\alpha, \nu$ are suitably chosen as a function of $\lambda$ (analytic in the same domain) and thus one defines, by removing the cutoffs, a Gibbs state in the above sense.

As long as other cut offs, besides $\beta, L$, are removed first, the already remarked and obvious analyticity in $\lambda, \alpha, \nu$ at fixed $\beta, L$ guarantees that the functions obtained in this way do have the required positivity property necessary to interpret them as Schwinger functions for a Gibbs state (namely the reflection positivity). In fact the series expansions for real $\lambda, \alpha, \nu$ must coincide with the non perturbative definitions of the same expressions by analyticity and the latter, of course, have the reflection positivity property.

The most convenient representation of the Schwinger functions, for the above purposes, is the *Euclidean functional integral representation.* Such a representation is set up with the help of two extra regularization parameters that we call $R, U$, with $R \le U$, and of a family of Grassmanian variables.

Here the Grassmanian variables will be denoted $A^{h\sigma}_{k\omega}, \varepsilon^\sigma_k$ and they bear labels $h \in \mathbf{Z}, \omega = \pm 1, \sigma = \pm 1, k = (k_0, \vec{k})$ such that:

$$e^{-ik_0\beta} = -1, \qquad e^{i\vec{k}L} = +1 \tag{2.13}$$

They must verify anticommutation rules:

$$\{A, A'\} = 0, \quad \{A, \varepsilon\} = 0, \quad \{\varepsilon, \varepsilon'\} = 0 \tag{2.14}$$

It is most convenient to think of the $A, \varepsilon$ as concrete objects by using a representation on a Hilbert space $\mathbf{h}$. The best Hilbert space is probably the countable tensor product of two dimensional spaces $\mathbf{C}^2$: $\mathbf{h} = \otimes_{j=1}^\infty \mathbf{C}^2$. Then we order (absolutely arbitrarily) the variables labels, by replacing each of them with an integer label $j = 1, 2, \ldots$ and set the $j-$th Grassmanian variable to be:

$$[\otimes_{i<j} \sigma^z_i] \otimes \sigma^+_j \tag{2.15}$$

where $\sigma^z, \sigma^+$ are the usual Pauli matrices.

Hence the Grassmanian variables can be regarded as a set of creation operators (just creation and no annihilation) on a Fock space.

The $A, \varepsilon$ variables are norm 1 operators on $\mathbf{h}$. They will be used to define the *euclidean field on scale* $h$ and the *external field* $\varphi^\sigma_x$ as

$$\begin{aligned}
\psi^{h\sigma}_{x\omega} &= \sum_k \frac{e^{i\sigma(kx - p_F\omega\vec{x})}}{\sqrt{\beta L}} \frac{(e^{-\gamma^{-2h}\beta(k)} - e^{-\gamma^{-2h+2}\beta(k)})^{1/2}}{\sqrt{-ik_0 + e(\vec{k})}} \sqrt{\chi(\omega\gamma^{-h}\vec{k})} A^{h\sigma}_{k\omega} \\
\varphi^\sigma_x &= \sum_k e^{i\sigma kx} \varepsilon^\sigma_k
\end{aligned} \tag{2.16}$$

where $x = (\vec{x}, t)$, $\beta(k) = [k_0^2 + e(\vec{k})^2]p_0^{-2}$ and $\chi(t) = \pi^{-1/2}\int_{-\infty}^t ds \exp(-s^2)$ is a regularization of the step function. The motivation of such definition will be explained in the following sections.

We define the fields with ultraviolet cut off on scale $\gamma^{-U}$ and infrared on scale $\gamma^{-R}$ as:

$$\psi_{x\omega}^{[R,U]\sigma} = \sum_{h=R}^{U} \psi_{x\omega}^{h\sigma} \tag{2.17}$$

The fields $\psi_{x\omega}^{[R,U]\sigma}$ are bounded operators on $\mathbf{h}$, because the $A$'s have norm 1.

The fields $\psi_{x\omega}^{[R,U]\sigma}$ and $\psi_{x\omega}^{h\sigma}$ are *quasi particle fields*. We also define the *particle fields* simply by:

$$\psi_x^{h\sigma} = \sum_\omega e^{i\sigma p_F \omega \vec{x}} \psi_{x\omega}^{h\sigma} \quad , \quad \psi_x^{[R,U]\sigma} = \sum_\omega e^{i\sigma p_F \omega \vec{x}} \psi_{x\omega}^{[R,U]\sigma} \tag{2.18}$$

As a matter of fact, we shall not introduce the quasi particle fields and we shall introduce the cut off in a different way, in the *ultraviolet region* $h > 0$; therefore the particle fields will be defined in terms of suitable $A_k^{h\sigma}$ variables in a different way. However, in order to simplify the notation, we nevertheless proceed in a symmetric way in the ultraviolet and infrared region; it will be clear that our definitions would work also for the representation of the field used in the following sections.

The Grassmanian or fermionic functional integral is then defined as a linear functional on the operators on $\mathbf{h}$, in the algebra generated by the Grassmanian variables. The integration rule is simply the Wick rule based on the following "propagator":

$$\int A_{k\omega}^{h-} A_{k'\omega'}^{h'+} dP = \delta_{hh'}\delta_{kk'}\delta_{\omega\omega'} \tag{2.19}$$

while all the integrals of $A^+A^+$ and $A^-A^-$ vanish. This means that the integral of an arbitrary monomial in the $A^+$ and $A^-$ is obtained as a sum over the pairings of the factors into pairs with non zero propagator of the product of the propagators corresponding to the pairs times a sign $\pm$ equal to the parity of the permutation necessary to bring the considered pairs next to each other.

The above rule is just a linear functional and we may have problems in the integration of expressions which are not finite linear combination of products of $A$: but of course this is precisely the kind of operation that we shall wish to do.

Therefore it is convenient to define a class of operators on $\mathbf{h}$ on which we can operate the functional integral "absent-mindedly". It will be the class of *integrable operators*.

**Definition**: *An operator $O(\psi, \varphi)$ is said to be integrable if it has the form:*

$$\begin{aligned} O(\psi,\varphi) = \sum_{n,m,\underline{\omega},\underline{\omega}'} \int \left(\prod_{i=1}^n dx_i dy_i\right) \left(\prod_{j=1}^m du_i dv_i\right) O_{n,m}(\underline{x},\underline{y},\underline{u},\underline{v},\underline{\omega},\underline{\omega}')\cdot \\ \cdot [\mathcal{D}_1\psi_{u_1\omega_1}^+ \ldots \mathcal{D}_{2m}\psi_{v_m\omega_m'}^-]\,\varphi_{x_1}^+ \ldots \varphi_{y_n}^- \end{aligned} \tag{2.20}$$

*where the $O_{n,m}(\ldots)$ are the "kernels of $O(\psi,\varphi)$" and $\psi^\pm$ are quasi particle operators on various scales between two scales $R,U$, for all $n$, and $\mathcal{D}_1\ldots\mathcal{D}_{2m}$ are differential operators with constant coefficients (possibly dependent on $h,\omega$), and with order bounded by some $N$, for all $n$. Furthermore the $O_{n,m}$ should be measures (i.e. $\delta$ functions are allowed) and:*

$$\begin{aligned}|O(\psi,\varphi)|_b \equiv \sum_{n,m,\underline{\omega},\underline{\omega}'} b^{n+m}\int\left(\prod_{i=1}^n dx_i dy_i\right)\left(\prod_{j=1}^m du_i dv_i\right)\cdot\\ |O_{n,m}(\underline{x},\underline{y},\underline{u},\underline{v},\underline{\omega},\underline{\omega}')|<\infty \qquad \forall b>0\end{aligned} \tag{2.21}$$

*Then we define (consistently with (2.19), as it is possible to check):*

$$\begin{aligned}\int P(d\psi)O(\psi,\varphi) = \sum_{n,m,\omega,\omega'}\int\left(\prod_{i=1}^n dx_i dy_i\right)\left(\prod_{j=1}^m du_i dv_i\right) O_{n,m}(\underline{x},\underline{y},\underline{u},\underline{v},\underline{\omega},\underline{\omega}')\cdot\\ \cdot\,\mathcal{D}_1\,\mathcal{D}_2\ldots\mathcal{D}_{2m}\det\left[g^{[R,U]}_{\omega_i\omega'_j}(u_i-v_j)\right]\varphi^+_{x_1}\ldots\varphi^-_{y_n}\end{aligned} \tag{2.22}$$

*where the propagator $g^{[R,U]}(x-y)$ is $\sum_{h=R}^U g^{(h)}(x-y)$, with:*

$$g^{(h)}_{\omega\omega'}(x-y) = \frac{\delta_{\omega\omega'}}{\beta L}\sum_k e^{-i[k(x-y)-p_F\omega(\vec{x}-\vec{y})]}\frac{e^{-\gamma^{-2h}\beta(k)}-e^{-\gamma^{-2h+2}\beta(k)}}{-ik_0+e(\vec{k})}\chi(\omega\gamma^{-h}\vec{k}) \tag{2.23}$$

**Remark:** The r.h.s. of (2.22) is a well defined operator, thanks to (2.21), as a consequence of the Gramm-Hadamard inequality (see appendix 2):

$$|\mathcal{D}_1\ldots\mathcal{D}_{2m}\,\det\left[g^{[R,U]}_{\omega_i\omega'_j}(u_i-v_j)\right]|\le B^m_{R,U} \tag{2.24}$$

Furthermore the definition is meaningful since the representation (2.20) is unique if the kernels:

$$\mathcal{D}_1\,\mathcal{D}_2\ldots\mathcal{D}_{2m}O_{n,m}(\underline{x},\underline{y},\underline{u},\underline{v},\underline{\omega},\underline{\omega}') \tag{2.25}$$

are antisymmetric in the permutation between themselves of the $(u_i,\omega_i)$, of the $(v_i,\omega'_i)$, of the $x_i$ and of the $y_i$.

Several easy theorems follow. For instance, if $O$ is integrable also $\exp O$ is integrable: this is a key property that overcompensates the fact that the fermionic integration is not a positive functional in the sense of measure theory (and makes the world of fermionic integration look like a fairy tale compared to that of measure theory).

Also, if $O(\psi,\varphi)$ is integrable and if we write $\psi^{[R,U]}=\psi_1+\psi_2$ with $\psi_1=\psi^{[R,U_1]}$ and $\psi_2=\psi^{[U_1,U]}$, then $O(\psi_1+\psi_2,\varphi)=\sum O_1(\psi_1,\varphi)O_2(\psi_2,\varphi)$ and $O_i$ are integrable; moreover:

$$\int P(d\psi)O(\psi,\varphi)=\sum\int P(d\psi_1)O_1(\psi_1,\varphi)\int P(d\psi_2)O_2(\psi_2,\varphi) \tag{2.26}$$

i.e. "Fubini's theorem" holds.

The above obvious remarks constitute the theory of non commutative or fermionic Grassmanian integration.

Its interest lies in the fact that it is easy to see that the coefficients of the perturbation expansion of the Schwinger functions are generated by:

$$q_{R,U}(\varphi) = \log \int P(d\psi^{[R,U]}) e^{-V(\psi)+\int dx(\varphi_x^+\psi_x^- + \psi_x^+\varphi_x^-)} \tag{2.27}$$

via:

$$S^T(x_1,\sigma_1,...,x_n,\sigma_n) = \lim_{\substack{U\to\infty \\ R\to-\infty}} \frac{\delta^{2n} q_{R,U}(\varphi)}{\delta\varphi_{x_1}^+ \ldots \delta\varphi_{y_n}^-}\Big|_{\varphi=0} \tag{2.28}$$

Hence we shall confine ourselves to studying $q_{R,U}(\varphi)$ and reorganizing the expansion of $S^T$ in powers of $\lambda$ (with $\nu$, $\alpha$ also expanded in terms of $\lambda$) so that the expansion have analyticity properties in $\lambda$ uniform in $R, U$ as well as in $L, \beta$. We shall also use the expansion to infer the long distance behaviour of $S^T(x_1,\sigma_1,...,x_n,\sigma_n)$ (long means $O(L)$ in space and $O(\beta)$ in time).

**Remark:** $q_{R,U}(\varphi)$ has an expression like (2.20) (with $n = 0$), whose kernels are the *functional derivatives* appearing in the r.h.s. of (2.28). Furthermore one can define the $|q_{R,U}(\varphi)|_b$ norm as in (2.21) and it is possible to see (using (2.24) and some standard procedure to bound the truncated expectations, see last part of Appendix 3) that this norm is finite for $b \le b_0$, with $b_0$ depending on the strength of the interaction; this is sufficient to define $q_{R,U}(\varphi)$ as a bounded operator.

In order to simplify the notation, in the following sections we shall consider, for the propagator, only the limiting case $L = \beta = \infty$, by interpreting the functional integrals as a formal tool to represent in a convenient way the expansions of the Schwinger functions in powers of $\lambda$, $\alpha$, $\nu$. It will be clear that all our results are valid also for $L$, $\beta$ finite and that one can take the limit $L, \beta \to \infty$ without any further problem.

Moreover we shall change the meaning of the symbol $\psi_x^\sigma$ (see (2.3)), which from now on will denote the formal limit $R \to -\infty, U \to +\infty$ of the grassmanian field $\psi_x^{[R,U]\sigma}$ defined in (2.18). Then we can write the generating functional of the Schwinger functions, in the limit where all the cut off are removed, as:

$$q(\varphi) = \log \int P(d\psi) e^{-V(\psi)+\int dx(\varphi_x^+\psi_x^- + \psi_x^+\varphi_x^-)} \tag{2.29}$$

and we can say that $P(d\psi)$ is *grassmanian gaussian measure* with propagator:

$$g(x-y) = \int P(d\psi)\psi_x^-\psi_y^+ = \int \frac{dk_0 d\vec{k}}{(2\pi)^2} \frac{e^{-i[k_0((x_0-y_0)+0^-)+\vec{k}(\vec{x}-\vec{y})]}}{-ik_0 + e(\vec{k})} \tag{2.30}$$

where the $0^-$ in the exponential means that $g(0,\vec{x})$ must be interpreted as $\lim_{x_0\to 0^-} g(x_0,\vec{x})$. Moreover, if $\Lambda = L\times[0,\beta]$:

$$V(\psi) = \lambda\int_{\Lambda\times\Lambda} dx\,dy\,v(x-y)\psi_x^+\psi_y^+\psi_y^-\psi_x^- + (\nu-\mu\alpha)\int_\Lambda dx\,\psi_x^+\psi_x^- + \alpha\int_\Lambda dx\,\psi_x^+(-\Delta)\psi_x^-$$
$$v(x-y)\equiv\delta(x_0-y_0)\bar{v}(\vec{x}-\vec{y}) \tag{2.31}$$

where $\Delta = \partial_{\vec{x}}^2$ is the Laplacian in the space variables.

A very convenient object which is related to $q(\varphi)$ is the *effective potential* defined by:

$$e^{-V_{eff}(\varphi)} = \frac{1}{\mathcal{N}}\int P(d\psi)e^{-V(\psi+\varphi)} \tag{2.32}$$

where $\mathcal{N}$ is a normalization constant chosen so that $V_{eff}(0)=0$

The relation is, if $(g\varphi)^- = g*\varphi^-$ and $(g\varphi)^+ = \varphi^+ * g'$, where the $*$ denotes convolution and $g'(x) = g(-x)$, the following:

$$-V_{eff}(g\varphi) + (\varphi^+, g\varphi^-) = q(\varphi) \tag{2.33}$$

The above relations are formally trivial if one treats $\int P(d\psi)\cdot$ as an ordinary integral with respect to a grassmanian measure proportional to:

$$d\psi^+d\psi^- e^{-\int[\psi_x^+(\partial_t + (-\Delta+p_F^2)/2m]\psi_x^-\,dx} \tag{2.34}$$

and proceeding to the *change of variables* $\psi + g\varphi = \tilde{\psi}$.

The formal argument on the change of variables is meaningless as presented; however if one writes the above calculations (*i.e.* the change of variables) as relations between the power series in the fermion fields defining the fermionic integrals, one sees that they are indeed valid.

The equation (2.33) should allow us, in principle, to reduce the study of the Schwinger functions to that of the effective potential. However, because of the anomalous large distance behaviour, this is not so simple, in the sense that it is not possible to use directly (2.33), see [BGM]. In any event, the analysis of the effective potential will play an essential role; therefore, in the following three sections, we shall analyze the integral in the r.h.s. of (2.32) by an iterative procedure, based on the scale decomposition (2.17) of the field. This will allow us to define the effective potential on scale $\gamma^{-h}$, whose properties will be used in §6 to study the pair Schwinger function, by an expansion that will take the place of the relation (2.33). The same technique could be used also to study the other Schwinger functions, but we shall not do it explicitly.

*§3 Ultraviolet limit for the effective potential*

In this section we shall begin the analysis of the effective potential defined in equation (2.32), by studying the ultraviolet problem.

We start by decomposing $g(x)$ in its u.v. (ultraviolet) and its i.r. (infrared) part:

$$g(x) = g_{u.v.}(x) + g_{i.r.}(x) \tag{3.1}$$

with:

$$g_{u.v.}(x) = \int \frac{dk_0 d\vec{k}}{(2\pi)^2} \frac{1 - e^{-(k_0^2+e(\vec{k})^2)p_0^{-2}}}{-ik_0 + e(\vec{k})} e^{-i(k_0(x_0+0^-)+\vec{k}\vec{x})} \tag{3.2}$$

where $p_0^{-1}$ is the range of the potential, see (1.1) and (3.24) below.

It is easy to see that:

$$g(x) = \theta(x_0) e^{\frac{x_0 p_F^2}{2m}} \left(\frac{m}{2\pi x_0}\right)^{1/2} e^{-\frac{m\vec{x}^2}{2x_0}} - \int_{-p_F}^{p_F} \frac{d\vec{k}}{2\pi} e^{-i\vec{k}\vec{x} - x_0 \frac{\vec{k}^2 - p_F^2}{2m}} \tag{3.3}$$

where $\theta(x_0)$ is the step function. Hence we can write:

$$g_{u.v.}(x) = G(x) + R(x) \tag{3.4}$$

with:

$$G(x) = h(\vec{x}) h(x_0) \theta(x_0) e^{\frac{x_0 p_F^2}{2m}} \left(\frac{m}{2\pi x_0}\right)^{1/2} e^{-\frac{m\vec{x}^2}{2x_0}} \tag{3.5}$$

$$\begin{aligned} R(x) = {} & [1 - h(\vec{x})h(x_0)] g_{u.v.}(x) - h(\vec{x})h(x_0) g_{i.r.}(x) - \\ & - h(\vec{x})h(x_0) \int_{-p_F}^{p_F} \frac{d\vec{k}}{2\pi} e^{-i\vec{k}\vec{x} - x_0 \frac{\vec{k}^2 - p_F^2}{2m}} \end{aligned} \tag{3.6}$$

where $h(t)$, $t \in \mathbf{R}^1$, is a smooth function of compact support such that $h(t) = 1$, if $|t| \le 1$, and $h(t) = 0$, if $|t| \ge \gamma$, $\gamma$ being any number greater than 1, fixed once and for all.

It is easy to show that $R(x)$ is a smooth function on $\mathbf{R}^2$, such that, for suitable $A, \bar{\kappa}$:

$$|R(x)| \le A e^{-\bar{\kappa}|x|} \tag{3.7}$$

The equations (3.1), (3.4) and (2.32) imply that:

$$e^{-V_{eff}(\varphi)} = \frac{\bar{\mathcal{N}}^{(0)}}{\mathcal{N}} \int P^{(i.r.)}(d\psi^{(i.r.)}) e^{-\bar{V}^{(0)}(\psi^{(i.r.)}+\varphi)} \tag{3.8}$$

where

$$e^{-\bar V^{(0)}(\varphi)} = \frac{\mathcal{N}^{(0)}}{\bar{\mathcal{N}}^{(0)}} \int P^{(R)}(d\psi) e^{-V^{(0)}(\psi+\varphi)} \tag{3.9}$$

and

$$e^{-V^{(0)}(\varphi)} = \frac{1}{\mathcal{N}^{(0)}} \int P^{(G)}(d\psi) e^{-V(\psi+\varphi)} \tag{3.10}$$

with $\mathcal{N}^{(0)}$, $\bar{\mathcal{N}}^{(0)}$ defined so that $V^{(0)}(0) = \bar V^{(0)}(0) = 0$, and $P^{(i.r.)}(d\psi)$, $P^{(R)}(d\psi)$, $P^{(G)}(d\psi)$ are the grassmanian integrations with propagator $g_{i.r.}(x)$, $R(x)$ and $G(x)$, respectively.

In order to give a meaning to (3.10), we now introduce an u.v. cutoff by replacing $G$ with:

$$G_N(x) = \theta_N(x_0) h(\vec x) e^{\frac{x_0 p_F^2}{2m}} \left(\frac{m}{2\pi x_0}\right)^{d/2} e^{-\frac{m\vec x^2}{2x_0}} \tag{3.11}$$

where $\theta_N(t)$ is a smooth function with support in the interval $[\gamma^{-N}, \gamma]$ and $N$ is a large positive integer.

Note that the cut off is different from that introduced in §2, which has allowed us to present in a symmetric way the ultraviolet and the infrared problems. However, one can check that the results of this section do not depend on the choice of the cut off; in fact, one could add the new cut off to the previous one, parametrized by $U$, and note that all bounds are uniform in $U$. Furthermore, in this section we shall use only the particle field representation of the grassmanian integrations, see (2.18).

It is convenient to define more precisely $\theta_N(t)$ in the following way:

$$\theta_N(t) = \sum_{i=1}^{N} f(\gamma^i t) \tag{3.12}$$

where:

$$f(t) = [h(t/\gamma) - h(t)]\, \theta(t) \tag{3.13}$$

is a smooth function with support on $[1, \gamma^2]$. The function $\theta_N(t)$ has the claimed support properties and :

$$\theta(t) h(t) = \lim_{N\to\infty} \theta_N(t) \tag{3.14}$$

It is worth remarking that:

$$\lim_{N\to\infty} G_N(x) = G(x), \qquad \text{for all} \quad x \in \mathbf{R}^2 \tag{3.15}$$

because in the discontinuity point $x_0 = 0$, by definition, $G(0, x_1) = \lim_{x_0 \to 0^-} G(x_0, x_1) = 0 = G_N(0, x_1)$.

Two other consequences immediately follow from (3.15):

1) in (3.10) we can suppose that the potential (2.31) is Wick ordered w.r.t. $G_N$, since only products of fields at coinciding times appear in it;

2) all Feynman graphs with closed fermion loops in the perturbative expansion of $V^{(0)}(\varphi)$ vanish; furthermore, because of the $\delta(x_0-y_0)$ in (2.31), also the loops containing some lines $v(x-y)$ are forbidden, if the directions of the fermionic lines are compatible.

Then we define:

$$V^{(0)}(\varphi)=\lim_{N\to\infty}\log\frac{1}{\mathcal{N}^{(0)}}\int P^{(\le N)}(d\psi^{(\le N)})e^{-V(\psi^{(\le N)}+\varphi)} \tag{3.16}$$

where $P^{(\le N)}(d\psi^{(\le N)})$ is the grassmanian integration with propagator $G_N$.

We want to prove that the limit exists and that it is an analytic function of $z=(\lambda,\nu,\alpha)$ in a neighbourhood of $z=0$, in the sense that the kernels $O_n$ of the operator $O=V^{(0)}(\varphi)$, defined as in (2.20) (without the sum over $\omega,\omega'$), are analytic functions verifying, in their holomorphy domain, bounds like (2.21). We shall also prove that $V^{(0)}(\varphi)$ has some "exponential decay" properties (*i.e.* its kernels decay exponentially fast as the arguments separate to $\infty$). The extension of these results to $\bar V^{(0)}(\varphi)$ will be trivial. More precisely we shall prove the following theorem:

**Theorem 1**: *There exist $\varepsilon>0$ and $D>0$ such that $\bar V^{(0)}(\varphi)$ can be written, for $|z|\le\varepsilon$, if $z=(\alpha,\nu,\lambda)$, in the following way:*

$$\begin{aligned}\bar V^{(0)}(\psi)&=\lambda\int dx\,dy\,v(x-y)\psi^+_x\psi^+_y\psi^-_y\psi^-_x+2\lambda\int dx\,dy\,v(x-y)R(x-y)\psi^+_x\psi^-_y+\\&+(\nu-4\pi\lambda\hat v(0)R(0))\int dx\,\psi^+_x\psi^-_x+\alpha\int dx\,\psi^+_x(\frac{-\Delta-p_F^2}{2m})\psi^-_x+\\&+\int dx\,dy\,\psi^+_x\Delta\psi^-_y\tilde W_2(z,x-y)+\\&+\sum_{n=1}^\infty\sum_{\substack{n_1,n_2\\ n_1+n_2=2n}}\int dx_1\dots dx_{2n}\psi^+_{x_1}\dots\psi^+_{x_n}\psi^-_{x_{n+1}}\dots\psi^-_{x_{2n-n_2}}\cdot\\&\cdot\Delta\psi^-_{x_{2n-n_2+1}}\dots\Delta\psi^-_{x_{2n}}W_{n_1n_2}(z,x_1\dots x_{2n})\end{aligned}\tag{3.17}$$

*where the kernels $W_{n_1n_2}$ are products of suitable delta functions by smooth functions, which are analytic in $z$ if $|z|\le\varepsilon$, and satisfy, uniformly in $N$, the following estimate:*

$$\int dx_1\dots dx_{2n}|W_{n_1n_2}(z,x_1\dots x_{2n})|e^{\frac{\kappa}{2}d^{(0)}(x_1\dots x_{2n})}\le|\Lambda|(D|z|)^{\max\{2,n-1\}} \tag{3.18}$$

*while $\tilde W_2(z,x)$ singles out some "special" contributions (see discussion after (3.40) below) and satisfies (uniformly in $N$):*

$$\int dx\,|\tilde W_2(z,x)||x|e^{\frac{\kappa}{2}|x|}\le(D|z|)^2 \tag{3.19}$$

$$\int dx\,\tilde{W}_2(z,x) = 0 \tag{3.20}$$

*The r.h.s. of (3.18) is summable in n, for $|z|$ small enough and we shall take this property as definition of analyticity around $z=0$ for a function of the field of the general form (3.17), see also §2, (2.20), (2.21).*

We shall study the integral in (3.16) by decomposing the grassmanian integration $P^{(\le N)}(d\psi^{(\le N)})$ in the product of the independent integrations $P^{(h)}(d\psi^{(h)})$, $h=1,\ldots,N$, with propagator:

$$C_h(x) = f(\gamma^h x_0) h(\vec{x}) e^{\frac{x_0 p_F^2}{2m}} \left(\frac{m}{2\pi x_0}\right)^{1/2} e^{-\frac{m\vec{x}^2}{2x_0}} = \gamma^{h/2}\bar{C}_h(\gamma^h x_0, \gamma^{h/2}\vec{x}) \tag{3.21}$$

where $\bar{C}_h(x)$ is a smooth function such that, for suitable $A$ and $\bar{\kappa}$:

$$|\bar{C}_h(x)| \le A e^{-\bar{\kappa}|x|} \quad , \quad \forall\, h \ge 1 \tag{3.22}$$

and $\bar{\kappa}$ can be taken to be the same as in (3.7). In fact, by (3.12)

$$G_N(x) = \sum_{h=1}^{N} C_h(x) \tag{3.23}$$

We shall assume that $A$ is chosen so that also the following bound is satisfied:

$$|\bar{v}(\vec{x}-\vec{y})| \le A e^{-p_0|\vec{x}-\vec{y}|} \tag{3.24}$$

for a suitable $p_0$; we shall call $p_0^{-1}$ the *range* of the potential $\bar{v}$, see (1.1).

We shall integrate iteratively the fields $\psi^{(h)}$ in (3.16), by studying the properties of the *effective potential on scale* $\gamma^{-k}$, defined by:

$$V^{(k)}(\varphi) = \lim_{N\to\infty} \log \frac{1}{\mathcal{N}^{(k)}} \int P^{(k+1)}(d\psi^{(k+1)}) \ldots P^{(N)}(d\psi^{(N)}) e^{-V(\psi^{(k+1)}+\ldots\psi^{(N)}+\varphi)} \tag{3.25}$$

so that:

$$e^{-V^{(0)}(\varphi)} = \frac{\mathcal{N}^{(k)}}{\mathcal{N}^{(0)}} \int P^{(\le k)}(d\psi^{(\le k)}) e^{-V^{(k)}(\psi^{(\le k)}+\varphi)} \tag{3.26}$$

An essential role in our analysis will be plaid by the tree expansion (see Ref. [G]), with which we assume that the reader is familiar. We start with some definitions and notations.

Fig. 2

1) Let us consider the family of all trees which can be constructed by joining a point $r$, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree* (see Fig. 2). Two unlabeled trees that can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide, will be said to have the same *topological structure* and they will be regarded as equivalent.

The unlabeled trees are partially ordered from the root to the endpoints in the natural way (we shall use the symbol $<$ to denote the order); $n$ will be called the *order* of the unlabeled tree.

We shall consider also the *labeled trees* (which in general will be simply called trees in the following); they are defined by associating some *labels* with the unlabeled trees, as explained in the following items. We shall denote $\mathcal{T}_n$ the set of labeled trees of order $n$.

2) Given $\tau \in \mathcal{T}_n$, we associate with each endpoint one of the three terms of (2.31), which we denote $\tilde{V}_\alpha$, $\alpha$ being a suitable label, and which we represent pictorially by the following *graph elements*:

Fig. 3

We shall say that the three different graph elements are of type $4, 2, 2'$, respectively, and we shall call *space vertices* the corresponding integration variables (a more appropriate name would be "space inverse-temperature vertices", but this is too long).

3) We introduce a family of vertical lines, labeled by a *frequency index* $h$, which takes all the integer values between $k$ and $N+1$; the vertical lines are ordered from left to right as the frequency index increases. Furthermore the root of the labeled tree must belong to the line with index $k$, the endpoints must belong to the line index $N+1$ and, finally, any branch point must belong to a vertical line with index larger than $k$ and smaller than $N+1$.

We call *non trivial vertices* of $\tau$ its branch points (this set is empty if $n = 1$ and, in this case, there is only one unlabeled tree); we call *trivial vertices* the points where the branches *connecting two non trivial vertices* intersect the family of vertical lines; finally, we call vertices the trivial or non trivial vertices and the endpoints (see the dots in Fig. 2). Note that there are no vertices on the endbranches of the tree except the endpoints.

Given a vertex $v$, we denote $h_v$ the frequency index of the vertical line containing it; note that:

$$h_{v'} < h_v, \quad \text{if} \quad v' < v$$

The first vertex of the tree (having frequency index $k+1$) will be denoted $v_0$. Given a trivial or non trivial vertex $v$, $s_v$ will denote the number of lines branching from $v$ (then $s_v = 1$, if $v$ is a trivial vertex).

4) We can associate to the endpoints $4n_4 + 2n_2 + 2n_2' = n_\tau$ fields, if the number of different graph elements associated with the tree is, respectively, $n_4, n_2, n_2'$; we shall attach a label $f = 1, \dots, n_\tau$ to each field to distinguish them.

5) We shall denote $x_{2i-1}, x_{2i}$, $i = 1, \dots, n_4$, the $2n_4$ space vertices associated with the $n_4$ graphs elements of type 4 and $x_{2n_4+j}$, $j = 1, \dots, n - n_4$, those associated with the graph elements of type 2 and $2'$; $\underline{x}$ will be the set of all space vertices. We shall use also the notation $x_j = (t_j, \vec{x}_j)$ for the time and space components of $x_j$.

Note that each $x_j$ is associated with a pair of fields $\psi^+_{x_j}$ and $\psi^-_{x_j}$ or $\psi^+_{x_j}$ and $\Delta\psi^-_{x_j}$ and, conversely, to every field of label $f$, representing a line of a graph element, corresponds a point $x_{(f)}$, representing the vertex of the graph element from which it emerges, and a label $\sigma_f$, allowing to distinguish the three possibilities $\psi^+_{x_j}$, $\psi^-_{x_j}$ and $\Delta\psi^-_{x_j}$.

6) Finally, we denote $\mathcal{E}_h$ and $\mathcal{E}^T_h$ the expectation and the truncated expectation, respectively, with respect to the grassmanian integration $P^{(h)}(d\psi^{(h)})$.

As it is explained in [G], we can express the contribution to the effective potential on scale $\gamma^{-k}$ of order $n$ in the $z$ variables, say $V^{(k)}(N, n, \psi^{(\le k)} + \varphi)$, in the following way (setting $\varphi = 0$ for simplicity of notation):

$$V^{(k)}(N, n, \psi^{(\le k)}) = \sum_{\tau \in \mathcal{T}_n} V^{(k)}(N, \tau, \psi^{(\le k)}) \tag{3.27}$$

where:

$$V^{(k)}(N, \tau, \psi^{(\le k)}) = \left[ \prod_v \frac{1}{s_v!} \mathcal{E}^T_{h_v} \right] \prod_{i=1}^n \left[ \mathcal{E}_{h_i+1} \dots \mathcal{E}_N(\tilde{V}_{\alpha_i}(\psi^{(\le N)})) \right] \tag{3.28}$$

The tree definition and the corresponding combinatorial weights in (3.28) are adopted from [FG], rather than from [GN],[G].

By remark 1) after (3.15), the last expectations in the r.h.s. of (3.28) have the only effect of lowering the cutoff of the field to $h_i = h_{v_i}$ if $v_i$ is the top non trivial vertex out of which emerge the line representing $\tilde{V}_{\alpha_i}(\psi^{(\leq N)}$ (this is in fact one of the main properties of the Wick ordering); therefore we can write:

$$V^{(k)}(N, \tau, \psi^{(\leq k)}) = \left[\prod_v \frac{1}{s_v!}\mathcal{E}^T_{h_v}\right]\left[\prod_{i=1}^n \tilde{V}_{\alpha_i}(\psi^{(\leq h_i)})\right] \tag{3.29}$$

This is a symbolic notation; the $\mathcal{E}^T$ operations have to be thought as performed hierarchically starting from the higher non trivial vertices and going down along the tree toward the root.

We need to find a more explicit expression for the r.h.s. of (3.29). We begin by proving that $V^{(k)}(N, n, \psi^{(\leq k)})$ can be also written in the following way:

$$V^{(k)}(N, n, \psi^{(\leq k)}) = \int d\underline{x}^{(P_{v_0})} \sum_{\tau\in\mathcal{T}_n} \sum_{P_{v_0}} W^{(k)}(N, \tau, P_{v_0}, \underline{x}^{(P_{v_0})}) \tilde{\psi}^{(\leq k)}(P_{v_0}) \tag{3.30}$$

where:

$$W^{(k)}(N, \tau, P_{v_0}, \underline{x}^{(P_{v_0})}) = \int V^{(k)}(N, \tau, P_{v_0}, \underline{x})\, d(\underline{x}\backslash\underline{x}^{(P_{v_0})}) \tag{3.31}$$

$P_{v_0}$ is a non empty subset of $I_{v_0} = \{1, \ldots, n_\tau\}$, the field labels associated with the tree endpoints reachable from $v_0$ (*i.e.* all of them); the sum $\sum_{P_{v_0}}$ is the sum over such subsets and $\underline{x}^{(P_{v_0})}$ is the set of space vertices from which the fields labeled by $P_{v_0}$ emerge. Finally:

$$\tilde{\psi}^{(\leq k)}(P_{v_0}) = \prod_{f\in P_{v_0}} \psi^{\sigma_f\,(\leq k)}_{x(f)} \tag{3.32}$$

and the kernels $V^{(k)}(\cdot)$ in (3.31) are suitably chosen. Note that in (3.32) there is no Wick order: it is in fact convenient to work without it.

It is convenient to think of $I_{v_0}$ as an ordered set and of $P_{v_0}$ as an ordered subset: in this way it will be easier to keep track of the sign changes due to the anticommutation relations between the grassmanian variables.

If $\tau$ has no non trivial vertex, (3.30) is trivial; hence we can proceed by induction, by assuming that (3.30) is true for all the subtrees of $\tau$ branching from $v_0$ (if $s_{v_0} = 1$, there is only one subtree). By using (3.29), we can write:

$$\begin{aligned}&\sum_{P_{v_0}} V^{(k)}(N, \tau, P_{v_0}, \underline{x})\tilde{\psi}^{(\leq k)}(P_{v_0}) = \\ &= \frac{1}{s_{v_0}!} \sum_{P_{v_0^1},\ldots,P_{v_0^{s_{v_0}}}} \prod_{j=1}^{s_{v_0}} (V^{(k+1)}(N, \tau_j, P_{v_0^j}, \underline{x}_j)) \cdot \\ &\cdot \mathcal{E}^T_{k+1}(\tilde{\psi}^{(\leq k+1)}(P_{v_0^1}), \ldots, \tilde{\psi}^{(\leq k+1)}(P_{v_0^{s_{v_0}}}))\end{aligned} \tag{3.33}$$

where:

1) $\tau_j$, $j=1,\dots,s_{v_0}$, are the $s_{v_0}$ subtrees with root in $v_0$, whose frequency index is $k+1$;
2) $n_j$ is the degree of the tree $\tau_j$, (with $\sum_j n_j = n$);
3) $P_{v_0^j}$ is a non empty subset of the set $I_{v_0^j}$ of the $n_{\tau_j}$ fields in $\tau_j$;
4) $v_0^1,\dots,v_0^{s_{v_0}}$ are the vertices immediately following $v_0$ in the tree.

By using the identity:

$$\tilde\psi^{(\le k+1)}(P) = \sum_{Q\subseteq P} (-1)^{\Pi}\tilde\psi^{(k+1)}(P\backslash Q)\tilde\psi^{(<k+1)}(Q) \tag{3.34}$$

where $\Pi$ is the parity of the permutation necessary to pass from the ordered subset $P$ to the ordered subset obtained by writing first $Q$ followed by $P\backslash Q$; and $P\backslash Q$ denotes the complement of $Q$ in $P$. We can write:

$$\begin{aligned}(3.33) = \frac{1}{s_{v_0}!} &\sum_{P_{v_0^1},\dots,P_{v_0^{s_{v_0}}}} \prod_{j=1}^{s_{v_0}} (V^{(k+1)}(N,\tau_j,P_{v_0^j},\underline{x}_j))\cdot \\ \cdot &\sum_{Q_{v_0^1},\dots,Q_{v_0^{s_{v_0}}}} \mathcal{E}^T_{k+1}(\tilde\psi^{(k+1)}(P_{v_0^1}\backslash Q_{v_0^1}),\dots,\tilde\psi^{(k+1)}(P_{v_0^{s_{v_0}}}\backslash Q_{v_0^{s_{v_0}}}))\tilde\psi^{(\le k)}(P_{v_0})\end{aligned} \tag{3.35}$$

where $P_{v_0} = \bigcup_i Q_{v_0^i}$ is the set of fields which are not contracted and $Q_{v_0^i}\subset P_{v_0^i}$.

The latter relation proves our assumption (3.30) with:

$$\begin{aligned}V^{(k)}(N,\tau,P_{v_0},\underline{x}) = \frac{1}{s_{v_0}!} &\sum_{P_{v_0^1},\dots,P_{v_0^{s_{v_0}}}} \prod_{j=1}^{s_{v_0}} V^{(k+1)}(N,\tau_j,P_{v_0^j},\underline{x}_j)\cdot \\ \cdot\ &\mathcal{E}^T_{k+1}\left(\tilde\psi^{(k+1)}(P_{v_0^1}\backslash Q_{v_0^1}),\dots,\tilde\psi^{(k+1)}(P_{v_0^{s_{v_0}}}\backslash Q_{v_0^{s_{v_0}}})\right)\end{aligned} \tag{3.36}$$

where $Q_{v_0^i} = P_{v_0^i}\bigcap P_{v_0}$.

By using the expression (3.36) iteratively it is easy to find the general expression for $V(N,\tau,P_{v_0},\underline{x})$: one has simply to write explicitly $V^{(j)}(N,\tau_j,P_{v_0^j},\underline{x}_j)$ and so on, reaching finally the endpoints of the tree. This yields:

$$\begin{aligned}V(N,\tau,P_{v_0},\underline{x}) = &\left[\prod_{\substack{v>v_0\\ v \text{ not e.p.}}} \sum_{P_v}\right]\cdot \\ \cdot &\left\{\prod_{v\text{ not e.p.}} \frac{1}{s_v!}\mathcal{E}^T_{h_v}\left(\tilde\psi^{(h_v)}(P_{v^1}\backslash Q_{v^1}),\dots,\tilde\psi^{(\le h_v)}(P_{v^{s_v}}\backslash Q_{v^{s_v}})\right)\cdot\right. \\ \cdot &\left.(-\alpha)^{n_2'}(-\nu)^{n_2}\prod_{i=1}^{n_4}[-\lambda v(x_{2i-1}-x_{2i})]\right\}\end{aligned} \tag{3.37}$$

where:

1) $v^1, \ldots, v^{s_v}$ are the vertices immediately following $v$;
2) If $v$ is a trivial or non trivial vertex $P_v = \bigcup_j Q_{v^j}$ and $Q_{v^i} = P_v \bigcap P_{v^i}$, then $P_v$ is a subset of the set $I_v$ of $n_{\tau^{(v)}}$ fields in $\tau^{(v)}$; if $v$ is an endpoint of the tree, $P_v$ coincides with the set of fields appearing in the corresponding graph element.

If we expanded the expectations in (3.37) by Wick's theorem, we could represent the r.h.s. as a sum of *Feynman graphs* in the usual way (see, however, comments after (3.44) below). Such graphs have *internal lines* with *propagator* $C_{h_v}$ (and we shall say that they have frequency $h_v$), if they are *generated* in $v$ by the operation $\mathcal{E}^T_{h_v}$; the *external lines* are associated with the fields appearing in $\tilde\psi^{(\le k)}(P_{v_0})$. Furthermore, if $\mathcal{G}_\tau$ is the set of all Feynman graphs associated to $\tau$, given $g \in \mathcal{G}_\tau$, it is natural to associate a *subgraph* $g_v$ to the vertex $v$; the internal lines of $g_v$ are the lines generated in all vertices $\ge v$, while the external lines are those associated with the fields appearing in $\tilde\psi^{(\le h_v - 1)}(P_v)$.

If we insert (3.37) in (3.31), we obtain a rather explicit expression for the *kernel* $W^{(k)}$. It is an expression that we shall use to prove that the effective potential is an analytic function of $z \equiv (\lambda, \alpha, \nu)$ around $z = 0$ (in the sense of the theorem that we are proving), uniformly in $N$, and that it decays exponentially on scale $\gamma^{-k}$, as the distance between the space vertices $\underline{x}^{(P_{v_0})}$ goes to infinity. This will be the interpretation of the following *ultraviolet bound* stating that, for all $N, n, \tau, P_{v_0}$:

$$\int d\underline{x}^{(P_{v_0})} \chi_\tau(\underline{x}^{(P_{v_0})}) |W^{(k)}(N, \tau, P_{v_0}, \underline{x}^{(P_{v_0})})| e^{\frac{\kappa}{2} d^{(k)}(P_{v_0})} \le (C|z|)^n |\Lambda| \tag{3.38}$$

where (here and always in the following) $C$ denotes a suitable positive constant and $\kappa$ is the minimum between $\bar\kappa$ and $p_0$ (see (3.24), (3.7), (3.22) ); furthermore $d^{(k)}(P_{v_0})$ and $\chi_\tau(\underline{x}^{(P_{v_0})})$ are defined in the following way.

Let $\mathbf{T}$ be the set of all connected tree graphs joining the $m(P_{v_0}) = |\underline{x}^{(P_{v_0})}|$ space vertices; if $\mathbf{b} \in \mathbf{T}$, we call $b^{(1)}, \ldots, b^{(m(P_{v_0})-1)}$ its bonds and $b^{(i)}_j$, $j = 0, 1$, the two components of $b^{(i)}$ (0 is the index of the time component); then:

$$d^{(k)}(P_{v_0}) \equiv \min_{\mathbf{b} \in \mathbf{T}} \sum_{i=1}^{m(P_{v_0})-1} (\gamma^k |b^{(i)}_0| + |b^{(i)}_1|) \tag{3.39}$$

Let $\mathcal{T}^*_n \subset \mathcal{T}_n$ be the family of trees satisfying one of the following two conditions:

a) the graph elements associated with the endpoints of $\tau$ are all of type $2'$ and, as a consequence, $\tilde\psi^{(\le k)}(P_{v_0}) = \psi^{+(\le k)}_{x_1} \Delta \psi^{-(\le k)}_{x_n}$;

b) there are $(n-1)$ graph elements of type $2'$, while the other one is of type 2 and its $\psi^-$ line is an external line, so that $\tilde\psi^{(\le k)}(P_{v_0}) = \psi^{+(\le k)}_{x_1} \psi^{-(\le k)}_{x_n}$.

We define:

$$\chi_\tau(\underline{x}^{(P_{v_0})}) = \begin{cases} \gamma^k |t_n - t_1| + \gamma^{k/2} |\vec{x}_n - \vec{x}_1| & \text{if } \tau \in \mathcal{T}^*_n \\ 1 & \text{otherwise} \end{cases} \tag{3.40}$$

Note that, if $\tau \in \mathcal{T}_n^*$, the corresponding graph expansion of $V^{(k)}(N, \tau, \{h_v\}, P_{v_0}, \underline{x})$ contains only chains connecting $x_1$ to $x_n$, see Fig. 4.

Fig. 4

It is easy to see that their contribution has a singularity, as $|x_1 - x_n| \to 0$, whose $L_1$ norm is logarithmically divergent when $N \to \infty$; the $\chi_\tau$ factor in (3.38) is introduced to deal with the singularity, (see below).

The contribution to the effective potential of such trees can be easily summed; the result can be expressed in terms of the same two Feynman graphs of Fig. 4, where now the lines represent the full propagator $\sum_{h=k+1}^{N} C_h$. Let us consider, for example, the chain of item a) for $k = 0$; it is easy to see, by explicit calculation, that such graphs behave, when $|x_1 - x_n|$ is small, in the limit $N \to \infty$, as:

$$\alpha^n \vartheta(t_n - t_1) \frac{(t_n - t_1)^{n-2}}{(n-2)!} \frac{\partial^{n-1}}{\partial t_n^{n-1}} e^{-\frac{m(\vec{x}_1 - \vec{x}_n)^2}{2(t_n - t_1)}} \left( \frac{m}{t_n - t_1} \right)^{1/2} \tag{3.41}$$

which is not $L_1$.

The origin of this singularity can be easily understood. Suppose, in fact, that there is an infrared cutoff on scale 1, so that the full propagator coincides with $G(x)$. Hence the contribution of the chain to the two points Schwinger function $S_2(x - y)$ is obtained by substituting the two external lines with the full propagators $G(x - x_n)$ and $G(x_1 - y)$ and one finds that the leading contribution for $|x - y| \to 0$ behaves as:

$$\alpha^n \vartheta(t - t') \frac{(t - t')^n}{n!} \frac{\partial^n}{\partial t^n} e^{-\frac{m(\vec{x} - \vec{y})^2}{2(t - t')}} \left( \frac{m}{t - t'} \right)^{1/2} \tag{3.42}$$

The latter expression can be summed over $n$ and we get a function with the same behaviour of $G(x - y)$ with the substitution $m \to m/(1 + \alpha)$; this result should have been expected, since the term proportional to $\alpha$ in the interaction could be absorbed in the free grassmanian integration producing exactly such change in the *bare* mass of the particles.

The proof of equation (3.38) will make use of the fermionic nature of the fields and of the explicit form of the propagator defined in section 2. We shall need the following results for the fermionic expectations:

$$\frac{1}{s!}\Big|\mathcal{E}_h^T\big(\tilde\psi^{(h)}(P_1),\dots,\tilde\psi^{(h)}(P_s)\big)\Big| \le \gamma^{\frac{h}{4}\sum_j |P_j^1|}\gamma^{\frac{5}{4}h\sum_j |P_j^2|}C^{\sum_j |P_j|}\frac{1}{s!}\sum_T e^{-\kappa d_T^{(h)}(P_1,\dots,P_s)} \tag{3.43}$$

where $|P| = |P^1| + |P^2|$ is the number of elements in $P$, $|P^1|$ is the number of fields $\psi^{(\cdot)}$, $|P^2|$ is the number of fields $\Delta\psi^{(\cdot)}$. Furthermore $T$ is an *anchored tree graph* between the clusters of space vertices from which the fields labeled by $P_1,\dots,P_s$ emerge; this means that $T$ is a set of lines connecting two points in different clusters, which becomes a tree graph if one identifies all the points in the same cluster. If $b^1,\dots,b^s$ are the lines belonging to $T$ we define:

$$d_T^{(h)}(P_1\dots P_s) = \sum_{j=1}^s(\gamma^h|b_0^j| + \gamma^{h/2}|b_1^j|) \tag{3.44}$$

Note that, if $s = 1$, the sum over $T$ is void and must be understood as a trivial factor 1.

The proof of the bounds (3.43) is in Appendix 2; here we want to stress the absence of factorials in the number of fields, which is essentially linked to the fact that *we do not expand the l.h.s. in Feynman graphs.*

With the aid of (3.43) we can bound (3.37) as follows:

$$\begin{aligned}
|V^{(k)}(N,\tau,\{h_v\},P_{v_0},\underline{x})| \le &\left\{\prod_{\substack{v>v_0\\ v \text{ not e.p.}}}\sum_{P_v}\right\}\cdot\\
&\cdot\prod_{v \text{ not e.p.}}\prod_j\left[\gamma^{\frac{h_v}{4}[|P^1_{vj}|-|Q^1_{vj}|+5|P^2_{vj}|-5|Q^2_{vj}|]}\right]C^{\sum_j(|P_{vj}|-|Q_{vj}|)}\cdot\\
&\cdot\left\{\prod_{v \text{ not e.p.}}\frac{1}{s_v!}\sum_{T_v}e^{-\kappa d_{T_v}^{(h_v)}(P_{v1},\dots,P_{vs_v})}\right\}|\alpha|^{n_2'}|\nu|^{n_2}\prod_{i=1}^{n_4}|\lambda v(x_{2i-1}-x_{2i})|
\end{aligned} \tag{3.45}$$

where $Q^i_{vj} = P^i_v\bigcap P^i_{vj}$, $i = 1, 2$ and $j = 1,\dots,s_v$.

Now we have to integrate the expression (3.45) multiplied by the weight $e^{\frac{\kappa}{2}d^{(k)}(P_{v_0})}$. It is clear that in the r.h.s. of (3.45) $\underline{x}$ appears only in the last line; therefore we have to evaluate the expression:

$$\int\left\{\prod_{v \text{ not e.p.}}\frac{1}{s_v!}\sum_{T_v}e^{-\kappa d_{T_v}^{(h_v)}(P_{v1},\dots,P_{vs_v})}\right\}\prod_{i=1}^{n_4}|\lambda v(x_{2i-1}-x_{2i})|e^{\frac{\kappa}{2}d^{(k)}(P_{v_0})}d\underline{x} \tag{3.46}$$

Here we have to use the properties of $v(x-y)$: in fact a global tree graph (on all the scales) requires in general also the $v$'s to insure the connection. The property of $v$ that we need is (see (3.24) and (2.31)):

$$|\lambda v(x-y)| \le |\lambda|Ae^{-p_0|\vec{x}-\vec{y}|}\delta(t-t') \tag{3.47}$$

where $x = (t, \vec{x}), y = (t', \vec{y})$.

The latter inequality and a standard estimation of the integral allow to bound (3.46) by:

$$C^n |\Lambda| \prod_{v \geq v_0} \gamma^{-\frac{3}{2} h_v (s_v - 1)} |\lambda|^{n_4} \tag{3.48}$$

By (3.48) and (3.45) we have:

$$\begin{aligned} &\frac{1}{|\Lambda|} \int d\underline{x} |V^{(k)}(N, \tau, P_{v_0}, \underline{x})| e^{\frac{\kappa}{2} d^{(k)}(P_{v_0})} \leq C^n \, |\lambda|^{n_4} |\alpha|^{n_2'} |\nu|^{n_2} \cdot \\ &\cdot \left\{ \prod_{\substack{v > v_0 \\ v \text{ not e.p.}}} \sum_{P_v} \right\} \prod_{v \text{ not e.p.}} [\gamma^{\frac{h_v}{4}[\sum_j |P^1_{vj}| - |P^1_v| + 5 \sum_j |P^2_{vj}| - 5|P^2_v|]} \gamma^{-\frac{3}{2} h_v (s_v - 1)}] \end{aligned} \tag{3.49}$$

and we note that:

$$\begin{aligned} &\prod_{v \text{ not e.p.}} \gamma^{\frac{h_v}{4}[\sum_j |P^1_{vj}| - |P^1_v| + 5 \sum_j |P^2_{vj}| - 5|P^2_v|]} = \\ &= \left[ \prod_{v \text{ not e.p.}} [\gamma^{\frac{1}{4}[6 n_v^{2'} + 4 n_v^4 + 2 n_v^2 - |P^1_v| - 5|P^2_v|]} \right] \gamma^{\frac{k}{4}[6 n_{2'} + 4 n_4 + 2 n_2 - |P^1_{v_0}| - 5|P^2_{v_0}|]} \end{aligned} \tag{3.50}$$

and:

$$\prod_{v \text{ not e.p.}} \gamma^{-\frac{3}{2} h_v (s_v - 1)} = \left[ \prod_{v \text{ not e.p.}} \gamma^{-\frac{3}{2}(n_v - 1)} \right] \gamma^{-\frac{3}{2} k (n-1)} \tag{3.51}$$

Therefore we can rewrite the last factor of (3.49) as:

$$\left[ \prod_{v \text{ not e.p.}} \gamma^{-\frac{1}{4}[|P^1_v| + 5|P^2_v| + 2 n_v^4 + 4 n_v^2 - 6]} \right] \gamma^{-k/4[2 n_4 + 4 n_2 + |P^1_{v_0}| + 5|P^2_{v_0}| - 6]} \tag{3.52}$$

where $n_v$ is the number of endpoints which follow $v$ in the tree, while $n_v^4, n_v^2, n_v^{2'}$ are the numbers of endpoints of type 4,2,2' which follow $v$. where $n_v$ is the number of endpoints which follow $v$ in the tree, while $n_v^4, n_v^2, n_v^{2'}$ are the numbers of endpoints of type 4,2,2' which follow $v$.

Let us observe now that:

$$[|P^1_v| + 5|P^2_v| + 2 n_v^4 + 4 n_v^2 - 6] > 0 \tag{3.53}$$

(hence $\geq 1$), except in the following cases, that we discuss separately.

1) $|P^1_v| = 2, \quad n_v^4 = 2, \quad |P^2_v| = n_v^2 = 0$.

The only possible Feynman graphs associated with $\tau_v$ are, in this case:

Fig. 5

where the dots on the inner lines and on the external outgoing lines represent insertions of type $2'$ graph elements. However, their contribution is exactly zero by the remark 2) after (3.15), which is valid also for Feynman graphs with propagators of different frequencies.

2) $|P_v^1| = 2, \quad n_v^4 = 1, \quad |P_v^2| = n_v^2 = 0.$

This is the case of the graphs:

Fig. 6

which vanish for the same reason of the case 1).

3) $|P_v^1| = 1, \quad |P_v^2| = 1, \quad n_v^4 = n_v^2 = 0.$

This is the case of the trees, whose graph elements are all of type $2'$, so that only the chains of Fig. 7 are allowed.

Fig. 7

If $v \neq v_0$, one of the two lines external with respect to $v$ is internal to the non trivial vertex $v'$ preceding $v$. To be definite, let us suppose that this is the case for the line emerging from $x_m$ (the other case can be treated in the same way); then all terms contributing to the expansion in Feynman graphs of $W^{(k)}(N, \tau, P_{v_0}, \underline{x}^{(P_{v_0})})$ contain a factor of the type:

$$\int dx_2 \ldots dx_m \Delta_{x_2} C_{h_1}(x_1 - x_2) \ldots \Delta_{x_m} C_{h_{m-1}}(x_{m-1} - x_m)(\Delta_y)^\rho C_{h_{v'}}(x_m - y) \quad (3.54)$$

where $\rho = 0$ or $\rho = 1$.

Let us suppose first that all the lines have the same frequency, that is $h_i = h_v$, for $i = 1, \ldots, m-1$. Then, since $\int dx_m \Delta C_h(x_{m-1} - x_m) = 0$, we can substitute in (3.54) $C_{h_{v'}}(x_m - y)$ with:

$$C_{h_{v'}}(x_m - y) - C_{h_{v'}}(x_{m-1} - y) = (x_m - x_{m-1}) \int_0^1 dt \partial C_{h_{v'}}(x_m - y - t(x_m - x_{m-1})) \quad (3.55)$$

and it is easy to see that such substitution allows us to improve the bound by a factor $\gamma^{-(h_v - h_{v'})/2}$.

If the lines have different frequencies, *i.e.* if there are other non trivial vertices following $v$, we have to apply the previous argument iteratively starting from the higher vertices. The only change is that some covariance in (3.54) is substituted by its gradient calculated at an interpolated point as in the r.h.s. of (3.55); it is easy to see that the improvement for each non trivial vertex is always the same, *i.e.* $\gamma^{-1/2}$ raised to a power equal to the difference between the frequency of the vertex and that of the preceding non trivial one. Furthermore, there is at most a factor $|x - x'|$ for each line connecting $x$ and $x'$ and each covariance must be interpolated at most two times; so no dangerous factorials appear.

Of course, in order to improve the bound, we have to expand in Feynman graphs the subtree starting in the vertex $v$ and *extract* the propagator $C_{h_{v'}}(x_m - y)$ from the truncated expectation associated with $v'$. One could be afraid that this destroys the good combinatorial properties of (3.43), but this is not the case. In fact the subtree starting from $v$ belongs to $\mathcal{T}_m^*$ and it is easy to see that its expansion in Feynman graphs contains exactly $s_v!$ terms, which is compensated by making use of the $1/s_v!$ factors of (3.37); so there is no combinatorial problem here. The problem of the extraction of $C_{h_{v'}}(x_m - y)$ from the truncated expectation is not really present, since each term contributing to the r.h.s. of (3.43) has a factor equal to one of the external propagators of $v$ (see the proof of (3.43) in appendix 2).

We have still to consider the case $v = v_0$, but now $\tau \in \mathcal{T}_n^*$ and we can use the factor $\chi_\tau(\underline{x}^{(P_{v_0})})$ in (3.38) to improve the bound by a factor $\gamma^{-(h_v - k)/2}$.

4) $|P_v^1| = 2, \quad |P_v^2| = 0, \quad n_v^4 = 0, \quad n_v^2 = 1$

This is the case of the tree with an arbitrary number of type $2'$ graph elements and one of type 2. The same considerations of the case 3) apply, so that again we can improve the bound by a factor $\gamma^{-(h_v - h_{v'})/2}$.

We can summarize the discussion above, by saying that the last line of (3.49) can be replaced by the expression:

$$\left[\prod_{v \text{ not e.p.}} \gamma^{-\frac{1}{4}D_v}\chi(D_v > 0)\right]\gamma^{-\frac{k}{4}D_{v_0}} \tag{3.56}$$

where $\chi(D_v > 0)$ is the characteristic function of the set $\{D_v > 0\}$ (it recalls us that the graphs of items 1 and 2 above are not allowed) and

$$D_v = |P_v^1|+5|P_v^2|+2n_v^4+4n_v^2-6+2\delta_{|P_v^1|,1}\delta_{|P_v^2|,1}\delta_{n_v^4,0}\delta_{n_v^2,0}+2\delta_{|P_v^1|,2}\delta_{|P_v^2|,0}\delta_{n_v^4,0}\delta_{n_v^2,1} \tag{3.57}$$

The above discussion shows the essentially trivial renormalizability of this model. In fact, since the number of unlabeled trees with $n$ endpoints can be bounded by $4^n$, in order to prove the bound (3.38) it is sufficient to control the multiple sums in (3.49) and the sum over the labeled trees with a fixed topological structure. This can be easily done by using the factors $\gamma^{-\frac{1}{4}D_v}$ of (3.56).

We first observe that, given an unlabeled tree $\tilde{\tau}$, there are only $3^n$ corresponding families of labeled trees differing for the choice of the graph elements associated with the endpoints; hence it is sufficient to consider only one of such families, say $\tilde{\mathcal{T}}$. The trees $\tau \in \tilde{\mathcal{T}}$ can be distinguished by fixing the frequencies indices of the non trivial vertices, which we shall denote $\tilde{v}$. We can write:

$$\prod_{v \text{ not e.p.}} \gamma^{-\frac{1}{4}D_v} \leq \left[\prod_{\tilde{v}} \gamma^{-\frac{1}{8}(h_{\tilde{v}}-h_{\tilde{v}'})}\right]\left[\prod_{v \text{ not e.p.}} \gamma^{-\frac{1}{8}D_v}\right] \tag{3.58}$$

where $\tilde{v}'$ is the non trivial vertex immediately preceding $\tilde{v}$ or the root, if there is no such vertex.

The sum over the set $\tilde{\mathcal{T}}$ of the first factor in the r.h.s. of (3.58) can be bounded in a trivial way by a factor $C^n$. Furthermore, by (3.57), if $D_v > 0$:

$$D_v \geq \max\{1, |P_v|-2\} \geq \frac{|P_v|}{3} \tag{3.59}$$

Hence, in order to complete the proof of (3.38), it is sufficient to prove that:

$$\prod_{v \text{ not e.p.}} \sum_{P_v} \gamma^{-\frac{|P_v|}{24}} \equiv S(P_{v_0}, \tau, n) \leq C^n \tag{3.60}$$

where the sums over the sets $P_v$ are constrained by the condition that $P_v = \bigcup_j Q_{v^j}$ with $Q_{v^j}$ a subset, possibly empty, of $P_{v^j}$; furthermore $P_v$ is a fixed set with four or two elements, if $v$ is an endpoint, and we have eliminated the constraint that $P_{v_0}$ is a fixed subset of the fields associated with the tree graph elements.

The latter estimate, evident for large $\gamma$, can be proved in the general case $\gamma > 1$ in the following way. We note that:

$$S(P_{v_0}, \tau, n) \leq \prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{24}} C_v \tag{3.61}$$

where $C_v$ counts the number of ways of choosing a subset $P_v$ with $p_v$ elements, satisfying the constraints; hence it can be easily bounded by a binomial coefficient and we obtain:

$$S(P_{v_0}, \tau, n) \leq \prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{24}} \binom{\sum_{j=1}^{s_v} p_{vj}}{p_v} \tag{3.62}$$

Set $\bar{\gamma} = \gamma^{\frac{1}{24}}$ and let us denote with $\mathcal{P}$ a path from the root of the tree to an endpoint and with $l(\mathcal{P})$ the number of vertices lying on $\mathcal{P}$. It is easy to show, by simply performing the sums in (3.62) one after the other starting from $v_0$, that:

$$S(P_{v_0}, \tau, n) \leq \prod_{\mathcal{P}} \left( \sum_{n=0}^{l(\mathcal{P})} \bar{\gamma}^{-n} \right)^4 \leq \left( \frac{1}{1 - \bar{\gamma}^{-1}} \right)^{4n} \tag{3.63}$$

The bound (3.38) implies that we can sum, for $|z|$ small enough, say $|z| \leq \varepsilon$, and uniformly in $N$, the terms in the effective potential, which have the same dependence on the field (*i.e.* that have the same set of labels $\{\sigma_f, f \in P_{v_0}\}$). In fact, we have still to bound only the sum of all trees of order $n$ satisfying that condition: as mentioned above this gives simply another factor $\leq 4^n$, as the trees are "topological trees", see item 1) after Fig. 2.

We can now integrate also the field fluctuations associated with the regular part $R(x)$ of the u.v. covariance, see (3.4) and (3.9). The regularity of the propagator $R$ makes this a trivial repetition of, say, the last integration lowering the u.v. cut off from $h = 1$ to $h = 0$ and we do not have to perform it in detail.

The bounds of this section imply that $\bar{V}^{(0)}(\varphi)$ can be written, for $|z| \leq \varepsilon$, as in (3.17) and that a similar expression is valid for the effective potential on scale $\gamma^{-k}$. Furthermore the kernel $\tilde{W}_2(z, x)$ singles out the contributions coming from the trees in $\mathcal{T}_n^*$ (see discussion after (3.40) ) and therefore satisfies (uniformly in $N$) the bound (3.19) and the equation (3.20).

¿From the considerations of §2 it is almost obvious that the effective potentials can be given the expression (3.17) for $|z| < O(L^{-1}\gamma^{-N})$. The results just derived show that in fact the analyticity in $z$ of the kernels for the effective potentials can be extended to $|z| < \varepsilon$ for some $\varepsilon > 0$ and of order $O(1)$ and have a uniform exponential decay ($\Lambda, N$-independent): see (3.18),(3.19).

This means that we can sum the coefficients of give order in $z$ and that their sum admits good exponential bounds.

Note that this *is not sufficient* to guarantee the integrability in the sense of §2 of $\exp \bar V^{(0)}(\psi^{(i.r.)}+\varphi)$ with respect to the i.r. part of the grassmanian fields for $|z|<\varepsilon$.

We shall proceed by imagining that we have a u.v. cut off $N$ and perform the integrations down to the infrared cut off $R$: and we shall see that it is possible to perform a resummation of perturbation theory permitting to express the effective potentials as uniformly convergent power series in a sequence of constants $\underline{r}_h$, called the *running couplings*, which are themselves expressed as sums of series in the initial couplings $z$. The series for the running couplings will have very small $L,N,R$ dependent radii of convergence. But they will be related by a map permitting to express their values at scale $h$ in terms of the values at the preceding scales $h-1,\ldots,0$. We shall show that the relation is expressed by an analytic function, *the beta functional*, of the preceding couplings with a radius of convergence which is uniform in $N,R,L$. Thus if *by some other means*, see §7, one can be sure that the beta functional generates a sequence $\underline{r}_h$, $h=0,-1,\ldots$, of running couplings which stay small uniformly in the index $h$, then one will have shown the possibility of a resummation of the perturbation series for the full effective potential kernels, which is uniform in $R,N,L$ and a theory of the ground state will have been constructed (up to the *technicalities* analyzed in §6).

In the next section we begin the discussion on the beta functional and its analyticity properties.

## §4 *The effective potential in the infrared region. Failure of normal scaling.*

In this section we shall begin the analysis of the infrared problem, that is of the possibility of giving a meaning to the integration in (3.8) of the infrared fluctuations of the field, associated with the propagator:

$$g_{i.r.}(x)\equiv g^{(\le 0)}(x)=\int\frac{dk_0 d\vec k}{(2\pi)^2}\frac{e^{-ikx}}{-ik_0+e(\vec k)}e^{-[k_0^2+e(\vec k)^2]p_0^{-2}} \tag{4.1}$$

Note that the Fourier transform of $g^{(\le 0)}(x)$ has a linear divergence on the *Fermi surface* $k_0=0,\vec k=\pm p_F$, which cannot be treated by a naive multiscale decomposition as the one used for the u.v. problem, because of the presence of the built-in scale $p_F$. It is possible, however, to rewrite the problem in terms of *quasi particle fields* in the way presented in [BG], that we briefly summarize here.

We write the *particle field* $\psi_x^{\sigma(\le 0)}$ of covariance $g^{(\le 0)}(x)$ as a sum of independent *quasi* *particle fields*:

$$\psi_x^{\sigma(\leq 0)} \equiv \sum_{\omega=\pm 1} e^{i\sigma p_F \omega \vec{x}} \psi_{\omega,x}^{\sigma(\leq 0)} \tag{4.2}$$

and, as usual, the fields $\psi_{\omega,x}^{\sigma(\leq 0)}$, essentially describing the fluctuations around the two points of the Fermi surface, are decomposed as sums of independent fields in the following way:

$$\psi_{\omega,x}^{\sigma(\leq 0)} = \sum_{h=-\infty}^{0} \psi_{\omega,x}^{\sigma(h)} \tag{4.3}$$

where $\psi_{\omega,x}^{\sigma(h)}$ has covariance:

$$g_\omega^{(h)}(x) = e^{ip_F\omega\vec{x}} \int_{\gamma^{-2h}}^{\gamma^{-2h+2}} d\alpha \int \frac{dk}{(2\pi)^2} e^{-ik\cdot x}(ik_0 + e(\vec{k}))e^{-\alpha p_0^{-2}[k_0^2+e(\vec{k})^2]}\chi(\omega\gamma^{-h}\vec{k}) \tag{4.4}$$

Here $\chi(t) = \pi^{-1/2}\int_{-\infty}^t ds \exp(-s^2)$ is a regularization of the step function.

In App. 1 we show (see also [BG], App. A) that, for any integer $m \geq 0$:

$$|\partial^m g_\omega^{(h)}(x)| \leq C_m \gamma^{h(1+m)} e^{-\kappa\gamma^h |x|} \tag{4.5}$$

for some suitable constants $C_m$ and $\kappa$, independent of $h$.

In the following we shall use also the definitions:

$$\psi_{\omega,x}^{\sigma(\leq h)} = \sum_{k=-\infty}^{h} \psi_{\omega,x}^{\sigma(k)} \quad , \quad \psi_x^{\sigma(\leq h)} = \sum_{\omega=\pm 1} e^{i\sigma p_F\omega\vec{x}} \psi_{\omega,x}^{\sigma(\leq h)} \tag{4.6}$$

In order to evaluate $V_{eff}(\varphi)$, by (3.8) and (3.9), we should study the functional integral

$$\int P(d\psi^{\leq 0}) e^{-\bar{V}^{(0)}(\psi^{\leq 0}+\varphi)} \tag{4.7}$$

However, the analysis of this integral is more delicate in comparison to the analogous ultraviolet problem, because of the anomalous scaling. Therefore we split the problem into the simpler problem of defining the *running couplings* and into that of evaluating the effective potential. The first problem already emerges from the study of the integral (4.7) for $\varphi = 0$, *i.e.* from the study of the normalization constant in (3.8); this analysis will be performed in this section and in the following one. The second problem will be faced up in §6 indirectly, through the analysis of the Schwinger functions, which are the physically relevant quantities.

Setting $\psi = \psi^{(\leq 0)}$ to simplify the notation, we represent the potential $\bar V^{(0)}(\psi)$, see (3.17), in terms of quasi particle fields and we obtain:

$$
\begin{aligned}
\bar V^{(0)}(\psi) = &\lambda \int dx\, dy \sum_{\omega_1\ldots\omega_4} e^{ip_F[(\omega_1-\omega_2)\vec x + (\omega_3-\omega_4)\vec y]} \psi^+_{\omega_1 x}\psi^-_{\omega_2 x} v(x-y)\psi^+_{\omega_3 y}\psi^-_{\omega_4 y} + \\
&+ \nu \int dx \sum_{\omega_1,\omega_2} e^{ip_F(\omega_1-\omega_2)\vec x}\, \psi^+_{\omega_1 x}\psi^-_{\omega_2 x} + \\
&+ \alpha \int dx \sum_{\omega_1,\omega_2} e^{ip_F(\omega_1-\omega_2)\vec x}\, \psi^+_{\omega_1 x} i\beta\omega_2 \mathcal{D}^-_{\omega_2}\psi^-_{\omega_2 x} + \\
&+ \sum_{n=1}^{\infty} \sum_{\substack{n_1,n_2\\ n_1+n_2=2n}} \sum_{\substack{\omega_1\ldots\omega_n\\ \omega'_1\ldots\omega'_n}} \int dx_1\ldots dx_{2n} e^{ip_F\sum_{i=1}^n(\omega_i\vec x_i - \omega'_i \vec x_{n+i})} \cdot \\
&\cdot \psi^+_{\omega_1 x_1}\ldots\psi^+_{\omega_n x_n}\psi^-_{\omega'_1 x_{n+1}}\ldots\psi^-_{\omega'_{n-n_2} x_{2n-n_2}} \cdot \\
&\cdot i\omega'_{n-n_2+1}\mathcal{D}^-_{\omega'_{n-n_2+1}}\psi^-_{\omega'_{n-n_2+1} x_{2n-n_2+1}}\ldots i\omega'_n \mathcal{D}^-_{\omega'_n}\psi^-_{\omega_n x_{2n}} \bar W_{n_1 n_2}(z, x_1\ldots x_{2n})
\end{aligned}
\tag{4.8}
$$

where $\beta \equiv p_F/m$, the *covariant derivative* $\mathcal{D}^-_\omega$ is a differential operator acting only on the space coordinate, defined by:

$$
\mathcal{D}^-_\omega = \partial_{\vec x} + \frac{i\omega\partial^2_{\vec x}}{2p_F} \tag{4.9}
$$

and the contribution of the third line in (3.17) has been included in the last term (see discussion related to (4.38) and (4.39) below).

The $\mathcal{D}^-_\omega$ operator satisfies the following identity, which will play an important role in the following:

$$
\int dx \psi^{+(\leq h)}_x e(i\partial_{\vec x})\psi^{-(\leq h)}_x = \sum_{\omega_1,\omega_2}\int dx\, e^{ip_F(\omega_1-\omega_2)\vec x}\, \psi^{+(\leq h)}_{\omega_1 \vec x} i\beta\omega_2\mathcal{D}^-_{\omega_2}\psi^{-(\leq h)}_{\omega_2 \vec x} \tag{4.10}
$$

It is now very natural to define the *effective potential on scale* $\gamma^{-h}$, for $h < 0$, as in (3.25), through the expression:

$$
e^{-\bar V^{(h)}(\psi^{(\leq h)})} = \frac{1}{\mathcal{N}}\int P(d\psi^{(h+1)})\ldots\int P(d\psi^{(0)}) e^{-\overline{V}^{(0)}(\psi^{(\leq 0)})} \tag{4.11}
$$

We shall see in the following that *this is is not the correct definition*, because of the anomalous scaling properties of the model. However we proceed for the moment with this definition in order to show where and why the problem arises.

As explained in Ref. [BG], we can isolate the relevant part of the effective potential by introducing a localization operator $\mathcal{L}$ which acts linearly on the monomials in the fields of the form $\prod_i \psi^{\sigma_i}_{\omega_i x_i}$ and is zero on all monomials of degree $\geq 6$. Its action on the monomials of degree 2 and 4 is generated by linearity from:

$$\mathcal{L}(\psi^+_{\omega_1 x_1}\psi^+_{\omega_2 x_2}\psi^-_{\omega_3 x_3}\psi^-_{\omega_4 x_4}) = \frac{1}{2}[\psi^+_{\omega_1 x_1}\psi^+_{\omega_2 x_1}\psi^-_{\omega_3 x_1}\psi^-_{\omega_4 x_1} + \psi^+_{\omega_1 x_2}\psi^+_{\omega_2 x_2}\psi^-_{\omega_3 x_2}\psi^-_{\omega_4 x_2}]$$
$$\mathcal{L}(\psi^+_{\omega_1 x_1}\psi^-_{\omega_2 x_2}) = \psi^+_{\omega_1 x_1}\psi^-_{\omega_2 x_1} + (x_2 - x_1)\psi^+_{\omega_1 x_1}\mathcal{D}_{\omega_2}\psi^-_{\omega_2 x_1} \tag{4.12}$$

where (see (4.9)):

$$\mathcal{D}_\omega \equiv (\partial_t, \mathcal{D}^-_\omega) \tag{4.13}$$

We used in the second line of (4.12) the covariant derivative (4.9) instead of the normal space derivative, which could perhaps look more natural, for a reason which will be explained later (see remark following (4.30) below); in any event our choice (4.12) differs from the other one only by an irrelevant term.

If $\mathcal{R} = 1 - \mathcal{L}$, we have also:

$$\mathcal{R}(\psi^+_{\omega_1 x_1}\psi^+_{\omega_2 x_2}\psi^-_{\omega_3 x_3}\psi^-_{\omega_4 x_4}) = \tag{4.14}$$
$$= \frac{1}{2}[\psi^+_{\omega_1 x_1}D^+_{21\omega_2}\psi^-_{\omega_3 x_3}\psi^-_{\omega_4 x_4} + \psi^+_{\omega_1 x_1}\psi^+_{\omega_2 x_1}D^-_{31\omega_3}\psi^-_{\omega_4 x_4} + \psi^+_{\omega_1 x_1}\psi^+_{\omega_2 x_1}\psi^-_{\omega_3 x_1}D^-_{41\omega_4}] +$$
$$+ \frac{1}{2}[D^+_{12\omega_2}\psi^+_{\omega_2 x_2}\psi^-_{\omega_3 x_3}\psi^-_{\omega_4 x_4} + \psi^+_{\omega_1 x_2}\psi^+_{\omega_2 x_2}D^-_{32\omega_3}\psi^-_{\omega_4 x_4} + \psi^+_{\omega_1 x_2}\psi +_{\omega_2 x_2}\psi^-_{\omega_3 x_2}D^-_{42\omega_4}]$$

where:

$$D^\sigma_{ji\omega_j} = \psi^\sigma_{\omega_j x_j} - \psi^\sigma_{\omega_j x_i} = (x_j - x_i)\int_0^1 dr\, \partial\psi^\sigma_{\omega_j x_{ji}(r)}$$
$$x_{ji}(r) = x_i + r(x_j - x_i)\ , \quad \partial \equiv (\partial_t, \partial_{\vec{x}}) \tag{4.15}$$

and for the quadratic term in the fields we have:

$$\mathcal{R}(\psi^+_{\omega_1 x_1}\psi^-_{\omega_2 x_2}) = -\frac{i\omega_2}{2p_F}(\vec{x}_2 - \vec{x}_1)\psi^+_{\omega_1 x_1}\partial^2_{\vec{x}}\psi^-_{\omega_2 x_1} +$$
$$+ (x_2 - x_1)^2\psi^+_{\omega_1 x_1}\int_0^1 dr \int_0^r ds \partial^2\psi^-_{\omega_2 x_{21}(s)} \tag{4.16}$$

where:

$$(x_2 - x_1)^2\int_0^1 dr \int_0^r ds \partial^2\psi^-_{\omega_2 x_{21}(s)} = \psi^-_{\omega_2 x_2} - \psi^-_{\omega_2 x_1} - (x_2 - x_1)\partial\psi^-_{\omega_2 x_1} \tag{4.17}$$

We plan to evaluate iteratively the integrals in the r.h.s. of (4.11), by rewriting at each step $\bar{V}^{(h)}$ in the form $\mathcal{L}\bar{V}^{(h)} + \mathcal{R}\bar{V}^{(h)}$. This implies that we have to consider the action of $\mathcal{L}$ also on other monomials of second and fourth order, besides those appearing in (4.8). We shall give now the complete list of the monomials that one has to take into account, for which the action of $\mathcal{L}$ does not give zero, together with the result of the application of $\mathcal{L}$ and $\mathcal{R}$, deduced from (4.12) by linearity.

In the case of the fourth order monomials there is only one more term on which $\mathcal{L}$ is not trivial, in principle; it is the one of the form $\psi^+_{x_1\omega_1}\partial\psi^+_{x_2\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4}$. This term can only appear if $x_1$ is an interpolated point, see (4.15), so that we really need the following equation:

$$\begin{aligned}&\mathcal{L}\psi^+_{x_1\vec\omega_1}D^+_{x_2x_{2'}\vec\omega_2}\psi^-_{x_3\vec\omega_3}\psi^-_{x_4\vec\omega_4} =\\ &\quad=\frac{1}{2}[\psi^+_{x_2\vec\omega_1}\psi^+_{x_2\vec\omega_2}\psi^-_{x_2\vec\omega_3}\psi^-_{x_2\vec\omega_4}-\psi^+_{x_{2'}\vec\omega_1}\psi^+_{x_{2'}\vec\omega_2}\psi^-_{x_{2'}\vec\omega_3}\psi^-_{x_{2'}\vec\omega_4}]\end{aligned}\tag{4.18}$$

By the anticommutation properties of the field, the r.h.s. can be different from zero only if $\omega_1=-\omega_2,\omega_3=-\omega_4$. However, in this case, the integration on the $x$-variables cancels it, because the monomial in the l.h.s. appears multiplied by a translation invariant function of the $x$-variables; furthermore the oscillating factor $e^{ip_F(\omega_1\vec x_1+\omega_2\vec x_2-\omega_3\vec x_3-\omega_4\vec x_4)}$ is also translation invariant, if $\omega_1=-\omega_2,\omega_3=-\omega_4$.

Hence, for our purposes:

$$\mathcal{L}\psi^+_{x_1\omega_1}\partial\psi^+_{x_2\omega_2}\psi^-_{x_3\omega_3}\psi^-_{x_4\omega_4}=0\tag{4.19}$$

and we do not have to consider any other localization operation on the fourth order monomials, besides that of (4.12).

In the case of the second order monomials, we have to consider the following localization operations:

$$\begin{aligned}\mathcal{L}(\psi^+_{\omega_1x_1}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_2})&=\psi^+_{\omega_1x_1}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_1}\\ \mathcal{L}(\psi^+_{\omega_1x_1}\partial\psi^-_{\omega_2x_{2'}})&=\psi^+_{\omega_1x_1}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_1}\\ \mathcal{L}(\partial\psi^+_{\omega_1x_{1'}}\psi^-_{\omega_2x_2})&=\partial(\psi^+_{\omega_1x_{1'}}\psi^-_{\omega_2x_{1'}})-\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}}+\\ &\quad+(x_2-x_{1'})\partial(\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}})\\ \mathcal{L}(\partial\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_2})&=\partial(\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}})\\ \mathcal{L}(\partial\psi^+_{\omega_1x_{1'}}\partial\psi^-_{\omega_2x_{2'}})&=\partial(\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}})\end{aligned}\tag{4.20}$$

and the corresponding $\mathcal{R}$ operations:

$$\begin{aligned}\mathcal{R}(\psi^+_{\omega_1x_1}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_2})&=(x_2-x_1)\psi^+_{\omega_1x_1}\int_0^1 dr\mathcal{D}_{\omega_2}\partial\psi^-_{\omega_2x_{21(r)}}\\ \mathcal{R}(\psi^+_{\omega_1x_1}\partial\psi^-_{\omega_2x_{2'}})&=-\frac{i\omega_2}{2p_F}\psi^+_{\omega_1x_1}\partial^2_{\vec x}\psi^-_{\omega_2x_1}+(x_{2'}-x_1)\psi^+_{\omega_1}(x_1)\int_0^1 dr\partial\partial\psi^-_{\omega_2x_{2'1(r)}}\\ \mathcal{R}(\partial\psi^+_{\omega_1x_{1'}}\psi^-_{\omega_2x_2})&=(x_2-x_{1'})\partial\psi^+_{\omega_1x_{1'}}\int_0^1 dr\partial\psi^-_{\omega_2x_{21'(r)}}-\frac{i\omega_2}{2p_F}\psi^+_{\omega_1x_{1'}}\partial^2_{\vec x}\psi^-_{\omega_2x_{1'}}-\\ &\quad-(x_2-x_{1'})\partial\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}}-(x_2-x_{1'})\psi^+_{\omega_1x_{1'}}\partial\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}}\\ \mathcal{R}(\partial\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_2})&=\partial\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_2}-\partial(\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}})\\ \mathcal{R}(\partial\psi^+_{\omega_1x_{1'}}\partial\psi^-_{\omega_2x_{2'}})&=\partial\psi^+_{\omega_1x_{1'}}\partial\psi^-_{\omega_2x_{2'}}-\partial(\psi^+_{\omega_1x_{1'}}\mathcal{D}_{\omega_2}\psi^-_{\omega_2x_{1'}})\end{aligned}\tag{4.21}$$

where the symbol $x_{j'}$ is used to stress that $x_{j'}$ is a point on the segment connecting $x_j$ and some other point.

In the r.h.s. of the last three of equations (4.20), appear some new local terms with respect to the second relation in the r.h.s. of (4.12). However, the field $\partial\psi^+_{\omega_1 x_{1'}}$ in the l.h.s. of (4.20) can appear only through a field $D^+_{12}$ by interpolation, see (4.15). Hence one has really to consider the following localization operations:

$$\begin{aligned}\mathcal{L}(D^+_{12\omega_1}\psi^-_{x_3\omega_2}) &= \psi^+_{x_1\omega_1}\psi^-_{x_1\omega_2} - \psi^+_{x_2\omega_1}\psi^-_{x_2\omega_2} + \\ &\quad + (x_3 - x_1)\psi^+_{x_1\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_1\omega_2} - (x_3 - x_2)\psi^+_{x_2\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_2\omega_2} \\ \mathcal{L}(D^+_{12\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_3\omega_2}) &= \psi^+_{x_1\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_1\omega_2} - \psi^+_{x_2\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_2\omega_2} \\ \mathcal{L}(D^+_{12\omega_1}D^-_{34\omega_2}) &= (x_3 - x_4)(\psi^+_{x_1\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_1\omega_2} - \psi^+_{x_2\omega_1}\mathcal{D}_{\omega_2}\psi^-_{x_2\omega_2})\end{aligned} \tag{4.22}$$

And we can conclude that, as a result of the localization operation on the effective potential, we get, for each scale, the following local monomials:

$$\psi^+_{x+}\psi^+_{x-}\psi^-_{x+}\psi^-_{x-}, \qquad \psi^+_{x\omega}\psi^-_{x\omega'}, \qquad \psi^+_{x\omega} i\omega'\beta\mathcal{D}^-_{\omega'}\psi^-_{x\omega'}, \qquad \psi^+_{x\omega}\partial_t\psi^-_{x\omega'} \tag{4.23}$$

multiplied by some constants, the *running coupling constants* of the model, that we shall indicate, respectively, with $\lambda_h$, $\gamma^h\nu_h$, $\alpha_h$, $\zeta_h$.

At first sight, the running coupling constants depend on the $\omega$ variables; however, we shall see that they are actually $\omega$-independent.

The fourth order local part must have the form:

$$\sum_{\omega}\int dx\, \lambda_h(\omega_1,\omega_2,\omega_3,\omega_4)\, e^{ip_F(\omega_1+\omega_2-\omega_3-\omega_4)\vec{x}}\psi^{+(\le h)}_{\omega_1 x}\psi^{+(\le h)}_{\omega_2 x}\psi^{-(\le h)}_{\omega_3 x}\psi^{-(\le h)}_{\omega_4 x} \tag{4.24}$$

and recalling the anticommutation properties of fermions, we can write:

$$\lambda_h(\omega_1,\omega_2,\omega_3,\omega_4) = -\frac{\lambda_h}{4}\omega_1\omega_3\delta_{\omega_1,-\omega_2}\delta_{\omega_3,-\omega_4} \tag{4.25}$$

Hence we can rewrite the quartic relevant part in the simpler way:

$$\lambda_h\int dx\, \psi^{+(\le h)}_{+1x}\psi^{+(\le h)}_{-1x}\psi^{-(\le h)}_{-1x}\psi^{-(\le h)}_{+1x} \tag{4.26}$$

Let us now investigate the $\omega$-dependence of the running coupling constants associated with the quadratic terms in the effective potential on scale $\gamma^{-h}$. By the linearity of $\mathcal{L}$, we can calculate the local part in a different way. First we can do all the integrations in (4.11) without introducing the quasi particle field representation; then we represent the effective potential in terms of the quasi particle fields and finally we apply the localization operator. After the first step, the quadratic part of the effective potential on scale $\gamma^{-h}$, expressed in terms of particle fields, looks as follows:

$$\bar{V}^{[2](h)} = \int dxdy\; v_h(x-y)\psi^+(x)\psi^-(y) + \int dxdy\; w_h(x-y)\psi^+(x)e(i\partial_{\vec{y}})\psi^-(y) \tag{4.27}$$

where $v_h$ and $w_h$ are rotation invariant kernels (this means, in one dimension, that they are even functions in the spatial coordinate); such property follows from the fact that the free propagator of the theory and the interaction are indeed rotation invariant.

We represent now $\bar{V}^{[2](h)}$ in terms of quasi particle fields:

$$\begin{aligned}\bar{V}^{[2](h)} = \sum_{\omega,\omega'} \Big[ &\int dxdy\ v_h(x-y) e^{ip_F(\omega\vec{x}-\omega'\vec{y})} \psi^+_{\omega x}\psi^-_{\omega' y} + \\ &+ \int dxdy\ w_h(x-y) e^{ip_F(\omega\vec{x}-\omega'\vec{y})} \psi^+_{\omega x} i\beta\omega' \mathcal{D}^-_{\omega'} \psi^-_{\omega' y} \Big]\end{aligned} \tag{4.28}$$

Hence the second order local part has the form:

$$\begin{aligned}\mathcal{L}\bar{V}^{[2](h)} = &\gamma^h \nu_h \sum_{\omega\omega'} \int dx e^{ip_F(\omega-\omega')\vec{x}} \psi^+_{\omega x}\psi^-_{\omega' x} + \\ &+ \alpha_h \sum_{\omega\omega'} \int dx e^{ip_F(\omega-\omega')\vec{x}} \psi^+_{\omega x} i\beta\omega' \mathcal{D}^-_{\omega'} \psi^-_{\omega' x} + \zeta_h \sum_{\omega\omega'} \int dx e^{ip_F(\omega-\omega')\vec{x}} \psi^+_{\omega x} \partial_t \psi^-_{\omega' x}\end{aligned} \tag{4.29}$$

where, if $\vec{z}$ is the spatial part of the two dimensional space-time vector $z$ and $z_0$ is its time component:

$$\begin{aligned}\gamma^h \nu_h &= \int dz v_h(z) e^{ip_F\omega'\vec{z}} \\ \alpha_h &= \int dz e^{ip_F\omega'\vec{z}} [w_h(z) + \frac{i}{\beta}\omega'\vec{z} v_h(z)] \\ \zeta_h &= \int dz e^{ip_F\omega'\vec{z}} (-z_0) v_h(z)\end{aligned} \tag{4.30}$$

The latter definitions immediately imply that $\nu_h$, $\alpha_h$ and $\zeta_h$ are independent of the $\omega$'s, as a consequence of the rotation invariance of the theory.

The previous observation has another consequence, which will play an important role in the following analysis. The structure of (4.22) is, in fact, not suitable for the dimensional bounds that we want to discuss: the r.h.s. of (4.22) is written as a sum of terms which do not vanish when $x_1 = x_2$, i.e. we loose track of the fact that the l.h.s. of (4.22) vanishes for $x_1 = x_2$, a property which is manifest in the l.h.s. through the field $D^+_{12\omega_1}$; this is disappointing because the property of vanishing of the l.h.s. must be used to regularize the vertex where the field $D^+_{12\omega_1}$ appeared at a previous scale, along the iterative construction of $\bar{V}^{(h)}$.

As a consequence we cannot have good bounds for the contributions to $\nu_h, \alpha_h, \zeta_h$, coming from the individual terms in the r.h.s. of (4.22). However, if $\omega_1 = \omega_2$, it is easy to see that the contributions arising from the second and the third of (4.22) cancel out, because of the translation invariance of the theory, by an argument similar to that used in the remark following (4.18) and leading to the "effective validity" of (4.19) (see also [BG], Sect. 11).

In the first of (4.22) the translation invariance implies that, if $\omega_1 = \omega_2 = \omega$, the r.h.s. can be replaced by $(x_1 - x_2)\psi^+_{x_1\omega}\mathcal{D}_\omega\psi^-_{x_1\omega}$, and in this way the needed $(x_1 - x_2)$ factor is explicitly exhibited.

To summarize, if $\omega_1 = \omega_2 = \omega$, we can replace (4.22) with:

$$\mathcal{L}(D^+_{12\omega}\psi^-_{x_3\omega}) = (x_1 - x_2)\psi^+_{x_1\omega}\mathcal{D}_\omega\psi^-_{x_1\omega} \;, \quad \mathcal{L}(D^+_{12\omega}\mathcal{D}\psi^-_{x_3\omega}) = \mathcal{L}(D^+_{12\omega}D^-_{34\omega}) = 0 \tag{4.31}$$

The previous properties are not valid anymore, if $\omega_1 \neq \omega_2$; hence there would be a serious problem, if we had to bound the contributions to the effective potential associated with the local terms in the r.h.s. of (4.22) for all $\omega_1, \omega_2$. But this is not the case, since we know *a priori* that $\nu_h, \alpha_h, \zeta_h$ are *independent* of $\omega_1, \omega_2$ and we are not interested in the single contributions building the running coupling constants expansions, but only in their sums. Hence we can choose to compute the running coupling constants via their expansions valid for $\omega_1 = \omega_2$, which does not give any trouble, as we shall see.

Before starting the inductive evaluation of (4.11), we write:

$$\bar V^{(0)}(\psi^{\leq(0)}) = \mathcal{L}\bar V^{(0)}(\psi^{\leq(0)}) + \mathcal{R}\bar V^{(0)}(\psi^{\leq(0)}) \tag{4.32}$$

It is easy to see that:

$$\begin{aligned}
\mathcal{L}\bar V^{(0)}(\psi^{\leq(0)}) = \lambda_0 \int dx\, \psi^{+(\leq 0)}_{+1x}\psi^{+(\leq 0)}_{-1x}\psi^{-(\leq 0)}_{-1x}\psi^{-(\leq 0)}_{+1x} + \\
+ \nu_0 \int dx \sum_{\omega_1\omega_2} e^{ip_F(\omega_1-\omega_2)\vec x}\psi^{+(\leq 0)}_{\omega_1 x}\psi^{-(\leq 0)}_{\omega_2 x} + \\
+ \alpha_0 \int dx \sum_{\omega_1\omega_2} e^{ip_F(\omega_1-\omega_2)\vec x}\psi^{+(\leq 0)}_{\omega_1 x} i\beta\mathcal{D}^-_{\omega_2}\psi^{-(\leq 0)}_{\omega_2 x} + \\
+ \zeta_0 \int dx \sum_{\omega_1\omega_2} e^{ip_F(\omega_1-\omega_2)\vec x}\psi^{+(\leq 0)}_{\omega_1 x}\partial_t\psi^{-(\leq 0)}_{\omega_2 x}
\end{aligned} \tag{4.33}$$

for suitably chosen $\lambda_0, \nu_0, \alpha_0, \zeta_0$, and:

$$\mathcal{R}\bar V^{(0)}(\psi^{\leq(0)}) = \sum_{n=1}^{\infty}\sum_{\rho\in I_n}\int d\underline{x}\bar W_\rho(z,\underline{x})M_\rho(\psi^{\leq(0)}) \tag{4.34}$$

Here $I_n$ is the finite set of different monomials of the form:

$$M_\rho(\psi) = (\prod_{i=1}^{n}\Phi^+_i)(\prod_{j=1}^{n}\Phi^-_j) \tag{4.35}$$

where $\Phi^+_i$ has to be chosen between the fields (see (4.21)):

$$e^{ip_F\omega\vec x}\psi^+_{\omega x} \quad , \quad e^{ip_F\omega\vec x_2}\int_0^1 dr\partial\psi^+_{\omega x_{21}(r)} \tag{4.36}$$

and $\Phi_j^-$ has to be chosen between the fields:

$$\begin{aligned}&e^{-ip_F\omega\vec{x}}\psi^-_{\omega x}\ ,\ e^{-ip_F\omega\vec{x}}\mathcal{D}_\omega\psi^-_{\omega x}\ ,\ (\frac{-i\omega}{2p_F})e^{-ip_F\omega\vec{x}_2}\partial^2_{\vec{x}}\psi^-_{\omega x_1}\ ,\ e^{-ip_F\omega\vec{x}_2}\int_0^1 dr\partial\psi^-_{\omega x_{21}(r)}\\ &e^{-ip_F\omega\vec{x}_2}\int_0^1 dr\mathcal{D}_\omega\partial\psi^-_{\omega x_{21}(r)}\ ,\ e^{-ip_F\omega\vec{x}_2}\int_0^1 dr\int_0^r ds\partial^2\psi^-_{\omega x_{21}(s)}\end{aligned} \tag{4.37}$$

Moreover, in (4.34) $\underline{x}$ represents the set of points appearing as labels of the fields in the monomial $M_\rho$.

**Remark**: the running couplings $\lambda_0$, $\nu_0$, $\alpha_0$ and $\zeta_0$ are in fact convergent series of the *bare couplings* $z = (\lambda, \nu, \alpha)$, uniformly in the u.v. cutoff $N$. This follows from (3.18) for the contributions coming from $\bar{W}_{n_1 n_2}$, with $n_1 + n_2 = 2$ or 4, but there is, at first sight, a problem for the contributions to $\nu_0$, $\alpha_0$ and $\zeta_0$, coming from $\tilde{W}_2$. However we can use here (3.20), which implies, for example, that the contribution of $\tilde{W}_2$ to $\alpha_0$ is:

$$-2m\int dx e^{ip_F\vec{x}}\tilde{W}_2(x) = 2m\int dx[1-e^{ip_F\vec{x}}]\tilde{W}_2(x) \tag{4.38}$$

which can be bounded by:

$$2mp_F\int dx|\vec{x}||\tilde{W}_2(x)| \tag{4.39}$$

a finite bound uniformly in $N$ by (3.19).

It is also important to stress that, by (3.18) and (3.19), the kernels of (4.34) are convergent series of $z$, which satisfy for $|z|$ small enough the bound:

$$\int d\underline{x}|\bar{W}_\rho(z,\underline{x})|e^{\frac{\kappa}{2}d^{(0)}(\underline{x})} \le |\Lambda|(C|z|)^{\max\{1,n-1\}} \tag{4.40}$$

and the power on the r.h.s. can be really 1 only in the case of the term coming from the action of $\mathcal{R}$ on the first term of (4.8).

The first order in the bare constants gives:

$$\begin{aligned}\lambda_0 &= 2\lambda\int d\vec{x}\bar{v}(\vec{x})[1-\cos(2p_F\vec{x})]\\ \nu_0 &= \nu + 2\lambda\int d\vec{x}\bar{v}(\vec{x})[e^{ip_F\vec{x}}R(0,\vec{x})-R(0)]\\ \alpha_0 &= \alpha + 2\lambda\frac{i}{\beta}\int d\vec{x}\bar{v}(\vec{x})R(0,\vec{x})\vec{x}e^{ip_F\vec{x}}\ ,\qquad \zeta_0 = 0\end{aligned} \tag{4.41}$$

We can now start the inductive evaluation of (4.11), by applying at each step the localization operator to the effective potential. We will obtain for $\mathcal{L}V^{(h)}$ a formula like (4.33), with $\lambda_0,\ \nu_0,\ \alpha_0,\ \zeta_0$ replaced by $(\lambda_h, \gamma^h\nu_h, \alpha_h, \zeta_h)$; and $(\lambda_h, \nu_h, \alpha_h, \zeta_h) \equiv r_h$ will be called the *running coupling constants of frequency h*. The $r_h$ can be expressed as a series of the running coupling constants of frequencies $k \geq h+1$, i.e. $r_{h+1} \dots r_0$. We could show that this series, called the *beta functional*, is convergent if all the running coupling constants $r_{h+1}, \dots, r_0$ stay bounded within a certain radius of convergence, and we could show as well that the irrelevant part of the effective potential can be written as a convergent series of $r_{h+1}, \dots, r_0$ (for a general discussion on the beta-functional see for example [G]).

Of course, in order to use this result, we would also have to prove that the running constants *really do stay bounded*, at least if the bare constants are small enough. However, if we try to pursue this program, we immediately find a difficulty. In fact, if we calculate the beta functional at second order, we find ([BG], [G]):

$$\begin{aligned} \lambda_{h-1} &= \lambda_h \\ \alpha_{h-1} &= \alpha_h + \beta_2 \lambda_h^2 + O(\gamma^h) \\ \zeta_{h-1} &= \zeta_h + \beta_2 \lambda_h^2 + O(\gamma^h) \end{aligned} \tag{4.42}$$

with $\beta_2 \neq 0$.

The latter equations imply that, at the second order, $\lambda_h$ neither does increase nor does decrease; so we need the third order to decide what happens to $\lambda_h$. However, even if we suppose that the third order for $\lambda_h$, once calculated, will imply that $\lambda_h$ goes to zero when $h \to -\infty$, the best that we can hope to find for its behaviour is clearly a rate $\sqrt{1/|h|}$. Looking at second order equations for $\alpha_h$ and $\zeta_h$, this implies that $\alpha_h$ and $\zeta_h$ go to infinity at least as $\sum_h 1/|h|$, *i.e.* we get out of the established domain of convergence of the beta functional in a finite number of steps.

¿From the mathematical point of view this is a big trouble, because it makes impossible to construct a perturbation theory for the model; from the physical point of view this means, as it is well known, that the expectation of the number of particles with fixed momentum, in the one dimensional Fermi gas, has a singularity, at the Fermi momenta $\pm p_F$, of a different kind with respect to the free case, where it is simply discontinuous.

Hence we need to introduce a different type of scaling, allowing us to study the nature of the singularity on the Fermi surface via a consistent perturbation theory.