The Maxwell-Dirac equations : Asymptotic completeness and the infrared problem

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Dedicated to Elliott Lieb with our affection.

Abstract : In this article we present an announcement of results concerning :

a) A solution to the Cauchy problem for the M-D equations, namely global existence, for small initial data at $t = 0$, of solutions for the M-D equations.

b) Arguments from which asymptotic completeness for the M-D equations follows.

c) Cohomological interpretation of the results in the spirit of nonlinear representation theory and its connection to the infrared tail of the electron in M-D classical field theory.

The full detailed results will be published elsewhere.

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1. Introduction : statement of the problem.

The classical Maxwell-Dirac (M-D) equations in $1+3$ dimensions read in conventional notations (electron charge $e=1$; Dirac matrices γ^μ, $0\leq\mu\leq 3$; metric tensor $g^{\mu\nu}$, $g^{00}=1$, $g^{ii}=-1$ for $1\leq i\leq 3$ and $g^{\mu\nu}=0$ for $\mu\neq\nu$, ; $\gamma^\mu\,\gamma^\nu+\gamma^\nu\,\gamma^\mu=2g^{\mu\nu}$; $\square=\partial_\mu\,\partial^\mu$) :

$$\square\, A_\mu=\bar{\psi}\,\gamma_\mu\,\psi,\ 0\leq\mu\leq 3, \tag{1.1a}$$

$$(i\gamma^\mu\,\partial_\mu+m)\,\psi=A_\mu\,\gamma^\mu\,\psi,\ m>0, \tag{1.1b}$$

$$\partial_\mu\,A^\mu=0, \tag{1.1c}$$

where $\bar{\psi}=\psi^+\,\gamma^0$, ψ^+ being the Hermitian conjugate of the Dirac spinor ψ.

Let us introduce some notations : Let E be a Hilbert space of vectors $(f,\dot{f},\alpha)$ where the functions $f:\mathbb{R}^3\to\mathbb{R}^4$, $\dot{f}:\mathbb{R}^3\to\mathbb{R}^4$, $\alpha:\mathbb{R}^3\to\mathbb{C}^4$ belong to spaces to be specified in paragraph 2. M (resp. D) is the space of vectors $(f,\dot{f})$ (resp. α) for $(f,\dot{f},\alpha)\in E$. As will be explained in detail in paragraph 2 there is a linear strongly continuous representation U^1 of the Poincaré group in E, which is defined by the linear part of equations (1.1a) and (1.1b) and $U^1=U^{M1}\oplus U^{D1}$ where U^{M1} acts on M and U^{D1} on D. E_∞ denotes the space of differentiable vectors in E for the representation U^1. It contains vectors $(f,\dot{f},\alpha)$, such that f is a long range potential decreasing as $(1+|x|)^{\rho-3/2}$, $x\in\mathbb{R}^3$ and $\dot{f}$ decreasing as $(1+|x|)^{\rho-5/2}$ for some $1/2<\rho<1$. We shall consider solutions of equations (1.1a), (1.1b) and (1.1c) which are in E_∞ in the sense that $(A(t),\dot{A}(t),\psi(t))\in E_\infty$ at time t, where $\dot{A}_\mu(t)=\frac{d}{dt}\,A_\mu(t)$.

It is well-known that the construction of the observables on the Fock space of QED (Quantum Electrodynamics) requires infrared corrections to eliminate the infrared divergencies in the perturbative expression of the quantum scattering operator. These corrections are introduced by hand with the purpose to give *a posteriori* a finite theory. In this paper we shall announce rigorous results which we have obtained concerning the infrared problem for the (classical) M-D equations. Our belief is that such results can *a priori* be of interest for QED, especially for the infrared regime and combined with the deformation-quantization approach [1], [2], [3].

On the classical level the infrared problem consists of determining to which extent the long range interaction created by the coupling $A^\mu\,j_\mu$ between the electromagnetic potential A_μ and the electric current $j_\mu=\bar{\psi}\,\gamma_\mu\,\psi$ is an obstruction for the separation, when $|t|\to\infty$, of the nonlinear relativistic system into two asymptotic isolated relativistic systems, one for the electromagnetic potential A_μ and one for the Dirac field ψ. It will be seen in this article that there is such an obstruction, which in particular implies that asymptotic in and out states do not transform according to a linear representation of the Poincaré group. This constitutes a serious problem for the second quantization of the asymptotic (in and outgoing) fields, since the particle interpretation usually requires free relativistic fields, i.e. at least a linear representation of the Poincaré group (U^1 in our case). Therefore we have to introduce nonlinear representations U^{as} of the Poincaré group $\mathcal{P}_0$, in the sense of [4], which can give the transformation of the asymptotic in and out states under $\mathcal{P}_0$ and which can permit a particle interpretation.

There are two reasons which permit to determine the class of asymptotic representations U^{as}. First, the classical observables, 4-current density, 4-momentum and 4-angular momentum are invariant under gauge transformations. Second, if the evolution equations of the asymptotic fields become linear after a gauge transformation one can use the freedom of gauge in the second quantization of the fields. It is therefore reasonable to postulate that asymptotic representations $g \mapsto U_g^{as}$ are linear modulo a nonlinear gauge transformation depending on g and respecting the Lorentz gauge condition. Since we compare solutions of the M-D system and the asymptotic system only for large time we can from a technical point of view, using stationary phase methods, define the asymptotic representations in the following way:

$$U_g^{as}(u) = (U_g^{asM}(u), U_g^{asD}(u)),\ U^{asM} = U^{M1}, \tag{1.2a}$$

$$(U_g^{asD}(u))^\wedge\,(k) = \sum_{\varepsilon=\pm} e^{i\varphi_g^\varepsilon(u,k)}\ P_\varepsilon(k)\ (U_g^{D1}\ \alpha)^\wedge\,(k),\ g \in \mathcal{P}_0, \tag{1.2b}$$

where $u = (f, \dot{f}, \alpha) \in E$, P_ε is the orthogonal projector in D on the subspace of initial of data with energy sign $\varepsilon = \pm$, the symbol $\wedge$ denotes Fourier transform and the function $(g, u, k) \mapsto \varphi_g^\varepsilon(u, k)$ from $\mathcal{P}_0 \times E_\infty \times \mathbb{R}^3$ to $\mathbb{R}$ is C^∞ . We shall call such nonlinear representations "gauge projectively linear".

The nonlinear relativistic system is defined by a nonlinear representation T of the Poincaré Lie algebra $\mathbf{p}$ (in the sense of [4]) on a manifold $V_\infty \subset E_\infty$ of initial conditions satisfying the Lorentz gauge condition, which in the case of the M-D system (1.1a), (1.1b) and (1.1c) is in fact a reformulation of the relativistic covariance of the system. If $g \mapsto U_g$ is a nonlinear representation of the Poincaré group $\mathcal{P}_0$, with the corresponding Lie algebra representation T we can conclude that the classical infrared problem consists of determining gauge projectively linear representations $U^{(+)}$ and $U^{(-)}$ equivalent to U by the (modified) wave operators Ω_+ and Ω_-, i.e.

$$U_g^{(\varepsilon)} = \Omega_\varepsilon^{-1}\ \circ\ U_g\ \circ\ \Omega_\varepsilon,\ g \in \mathcal{P}_0,\ \varepsilon = \pm, \tag{1.3a}$$

where $U_{exp(tP_0)}\ (\Omega_\varepsilon(u))$, P_0 being the time evolution generator in $\mathcal{P}_0$, converges in a sense that we shall specify to $U^{(\varepsilon)}_{exp(tP_0)}\ (u)$ as $\varepsilon t \to \infty$, $\varepsilon = \pm$. What is important is that the limit takes place for the observables, i.e. at least modulo gauge transformations (even not respecting the Lorentz condition). Therefore we impose that, for $u = (f, \dot{f}, \alpha) \in E_\infty$ and satisfying the Lorentz condition for the linear part of the M-D equations :

$$\lim_{\eta t \to \infty} (\|U^M_{exp(tP_0)}\ (\Omega_\eta(u)) - U^{M1}_{exp(tP_0)}\ (f, \dot{f})\|_M \tag{1.3b}$$

$$+\|U^D_{exp(tP_0)}\ (\Omega_\eta(u)) - \sum_{\varepsilon=\pm} e^{is_\varepsilon^{(\eta)}\ (u,t,-i\partial)}\ P_\varepsilon\ (-i\partial)\ U^{(\eta)D}_{exp(tp_0)}(u)\|_D) = 0,$$

$\eta = \pm$, where $(u, t, k) \mapsto s_\varepsilon^{(\eta)}\ (u, t, k)$ is a real C^∞ function and $s_\varepsilon^{(\eta)}\ (u, t, -i\partial)$ is the operator defined by the inverse Fourier transform of the multiplication operator $k \mapsto s_\varepsilon^{(\eta)}\ (u, t, k)$. We also impose growth conditions on the phase $s_\varepsilon^{(\eta)}$, so that the major contribution comes from $U^{(\eta)D}$:

$$|s_\varepsilon^{(\eta)}\ (u, t, k)| \le C\ w(k)\ (1 + |t|)^\delta,\ |\frac{d}{dt}\ s_\varepsilon^{(\eta)}\ (u, t, k)| \le C\ w(k)\ (1 + |t|)^{\delta - 1}, \tag{1.3c}$$

where $w(k) = (m^2 + |k|^2)^{1/2}$ and C is a constant depending on u. We note that the equivalence (1.3a) requires the existence of Ω_η and its inverse, i.e. asymptotic completeness. For future reference let us denote

$$W_t^{(\eta)}(u) = \sum_{\varepsilon=\pm} e^{i\, s_\varepsilon^{(\eta)}\, (u,t,-i\partial)}\, P_\varepsilon(-i\partial). \tag{1.3d}$$

Next we shall study the cohomological interpretation of conditions (1.3a) and (1.3b). We only consider the case $t \to \infty$. A necessary condition for $U^{(+)}$ and Ω_+ to be a solution of equation (1.3a) is that the formal power series development of $U_g^{(+)}$, U_g and Ω_+ in the initial conditions satisfy the cohomological equations defined in [4] and [5]. In particular (after trivial transformations) the second order terms $U_g^{(+)2}$, U_g^2 and Ω_+^2 shall satisfy

$$\delta\, \Omega_+^2 = C^2,\ \delta\, R^{(+)2} = 0 \tag{1.4}$$

where δ is the coboundary operator defined by the representation $g \mapsto U_g^1\, A^2\, (\otimes^2\, U_{g^{-1}}^1)$ of the Poincaré group $\mathcal{P}_0$ on bilinear maps, $C^2 = R^{(+)2} - R^2$ is the cocycle defined by $R_g^2 = U_g^2\, (U_{g^{-1}}^1 \otimes U_{g^{-1}}^1)$ and $R_g^{(+)2} = U_g^{(+)2}\, (U_{g^{-1}}^1 \otimes U_{g^{-1}}^1)$, $g \in \mathcal{P}_0$. Equation (1.4) shows that the cochain $R^{(+)2}$ has to be a cocycle and then that the cocycle C^2 has to be a coboundary. This is equivalent to the existence of a solution $U^{(+)}$, Ω_+ modulo terms of order at least three. There are similar equations for higher order terms

$$\delta\, \Omega_+^n = C^n,\ \delta\, R^{(+)n} = 0, \tag{1.5}$$

where C^n and $R^{(+)n}$ are functions of $\Omega_+^2, ..., \Omega_+^{n-1}$ and $U^{(+)2}, ..., U^{(+)n}$. In a previous paper [6] we proved that there exists a modified wave operator and global solutions of the M-D equations for a set of scattering data $(f, \dot{f}, \alpha)$ satisfying $\hat{f}_\mu(k) = \hat{\dot{f}}_\mu(k) = 0$ for k in a neighbourhood of zero, i.e. f_μ and $\dot{f}_\mu$ have no low frequencies component. It follows from that paper that the usual wave operator (i.e. $U^{(+)} = U^1$ and $s_\varepsilon^{(+)} = 0$ in (1.3b)) does not exist, even in this case where there are no low frequencies. In fact there is an obstruction to the existence of such a solution of equation (1.5) for $n = 3$ due to the self-coupling of ψ with the electromagnetic potential created by the current $\bar{\psi}\, \gamma_\mu\, \psi$. However, as was proved in [6] there exists a modified wave operator Ω_+ satisfying (1.5) for $n \geq 2$, with $U^{(+)} = U^1$. Moreover it was proved that the phase functions $s_\varepsilon^{(+)}$ are given by (see formulas (1.5), (3.28) and (3.33) of [6])

$$s_\varepsilon^{(+)}\, (u,t,k) = S_\varepsilon(u,t,k) - \varepsilon\, w(k)\, t, \tag{1.6}$$

where S_ε is a certain approximate solution of the Hamilton-Jacobi equation for a relativistic electron in an external electromagnetic potential :

$$(\frac{\partial}{\partial t}\, \mathcal{S}_\varepsilon(u,t,k) + A_0\, (t, -\nabla_k\, \mathcal{S}_\varepsilon(u,t,k)))^2 - \sum_{i=1}^{3}\, (k_i + A_i(t, -\nabla_k\, \mathcal{S}_\varepsilon(u,t,k)))^2 = m^2. \tag{1.7}$$

With the spaces defined in [6] it was possible to choose (formula (3.33a) of [6])

$$S_\varepsilon(u,t,k) = \varepsilon t w(k) - \theta(A^{(+)}, t, -\varepsilon t k w(k)^{-1}), \tag{1.8}$$

where $w(k) = (m^2 + |k|^2)^{1/2}$, and $\theta(A^{(+)}, y) = \int_{L(y)} A_\mu^{(+)}(z)\, dz^\mu$,

$$L(y) = \{z \in \mathbb{R}^4 | z = sy,\ 0 \leq s \leq 1\},$$

$A^{(+)}$ being a certain approximate solution of equation (1.1a) containing the long range part of A. Thus in the absence of low frequencies we can choose Ω_+ such that it intertwines the linear and the nonlinear representation of $\mathcal{P}_0$. Now if $f_\mu, \dot{f}_\mu$ have a non-zero low frequency part, as is the case of the Coulomb potential ($\hat{f}_\mu(k) \sim |k|^{-2}$) then there is a cohomological obstruction already for $n = 2$ if $U^{(+)2} = 0$. The essential point now is that the cocycle R^2 can be split into a trivializable part $-C^2$ (a coboundary) and into a nontrivial part $R^{(+)2}$ which defines $U^{(+)2}$, and therefore the whole representation $U^{(+)}$. We shall call this nontrivial part the infrared cocycle. As we shall see in paragraph 2, $U_g^{(+)}$ is trivial on the Lorentz subgroup $SL(2,\mathbb{C})$ (i.e. $\varphi_g^\varepsilon = 0$ for $g \in SL(2,\mathbb{C})$ in (1.2b)). Moreover denoting by P_μ, $0 \leq \mu \leq 3$, the translation generators in $\mathbf{p} = \mathbb{R}^4 + \mathbf{sl}(2,\mathbb{C})$ and denoting $\tilde{T}_{P_\mu}^{(+)D}(u) = \frac{d}{ds}\, U^{D1}_{exp(-sP_\mu)}\, U^{(+)D}_{exp(sP_\mu)}(u)|_{s=0}$, we have for $u = (f, \dot{f}, \alpha)$:

$$(\tilde{T}_{P_\mu}^{(+)D}(u))^\wedge(k) = -i \sum_{\varepsilon=\pm} \theta^\infty\, (\chi' \partial_\mu\, B,\ l_\varepsilon)\, P_\varepsilon(k)\, \hat{\alpha}(k), \tag{1.9a}$$

where

$$\theta^\infty\, (H, y) = \int_0^\infty y^\nu\, H_\nu(\tau y)\, d\tau,\ y \in \mathbb{R}^4,\ H = (H_0, H_1, H_2, H_3), \tag{1.9b}$$

where B_ν is the free field with data $f_\nu, \dot{f}_\nu$, $l_\varepsilon = (w(k), -\varepsilon k)$, $k \in \mathbb{R}^3$ is the on-shell four-momentum and $z \mapsto \chi'(z)$ a cutoff function equal to zero for $(z_0^2 - (z_1^2 + z_2^2 + z_3^2))^{1/2} \leq 1$ and equal to one for $(z_0^2 - (z_1^2 + z_2^2 + z_3^2))^{1/2} \geq 2$.

Let S_ε, $\varepsilon = \pm$ be the functions defined by

$$e^{iS_\varepsilon(u,t,-i\partial)}\, P_\varepsilon(-i\partial)\alpha = e^{is_\varepsilon^{(+)}\,(u,t,-i\partial)}\, P_\varepsilon\,(-i\partial)\, U^{(+)D}_{exp(tP_0)}(u), \tag{1.10}$$

where $u = (f, \dot{f}, u)$. Then S_ε is given by (1.8). When the limit $\theta(\chi' B, t, -\varepsilon k\, w(k)^{-1} t) \to \theta^\infty\, (\chi' B, l_\varepsilon)$, $l_\varepsilon = (w(k), -\varepsilon k)$ exists for $t \to \infty$, then the infrared cocycle is in fact a coboundary. In particular, this is the case when $\hat{f}_\mu, \hat{\dot{f}}_\mu$ are equal to zero in a neighbourhood of zero.

In paragraph 2 we shall outline the proof of the facts mentioned previously as well as that of the existence of the nonlinear representation $g \mapsto U_g$ of the Poincaré group on an invariant domain of initial data $\mathcal{O}_0$, that of the existence of modified wave operators $\Omega_\pm : \mathcal{O}_\pm \to \mathcal{O}_0$ and that of asymptotic completeness. To prove these facts we construct as in [6], [9], [10], [11] an approximate solution of the nonlinear evolution equation for a given scattering data $(f, \dot{f}, \alpha)$. The enveloping algebra techniques in [11] and new energy estimates and $L^2 - L^\infty$ estimates for the Dirac equation permit to prove that the rest term (difference between what will be the solution and the approximate solution) exists and is a decreasing function of time in L^2 spaces. We obtain a solution of the M-D equations with scattering data $u_+ = (f, \dot{f}, \alpha)$. The corresponding initial data will be denoted by u. The seminorms of u considered as a C^n-vector of U^1 are proved to be bounded by the norms of $T_Y(u),\ Y \in \mathcal{U}(\mathbf{p})$, where $T_Y(u)$ is the extension of $T_X(u),\ X \in \mathbf{p}$ to the enveloping algebra $\mathcal{U}(\mathbf{p})$ defined in [11]. We can now conclude that modified wave operators $\Omega_\varepsilon : \mathcal{O}'_\varepsilon \to \mathcal{O}'_0,\ \varepsilon = \pm$ exist on small open neighbourhoods $\mathcal{O}'_\varepsilon,\ \varepsilon = \pm$ of the scattering data space which is here the Fréchet space of C^∞-vectors of U^1. The maps Ω_ε are proved to satisfy the hypothesis of the implicit functions theorem for Fréchet spaces. This proves that $\mathcal{O}'_\varepsilon,\ \varepsilon = \pm$ and $\mathcal{O}_0$ can be chosen such that Ω_ε are invertible. This gives the asymptotic completeness and the existence of U_g on a Poincaré invariant domain. The explicit expression of $U^{(+)}$ is found by a suitable choice of $W^{(+)}(u)$ in (1.3d).

Before stating the results and outlining the proofs in paragraph 2 we make the following remarks :

i) It can be natural to think of the underlying classical theory of QED not as the M-D equations with c-number spinor components but as a theory with anticommuting spinor components. Since the second order terms in the spinor component of the wave operator originates in the coupling $A_\mu\, \psi$ of a c-number free electromagnetic potential A_μ and a free Dirac-field, we believe that the infrared cocycle will remain the same for a theory with anticommuting spinor components. Further for higher order terms, the nilpotency property $\psi_\alpha(x)^2 = 0$ can ameliorate the infrared problems arising from the self-coupling.

ii) The observables, 4-current, 4-momentum and 4-angular momentum, defined for the asymptotic representation $U^{(+)}$, which is "gauge projectively linear", converge when $t \to \infty$ to the usual free field observables. Therefore there should be no observable phenomenas which distinguish $U^{(+)}$ from U^1, at least as far as these observables are concerned. Since the representation U is equivalent to $U^{(+)}$ by Ω_+, we shall call U a "gauge projectively linearizable" nonlinear representation. Of course if the Dirac field itself was an observable, then this would no longer be true, i.e. it would then be possible to distinguish $U^{(+)}$ and U^1 by the observables.

iii) One should note that the phase-factor (3.33a of [6]), which looks as a very familiar factor in abelian and nonabelian gauge theories, was obtained in [6] in the different context of the Hamilton-Jacobi equation associated with the full Maxwell-Dirac equations. Would one have taken initial data for the A_μ field decreasing slower than $r^{-1/2}$ and suitable initial data for $\frac{d}{dt}A_\mu$, one would obtain phase factors with higher powers of A_μ.

2. Statement of results and outline of proofs.

Let $M^\rho,\ -1/2 < \rho < \infty$ be the completion of $S(\mathbb{R}^3, \mathbb{R}^4) \oplus S(\mathbb{R}^3, \mathbb{R}^4)$ with respect to the norm

$$\|(f, \dot{f})\|_{M^\rho} = (\||\nabla|^\rho f\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)} + \||\nabla|^{\rho-1}\dot{f}\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)})^{1/2}. \tag{2.1a}$$

M^ρ is a Hilbert space. Let $D = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and let $E^\rho = M^\rho \oplus D,\ -1/2 < \rho < \infty$, be the Hilbert space with norm

$$\|(f, \dot{f}, \alpha)\|_{E^\rho} = (\|(f, \dot{f})\|^2_{M^\rho} + \|\alpha\|^2_D)^{1/2}. \tag{2.1b}$$

When there is no possibility of confusion we write E (resp. M) instead of E^ρ (resp. M^ρ) for $1/2<\rho<1$. $M^{o\rho}$ is the closed subspace of elements $(f,\dot f)\in M^\rho,\ -1/2<\rho<\infty$ such that

$$\dot f_0=\sum_{1\leq i\leq 3}\partial_i\, f_i,\ f_0=-\sum_{1\leq i\leq 3}|\nabla|^{-2}\,\partial_i\,\dot f_i. \tag{2.2c}$$

A solution B_μ of $\Box B_\mu=0,\ 0\leq\mu\leq 3$ with initial conditions $(f,\dot f)\in M^\rho$ satisfies the gauge condition $\partial_\mu\, B^\mu=0$ if and only if $(f,\dot f)\in M^{o\rho}$.

Let $\Pi=\{P_\mu,\ M_{\alpha\beta}|0\leq\mu\leq 3,\ 0\leq\alpha<\beta\leq 3\}$ be a standard basis of the Poincaré Lie algebra $\mathbf{p}=\mathbb{R}^4+\mathbf{sl}(2,\mathbb{C})$. P_0 is the time translation generator, $P_i,\ 1\leq i\leq 3$ the space translation generators, $M_{ij},\ 1\leq i<j\leq 3$ the space rotation generators and $M_{oj},\ 1\leq j\leq 3$ the boost generators. We define $M_{\alpha\beta}=-M_{\beta\alpha}$ for $0\leq\beta\leq\alpha\leq 3$. There is a linear (strongly) continuous representation U^1 of $\mathcal{P}_0=\mathbb{R}^4\ .\ SL(2,\mathbb{C})$ in $E^\rho,\ -1/2<\rho<\infty$ with differentiable vectors E^ρ_∞ such that its differential is the following linear representation T^1 of $\mathbf{p}$ on E^ρ_∞ :

$$T^1_{P_0}\,(f,\dot f,\alpha)=(\dot f,\Delta f,\mathcal{D}\alpha),\Delta=\sum_{1\leq i\leq 3}\partial_i^2,\ \mathcal{D}=-\sum_{1\leq j\leq 3}\gamma^0\gamma^j\partial_j+i\gamma^0\,m\ ; \tag{2.3a}$$

$$T^1_{P_i}(f,\dot f,\alpha)=\partial_i(f,\dot f,\alpha),\ 1\leq i\leq 3\ ; \tag{2.3b}$$

$$T^1_{M_{ij}}(f,\dot f,\alpha)\,(x)=(x_i\partial_j-x_j\partial_i)\,(f,\dot f,\alpha)\,(x)+(n_{ij}f,\ n_{ij}\dot f,\ \sigma_{ij}\,\alpha)\,(x), \tag{2.3c}$$
$$1\leq i<j\leq 3,\ \sigma_{ij}=-1/2\ \gamma_i\gamma_j\in\mathbf{su}(2),\ n_{ij}\in\mathbf{so}(3)\ ;$$

$$T^1_{M_{oi}}(f,\dot f,\alpha)\,(x)=(x_i\,\dot f(x),\ \sum_{j=1}^{3}\partial_j x_i\,\partial_j f(x),\ x_i\mathcal{D}\alpha(x))+(n_{oi}\,f,\ n_{oi}\,\dot f,\ \sigma_{oi}\alpha)(x), \tag{2.3d}$$
$$1\leq i\leq 3,\ \sigma_{oi}=\frac{1}{2}\,\gamma_0\,\gamma_i\in\mathbf{sl}(2,\mathbb{C}),\ n_{oi}\in\mathbf{so}(3,1).$$

The explicit form of $n_{\alpha\beta},\ 0\leq\alpha<\beta\leq 3$ defining a vector representation of $\mathbf{p}$ on $\mathbb{R}^4$ is not important in this paper. We note that U^1 leaves $E^{o\rho}=M^{o\rho}\oplus D$ invariant.

Following closely [11] we introduce the graded sequence of Hilbert spaces $E^\rho_i,\ i\geq 0,\ E^\rho_0=E^\rho$, where $E^\rho_\infty\subset E^\rho_j\subset E^\rho_i$ for $i\leq j$. E^ρ_i is the space of C^i-vectors of the representation U^1. Suppose given an ordering $X_1<X_2<...<X_{10}$ on Π. Then, in the universal enveloping algebra $\mathcal{U}(\mathbf{p})$ of $\mathbf{p}$, the subset of all products $X_1^{\alpha_1}...X_{10}^{\alpha_{10}},\ 0\leq\alpha_i,\ 1\leq i\leq 10$ is a basis Π' of $\mathcal{U}(\mathbf{p})$. If $Y=X_1^{\alpha_1}...X_{10}^{\alpha_{10}}\in\Pi'$, we define $|Y|=|\alpha|=\sum_{0\leq i\leq 10}\alpha_i$. Let $E^\rho_i,\ i\in\mathbb{N}$ be the completion of E^ρ_∞ with respect to the norm

$$\|u\|_{E^\rho_i}=(\sum_{Y\in\Pi',\ |Y|\leq i}\|T^1_Y(u)\|^2_{E^\rho})^{1/2}, \tag{2.4}$$

where $T^1_Y,\ Y\in\mathcal{U}(\mathbf{p})$ is defined by the canonical extension of T^1 to the enveloping algebra $\mathcal{U}(\mathbf{p})$ of $\mathbf{p}$. Let $M^{o\rho}_i=M^\rho_i\cap M^{o\rho},\ E^{o\rho}_i=E^\rho_i\cap E^{o\rho}$ and similarly for $E^{o\rho}_\infty,\ M^{o\rho}_\infty$.

To understand better what the elements of the spaces E^ρ_i are we introduce the following seminorms $q_n, \ n \geq 0$, on E^ρ_∞ :

$$q_n(u) = (q^M_n(v)^2 + q^D_n(\alpha)^2)^{1/2}, \ u = (v, \alpha) \in E^\rho_\infty,$$

$$v \in M^\rho_\infty, \ \alpha \in D_\infty,$$

$$q^M_n(v) = (\sum_{|\mu| \leq |\nu| \leq n} \|M_\mu \ \partial^\nu v\|^2_{M^\rho})^{1/2} \tag{2.5a}$$

and

$$q^D_n(\alpha) = (\sum_{|\mu| \leq n, \ |\nu| \leq n} \|M_\mu \ \partial^\nu \ \alpha\|^2_D)^{1/2}, \tag{2.5b}$$

where $\mu = (\mu_1, \mu_2, \mu_3), \ \nu = (\nu_1, \nu_2, \nu_3)$ are multi-indices and $M_\mu(x) = x^\mu = x_1^{\mu_1} \ x_2^{\mu_2} \ x_3^{\mu_3}$.

We have the following result, which is obtained by a close study of the explicit expression for $T^1_Y, \ Y \in \mathcal{U}(\mathbf{p})$:

Proposition 2.1 : *There are constants $C_n > 0$ such that*

$$C_n^{-1} \ \|u\|_{E^\rho_n} \leq q_n(u) \leq C_n \ \|u\|_{E^\rho_n}, \ n \geq 0.$$

The decrease properties of elements $u = (f, \dot{f}, \alpha) \in E^\rho_\infty$ follow from this proposition; $\alpha \in S(\mathbb{R}^3, \mathbb{C}^4)$ and for $1/2 < \rho < 1$ we obtain using Sobolev estimates :

Proposition 2.2 : *If $(f, \dot{f}) \in M^\rho_\infty, \ 1/2 < \rho < 1$ then*
i) $(1 + |x|)^{3/2-\rho} \ |f(x)| \leq C \ \|(f, 0)\|_{M^\rho_1}$,
ii) $(1 + |x|)^{5/2-\rho+|\nu|} \ (|\partial^\nu \partial_i \ f(x)| + |\partial^\nu \dot{f}(x)|) \leq C_{|\nu|} \ \|(f, \dot{f})\|_{M^\rho_{|\nu|+2}}, |\nu| \geq 0,$
iii) If in addition $x^\alpha \partial^\beta \partial_i f \in L^p$ and $x^\alpha \partial^\beta \dot{f} \in L^p, \ p = 6(5 - 2\rho)^{-1}, \ |\alpha| \leq |\beta| \leq n$, $1 \leq i \leq 3$ then

$$\|(f, \dot{f})\|_{M^\rho_n} \leq C_n \sum_{0 \leq |\alpha| \leq |\beta| \leq n} (\sum_{0 \leq i \leq 3} \|x^\alpha \partial^\beta \partial_i f)\|_{L^p} + \|x^\alpha \partial^\beta \dot{f}\|_{L^p}).$$

This proposition shows in particular that long range potentials satisfying

$$|\partial^\nu \ f(x)| \leq C_\nu \ (1 + |x|)^{\rho-3/2-|\nu|-\varepsilon}, \ |\partial^\nu \ \dot{f}(x)| \leq C_\nu \ (1 + |x|)^{\rho-5/2-|\nu|-\varepsilon} \tag{2.6}$$

for all $|\nu| \geq 0$ and some $\varepsilon > 0$, are in $M^\rho_\infty, \ 1/2 < \rho < 1$.

A nonlinear representation T of $\mathbf{p}$ on E^ρ_∞, in the sense of [4], is obtained by the fact that the M-D equations are manifestly covariant :

$$T_X = T^1_X + T^2_X, \ X \in \mathbf{p}, \tag{2.7}$$

where T^1 is given by (2.3a) - (2.3d) and for $u = (f, \dot{f}, \alpha) \in E^\rho_\infty$:

$$T^2_{P_0}(u) = (0, \bar{\alpha} \ \gamma \ \alpha, \ -i \ f_\mu \ \gamma^0 \ \gamma^\mu \ \alpha), \ \gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3); \tag{2.8a}$$

$$T^2_{P_i} = 0,\ 1 \leq i \leq 3\ ;\ T^2_{M_{ij}} = 0,\ 1 \leq i < j \leq 3\ ; \tag{2.8b}$$

$$T^2_{M_{oj}}(u) = x_j\ T^2_{P_0}(u),\ 1 \leq i \leq 3. \tag{2.8c}$$

The gauge condition (1.1c) takes on initial data $u = (f, \dot{f}, \alpha)$ the form (cf. [7])

$$\dot{f}_0 = \sum_{i=1}^{3} \partial_i f_i,\ \Delta f_0 = \sum_{i=1}^{3} \partial_i \dot{f}_i - |\alpha|^2. \tag{2.8d}$$

Let V^ρ_N be the subset of elements in E^ρ_N satisfying (2.8d). It is easy to establish the following result :

Proposition 2.3 : *There is a continuous polynomial of degree two $F : E^\rho_N \to E^\rho_N$, $1/2 < \rho < 1$, $N \geq 1$, which is a diffeomorphism from E^ρ_N onto E^ρ_N and which is such that $V^\rho_N = F^{-1}\ (E^{o\rho}_N)$.*

Continuing to follow [11], we extend the linear map $X \mapsto T_X$ from $\mathbf{p}$ to the vector space of all differentiable maps from E_∞ to E_∞, to the enveloping algebra $\mathcal{U}(\mathbf{p})$ by defining inductively $T_{\mathbf{1}} = I$, where $\mathbf{1}$ is the identity in $\mathcal{U}(\mathbf{p})$, and T_{YX} by

$$T_{YX} = DT_Y.T_X, \quad \text{for } Y \in \mathcal{U}(\mathbf{p}),\ X \in \mathbf{p}, \tag{2.9}$$

where $(DA.B)(f)$ is the Fréchet derivative of A at the point f in the direction $B(f)$. It was proved in [11] that this defines a linear map from $\mathcal{U}(\mathbf{p})$ to the vector space of differentiable maps from E_∞ to E_∞ (see formula (1.10) of [11] and the sequel) and that

$$\frac{d}{dt}\ T_{Y(t)}\ (u(t)) = T_{P_0 Y(t)}\ (u(t)),\ Y \in \mathcal{U}(\mathbf{p}), \tag{2.10a}$$

where

$$Y(t) = exp(tP_0)\ Y\ exp(-tP_0), \tag{2.10b}$$

if

$$\frac{d}{dt}\ u(t) = T_{P_0}\ (u(t)). \tag{2.10c}$$

Next we introduce the nonlinear representations $U^{(+)}$. Let $u_+ = (f, \dot{f}, \alpha) \in E^{o\rho}_\infty$ and let

$$J^{(+)}_\mu\ (t, x) = (\frac{m}{t})^3\ (\frac{t}{\delta(t, x)})^5 \times$$

$$\sum_{\varepsilon = \pm} ((P_\varepsilon\ (-i\partial)\alpha)^\wedge\ (-\frac{\varepsilon\ m\ x}{\delta\ (t, x)}))^+\ \gamma^0\ \gamma_\mu\ (P_\varepsilon(-i\partial)\alpha)^\wedge\ (-\frac{\varepsilon\ m\ x}{\delta\ (t, x)}), \tag{2.11}$$

where $\delta(t, x) = (t^2 - |x|^2)^{1/2} > 0,\ t > 0$ and

$$P_\varepsilon(k) = \frac{1}{2}\ (I + \varepsilon\ w(k)^{-1}\ (-\sum_{j=1}^{3}\ \gamma^0\ \gamma^j\ k_j + m\ \gamma^0)) \tag{2.12}$$

defines the orthogonal projectors in D on the subspace of energy sign ε. The Fourier transform is defined by

$$\hat{f}(k) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ikx} \, f(x) \, dx,$$

and by $(P_\varepsilon(-i\partial)\alpha)^\wedge(-\frac{\varepsilon mx}{\delta(t,x)})$ we mean the value of the function $(P_\varepsilon(-i\partial)\alpha)^\wedge$ for the argument $k = -\frac{\varepsilon mx}{\delta(t,x)}$. Since $\alpha \in S(\mathbb{R}^3, \mathbb{C}^4)$, (2.11) defines for a given $t > 0$ a function $J_\mu^{(+)}(t,.)$ in $S(\mathbb{R}^3, \mathbb{R})$, vanishing for $|x| \geq t$. $J_\mu^{(+)}$ is the leading contribution obtained from the stationary phase method to $\sum_\varepsilon (e^{\mathcal{D}t} \, P_\varepsilon(-i\partial)\alpha)^+ \, \gamma^0 \, \gamma_\mu \, (e^{\mathcal{D}t} \, P_\varepsilon(-i\partial)\alpha)$, when $t \to \infty$. Choose $\chi \in C^\infty(\mathbb{R}), \chi(\tau) = 0$ for $\tau \leq 1, \; \chi(\tau) = 1$ for $\tau \geq 2$ and $0 \leq \chi(\tau) \leq 1$ for $\tau \in \mathbb{R}$. We define

$$A_\mu^{(+)} \, (t,x) = \chi(\delta(t,x)) \, B_\mu^{(+)}(t,x), \quad \text{for } \delta(t,x) \geq 1, \; t \geq 0, \tag{2.13a}$$

$$A_\mu^{(+)} \, (t,x) = 0 \text{ otherwise,}$$

$$B_\mu^{(+)} \, (t,.) = B_\mu^{(+)1}(t,.) - \int_t^\infty |\nabla|^{-1} \sin \, (|\nabla|(t-s)) \; J_\mu^{(+)}(s,.) \, ds, \tag{2.13b}$$

where $t > 0$, and

$$B_\mu^{(+)1}(t,.) = \; \cos \, (|\nabla|t) \; f_\mu + |\nabla|^{-1} \; \sin \, (|\nabla|t) \; \dot{f}_\mu.$$

The cutoff function $\chi' : \, (t,x) \mapsto \chi(\delta(t,x))$ has been introduced to exclude, in a Lorentz invariant way the points $(0,x), \; x \in \mathbb{R}^3$ from the support of $A_\mu^{(+)}$.

Then we define the nonlinear representation $U^{(+)}$ of $\mathcal{P}_0$ on $E_\infty^{o\rho}$ by

$$U_g^{(+)} \, (u_+) = (I \oplus I \oplus e_g \; (f, \dot{f})) \; U_g^1 \; u_+, \;\; g \in \mathcal{P}_0, u_+ \in E_\infty^{o\rho}, \tag{2.14}$$

where $e_g(f,\dot{f})$ is the linear operator from D to D given by

$$e_g(f,\dot{f}) = I \text{ for } g \in SL(2,\mathbb{C}) \tag{2.15a}$$

and

$$(e_g(f,\dot{f}) \; \alpha)^\wedge \; (k) = exp(i \sum_{\varepsilon = \pm} \theta^\infty \; (\chi'(B^{(+)1} \; (. + a) - B^{(+)1}(.)), \; l_\varepsilon(k)) \tag{2.15b}$$

for $g = exp(a^\mu \; P_\mu), \; l_\varepsilon(k) = (w(k), -\varepsilon k), \; k \in \mathbb{R}^3$.

Finally we introduce, for given $u_+ = (f, \dot{f}, \alpha) \in E_\infty^{o\rho}$, the operator $W_t^{(+)} \; (u_+)$ satisfying (1.3a) - (1.3d) by

$$(W_t^{(+)} \; (u_+)\beta)^\wedge(k) = \sum_{\varepsilon = \pm} \; e^{is_\varepsilon^{(+)}(u_+,t,k)} P_\varepsilon(k) \; \hat{\beta}(k), \tag{2.16}$$

where $s_\varepsilon^{(+)}(u_+, t, k) = -\theta(A^{(+)}, t, -\varepsilon t k w(k)^{-1}) - \theta^\infty(\chi'(B^{(+)}(. + b(t)) - B^{(+)}(.),\ l_\varepsilon(k))$, and $b(t) = (t, 0, 0, 0)$.

Proposition 2.4 : *If $n \geq 4$ then $U^{(+)}$ is a continuous nonlinear (analytic) representation of $\mathcal{P}_0$ on $E_n^{o\rho}$, $1/2 < \rho < 1$ and it is a C^∞ nonlinear representation of $\mathcal{P}_0$ on $E_\infty^{o\rho}$.*

We now state the main results. There is an invariant neighbourhood in V_∞^ρ on which $U_g,\ g \in \mathcal{P}_0$ exists :

Theorem 2.5 : *There exists a neighbourhood $\mathcal{O}_0$ of zero in V_∞^ρ, $1/2 < \rho < 1$ such that T is the differential of a unique C^∞ nonlinear representation U of $\mathcal{P}_0$ on $\mathcal{O}_0$. In particular the M-D equations have global solutions for initial data in $\mathcal{O}_0$.*

Theorem 2.5 follows from the fact that the modified wave operators $\Omega_\pm$ exist and are invertible. We state the result only for the case of Ω_+.

Theorem 2.6 : *Let $1/2 < \rho < 1$. Then there exists a neighbourhood $\mathcal{O}_0$ of zero in V_∞^ρ, a neighbourhood $\mathcal{O}_+$ of zero in $E_\infty^{o\rho}$ and a C^∞-diffeomorphism $\Omega_+ : \mathcal{O}_+ \to \mathcal{O}_0$ satisfying :*

i) $U_g \circ \Omega_+ = \Omega_+ \circ U_g^{(+)}$, for $g \in \mathcal{P}_0$,

ii) If $h(t) =$ exp $(t\ P_0)$, then for $u_+ \in \mathcal{O}_+$:

$$\lim_{t\to\infty} \|U_{h(t)}\ (\Omega_+(u_+)) - W_t^{(+)}(u_+)\ U_{h(t)}^{(+)}\ (u_+)\|_E = 0.$$

We shall outline the crucial steps of the proof of Theorem 2.6. As in the case of the Klein-Gordon equation [11], the linear operators $T_Y^1,\ Y \in \mathcal{U}(\mathbf{p})$ are bounded by the nonlinear operators $T_Y,\ Y \in \mathcal{U}(\mathbf{p})$. Namely, with the notation (cf. (2.38) of [11])

$$\mathcal{P}_N(a) = (\sum_{Y\in\Pi',\ |Y|\leq N} \|a_Y\|_E^2)^{1/2},\ N \geq 0, \tag{2.17}$$

where $Y \mapsto a_Y$ is a linear map from $\mathcal{U}(\mathbf{p})$ to E we have the following result :

Proposition 2.7 : *There exists a neighbourhood $\mathcal{O}$ of zero in E such that for $u \in E_N \cap \mathcal{O}$:*

$$\mathcal{P}_N\ (T(u)) \leq C_N\ \|u\|_{E_N} \leq F_N\ (\mathcal{P}_1(u))\ \mathcal{P}_N(u),\ N \geq 0,$$

where C_N is independent of u, F_N is a nondecreasing function on $\mathbb{R}^+$ and for $N = 0, 1$, F_N is a constant.

The next step in the proof consists of establishing approximate solutions of the M-D equations for given scattering data $u_+ = (f, \dot{f}, \alpha) \in E_\infty^{o\rho}$, $1/2 < \rho < 1$. This is done by iterating

$$A_{n+1,\mu}(t) = A_{0,\mu}(t) - \int_t^\infty\ |\nabla|^{-1}\ sin(|\nabla|\ (t-s))\ \sum_{\varepsilon=\pm} (\phi_n'^{\varepsilon})^+\ \gamma^0\ \gamma_\mu\ \phi_n'^{\varepsilon}(s)\ ds, \tag{2.18a}$$

$$\phi_{n+1}'^{\varepsilon}(t) = \phi_0'^{\varepsilon}(t) + i \int_t^\infty\ e^{(t-s)\mathcal{D}}\ (A_{n,\mu} + B_{n,\mu})\ \gamma^0\ \gamma^\mu\ \phi_n'^{\varepsilon}(s)\ ds,\ t > 0 \tag{2.18b}$$

where $B_{n,\mu}(y) = -\partial_\mu \theta(A_n, y),\ y \in \mathbb{R}^4,\ \phi_0'^{\varepsilon}(t) = P_\varepsilon(-i\partial) e^{\mathcal{D}t}\ \alpha,\ A_{0,\mu}(t) = \cos(|\nabla| t)\ f_\mu + |\nabla|^{-1}\ \sin(|\nabla| t)\ \dot{f}_\mu$. By using stationary phase methods it follows that for n sufficiently large and $1/2 < \rho < 1,\ 0 \leq \rho' \leq 1$:

$$(1+t)^{\rho'-\rho+1}\ \|(A_{n+1}(t) - A_n(t),\ \dot{A}_{n+1}(t) - \dot{A}_n(t))\|_{M^{\rho'}} \tag{2.19}$$

$$+(1+t)^{3/2-\rho}\ \|\phi'_{n+1}(t) - \phi'_n\|_D +$$

$$+(1+t+|x|)^{5/2-\rho+|\nu|-\varepsilon}\ (1+t)^\varepsilon\ |\frac{\partial^{\nu_0}}{\partial t^{\nu_0}}\ \partial_1^{\nu_1}\ \partial_2^{\nu_2}\ \partial_3^{\nu_3}\ (A_{n+1}(t,x) - A_n(t,x))|$$

$$+(1+t+|x|)^{3-\rho}\ |\phi'_{n+1}(t,x) - \phi'_n(t,x)| \leq C_\varepsilon\ \|u\|_{E_{N_0}},$$

for some N_0 depending on ρ, the constants C_ε depending on ρ and $\|u\|_{E_{N_0}}$. Similar estimates are satisfied for the derivatives of $A_n, \dot{A}_n, \phi'_n$ with respect to the scattering data and the terms obtained by the action of the enveloping algebra $\mathcal{U}(\mathbf{p})$ on these derivatives.

Let M be such that (2.19) is satisfied with $n = M-1$ and let

$$K_\mu(t) = A_\mu(t) - A_{M,\mu}(t),\ \Phi(t) = e^{i\theta(A,t)}\ \psi(t) - \phi'_M(t), \tag{2.20}$$

$t \geq 0$, where A_μ and ψ are the unknown variables in the M-D equations. To prove the existence of a solution of the equation for the new variables (K, Φ) we shall need a new energy estimate. If $G_\mu, \partial_\nu G_\mu \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R}^4) \cap C^0(\mathbb{R}^+, (1-\Delta)^{-1/2} M^1), 0 \leq \mu, \nu \leq 3$, it then follows from [8] that the equation $(i\gamma^\mu \partial_\mu + m)\ h = \gamma^\mu G_\mu h$ defines a unitary evolution operator family $w(t,s),\ t, s \geq 0$, strongly continuous on D and continuously differentiable on $W^{1,2}\ (\mathbb{R}^3, \mathbb{C}^4)$. Therefore the solution in D of the equation

$$(i\gamma^\mu \partial_\mu + m)\ h - G_\mu \gamma^\mu h = F_\mu \gamma^\mu r (\in D) \tag{2.21a}$$

is obtained by integration with $w(t,s)$. Suppose that r satisfies the equation

$$(i\gamma^\mu \partial_\mu + m)\ r = q. \tag{2.21b}$$

Proposition 2.8 : *Let $Q(t,x) = t\ F_0(t,x) + \sum_{1 \leq i \leq 3}\ x_i\ F_i(t,x),\ \xi_{M_{oi}} = x_i \frac{\partial}{\partial t} + t\partial_i$. If $0 \leq t \leq t'$ and $0 \leq a \leq 1$ then*

$$|\ \|h(t)\|_D - \|h(t')\|_D\ | \leq \frac{2}{m} \sup_{t \leq s \leq t'}\ \|\gamma^\mu\ F_\mu(s)\ r(s)\|_D$$

$$+\frac{1}{m} \int_t^{t'} (\|r(s)\ \partial_\mu F^\mu(s) + \frac{1}{2}\ r(s)\ \gamma^\mu \gamma^\nu\ (\partial_\mu F_\nu(s) - \partial_\nu F_\mu(s))$$

$$+i\gamma^\mu \gamma^\nu\ G_\mu(s)\ F_\nu(s) + i\gamma^\mu F_\mu(s)\ q(s)\|_D$$

$$+2\ \|(1+s)^{a-1}\ |(Q(s) + F_0(s))\ \frac{d}{ds}\ r(s) - \sum_{1 \leq i \leq 3}\ F_i(s)\ (\xi_{M_{oi}}\ r)(s)|^{1-a}$$

$$\times |F^\mu(s)\, \partial_\mu\, r(s)|^a \|_D)\, ds,$$

where the summation convention is used for the repeated upper and lower indices μ and ν.

This energy estimate permits to prove that the rest term and its derivatives exist in L^2 for each $t \geq 0$.

Using the fact that there exists a smothing operator on E^ρ_∞ we can now prove that the wave operator exists : $\Omega_+(u_+) = u(0) = (A(0), \dot{A}(0), \psi(0))$.

Theorem 2.9 : *Let $1/2 < \rho < 1$. There exists $N_0 \geq 0$ and a neighbourhood $\mathcal{O}_{N_0}$ of zero in $E^{o\rho}_{N_0}$ such that $\Omega_+ : \mathcal{O}_{N_0} \cap E^{o\rho}_\infty \to E^{o\rho}_\infty$ is a one-to-one C^∞ map satisfying*

$$\|(D^l\, \Omega_+)\, (u; u_1, ..., u_l)\|_{E_L} \leq C_{L,l}\, \|u\|_{E_{N_0+l+L}}\, \|u_1\|_{E_{N_0}} ... \|u_l\|_{E_L}$$

$$+ C'_{L,l}\, \|u_1\|_{E_{N_0+l+L}}\, \|u_2\|_{E_{N_0}} ... \|u_l\|_{E_{N_0}} + ... + C'_{L,l}\, \|u_1\|_{E_{N_0}} ... \|u_{l-1}\|_{E_{N_0}}\, \|u_l\|_{E_{N_0+l+L}},$$

for each $L \geq 0$, $l \geq 0$, $u_1, ..., u_l \in E^{o\rho}_\infty$, where $C_{L,l}$ and $C'_{L,l}$ are constants depending only of $\|u\|_{E_{N_0}}$. Moreover $\Omega_+(\mathcal{O}_{N_0} \cap E^{o\rho}_\infty) \subset V^\rho_\infty$.

D^l denotes here the derivative of order l. Finally, using once more the energy estimate in Proposition 2.8 one proves that the derivative $D\, \Omega_+$ has a right inverse satisfying an inequality similar to that in Theorem 2.9. The existence of the inverse Ω_+^{-1} now follows from the implicit functions theorem in a Fréchet space.

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