Star Products, Quantum Groups, Cyclic Cohomology and Pseudodifferential Calculus.

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ABSTRACT. We start with a short historical overview of the developments of deformation (star) quantization on symplectic manifolds and of its relations with quantum groups. Then we briefly review the main points in the deformation-quantization approach, including the question of covariance (and related star-representations) and describe its relevance for a cohomological interpretation of renormalization in quantum field theory. We concentrate on the newly introduced notion of closed star product, for which a trace can be defined (by integration over the manifold) and is classified by cyclic (instead of Hochschild) cohomology ; this allows to define a character (the cohomology class of cocycle in the cyclic cohomology bicomplex). In particular we show that the star product of symbols of pseudodifferential operators on a compact Riemannian manifold is closed and that its character coincides with that given by the trace, thus is given by the Todd class, while in general not satisfying the integrality condition.

In the last section we discuss the relations between star products and quantum groups, showing in particular that "quantized universal enveloping algebras" (QUEAs) can be realized, essentially in a unique way (using a strong star-invariance condition), as star product algebras with a different quantization parameter. Finally we show (in the $sl(2)$ case) that these QUEAs are dense in a model Fréchet-Hopf algebra, stable under bialgebra deformations, containing all of them (for different parameter values) and that they have the same product and equivalent coproducts with the original algebra.

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1. Introduction.

1.1. Historical background. The mathematical theory of deformations is inherent to the development of modern physics. However this fact has essentially been recognized a posteriori [1]. For example Newtonian mechanics is invariant under the 10-parameter Galilei group of transformations of space-time, while relativistic mechanics brings in its deformation, the Poincaré group, obtained by attributing a non-zero value to the inverse of the velocity of light. If in addition one gives a (non-zero) constant curvature to space-time one gets the De Sitter groups and (since they are simple) the buck stops there - as far as Lie groups are concerned. Considering the Lie bialgebra structure permits, as we shall see below, to get "quantum groups" by deforming the coproduct - but then the "transformation group" point of view is lost, or at least needs to be seriously revised (which wasn't done yet). Had the theory of deformations of Lie groups been developed a century ago, one could have already then found mathematically relativistic theories, before any experimental fact.

Quantum mechanics can in a similar way be considered as a deformation of classical mechanics. However this fact was made explicit and rigorous only 15 years ago [2], while again it might have been developed mathematically a century ago (had the mathematical tools been developed then). In 1927, H. Weyl [3] gave an integral formula (a Fourier transform with operatorial kernel) for a passage from classical to quantum observables, and E. Wigner gave in 1931 an inverse formula [4]. At the end of the 40's, Moyal and Groenewold [5] used the latter to write the symbols of a bracket and a product (respectively) of quantum observables, in a form from where the structure of deformation could have been read off, had the theory of deformation of algebras been developed and had they looked for its physical interpretation (instead of trying to interpret somehow the symbols as probability densities, which cannot be done). Many other works followed along similar lines including some that relied implicitely on the idea of deformation (e.g. the so-called semi-classical approximations), but none "hit the nail on the head" until the mid 70's.

Around 1970, coming from the representation theory of Lie groups, a theory called "geometric quantization" became quite popular [6]. The idea is to do for more general Lie groups (than the Heisenberg groups) and more general symplectic manifolds (than the flat ones) what the Weyl transform does, map functions on the symplectic manifold to operators on a Hilbert space of functions on "half" of the variables. While very interesting for representation theory, this approach proved not very effective physically; in particular polarizations (which in many cases could be found only in the complex domain) had to be introduced to get "half" of the variables, and relatively few classical observables could be thus quantized.

1.2. Star-products and quantum groups. Our approach [2], sometimes called "deformation quantization", was different - though starting from similar premises. We did not look for polarizations or operators, but had the quantum theory "built in" the algebra of classical observables by deforming the composition laws, product and Poisson bracket of functions, to what we called star-products and the associated commutators. We could develop in an autonomous manner, even on general symplectic manifolds, a quantum theory that can be mapped into the usual one by a Weyl mapping (when there exists one). And this has naturally lead to a parallel development of representation theory of Lie groups, without operators (cf. a partial summary in [7]). Furthermore the same approach can be developed for infinite-dimensional phase-spaces, giving rise to a star-product approach to quantum field theories and in particular to a cohomological interpretation of cancellation of infinities [8].

Around 1980, a new mathematical notion appeared in the quantization of 2-dimensional integrable models [9], and has gained tremendous popularity after Drinfeld [10] coined the (somewhat misleading but very effective) term "quantum group" to qualify it. The basic point is that, on a Lie group with a compatible Poisson structure, the functions have a natural Hopf algebra structure that can be deformed by replacing the usual product by a star-product; in the dual approach of Jimbo [11] one deforms the coproduct on a completion of the enveloping algebra of the Lie algebra (which has a bialgebra structure) and gets in particular the strange commutation relations obtained in [9]. A realization of these structures was given by Woronowicz [12] using the basic representations of compact Lie groups while taking coefficients in C^*-algebras (with some relations); this was inspired by the "non-commutative geometry" of A. Connes [13].

1.3. Main new points. In this paper we shall first briefly review the main points of the star-product theory, including the new developments of star-quantization in field theory. Then we shall develop the new notion of closed star-product [14], for which a trace can be defined by integration over phase-space, show its relation to cyclic cohomology and apply it to the pseudodifferential calculus on a compact Riemannian manifold : for the corresponding star-product a character can be defined (in the cyclic cohomology) that is given by the Todd class of the manifold (via the index theorem).

Then we shall concentrate on the relations between star-products and quantum groups. In particular we shall show, by the example of $sl(2)$, that "quantum groups" are neither quantum nor groups, but examples of star-products requiring another star-product in the background (with another parameter $\hbar$) in order to realize the quantized universal enveloping algebras (QUEAs) as star-product algebras and that this realization is then essentially unique. Finally, we shall indicate that there exists a universal Fréchet-Hopf algebra (containing densely the QUEAs), rigid as bialgebra, the dual of which contains the original (simple compact) group as a "hidden group".

2. Star-products, star quantization and closed star-products.

2.1. The framework. *a. Phase-space.* The phase-space is a Poisson manifold, i.e. a manifold where a Poisson bracket can be defined. In the case of a finite-dimensional manifold W the Poisson structure is given by a contravariant skew-symmetric 2-tensor Λ satisfying $[\Lambda, \Lambda] = 0$ in the sense of the supersymmetric Schouten brackets; the latter condition is equivalent [2] to the fact that the Poisson bracket :

$$P(u,v) = i(\Lambda)(du \wedge dv) \;\; ; \;\; u,v \in C^\infty(W) = N \tag{1}$$

satisfies the Jacobi identity. When Λ is everywhere non degenerate, W is a symplectic manifold, of even dimension $2l$ with a closed 2-form ω. Poisson structures can however be defined also for infinite-dimensional (e.g. Banach or Fréchet) manifolds, for instance on the space of initial conditions of a wave equation such as Klein-Gordon ([15, 16]).

b. Star-products, deformations of algebras and cohomologies. The relation between associative (resp. Lie) algebra deformations and Hochschild (resp. Chevalley) cohomology is well known [17], and can be made more precise (at each step) both for the existence question (determined by the 3-cohomology of the algebra valued in itself) and the equivalence question (classified by the 2-cohomology) [2]. For the associative (resp. Lie) algebra N it is consistent [2] to restrict oneself to differentiable cohomologies and to a formal series of differential operators for the equivalence.

DEFINITION. *A* **star-product** *is an associative deformation of N with a complex parameter ν :*

$$u * v = \sum_{r=0}^{\infty} \nu^r \, C_r(u,v) \quad ; \quad u, v \in N \tag{2}$$

$$C_0(u,v) = uv, \quad C_1(u,v) - C_1(v,u) = 2P(u,v)$$

where the C_r are bidifferential operators.

We thus get a deformation of the Lie algebra (N, P) by the "star-commutator" :

$$\frac{1}{2\nu}(u * v - v * u) \equiv [u,v]_\nu = P(u,v) + \sum_{r=1}^{\infty} \nu^r \, C'_r(u,v) \tag{3}$$

for which the relevant cohomologies are finite-dimensional (the Chevalley 2-cohomology has dimension $1 + b_2(W)$, where $b_2(W)$ is the second Betti number of W). Equivalence between two star-products $*$ and $*'$ (resp. brackets) is given by a formal series $T = I + \sum_{s=1}^{\infty} \nu^s \, T_s$ (where the T_s are necessarily differential operators [2]) such that $T(u *' v) = Tu * Tv$ (and similarly for the brackets). All this extends naturally from the algebra N of functions to $\mathcal{A} = N[[\nu]]$, the formal series in ν with coefficients in N.

For quantum groups one needs to consider Hopf algebras [10] where the essential ingredients are the product and coproduct, that together define a bialgebra structure for which a similar theory can be developed [18,19].

c. Example. The typical example is $W = \mathbb{R}^{2l}$ with the Moyal star-product for which one takes the r^{th} powers of the bidifferential operator P :

$$r! C_r(u,v) = P^r(u,v) = \Lambda^{i_1 j_1} \ldots \Lambda^{i_r j_r} (\partial_{i_1 \ldots i_r} u)(\partial_{j_1 \ldots j_r} v) \tag{4}$$

so that the star-product and bracket can be written :

$$u * v = \exp(\nu P)(u,v) \tag{5}$$

$$M(u,v) = \nu^{-1} \sinh(\nu P)(u,v) \tag{6}$$

The Weyl maps can be defined, for $u \in N$, by

$$\Omega_w(u) = \int_{\mathbb{R}^{2l}} \mathcal{F}(u)(\xi,\eta)w(\xi,\eta)\ \exp\ ((\xi.P+\eta.Q)/2\nu)d^l\xi\ d^l\eta \tag{7}$$

where $\mathcal{F}$ is the inverse Fourier transform, (P_α, Q_α, I) are generators of the Heisenberg Lie algebra $\mathcal{H}_l$ in the Von Neumann representation on $L^2(\mathbb{R}^l)$ satisfying $[P_\alpha, Q_\beta] = 2\nu\delta_{\alpha\beta}I$ $(\alpha, \beta = 1, ..., l)$, and (in the Moyal case) the weight function $w = 1$ and $2\nu = i\hbar$. Then Ω_1 will map functions in $C^\infty(\mathbb{R}^{2l})$ into operators in $L^2(\mathbb{R}^l)$, star-products into operator product, and the Moyal bracket M into commutators. The inverse map can be defined by a trace :

$$u = (2\pi\hbar)^{-1}Tr(\Omega_1(u)e^{(\xi.P+\eta.Q)/i\hbar}) \tag{8}$$

is one-to-one between $L^2(\mathbb{R}^{2l})$ and Hilbert-Schmidt operators on $L^2(\mathbb{R}^l)$, and can be defined on larger spaces of functions (cf.e.g.[20]). One then has, for $\Omega_1(u)$ trace-class :

$$Tr(\Omega_1(u)) = (2\pi\hbar)^{-l}\int_{\mathbb{R}^{2l}} u\ \omega^l \equiv Tr_M(u) \tag{9}$$

where $\omega = \sum_{\alpha=1}^l dp_\alpha \wedge dq_\alpha$ is the usual symplectic form on $\mathbb{R}^{2l}$, so that

$$Tr_M(u * v) = Tr_M(v * u).$$

The latter property can be seen directly since (due to the skew-symmetry of Λ, by integration by parts), for all r and C_r defined by (4) :

$$\int_W C_r(u,v)\omega^l = \int_W C_r(v,u)\omega^l. \tag{10}$$

For other orderings (other weight functions w in (7)), formula (9) is only approximate, i.e. valid modulo higher powers of $\hbar$. This is in particular true of the so-called standard ordering (all q's on the left, as in the usual way of writing differential operators) for which [21] $w(\xi,\eta) = exp(\frac{1}{2}i\xi\eta)$, that corresponds to the pseudodifferential calculus; if we denote by Ω_S the corresponding Weyl map, then [14], whenever defined :

$$Tr(\Omega_S(u)) = (2\pi\hbar)^{-l}\int_{\mathbb{R}^{2l}} u\omega^l + O(\hbar^{1-l}). \tag{11}$$

To define a trace on $\mathcal{A}$ with integration on W one is thus lead to look at the coefficient of $\hbar^l$ in $u * v$ for $u, v \in \mathcal{A}$.

This will motivate our definition of closed star products.

2.2. Closed star-products and cyclic cohomology.

a. DEFINITION. *Let W be a symplectic manifold of dimension $2l$, and $*$ be a star-product on N defined by (2). The star-product is said* **closed** *if (10) holds for all $u, v \in N$ and $r \leq l$, and* **strongly closed** *if (10) holds for all r.*

Equivalently, if $u = \sum_{k=0}^{\infty} \nu^k u_k \in \mathcal{A}$ and $v = \sum_{j=0}^{\infty} \nu^j v_j \in \mathcal{A}$, then the coefficient $a_l(u,v)$ of ν^l in $u * v$ is $\sum_{r+j+k=l} C_r(u_k, v_j)$ and the condition for closedness is

$$\int_W a_l(u,v)\, \omega^l = \int_W a_l(v,u)\, \omega^l. \tag{12}$$

Note that (10) is always true for $r = 0$ and 1 (thus all star-products on 2-dimensional symplectic manifolds are closed). If the star-product is closed, then the map τ defined by

$$\mathcal{A} \ni u = \sum_{k=0}^{\infty} \nu^k\, u_k \mapsto \tau(u) = \int_W u_l\, \omega^l \tag{13}$$

has the properties of a trace for the algebra $\mathcal{A}$ with the product $*$:

$$\tau(u * v) = \tau(v * u)$$

whenever both sides are defined.

b. Cyclic cohomology [13,14]. If M is a N-module, the coboundary operation b in the Hochschild cohomology can be defined on a p-cochain C by :

$$bC(f_0, ..., f_p) = f_0\, C(f_1, ..., f_p) - C(f_0 f_1, ..., f_p) + ...$$

$$+(-1)^p\, C(f_0, ..., f_{p-1} f_p) + (-1)^{p+1} C(f_0, ..., f_{p-1}) f_p. \tag{14}$$

To every $f \in \mathcal{A}$ one can associate an element $\tilde{f}$ in the dual $\mathcal{A}^*$ by :

$$\tilde{f} : g \mapsto \int fg\, \omega^l.$$

The action of $\mathcal{A}$ on $\mathcal{A}^*$ is given by $(x\varphi y)(a) = \varphi(yax)$ when $\varphi \in \mathcal{A}^*$ and $a, x, y \in \mathcal{A}$. The map $f \mapsto \tilde{f}$ gives then a map of p-cochains :

$$C^p(\mathcal{A}, \mathcal{A}) \to C^p(\mathcal{A}, \mathcal{A}^*)$$

that is compatible with the coboundary operation : if $\tilde{C}_r$ denotes the image in $C^p(\mathcal{A}, \mathcal{A}^*)$ of $C_r \in C^p(\mathcal{A}, \mathcal{A})$, then one has $\widetilde{bC_r} = b\tilde{C}_r$. If we define a bicomplex by $\{0\}$ for $n < m$ and, for $n \geq m$:

$$C^{n,m} = C^{n-m}(\mathcal{A}, \mathcal{A}^*) \tag{15}$$

the Hochschild coboundary b is of degree 1. We can, in addition, define another operation B of degree -1 that anticommutes with b ($bB = -Bb$, with $B^2 = 0 = b^2$) as $B = A_S B_0$ where A_S is the cyclic antisymmetrization and, for $\tilde{C} \in C^{n+1}(\mathcal{A}, \mathcal{A}^*)$:

$$B_0 \tilde{C}(f_0, ..., f_{n-1}) = \tilde{C}(1, f_0, ..., f_{n-1}) + (-1)^n\, \tilde{C}(f_0, ..., f_{n-1}, 1). \tag{16}$$

Let us denote by $C^n_\lambda \subset C^n(\mathcal{A}, \mathcal{A}^*)$ the space of cochains $\tilde{C}$ satisfying the cyclicity condition:

$$\tilde{C}(f_1, ..., f_n)(f_0) = (-1)^n \, C(f_2, ..., f_n, f_0)(f_1). \tag{17}$$

DEFINITION. *The* **cyclic cohomology** *of $\mathcal{A}$, denoted by $HC^n(\mathcal{A})$, is the cohomology of the complex (C^n_λ, b).*

A fundamental (and non trivial) property of cyclic cohomology is :

PROPOSITION. *At each level n, one has*

$$HC^n(\mathcal{A}) = (ker b \cap ker B) \; / \; b(ker B). \tag{18}$$

c. Classification of closed star-products. For $r = 2$, the closedness condition (10) can be written $B\tilde{C}_2 = 0$. (This condition is necessary; it is sufficient if $dimW = 4$). If we start with a Hochschild 2-cocycle C_1 (e.g. $C_1 = P$), standard deformation theory [2, 14, 17] gives a 3-cocycle E_2 (determined by C_1) that has to be equal to bC_2. Therefore $b\tilde{E}_2 = 0 = B\tilde{E}_2$ and $b\tilde{C}_2 = \tilde{E}_2 \in ker b \cap ker B$. From (18) we therefore get, since the same can be done [14] successively at each order of deformation, and in one degree less for the equivalence operators T :

PROPOSITION. *At each order, the obstructions to the existence of closed star-products are classified by $HC^3(\mathcal{A})$, and the obstructions to equivalence by $HC^2(\mathcal{A})$.*

If C is a non-closed current on W $(dC \neq 0)$, $\tilde{C}^2(f_1, f_2)(f_0) = < C, f_0 df_1 \wedge df_2 >$ will give $B\tilde{C}_2 \neq 0$ and therefore we get in this way a non-closed star-product.

2.3. Existence, uniqueness and rigidity of star-products.

a. THEOREM [22]. *On any symplectic manifold W there exists a strongly closed star-product.*

The Moyal product is closed, and can be defined on any canonical chart of W. The problem is to "glue" together these products, and to do it in a way that preserves the integrals of functions. The authors of [22] do both things together by defining globally on W a locally trivial algebra bundle (called Weyl manifold) giving a (globally defined) formal completion of the Heisenberg enveloping algebra; functions in $\mathcal{A}$ are then extended to sections of this bundle (called Weyl functions) in an integration-preserving way.

Previous existence proofs of star-products and brackets were first done by supposing $b_3(W) = 0$, to control the multiple intersections of charts. Then, after other generalizations, Lecomte and De Wilde gave a very abstract existence proof in the general case, and later simplified a first existence proof by Omori, Maeda and Yoshioka by showing that Moyal products on canonical charts can be transformed by equivalences so as to coincide on intersections [23] - but these equivalences were not constructed in a way that obviously preserved closedness. All these proofs extend to regular Poisson manifolds (where the symplectic leaves have all the same dimension).

b. Uniqueness. The Hochschild differentiable p-cohomology spaces $\tilde{H}^p(N)$ for the associative algebra $N = C^\infty(W)$ are isomorphic [24] to $\Lambda_p(W)$, the space of skew-symmetric contravariant p-tensors on W. More precisely [25], any differentiable p-cocycle can be written as such a tensor plus the coboundary of a differentiable $(p-1)$ cochain. Therefore they are huge; except for $dim\ W = 2$, the obstructions to existence (classified by $\tilde{H}^3(N)$) make the previous theorem highly non trivial.

However consideration of the Lie algebra (N, P) reduces the choice. We shall call [2] **Vey product** a star-product (2) for which the cochains C_r are bidifferential operators that vanish on constants (i.e. have no constants terms), have the same parity as r (thus $C_1 = P$) and have the same principal part as P^r in any canonical chart (a star-product satisfying all these conditions except the last one can always be brought into the Vey form by equivalence [26]).

We then get associated brackets where only odd cochains C_{2r+1} appear. The relevant Chevalley cohomology spaces $H^p(N)$ are finite-dimensional. H^2 and H^3 can be explicitely computed [25], and in particular $dim\ H^2(N) = 1 + b_2(W)$. Explicit expressions for the cochains up to C_4 in a general Vey product can also be given ([25, 26]), in terms of a symplectic connection Γ and some tensors (the general expressions for C_5 and above could not be written in such terms; however the existence proof of Fedosov [23], which remained obscure until his MIT preprint of December 1992, gives a recurrence algorithm showing that a star-product can be constructed globally in terms of a symplectic connection and its covariant derivatives). In particular one has, for a Vey product [26] :

$$C_2 = P^2_\Gamma + bH \quad , \quad C_3 = S^3_\Gamma + T + 3\partial H \tag{19}$$

where H is a differential operator of maximal order 2, T a 2-tensor corresponding to a closed 2-form, ∂ the Chevalley coboundary operator, and P^2_Γ and S^3_Γ are given (in canonical coordinates) by expressions similar to (4) where, in P^2, usual derivatives are replaced by covariant derivatives and, in P^3, by the relevant components of the Lie derivative of Γ in the direction of the vector field associated to the function (u or v).

It is then possible [2] to work by steps of 2 and to reduce the possible choices of Vey products from the infinity suggested by the Hochschild cohomology to the finite alternatives of Vey brackets. In particular :

PROPOSITION. *If $b_2(W) = 0$, the Moyal-Vey bracket is unique up to (mathematical) equivalence.*

The more general definition of a star-product given here is required in particular because normal and standard orderings are star-products in this sense, equivalent to Moyal (on flat space) but not Vey products (the cochains C_r are not of the same parity as r). Consideration of the associated Lie algebra is however still possible, and will give information on the skew-symmetric parts of the cochains C_r. The analysis developed in [2] for Vey products can however be carried over to this more general definition.

c. Rigidity. A natural question then arises : is it the end of the story, and in particular can the Moyal-Vey product be deformed ?

The answer to the latter is : essentially no. More precisely one can show [27] that any (Vey) star-product of the form

$$u *_{\nu\rho} v = \sum_{r,s=0}^{\infty} \nu^r \rho^s \, C_{rs}(u,v), \quad C_{rs}(u,v) = (-1)^r C_r(u,v) \tag{20}$$

can be transformed by equivalence to $*_\nu$, by deforming the 2-tensor Λ with the parameter ρ.

However if the requirement that $C_1 = P$ is relaxed that result is no longer true. Examples can be given where an intertwinning operator will still exist, but will transform the Poisson into a Jacobi bracket (associated with a conformal symplectic structure) [28]: one writes

$$u *_{\nu\rho} v = u *_\nu f_{\nu\rho} *_\nu v \tag{21}$$

where $f_{\nu\rho} = \sum_{r=0}^{\infty} \nu^{2r} \, f_{2r,\rho} \in (N[[\nu]])[[\rho]]$ with $f_{0,\rho}$ invertible and $f_{0,0} = 1$. In particular one can take for $f_{\nu\rho}$ the $*_\nu$ exponential of ρH, $H \in N$ (and ρ proportional to the inverse temperature, in applications) : the so-called KMS conditions in statistical mechanics take then a simple expression (the condition for a state to be KMS is similar to a trace).

Quantum groups are also, in a sense, deformations of the usual star-products where the covariance enveloping algebra is deformed. We shall now study more closely that notion of covariance of a star-product.

2.4. Invariance and covariance of star-products. Star representations. *a. Invariance.* The Poisson bracket P is (by definition) invariant under all symplectomorphisms. However its powers P^r $(r \geq 2)$ given by (4) are invariant only under the vector fields generated by the polynomials of degree at most 2. For general Vey products the cochains C_2 and C_3 given by (19) are invariant under a subgroup of the finite-dimensional Lie group of symplectomorphisms preserving the symplectic connection Γ. That is also the case in $\mathbb{R}^{2l}$ for orderings other than Moyal (only the linear polynomials remain, possibly supplemented - e.g. for standard and normal orderings - by one second-degree polynomial). We thus have [2] a finite-dimensional Lie algebra $\mathcal{O}$ of "**preferred observables**", under which the star-product will be **invariant** :

$$\mathcal{O} = \{a \in N \; ; \; [a,u]_\nu = P(a,u) \quad \forall u \in N\}. \tag{22}$$

b. Covariance. A star-product will be said **covariant** under a Lie algebra $\mathcal{O}$ of functions if $[a,b]_\nu = P(a,b)$ for all $a, b \in \mathcal{O}$. It can then be shown [29] that $*$ is $\mathcal{O}$-covariant iff there exists a representation τ of the Lie group G whose Lie algebra is $\mathcal{O}$ into $Aut\ (\mathcal{A}; *)$ such that

$$\tau_g u = (Id_N + \sum_{r=1}^{\infty} \nu^r \, \tau_g^r)(g.u) \tag{23}$$

where $g \in G, u \in N$, G acts on N by the natural action induced by the vector fields associated with $\mathcal{O}$, $(g.u)(x) = u(g^{-1}x)$, and where the τ_g^r are differential operators on W. Invariance of course means that the geometric action preserves the star-product : $g.u * g.v = g(u * v)$.

c. Star representations. Let G be a Lie group (connected and simply connected), acting by symplectomorphisms on a symplectic manifold W (e.g. coadjoint orbits in the dual of the Lie algebra $\mathcal{L}$ of G). The elements $x, y \in \mathcal{L}$ will be supposed realized by functions u_x, u_y in N so that their Lie bracket $[x,y]_{\mathcal{L}}$ is realized by $P(u_x, u_y)$. Let us take a G-covariant star-product $*$, so that $P(u_x, u_y) = [u_x, u_y]_\nu$. We can now define the **star exponential** :

$$E(e^x) = Exp(x) \equiv \sum_{n=0}^{\infty} (n!)^{-1} (u_x/2\nu)^{*n} \tag{24}$$

where $x \in \mathcal{L}$, $e^x \in G$ and the power $*n$ denotes the n^{th} star-power of the corresponding function. By the Campbell-Hausdorff formula one can extend E to a **group homomorphism** :

$$E : G \to (N[[\nu, \nu^{-1}]], *) \tag{25}$$

where, in the formal series, ν and ν^{-1} are treated as independent parameters for the time being. Alternatively, the values of E can be taken in the algebra $(\mathcal{P}[[\nu^{-1}]], *)$, where $\mathcal{P}$ is the algebra generated by $\mathcal{L}$ with the $*$-product (a representation of the enveloping algebra).

We now call **star representation** [2] of G a distribution $\mathcal{E}$ (valued in $Im\ E$) on W defined by

$$D \ni f \mapsto \mathcal{E}(f) = \int_G f(g)\ E(g^{-1}) dg \tag{26}$$

where D is some space of test-functions on G. The corresponding **character** χ is the (scalar-valued) distribution defined by

$$D \ni f \mapsto \chi(f) = \int_W \mathcal{E}(f)\ d\mu \tag{27}$$

where $d\mu$ is a quasi-invariant measure on W. The character permit a comparison with usual representation theory.

This theory is now very developed, and parallels in many ways the usual (operatorial) representation theory. For a review of developments (and of star-product theory) until 1986, see e.g. [30] and references quoted therein. Among notable results one may quote :

i) An exhaustive treatment of **nilpotent** groups and of solvable groups of exponential type [31]. The coadjoint orbits are there symplectomorphic to $\mathbb{R}^{2l}$, and one can lift the Moyal product to the orbits in a way that is adapted to the Plancherel formula. Polarizations are not required, and "star-polarizations" can always be introduced to compare with usual theory.

ii) For **semi-simple** Lie groups an array of results is already available, including a complete treatment of the holomorphic discrete series [32] (that includes the case of compact Lie groups) and scattered results for specific examples.

iii) For semi-direct products, and in particular the Poincaré and Euclidean groups, an autonomous theory has also been developed (see e.g.[33]).

Comparison with the usual results of "operatorial" theory of Lie group representations can be performed in several ways, in particular by constructing an invariant Weyl transform generalizing (7), finding "star-polarizations" that always exist, in contradistinction with the geometric quantization approach (where at best one can find complex polarizations), study of spectra (of elements in the center of the enveloping algebra and of compact generators) in the sense of the next subsection, comparison of characters, etc. But our main insistence is that the theory of star-representations is an **autonomous** one that can be formulated completely within this framework, based on coadjoint orbits (and some additional ingredients when required).

2.5. Star-quantization.

a. DEFINITION. *Let W be a symplectic (or Poisson) manifold and N an algebra of classical observables (functions, possibly including distributions if proper care is taken for the product). We shall call* **star-quantization** *a star-product on N invariant under some Lie algebra $\mathcal{O}$ of "preferred observables".*

Invariance of the star-product ensures that the classical and quantum evolutions of observables under a Hamiltonian $H \in \mathcal{O}$ will coincide [2]. The typical example is the case described in (2.1.c), the Moyal product on $W = \mathbb{R}^{2l}$. Physicists often prefer to work with the so-called **normal ordering**, for which the Weyl map (7) is taken [21] with a weight $w(\xi,\eta) = exp(-\frac{1}{4}(\xi^2+\eta^2))$.

b. Spectrality. Physicists want to get numbers that match experimental results, e.g. for energy levels of a system. That is usually achieved by describing the spectrum of a given Hamiltonian $\hat{H}$ supposed to be a self-adjoint operator so as to get a real spectrum and so that the evolution operator (the exponential of $it\hat{H}$) is unitary (thus preserves probability). A similar spectral theory can be done here, in an autonomous manner. The most efficient way to achieve it is to consider [2] the star exponential (corresponding to the evolution operator)

$$Exp(Ht) \equiv \sum_{n=0}^{\infty} (n!)^{-1}\ (t/i\hbar)^n\ (H*)^n \tag{28}$$

where $(H*)^n$ means the n^{th} star power of $H \in N$ (or $\mathcal{A}$). Then one writes its Fourier-Stieltjes transform $d\mu$ (in the distribution sense), formally

$$Exp(Ht) = \int\ e^{\lambda t/i\hbar} d\mu(\lambda). \tag{29}$$

DEFINITION. *The spectrum of $(H/\hbar)$ is the support S of $d\mu$.*

In the particular case that H has discrete spectrum, the integral can be written as a sum : the distribution $d\mu$ is a sum of "delta functions" supported at the points of S, multiplied by the symbols of the corresponding eigenprojectors.

In different "orderings" given by (7) with various weight funtions w, one gets in general different operators for different classical observables H, thus different spectra. For $W = \mathbb{R}^{2l}$ all those are mathematically equivalent (to Moyal under the Fourier transform T_w of the weight function w of (7)). This means that every observable H will have the same spectrum under Moyal ordering as T_wH

under the equivalent ordering. But this does not imply "physical equivalence", i.e. the fact that H will have the same spectrum under both orderings. In fact, the opposite is true :

PROPOSITION [34]. *If two equivalent star-products are isospectral (give the same spectrum for a "large family" of observables and all ν) they are identical.*

It is worth mentioning that our definition of spectrum permits to define a spectrum even for symbols of non-spectrable operators, such as the derivative on a half-line (that has different deficiency indices). That is one of the many advantages of our autonomous approach to quantization.

c. Applications. i) In **quantum mechanics** it is preferable to work (for $W = \mathbb{R}^{2l}$) with the star-product that has maximal symmetry, i.e. $sp(\mathbb{R}^{2l}).\mathcal{H}_l$ as algebra of preferred observables : the Moyal product (5). One indeed finds [2] that the star exponential of these observables (polynomials of order ≤ 2) is proportional to the usual exponential. More precisely, if $X = \alpha p^2 + \beta pq + \gamma q^2 \in sl(2)\quad (p, q \in \mathbb{R}^l\ ;\ \alpha, \beta, \gamma \in \mathbb{R})$, setting $d = \alpha\gamma - \beta^2$ and $\delta = |d|^{1/2}$ one gets (as distributions)

$$(30)\qquad Exp(Xt) = \begin{cases} (\cos\,\delta t)^{-l}\exp((X/i\hbar\delta)\,\tan(\delta t)) & \text{for } d > 0 \\ \exp(Xt/i\hbar) & \text{for } d = 0 \\ (\cosh\,\delta t)^{-l}\exp((X/i\hbar\delta)\,\tanh(\delta t)) & \text{for } d < 0 \end{cases}$$

hence the Fourier decompositions

$$(31)\qquad Exp(Xt) = \begin{cases} \sum_{n=0}^{\infty} \Pi_n^{(l)}\ e^{(n+\frac{l}{2})t} & \text{for } d > 0 \\ \int_{-\infty}^{\infty} e^{\lambda t/i\hbar}\Pi(\lambda, X)d\lambda & \text{for } d < 0. \end{cases}$$

We thus get the discrete spectrum $(n+\frac{l}{2})\hbar$ of the **harmonic oscillator** and the continuous spectrum $\mathbb{R}$ for the dilation generator pq. The "eigenprojectors" $\Pi_n^{(l)}$ and $\Pi(\lambda, X)$ are given by some special functions [2].

Other examples can be brought to this case by some functional manipulations [2]. For instance the Casimir element C of $so(l)$ representing angular momentum, which can be written $C = p^2q^2 - (pq)^2 - l(l-1)\frac{\hbar^2}{4}$, has $n(n+(l-2))\hbar^2$ for spectrum. For the **hydrogen atom,** with Hamiltonian $H = \frac{1}{2}p^2 - |q|^{-1}$, the Moyal product on $\mathbb{R}^{2l+2}$ ($l = 3$ in the physical case) induces a star product on $W = T^*S^l$ and the energy levels, solutions of $(H - E) * \phi = 0$, are found to be (from (31), for $l = 3$) :

$E = \frac{1}{2}(n+1)^{-2}\hbar^{-2}$ for the discrete spectrum

$E \in \mathbb{R}_+$ for the continuous spectrum.

It is worth noting that the term $\frac{l}{2}$ in the harmonic oscillator spectrum, obvious source of divergences in the infinite-dimensional case, disappears if the normal star-product is used instead of Moyal - which is one of the reasons it is preferred in field theory.

ii) **Path integrals** are intimately connected to star exponentials. In fact, in quantum mechanics the path integral of the action is nothing but the partial Fourier transform of the star exponential (28) with respect to the momentum variables, for $W = \mathbb{R}^{2l}$ as phase space with the Moyal star-product [35]. For compact groups the star exponential E given by (25) can be expressed in terms of unitary characters using a global coherent state formalism [36] based on the Berezin dequantization of compact group representation theory used in [32] (that gives star-products somewhat similar to normal ordering); the star exponential of any Hamiltonian on G/T (where T is a maximal torus in the compact group G) is then equal to the path integral for this Hamiltonian.

iii) For **field theory** similar results hold. In particular [8] the star exponential of the Hamiltonian of the free scalar field (for the normal star-product) is equal to a path integral. That is only part of the results obtained for field theory [8]. It has indeed been found that other star-products "close to normal" enjoy similar properties and permit nonstandard quantizations of the Klein-Gordon equation (not necessarily leading to a free field theory). And for interacting fields one can show that by transforming the normal star-product by a suitable equivalence one can remove some of the divergences occurring in $\lambda\phi_2^4$ theory, which indicates that **the processus of renormalization is cohomological in essence** (removing an infinite coboundary to get a finite cocycle).

2.6. Pseudodifferential calculus and character of a star-product.

a. Closedness of the star-product arising in pseudodifferential calculus.

Let $g \in \mathcal{A},\ g = \sum_{r=0}^{\infty} \nu^r\ g_r\ (g_r \in N, \nu = \frac{1}{2} i\hbar)$. Let us define

$$\tau(g) = \int g_l\ \omega^l. \tag{32}$$

From (11) we see that, in the pseudodifferential calculus on $\mathbb{R}^{2l}$, $(2\pi)^l\ Tr(\Omega_S(g))$ is equal to $\tau(g)$ modulo multiples of $\hbar$ (and $\hbar^{-1}$).

Therefore τ enjoys the same properties as a trace with respect to the standard star product, which is thus closed. More precisely, for a pseudodifferential operator D (with compact support) on $\mathbb{R}^l$ one can define its symbol (on $W = T^*\mathbb{R}^l$) by [37]

$$\sigma_D(q_0, p) = D\ e^{i(q-q_0).p}|_{q=q_0} \tag{33}$$

so that one has for a product :

$$\sigma_{BD} = \sigma_B\ *_S\ \sigma_D = \sum_{r=0}^{\infty} \frac{i^{-r}}{r!} \frac{\partial^r \sigma_B}{\partial p^r} \frac{\partial^r \sigma_D}{\partial q^r}. \tag{34}$$

These formulas extend [37] to $W = T^*M, M$ Riemannian compact manifold, using a globally defined function $L \in C^\infty(T^*M \times M)$ that can, on canonical charts, be written $L(q_0; p, q) = p.(q - q_0)$. As for standard ordering one can then show (by integration by parts e.g.) that the corresponding star-product on $W = T^*M$ is closed.

b. The character. The deviation from algebra homomorphism of the identity map on $\mathcal{A}$ between the algebras $\mathcal{A}_0$ ($\mathcal{A}$ endowed with commutative product) and $\mathcal{A}_\nu$ ($\mathcal{A}$ endowed with $*_\nu$ product) is expressed by

$$\sigma(f, g) = f * g - fg \quad (f, g \in \mathcal{A}). \tag{35}$$

As for operator algebras [13], since the map τ behaves like a trace, one is thus lead to define (for $f_i \in \mathcal{A}$) :

$$\varphi_{2k}(f_0, f_1, ..., f_{2k}) = \tau(f_0 * \sigma(f_1, f_2) * ... * \sigma(f_{2k-1}, f_{2k})) \tag{36}$$

and $\varphi_{2k} = 0$ except for $l \leq 2k \leq 2l$. This makes sense for a general closed star-product.

DEFINITION. *The character of a closed star-product $*$ is the cohomology class of the cocycle φ with components φ_{2k} defined by (38) in the cyclic cohomology bicomplex.*

The previous considerations then imply :

PROPOSITION. *On $W = T^*M$, M Riemannian compact, the character φ defined by (36) (with standard ordering) coincides with the character defined by the trace on pseudodifferential operators.*

c. Examples. The top component φ_{2l} of φ is given by [14]

$$\varphi_{2l}(f_0, f_1, ..., f_{2l}) = \int f_0 P(f_1, f_2) ... P(f_{2l-1}, f_{2l}) \omega^l$$

$$= \int f_0 \ df_1 \wedge ... \wedge df_{2l}.$$

If $dim\ W = 4$, the computation of φ can be carried out easily and one gets in addition $\varphi_2 = \tilde{C}_2$ (in the notations of (2.2.b)) and thus $b\varphi_2 = -\frac{1}{2}B\varphi_4$. Now, since $HC^2(N) = Z^2(W,\mathbb{C}) \oplus \mathbb{C}$ where Z^2 stands for the closed 2-currents, a change in C_2 will affect the class in HC^4 thus the class of φ, which has then no reason to be an integer - as is the case in the traditional approaches to quantization.

d. Consequences. Combining the above proposition with the algebraic index theory of A. Connes et al. [13] one gets [14] :

THEOREM. *The character φ of the star product of the pseudodifferential calculus belongs to $HC^{ev}(T^*M)$ and is given by the Todd class $Td(T^*M)$ as a current over T^*M.*

Since the trace on compact operators is unique, whenever a star product has an associated Weyl map that includes them in the image, the cyclic cocycle φ will necessarily be proportional to an integral one. But we have seen above that the character is not necessarily integral. Therefore on one hand star quantization (with a closed star-product) can be applied in cases where there are obstructions to the traditional (Bohr-Sommerfeld) quantization; and on the other hand it paves the way for a **generalized index theory where the index is no more necessarily an integer.**

For finite-dimensional manifolds, because the trace formula (9) is exact (not modulo $\hbar$ terms) in the Moyal case, the corresponding "symmetric pseudodifferential calculus" should be of special interest. And in the infinite-dimensional case since orderings "close to normal" permit a cohomological interpretation of cancellation of infinities [8], the corresponding "complex pseudodifferential calculus" appears to be an appropriate framework.

3. Quantum groups and star-products.

3.1. The background. *a. Yang-Baxter equations and Hopf algebras.*

As mentioned in the introduction, "quantum groups" appeared around 1980 in the quantization of 2-dimensional integrable models [9]. A basic ingredient there is an equation which the LOMI group called **Yang-Baxter equation.** If $R \in End(V \otimes V)$, where V is a vector space, and if we define $R^{12} = R \otimes I \in End(V \otimes V \otimes V)$ and similarly for R^{13}, R^{23}, that equation writes

$$R^{12}\, R^{13}\, R^{23} = R^{23}\, R^{13}\, R^{12} \text{ with } R^{12}\, R^{21} = I. \tag{37}$$

This is the quantum Yang-Baxter equation (QYBE). Writing

$$R = I + \sum_{j=1}^{\infty} \nu^j\, r_j \in End(V \otimes V)[[\nu]] \tag{38}$$

one gets (at order 2) the classical Yang-Baxter equation (CYBE) :

$$[r_1^{12}, r_1^{13}] + [r_1^{12}, r_1^{23}] + [r_1^{13}, r_1^{23}] = 0 = r_1^{12} + r_1^{21} \tag{39}$$

Looking for solutions lead to a strange modification of the $sl(2)$ commutation relations, which later was extended to a wide array of Lie groups and became systematized by Drinfeld in the framework of **Hopf algebras** [10]. These are **bialgebras**, i.e. associative algebras $\mathcal{A}$ with a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, that posess in addition a counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ and antipode map $S : \mathcal{A} \to \mathcal{A}$, satisfying the expected (and well-known) relations and commutative diagrams.

b. Poisson-Lie groups and Lie bialgebras. The typical example is the associative algebra N of functions over a Lie group G with pointwise multiplication, comultiplication $\Delta : N \to N \otimes N$ defined (by duality) from the group multiplication $G \times G \to G$, counit defined by the value at the identity e of G and antipode by the value at the inverse (in G). The group is said to be a **Poisson-Lie group** when it has a Poisson structure Λ for which the above Δ is a Poisson morphism, i.e.

$$\Delta P(u, v) = P(\Delta u, \Delta v) \tag{40}$$

where P denotes the Poisson bracket in G and $G \times G$, defined by Λ.

The infinitesimal version (dual to it in the sense that the enveloping algebra $\mathcal{U}(g)$ can be considered as a space of distributions on G with support at the identity) is the notion of **Lie bialgebras.** It is a Lie algebra g with a "compatible" bracket φ^* on its dual g^* (i.e. $\varphi : g \to g \otimes g$ is a 1-cocycle for the adjoint action of g on $g \otimes g$).

A case of special interest (triangular Poisson-Lie group [10]) is when φ is the coboundary of $r \in g \wedge g$ such that the Poisson-Lie structure Λ on the corresponding group G is the difference of the left and right invariant skew-symmetric 2-tensors defined by r : this r is solution of the CYBE (39).

c. Their quantization. It is now natural to try and "quantize" (40). This can indeed be done. More precisely [38,39] on $N = C^\infty(G)$ where G is a triangular Poisson Lie group there exists an invariant star product defined by invariant bidifferential operators $F_k(x,y)$, i.e. by a formal series $F = I + \sum_{k=1}^{\infty} \nu^k F_k$ such that the associativity condition for the star product gives a solution $S(x,y) = F^{-1}(y,x)F(x,y)$ of the QYBE (37) on $\mathcal{U}(g)[[\nu]]$.

Moreover, from that invariant star product one can build [38] an equivalent one ($*'$, non invariant) such that (with the same coproduct Δ on N) :

$$\Delta(u *' v) = \Delta u *' \Delta v. \tag{41}$$

These results were more or less implicit in Drinfeld's work but precise statements and proofs can be found in [39], together with related results.

In the dual approach [11] it is the coproduct Δ, on a completion $\mathcal{U}_\nu(g)$ of the enveloping algebra $\mathcal{U}(g)$, that is deformed to Δ_ν. The simplest example is when $g = sl(2)$ with generators H and $E^\pm$ (or more generally H_α and $E_\alpha^\pm$ for simple roots α on a simple Lie algebra g) where

$$\Delta_\nu(H) = H \otimes 1 + 1 \otimes H$$

$$\Delta_\nu(E^\pm) = E^\pm \otimes \exp(\frac{1}{2}\nu H) + \exp(-\frac{1}{2}\nu H) \otimes E^\pm$$

(see [40] for a very detailed exposition).

A quantum group can therefore be seen as a non commutative star-product deformation of the Hopf algebra of functions on a Poisson-Lie group, or as a non cocommutative deformation of its dual.

3.2. Quantized Universal Enveloping Algebras as star-product algebras. *a. Example* [41]. First let us remark that the following functions on $\mathbb{R}^2$:

$$H = pq, \quad J_+ = q, \quad J_-^\eta = (1 - \cos \eta pq)/\eta^2 q \tag{42}$$

satisfy, with respect to Poisson brackets, the same commutation relations as those of the QUEA $\mathcal{U}_\eta(sl(2))$. Since H and J_+ are preferred observables this is also true for Moyal brackets and in particular

$$M(J_+, J_-^\eta) = -\eta^{-1}\sin(\eta pq) \tag{43}$$

Now from (30) one gets that a strong invariance condition

$$f(*tX/i\hbar) = \alpha(t)\ f(\beta(t)X/i\hbar) \tag{44}$$

holds when f is an exponential or a trigonometric function (sine or cosine), where on the left-hand side we mean that we take (Moyal) star powers in the power series expansion of f. Therefore (after a slight rescaling) (43) can be rewritten with ($\sin *$) instead of sin, and the defining QUEA relations can be expressed entirely in the $*$-product algebra generated by the Lie generators.

b. Generalizations. The same can be done for all higher rank simple algebras, e.g. the A_n series [42]. The cubic Serre relations are also expressed in the $*$-product algebra.

One can show [41] that **for Moyal product, the only power series for which (44) holds, are sine, cosine and exponential** (with exception of these series truncated at order two). Therefore if one requires that the "quantum" commutation relations be a deformation of the "classical" ones, the strong invariance condition (44) alone shows that the algebra found in the litterature is unique (except for the truncated sine $f(x) = a_1 x + a_3 x^3$).

For other star-products than Moyal there are even less functions f satisfying (44). For instance [42] for a family of star-products interpolating between standard and antistandard ordering, the only function is the exponential.

In conclusion, we have seen that the QUEAs can be realized essentially uniquely, as star-product algebras with a quantization parameter $\hbar$ (in the star-product) different from the QUEA parameter. We have in fact a **2-parameter deformation**.

3.3. A universal rigid model for quantum groups [43].

a. The model. Let us start with $G = SU(2)$. Denote by (π_n, V_n) its $2n+1$-dimensional representation ($2n$ an integer) by matrices in $\mathcal{L}(V_n)$. The direct product $A = \Pi_{n=0}^{\infty} \mathcal{L}(V_n)$, endowed with the product topology and algebra law, becomes a Fréchet algebra into which both G and the enveloping algebra $\mathcal{U}(g)$ can be imbedded by $x \mapsto (\pi_n(x))$. These imbeddings are total because A is the bicommutant of the direct sum representation $\pi = \sum \pi_n$ on $V = \sum V_n$, and every π_n extends by continuity to a representation of A.

b. Universality. We have :

PROPOSITION. *If $t \notin 2\pi\mathbf{Q}$, the QUE algebras $\mathcal{U}_t(g)$ can be imbedded into a dense subalgebra A_t of A; π_n are still a complete set of representations of A_t, and the Hopf structures (coproduct Δ_t, counit ϵ_t, and antipode S_t) on $\mathcal{U}_t(g)$ have unique extensions to topological Hopf structures (with equivalent coproducts) on A, when $A \otimes A$ is endowed with the completed projective tensor topology so that*

$$A \otimes A \approx \Pi_{n,p}\ \mathcal{L}(V_n \otimes V_p).$$

The classical limit is $\mathcal{U}_0(g) \approx \mathcal{U}(g) \otimes \theta$ where $\theta^2 = 1$ (a parity), with coproduct Δ_0 equivalent to Δ_t : there exists $P(t) \in A \otimes A$ such that $\Delta_t = P(t)\ \Delta_0\ P(t)^{-1}$. The R-matrix is $R(t) = P(-t)\ P(t)^{-1}$ and can be chosen so that A_t is a quasitriangular Hopf algebra ($q = e^{it}$ in the usual notations).

c. The star-product. The topological dual A^* is isomorphic to $\oplus\mathcal{L}(V_n) = H$, endowed with the inductive topology, and can thus be viewed as polynomial functions on the complex extension $G_{\mathbf{C}}$ of G; as Hopf algebra,

$$H \approx \mathbf{C}[a, b, c, d]/(ad - bc = 1)$$

where the product is that of functions and the coproduct is defined by the product in G. Each co-associative coproduct Δ_t induces an associative star-product on H such that $(a+d)^{*k} = (a+d)^k$, $\forall k \in \mathbf{N}$, and conversely if this is true for a $*$-product compatible with the usual coproduct Δ_0, this $*$ is induced by a coproduct $\Delta_t = P(t)\Delta_0 P(t)^{-1}$ on A : *we recover Manin's description of functions of non-commutative arguments.*

d. Rigidity as bialgebra. Let B be a bialgebra, an associative algebra B with coproduct $\Delta : B \to B \otimes B$. We then define [18] in a natural way the notion of isomorphism of bialgebras (associative algebra isomorphism with equivalence of coproducts), and of deformations of such structures, where we denote by $H(B,\Delta)$ the relevant 2-cohomology.

It can be shown [43] that if (for Hochschild cohomologies)

$$\tilde{H}^2(B) = 0 = \tilde{H}^1(B, B \otimes B)$$

then $H(B,\Delta) = 0$. Whitehead's lemmas therefore prove that (A, Δ_t) **is rigid for all** $t \notin 2\pi\mathbf{Q}$.

REMARK. These results can be extended to all simple compact groups. From this follows that (in the generic case) the QUEAs associated with such groups have a hidden group structure, that of the original group, included in a universal (reflexive) topological Hopf algebra completion.

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