A Group Algebra for Inductive Limit Groups.
Continuity Problems of the Canonical Commutation Relations.

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Abstract *Given an inductive limit group $G = \lim\limits_{\to} G_\beta$, $\beta \in \Gamma$ where each G_β is locally compact, and a continuous two–cocycle $\rho \in Z^2(G,\,\mathbf{T})$, we construct a C^*–algebra $\mathcal{L}$ for which the twisted discrete group algebra $C^*_\rho(G_d)$ is imbedded in its multiplier algebra $M(\mathcal{L})$, and the representations of $\mathcal{L}$ are identified with the strong operator continuous ρ–representations of G . If any of these representations are faithful, the above imbedding is faithful. When G is locally compact, $\mathcal{L}$ is precisely $C^*_\rho(G)$, the twisted group algebra of G , and for these reasons we regard $\mathcal{L}$ in the general case as a twisted group algebra for G . Applying this construction to the CCR-algebra over an infinite dimensional symplectic space $(S,\,B)$, we realise the regular representations as the representation space of the C^*–algebra $\mathcal{L}$, and show that pointwise continuous symplectic group actions on $(S,\,B)$ produce pointwise continuous actions on $\mathcal{L}$, though not on the CCR–algebra. We also develop the theory to accommodate and classify "partially regular" representations, i.e. representations which are strong operator continuous on some subgroup H of G (of suitable type) but not necessarily on G , given that such representations occur in constrained quantum systems.*

Keywords: twisted group algebra, inductive limit group, discontinuous group representation, CCR–algebra, symplectic action, quantum field.

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Running headline: Group Algebra for Inductive Limit Groups

1. Introduction.

It is a well known fact of life that the C*–algebra of the CCRs has a continuity problem in that it admits nonregular representations, and the natural C*–dynamical systems on it, defined from nondiscrete symplectic actions, do not have pointwise norm continuity. These problems have inspired a long line of papers (cf. [1–3] and references therein) of which this is one. There is however a simple idea which explains the continuity problems on a structural level and allows one to circumvent it, using only standard C*–techniques, and it appears not to have been fully exploited for the CCR discontinuity problem yet. Part of the construction seems to be known, having appeared in various disguises, e.g. in Segal [18], and at greater length in Kastler [19], using twisted convolution algebras;– here we follow that line of thought through the use of twisted group algebras. It reduces to the problem of defining a a suitable " twisted group C*–algebra" for groups which are inductive limits of locally compact groups.

We hasten to remark that whilst the continuity problem of the CCRs is generally regarded as a blemish on the C*–theory of bosons (it does not occur for the CAR algebra), we did put it to good use in [10] (with C.A. Hurst) where it was demonstrated that in the presence of selfadjoint constraints, linear in the fields, the physical representations must be nonregular on the nonphysical parts of the theory but regular on the physical part. This was in order to avoid continuous spectrum problems for the constraint conditions. Hence in some contexts it is imperative that there are such "partially regular" representations on the field algebra. In this note we also wish to start a structure theory for these representations.

We begin with the CCR–algebra. Let S be a separable real linear topological space with a nondegenerate continuous symplectic form B given on it. The CCR algebra $\overline{\Delta(S,\,B)}$ is then the unique (simple) C*–algebra generated by unitaries $\big\{\,\delta_x\;\big|\;x\in S\,\big\}$ which satisfy the Weyl relations:

$$\delta_x\delta_y = e^{iB(x,\,y)/2}\delta_{x+y}\qquad \forall\,x,\;y\in S$$

(cf. [4, 5] for the uniqueness proof). Then $\|\delta_x-\delta_y\|=2$ if $x\neq y$ and hence for a fixed $x\in S$ the maps $\lambda\to\delta_{\lambda x}$, $\lambda\in\mathbb{R}$ are not continuous with respect to the C*–norm. The regular representations of $\overline{\Delta(S,\,B)}$ are those representations

π for which the one parameter groups $\lambda \to \pi(\delta_{\lambda x})$ (with fixed x) are weak operator continuous for all $x \in S$. (Hence in a regular representation Stone's theorem can be applied to obtain selfadjoint generators –the quantum fields for some S , B – for these groups). The regular states of $\overline{\Delta(S,\, B)}$ are those states with regular GNS–representations. However the regular states do not comprise the state space of $\overline{\Delta(S,\, B)}$ and the question arises about the nature and existence of a C*–algebra with state space exactly the regular states. We would also like to have a more algebraic characterisation of regular representations than the topological one above.

Let $\mathrm{Sp}\,(S,\, B)$ denote the group of linear symplectic transformations on S with the topology of pointwise convergence (i.e. given a net $\{T_\nu\} \subset \mathrm{Sp}\,(S,\, B)$ then it converges to $T \in \mathrm{Sp}\,(S,\, B)$ if $T_\nu(x) \to T(x) \quad \forall\, x \in S$). In practice we are frequently given a continuous group homomorphism $\gamma : \mathcal{G} \to \mathrm{Sp}\,(S,\, B)$ where $\mathcal{G}$ is some topological group of physical symmetries. Such a homomorphism defines an action $\alpha : \mathcal{G} \to \mathrm{Aut}\,\overline{\Delta(S,\, B)}$ by $\alpha_g(\delta_x) := \delta_{\gamma_g(x)} \quad \forall\, g \in \mathcal{G},\ x \in S$. However since $\|\delta_x - \delta_y\| = 2$ if $x \neq y$ the action α will only be pointwise norm continuous if $\mathcal{G}$ is a discrete group (assuming γ is injective). This has given rise to the construction of an auxiliary C*–algebra on which α is norm continuous [1], but here we consider construction more natural for the CCRs.

Briefly then, in this note we aim to find a C*–algebra with state space exactly the regular states, and on which the continuous symplectic actions define pointwise norm continuous actions. We would also like to characterise regular states and representations algebraically, and give a framework to deal with partially regular states and representations. This will be almost trivial when $\dim S < \infty$, and the main task is to extend the finite dimensional construction to the infinite dimensional case. We will develop the basic construction in a more general context than is warranted by the CCR algebra alone, because of the possible utility for some infinite dimensional groups. The content of Sect. 2 is well–known, with the only (presumed) novelty the application to the regularity question of the CCRs. Readers in a hurry can start with Sect. 3 after skimming Sect. 2 up to theorem 2.4.

2. The Basic Idea: Finite Dimensional Case.

Assume in this section that $\dim S < \infty$, so $S \cong \mathbb{R}^n \times \mathbb{R}^n$ and the symplectic form is

$$B((x_1,\, x_2)\,,\,(y_1,\, y_2)) = x_1 \cdot y_2 - x_2 \cdot y_1 \qquad \text{for} \quad x_i,\, y_i \in \mathbb{R}^n\,.$$

In this case we know by the Stone–Von Neumann theorem that there is a unique (up to unitary equivalence) irreducible regular representation π of $\overline{\Delta(S,\, B)}$. Denote: $\sigma(x,\, y) := \exp \frac{i}{2} B(x,\, y)$ and note that σ is a jointly continuous two–cocycle on the space S , regarded as an Abelian group. The regularity condition, that $\lambda \to \pi(\delta_{\lambda x})$ is weak operator continuous, is equivalent to the requirement that the σ–projective representation of S given by $\pi(x) := \pi(\delta_x)$, $x \in S$ is weak operator continuous. Nevertheless, there are many nonregular states and representations and many regular states of $\overline{\Delta(S,\, B)}$, although only one regular irreducible representation. These are best studied through twisted group algebras, which we next define, following [11, 6].

Let G be a nondiscrete locally compact group with Haar measure μ , and let $\rho \in Z^2(G,\, \mathbf{T})$ be a Borel 2–cocycle with $\rho(e,\, g) = \rho(g,\, e) = 1 \quad \forall\, g \in G$. We further assume that the maps $g \to \rho(g,\, x)$ are continuous for μ–almost all x . Start with the dual space $C_0(G)^*$, which we know by the Riesz–Markov theorem to be identified with the Banach space of bounded complex Baire measures on G . Each Baire measure has a unique extension to a regular Borel measure [21] on G . Denote this Banach space of regular Borel measures by $\mathcal{M}(G)$. Define twisted convolution and involution on $\mathcal{M}(G)$ by:

Def. 2.1 For $\gamma,\, \nu \in \mathcal{M}(G)$, $f \in C_0(G)$, define

$$\int f(t)\, d(\gamma * \nu)(t) := \int\!\!\int f(st)\rho(s,\, t)\, d\nu(t)\, d\gamma(s)$$

$$\text{and} \qquad \int f(t)\, d\gamma^*(t) := \overline{\int \overline{f}(t^{-1})\, \rho(t^{-1},\, t)\, d\gamma(t)}$$

then (with a further condition on ρ) these operations make $\mathcal{M}(G)$ into a Banach *–algebra. There are two distinguished subalgebras of $\mathcal{M}(G)$: identify $L^1(G)$ as the closed *–ideal of $\mathcal{M}(G)$ consisting of finite measures which are absolutely continuous w.r.t. the Haar measure μ of G . Then by writing $d\gamma(s) = f(s)d\mu(s)$, $f \in L^1(G)$ for these, we obtain from 2.1:

2.2 for $f,\, g \in L^1(G)$, twisted convolution and involution is:

$$f * g(x) = \int_G f(y)\, g(y^{-1}x)\, \rho(y,\, y^{-1}x)\, d\mu(y)$$
$$f^*(x) := \Delta(x)^{-1}\overline{\rho(x,\, x^{-1})}\ \overline{f(x^{-1})}\ ,$$

where Δ denotes the modular function.

Note that $f^{**} = f$ implies that $\rho(x^{-1},\, x) = \overline{\rho(x,\, x^{-1})}$. Then $L^1(G)$ becomes a Banach *–algebra, denoted $L^1_\rho(G)$, the closure of which in the enveloping C*–norm is the twisted group algebra of G , denoted $C^*_\rho(G)$. Since G is nondiscrete, $C^*_\rho(G)$ is not unital. The other distinguished subalgebra of $\mathcal{M}(G)$ is the Banach *–algebra generated by the Dirac measures $\left\{\, \delta_x \;\middle|\; x \in G \,\right\}$, which clearly is just $L^1(G_d) = \ell^1(G)$, and where G_d denotes G with the discrete topology. (The notation δ_x here extends the previous notation for the generating unitaries of the CCR–algebra, as we shall shortly see). The multiplication in $\mathcal{M}(G)$ specializes to:

2.3
$$(\delta_y * f)(x) = f(y^{-1}x)\, \rho(y,\, y^{-1}x)\ ,$$
$$(f * \delta_y)(x) = f(xy^{-1})\, \rho(xy^{-1},\, y)\, \Delta(y)^{-1}\ , \qquad \forall\, f \in L^1_\rho(G),\ \ x,\, y \in G$$

and
$$\delta_x * \delta_y = \rho(x,\, y)\, \delta_{xy}\ , \qquad \delta^*_x = \overline{\rho(x^{-1},\, x)}\ \delta_{x^{-1}}\ .$$

Observe that

$$[\rho(x,\, x^{-1})]^2 = (\delta_x\delta_{x^{-1}})^2 = \delta_x(\delta_{x^{-1}}\delta_x)\delta_{x^{-1}} = \rho(x^{-1},\, x)\, \rho(x,\, x^{-1})$$
$$\text{so} \qquad \rho(x,\, x^{-1}) = \rho(x^{-1},\, x) = \overline{\rho(x,\, x^{-1})} = \pm 1\ .$$

This is the condition which we will henceforth assume for ρ to ensure $\mathcal{M}(G)$ is a *–Banach algebra.

Let $H \subset G$ be a proper closed locally compact subgroup of G . Then its measures $\mathcal{M}(H)$ are embedded in $\mathcal{M}(G)$ as those measures on G with support in H , and the multiplication operation in $\mathcal{M}(G)$ also produces an action of $\mathcal{M}(H)$ on $L^1_\rho(G)$. In particular it gives an L^1–continuous embedding of

$L^1_\rho(H)$ in $\mathcal{M}(G)$ (the restriction of ρ to H is still denoted by ρ) such that $L^1_\rho(H)$ is still in the relative multiplier of $L^1_\rho(G)$.

Both of the L^1–embeddings above will extend to their enveloping C*–algebras, using the universal property of these cf. Dixmier 1.3.7 [8], and since $L^1_\rho(G)$ contains an approximate identity for $\mathcal{M}(G)$, we actually obtain C*–continuous embeddings in the multiplier algebra of $C^*_\rho(G)$, i.e.

$$C^*_\rho(G_d) \subset M(C^*_\rho(G)), \qquad \text{and} \qquad C^*_\rho(H) \subset M(C^*_\rho(G)) \; .$$

We will always assume that G is nondiscrete, in which case $C^*_\rho(G)$ does not have an identity. Further information about $C^*_\rho(G)$ is in [6], and the main aspect of interest here is the universal property that there is a bijection between the set of nondegenerate representations of $C^*_\rho(G)$ and the set of weak operator continuous ρ–projective representations of G (cf. Theorem 2.4 below).

In particular, if $G = S$, $\rho = \sigma$, then $C^*_\sigma(S_d) = \overline{\Delta(S,\, B)}$, indeed definition 2.1 reduces precisely to the definition given by Manuceau [7] for the CCR–algebra. For general locally compact groups G we will call those representations of $C^*_\rho(G_d)$ which come from weak operator continuous ρ–representations of G *regular* representations, the others are *nonregular* and a state is regular or not according to whether its GNS–representation is. This agrees with the usual physics terminology on $\overline{\Delta(S,\, B)}$, $\dim S < \infty$, and should not be confused with the left or right regular representations of G . We will also call a state or representation of $M(C^*_\rho(G))$ (non)regular if its restriction to $C^*_\rho(G_d)$ is (non)regular. Denote by $\mathrm{Rep}\,\mathcal{A}$ the set of nondegenerate representations of an algebra $\mathcal{A}$ and by $\mathrm{Rep}_\rho G$ the set of nondegenerate weak operator continuous unitary ρ–representations of a topological group G . If we relax the weak operator continuity, we use the notation $\mathrm{Rep}^0_\rho G$ $(= \mathrm{Rep}_\rho G_d$).

Say a representation $\pi \in \mathrm{Rep}^0_\rho G$ is Borel if it is a Borel map $\pi : G \to U(\mathcal{H}_\pi)$ where $U(\mathcal{H}_\pi)$ is equipped with the strong operator topology. We allow nonseparable representation spaces, hence Borel representations need not be strong operator continuous. For a Borel representation π we can use the Bochner integrals

$$\pi(\nu) = \int_G \pi(g)\, d\nu(g) \in \mathcal{B}(\mathcal{H}_\pi) \qquad \forall\, \nu \in \mathcal{M}(G)\; .$$

In the next theorem we give the well–known structure of representations on $M(C^*_\rho(G))$ (we include the proof of 2.4(1) for completeness, cf. Dixmier 13.3.1 [8] and Kastler [19]), but being explicit in our inclusion of nonregular representations in the analysis. Each ρ–representation π of G defines a representation of $C^*_\rho(G_d)$, and we define a "canonical extension" for π which will be central to the developments in Sect. 3.

Theorem 2.4. *Let G be a nondiscrete locally compact group, ρ as above,*

(1) *let $\pi \in \mathrm{Rep}^0_\rho G$ be a Borel representation, then the formula $\widetilde{\pi}(\nu) = \int_G \pi(g)\, d\nu(g)$, $\nu \in \mathcal{M}(G)$ defines a continuous *–representation $\widetilde{\pi} \in \mathrm{Rep}\,\mathcal{M}(G)$ on $\mathcal{H}_\pi$ such that*

  (*i*) $\widetilde{\pi}(\delta_g) = \pi(g) \quad \forall\, g \in G$,

  (*ii*) *$\widetilde{\pi} \in \mathrm{Rep}\,\mathcal{M}(G)$ is irreducible iff $\pi \in \mathrm{Rep}^0_\rho G$ is irreducible,*

  (*iii*) *$\widetilde{\pi}$ is nondegenerate on $L^1_\rho(G)$ iff $\pi \in \mathrm{Rep}^0_\rho G$ is strong operator continuous.*

(2) *Any $\pi \in \mathrm{Rep}\, C^*_\rho(G)$ extends uniquely on the same Hilbert space to a representation $\hat{\pi}$ of $M(C^*_\rho(G))$ preserving irreducibility, and it is regular on $C^*_\rho(G_d)$.*

(3 *i*) *Let $\mathcal{E} := C^*\big(C^*_\rho(G) \cup C^*_\rho(G_d)\big)$. Then a representation $\pi \in \mathrm{Rep}\,\mathcal{E}$ has a unique decomposition $\pi = \pi_1 \oplus \pi_2$ such that π_i are both nondegenerate and $C^*_\rho(G) \subseteq \mathrm{Ker}\,\pi_2$ and $\pi_1(C^*_\rho(G))$ is strongly dense in $\pi_1(\mathcal{E})$. Then*

$$\pi_1(f) = \int_G \pi_1(\delta_g)\, f(g)\, d\mu(g) \qquad \forall\, f \in L^1_\rho(G)\,,$$

where μ is the Haar measure on G .

(3 *ii*) *A representation $\pi \in \mathrm{Rep}\, C^*_\rho(G_d)$ has an extension on the same Hilbert space to a representation $\breve{\pi} \in \mathrm{Rep}\,\mathcal{E}$ with $C^*_\rho(G) \subseteq \mathrm{Ker}\,\breve{\pi}$. This extension is unique if π is irreducible and nonregular.*

(3 *iii*) *A representation $\pi \in \mathrm{Rep}\, C^*_\rho(G_d)$ has a* **canonical extension** *$\widetilde{\pi}$ to $\mathcal{E}$ on the same space $\mathcal{H}_\pi$, in the sense that for any other extension π' to $\mathcal{E}$ on $\mathcal{H}_\pi$ we have $\mathcal{H}_{\widetilde{\pi}_1} \supseteq \mathcal{H}_{\pi'_1}$ where $\widetilde{\pi} = \widetilde{\pi}_1 \oplus \widetilde{\pi}_2$ and $\pi' = \pi'_1 \oplus \pi'_2$ are the unique decompositions of (3 *i*) . This defines a*

decomposition $\pi = \pi_R \oplus \pi_N$ *where* $\pi_R := \widetilde{\pi}_1 \,\big|\, C^*_\rho(G_d)$ *is the* **regular part** *of* π *, and* $\pi_N := \widetilde{\pi}_2 \,\big|\, C^*_\rho(G_d)$ *is the* **irregular part.**

(3 *iv*) *A representation* $\pi \in \operatorname{Rep} C^*_\rho(G_d)$ *is regular iff its canonical extension* $\widetilde{\pi}$ *is nondegenerate on* $C^*_\rho(G)$ *, i.e.* $\widetilde{\pi}_2 = 0$ *.*

(4) *There are bijections between the following sets:*

$$\mathcal{P} := \big\{ \pi \in \operatorname{Rep} C^*_\rho(G_d) \;\big|\; \pi \text{ is regular} \big\}$$
$$\operatorname{Rep} C^*_\rho(G) := \big\{ \text{ nondegenerate representations of } C^*_\rho(G) \,\big\}$$
$$\operatorname{Rep}_\rho(G) := \{\text{weak operator continuous } \rho\text{–representations of } G\,\}$$

which preserve irreducibility, and these are obtained from the extensions above, so we have a bijection $t : \operatorname{Rep} C^*_\rho(G) \to \mathcal{P}$ *given by* $t(\pi) = \hat{\pi} \,\big|\, C^*_\rho(G_d)$ *where* $\hat{\pi}$ *is the unique extension of (2), and* $t^{-1}(\pi) = \widetilde{\pi} \,\big|\, C^*_\rho(G)$ *where now* $\widetilde{\pi}$ *is the canonical extension of* (3 *iii*) *.*

Proof: **(1)** Since $\nu \in \mathcal{M}(G)$,

$$\|\widetilde{\pi}(\nu)\| \le \sup_g \|\pi(g)\| \int d|\nu(g)| = \|\nu\|_1 < \infty$$

and hence $\widetilde{\pi}(\nu) \in \mathcal{B}(\mathcal{H}_\pi)$ and the map $\nu \to \widetilde{\pi}(\nu)$ is L^1–continuous .

$$\begin{aligned}\widetilde{\pi}(\gamma * \nu) &= \int \pi(g)\, d(\gamma * \nu)(g) = \int\!\!\int \pi(sg)\, \rho(s,\, g)\, d\nu(g)\, d\gamma(s) \\ &= \int\!\!\int \pi(s)\cdot\pi(g)\, d\nu(g)\, d\gamma(s) = \widetilde{\pi}(\gamma)\cdot\widetilde{\pi}(\nu)\end{aligned}$$

for all $\gamma,\, \nu \in \mathcal{M}(G)$, $\pi \in \operatorname{Rep}_\rho G$. Moreover for all $\nu \in \mathcal{M}(G)$ and $\eta,\, \xi \in \mathcal{H}_\pi$ we have

$$\begin{aligned}(\eta,\, \widetilde{\pi}(\nu^*)\xi) &= \int \big(\eta,\, \pi(g)\xi\big)\, d\nu^*(g) = \overline{\int \overline{(\eta,\, \pi(g^{-1})\xi)}\, \rho(g^{-1},\, g)\, d\nu(g)} \\ &= \overline{\int \big(\rho(g,\, g^{-1})\, \pi(g)^*\xi,\, \eta\big)\, \rho(g,\, g^{-1})\, d\nu(g)} \\ &= \overline{\int \big(\xi,\, \pi(g)\eta\big)\, d\nu(g)} = \big(\widetilde{\pi}(\nu)\eta,\, \xi\big)\end{aligned}$$

where we made use of $\pi(g^{-1}) = \rho(g,\, g^{-1})\pi(g)^*$ and $\rho(g,\, g^{-1}) = \overline{\rho(g^{-1},\, g)}$. Thus $\widetilde{\pi}(\nu^*) = \widetilde{\pi}(\nu)^*$, and so $\widetilde{\pi}$ is a continuous $*$–representation of $\mathcal{M}(G)$.

(i) Clearly $\widetilde{\pi}(\delta_g) = \int \pi(h)\, d\delta_g(h) = \pi(g)$.

(ii) That $\widetilde{\pi}(\nu)$ on $\mathcal{M}(G)$ is irreducible iff π is on G , follows from (i) .

(iii) If $\widetilde{\pi}$ is nondegenerate on $L^1_\rho(G)$, then the equation

$$\pi(g)\,\widetilde{\pi}(f) = \widetilde{\pi}(\delta_g * f) \quad \forall\, f \in L^1_\rho(G)\,,\ g \in G$$

determines the operator $\pi(g)$ uniquely on $\mathcal{H}_\pi$, and it is easily seen to be unitary. Now to show that π is strong operator continuous,

$$\begin{aligned}\big\|\big(\pi(g) - \pi(e)\big)\widetilde{\pi}(f)\,\xi\big\| &= \|\widetilde{\pi}(\delta_g * f - f)\xi\| \\ &\le \|\widetilde{\pi}(\delta_g * f - f)\| \cdot \|\xi\| \\ &\le \|\delta_g * f - f\|_{L^1} \cdot \|\xi\| \qquad (*)\end{aligned}$$

using the L^1–continuity of $\widetilde{\pi}$. But

$$\begin{aligned}\|\delta_g * f - f\|_{L^1} &= \int_G \big|f(g^{-1}x)\,\rho(g,\,g^{-1}x) - f(x)\big|\, d\mu(x) \\ &= \int_G \big|f(x)\,\rho(g,\,x) - f(gx)\big|\, d\mu(x)\ .\end{aligned}$$

Now for all $f \in C_c(G)$ (the continuous functions of compact support) we have:

$$\begin{aligned}\big|f(x)\,\rho(g,\,x) - f(gx)\big| &\le \big|f(x)\big(\rho(g,\,x) - 1\big) - \big(f(gx) - f(x)\big)\big| \\ &\le \big|f(x)\big| \cdot \big|\rho(g,\,x) - 1\big| + \big|f(gx) - f(x)\big| \\ &\longrightarrow 0 \quad \text{as} \quad g \to e\end{aligned}$$

for μ–almost all x , using the assumed continuity property of ρ . Thus, invoking the locally compactness of G to apply the Lebesgue dominated convergence theorem, we conclude that $\|\delta_g * f - f\|_{L^1} \to 0$ as $g \to e$ for all $f \in C_c(G)$. So since $\widetilde{\pi}\big(C_c(G)\big)\mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$ and $\pi(g) - 1$ is bounded, we obtain from the inequality $(*)$ that π is strong operator continuous on G .

Conversely, let $\pi \in \mathrm{Rep}^0_\rho G$ be strong operator continuous. Let J be a set of decreasing neighbourhoods of $e \in G$, ordered by inclusion, with intersection $\{e\}$ and let $u_i \in L^1_+(G)$ for $i \in J$ be a function with ess supp$(u_i) \subset i$ and $\int u_i(g)\, d\mu(g) = 1$. By strong operator continuity of π , we see that for each $\xi \in \mathcal{H}_\pi$, $\varepsilon > 0$, there is an $i \in J$ such that $\|(\pi(g) - 1)\xi\| < \varepsilon$ for all $g \in i$. Thus

$$\|(\widetilde{\pi}(u_i) - 1)\,\xi\| = \|\int u_i(g)\,(\pi(g) - 1)\xi\, d\mu(g)\|$$

$$\leq \sup_{g\in i} \|(\pi(g) - 1)\xi\| \int u_i(g)\, d\mu(g) < \varepsilon\ .$$

So $\widetilde{\pi}(u_i) \to 1$ in the strong operator topology and thus $\widetilde{\pi}\,\Big|\, L^1_\rho(G)$ is nondegenerate.

(2) Since $C^*_\rho(G)$ is a closed two sided ideal of $M(C^*_\rho(G))$, the assertion follows directly from Dixmier 2.11.1 [8] and (1) above. That it is regular on $C^*_\rho(G_d)$ follows from the first part of the proof of $(1\, iii)$.

(3i) Since $C^*_\rho(G)$ is an ideal of $\mathcal{E}$, the first part follows from Dixmier 2.11.1 [8]. To see that

$$\pi_1(f) = \int_G \pi_1(\delta_g)\, f(g)\, d\mu(g) \qquad \forall\, f \in L^1_\rho(G)\ :$$

observe that since π_1 is nondegenerate on $C^*_\rho(G)$, and we know that every nondegenerate representation of $C^*_\rho(G)$ comes from a regular ρ-representation of G via an integral as in the formula, there must be a regular representation π_0 of $C^*_\rho(G_d)$ on $\mathcal{H}_{\pi_1}$ such that

$$\pi_1(f) = \int_G \pi_0(\delta_g)\, f(g)\, d\mu(g) \qquad \forall\, f \in L^1_\rho(G)\ . \tag{$*$}$$

Thus $\pi_1 = \pi_0$ a.e. on G . From the uniqueness of extensions of nondegenerate representations from $C^*_\rho(G)$ to $\mathcal{E}$ on the same space, we conclude that $\pi_1 = \pi_0$.

(3ii) To see that an extension $\breve{\pi}$ exists, observe that

$$\mathcal{E} = \overline{L\big(C^*_\rho(G) \cup C^*_\rho(G_d)\big)} = \overline{L\big(L^1_\rho(G) \cup L^1_\rho(G_d)\big)}\,,$$

so since G is nondiscrete,

$$L^1_\rho(G) \cap L^1_\rho(G_d) = \{0\}\ ,$$

we have

$$L\big(L^1_\rho(G)\cup L^1_\rho(G_d)\big)/L^1_\rho(G)\cong L^1_\rho(G_d)$$

hence any $\pi\in\mathrm{Rep}\,L^1_\rho(G_d)=\mathrm{Rep}\,C^*_\rho(G_d)$ has a lifting $\breve{\pi}$ on the same space to $L(L^1_\rho(G)\cup L^1_\rho(G_d))$ for which $L^1_\rho(G)\subset\mathrm{Ker}\,\breve{\pi}$. By continuity, extend $\breve{\pi}$ to $\mathcal{E}$ to obtain an extension $\breve{\pi}$ of $\pi\in\mathrm{Rep}\,C^*_\rho(G_d)$ to $\mathcal{E}$ on the same space, and with $C^*_\rho(G)\subset\mathrm{Ker}\,\breve{\pi}$.
Given a $\pi\in\mathrm{Rep}\,C^*_\rho(G_d)$ which is irreducible and nonregular observe that any extension $\widetilde{\pi}$ of π to $\mathcal{E}$ on the same space will also be irreducible. Thus the decomposition of 3(i) becomes either $\widetilde{\pi}=\widetilde{\pi}_1$ or $\widetilde{\pi}=\widetilde{\pi}_2$. If it is nonregular, then $\widetilde{\pi}$ must be degenerate on $C^*_\rho(G)$, so the first alternative is impossible, so $\widetilde{\pi}=\widetilde{\pi}_2$ which vanishes on $C^*_\rho(G)$. Thus $\widetilde{\pi}$ can only be the lifting constructed above.
(3iii) Let $\pi\in\mathrm{Rep}\,C^*_\rho(G_d)$ and define

$$\mathcal{H}_B:=\big\{\,\psi\in\mathcal{H}_\pi\;\big|\;g\to\pi(\delta_g)\psi\quad\text{is Borel from}\quad G\quad\text{to}\quad\mathcal{H}\;\big\}$$

which is a closed linear space since sums and limits of sequences of Borel maps are Borel. Observe that $\mathcal{H}_B$ is preserved under $\pi\big(C^*_\rho(G_d))\big)$, using continuity assumption of ρ . Thus $\pi\,\big|\,\mathcal{H}_B$ is a subrepresentation, Borel and nondegenerate since π is nondegenerate and $\mathbb{1}\in C^*_\rho(G_d)$. So using 2.4(i), the representation $\pi\,\big|\,\mathcal{H}_B$ extends on $\mathcal{H}_B$ to a representation $\widetilde{\pi}$ of $\mathcal{M}(G)$ which is given on $L^1_\rho(G)$ by:

$$\widetilde{\pi}(f)\,\psi=\int_G\pi(\delta_g)\psi\,f(g)\,d\mu(g)\qquad\forall\,\psi\in\mathcal{H}_B\,,\;f\in L^1_\rho(G)\,.$$

On $\mathcal{H}_B^\perp$, set $\widetilde{\pi}(L^1_\rho(G))=0$ to obtain the extension $\widetilde{\pi}$ on $\mathcal{H}$.
The statement that the regular part of $\widetilde{\pi}$ is maximal, is clear by combining $(3i)$ and $(1iii)$.
(3iv) Clearly if π is regular, then $\mathcal{H}_B=\mathcal{H}_\pi$ and so by nondegeneracy of π , $\mathcal{H}_B=\mathcal{H}_{\widetilde{\pi}_1}=\mathcal{H}_\pi$, so $\widetilde{\pi}_2=0$.
Conversely, if $\widetilde{\pi}_2=0$, then $\widetilde{\pi}=\widetilde{\pi}_1$ is nondegenerate on $C^*_\rho(G)$ and so by part (2) is uniquely determined by its values on $C^*_\rho(G)$ and so by parts (1) and (3i) must be regular.

(4) This is well known, and easily seen from parts (1 iii) and (3), using Dixmier 2.7.4, though explicitly it is instructive to check that the map $t : \operatorname{Rep} C^*_\rho(G) \to \mathcal{P}$ by $t(\pi) = \hat{\pi} \,\big|\, C^*_\rho(G_d)$ is the inverse of $t^{-1}(\pi) = \widetilde{\pi} \,\big|\, C^*_\rho(G)$: for any $\pi \in \mathcal{P}$ we have:

$$\begin{aligned}\hat{\widetilde{\pi}}(\delta_h)\,\widetilde{\pi}(f) := \widetilde{\pi}(\delta_h * f) &= \int_G (\delta_h * f)(g)\,\pi(\delta_g)\,d\mu(g) \\ &= \int_G f(h^{-1}g)\,\rho(h,\,h^{-1}g)\,\pi(\delta_g)\,d\mu(g) \qquad \text{using 2.3} \\ &= \int_G f(g)\,\rho(h,\,g)\,\pi(\delta_{hg})\,d\mu(g) \\ &= \pi(\delta_h)\int_G f(g)\,\pi(\delta_g)\,d\mu(g) = \pi(\delta_h)\cdot\widetilde{\pi}(f)\end{aligned}$$

for all $f \in L^1_\rho(G)$, hence $\hat{\widetilde{\pi}} \,\big|\, C^*_\rho(G_d) = \pi$. ∎

Remarks: (*i*) Theorem 2.4 gives an algebraic characterization of regular representations which can replace the topological one for locally compact groups. Observe that the representation space $\mathcal{H}_{\widetilde{\pi}_1}$ of the regular part $\widetilde{\pi}_1$ of the canonical extension $\widetilde{\pi}$ can be characterised by

$$\mathcal{H}_R := \big\{\,\psi \in \mathcal{H}_\pi \;\big|\; g \to \pi(\delta_g)\psi \text{ continuous from } G \text{ to } \mathcal{H}\;\big\} \subseteq \mathcal{H}_B\,.$$

Hence we have on $C^*_\rho(G_d)$ an intrinsic method for obtaining the decomposition $\pi = \pi_R \oplus \pi_N$. [Proof that $\mathcal{H}_R = \mathcal{H}_{\widetilde{\pi}_1}$. It is obvious that $\mathcal{H}_{\widetilde{\pi}_1} \subseteq \mathcal{H}_R$. Conversely, let $\psi \in \mathcal{H}_R$, then since $\mathcal{H}_R \subset \mathcal{H}_B$, the integrals

$$\widetilde{\pi}(f)\,\psi = \int_G f(g)\,\pi(\delta_g)\psi\,d\mu(g)$$

are defined for all $f \in L^1_\rho(G)$, and must be nonzero for some, because the function $g \to (\psi,\,\pi(\delta_g)\psi)$ is continuous and nonzero. Hence ψ must have a component in the essential subspace of $\widetilde{\pi}(L^1_\rho(G))$ which is $\mathcal{H}_{\widetilde{\pi}_1}$. To see that this component is all of ψ , assume the contrary, i.e. $\psi = \psi_1 + \psi_0$, $\psi_1 \in \mathcal{H}_{\widetilde{\pi}_1}$ and $\psi_0 \perp \mathcal{H}_{\widetilde{\pi}_1}$. Then since $\pi(\delta_g)$ is continuous in g on both ψ and ψ_1 , it must be continuous on ψ_0 on which some $\pi(f)$ must therefore be nonzero, which is a contradiction. Thus $\psi \in \mathcal{H}_{\widetilde{\pi}_1}$.]

(ii) Note from 2.4(ii) and (iv) that a regular representation of $C^*_\rho(G_d)$ has more than one extension to $\mathcal{E}$. The author owes the extension $\breve{\pi}$ of a representation π and this observation to R. Schaflitzel.

(iii) Since $C^*_\rho(G)$ has as representation space only the regular representations, it appears to be a more convenient algebra than $C^*_\rho(G_d)$ (e.g. the CCR–algebra $\overline{\Delta(S,\,B)}$) if we only want to admit these representations into the theory. A further improvement of $C^*_\rho(G)$ over $C^*_\rho(G_d)$ is that $C^*_\rho(G)$ is separable if G is. The question considered in [2], of the existence of direct integral decompositions of regular representations into irreducible regular representations is obviously answered in the affirmative for the locally compact case, since these are now realized as the representation space of a separable C*–algebra. However in the present finite dimensional context, for the CCRs, the last statement is already obtainable by the Stone–Von Neumann uniqueness theorem.

(iv) In the case of the CCRs, where $G = S$, $\rho = \sigma$ nondegenerate, the algebra $C^*_\sigma(S)$ has been called the algebra of "regular" or "smooth" observables, cf. Segal [16, 18], and is well known to be isomorphic to the algebra of compact operators $\mathcal{K}$ on a separable Hilbert space, [16], and Folland theorem 1.30 [17]. Thus theorem 2.4(4) yields an immediate (and known) proof of the Stone–von Neumann theorem, using the fact that $\mathcal{K}$ has only one unitary equivalence class of irreducible representations. Moreover, $M(C^*_\sigma(S)) \cong M(\mathcal{K}) = \mathcal{B}(\mathcal{H})$, so if $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}$ is the canonical factorization map to the Calkin algebra, then $\varphi \circ \pi$ produces a faithful embedding of the CCR–algebra into the Calkin algebra where π can be any irreducible regular representation of $\overline{\Delta(S,\,B)}$. (It is faithful because the CCR algebra is simple– for nondegenerate B). Thus in an irreducible regular representation π of $\overline{\Delta(S,\,B)}$, the essential spectrum of any $A \in \pi\big(\overline{\Delta(S,\,B)}\big)$ is the same as its ordinary spectrum, and also, the Fredholm index theory on $\pi\big(\overline{\Delta(S,\,B)}\big)$ simplifies. Furthermore, since the states of $\mathcal{K}(\mathcal{H}) \cong C^*_\rho(G)$ are given by normalised "density matrices" i.e. trace–class operators D by

$$\omega(A) = \mathrm{Tr}\,(DA) \qquad \forall\, A \in \mathcal{K}(\mathcal{H})\,,$$

this shows the well–known fact that for finite dimensional quantum mechanics, the regular states are all given by density matrices. Thus the inclusion $\overline{\Delta(S,\,B)} \subset M\big(C^*_\sigma(S)\big)$ expresses in compact form several important facts of quantum mechanics.

(v) If $\pi \in \mathrm{Rep}\, C^*_\rho(G_d)$ is Borel w.r.t. the map $g \to \pi(\delta_g)$, then the canonical extension $\widetilde{\pi}$ of Theorem 2.4(3) is exactly the restriction to $\mathcal{E}$ of the representation of $\mathcal{M}(G)$ constructed in part 2.4(1) by the formula $\widetilde{\pi}(\nu) = \int \pi(\delta_g)\, d\nu(g)$. Now by the von Neumann theorem a representation of a one–parameter group on a separable Hilbert space is Borel iff it is strong operator continuous [21]. Hence Borel representations of $\mathbb{R}$ can only be nonregular on nonseparable Hilbert spaces. If $\pi \in \mathrm{Rep}\,\overline{\Delta(S,\,B)}$ is Borel but not regular, there is an $x \in S$ for which the one–parameter group $\lambda \to \pi(\delta_{\lambda x})$ is not strong operator continuous, and so $\mathcal{H}_\pi$ must be nonseparable. One example of such a nonregular representation, is the GNS–representation of the tracial state of the CCRs, i.e. the state ω_0 on $\overline{\Delta(S,\,B)}$ defined by $\omega_0(\delta_x) = 0$ if $x \neq 0$, and $\omega_0(\delta_0) = 1$. This state ω_0 is pure (and the unique trace) iff B is nondegenerate. A more interesting class of such states (and the motivation for this paper) is the Dirac states for a linear Bosonic system with linear selfadjoint constraints, cf. [10]. Given a CCR-algebra $\overline{\Delta(S,\,B)}$ and a specified constraint subspace $\mathcal{C} \subset S$ for which $B(\mathcal{C},\,\mathcal{C}) = 0$, such states by definition satisfy $\omega(\delta_x) = 1$ for all $x \in \mathcal{C}$. This then implies that for these states we have

$$\omega(\delta_y) = 0 \qquad \text{for all } \; y \in S \; \text{ with } \; B(y, \mathcal{C}) \neq 0\,.$$

and hence $\lambda \to \omega(\delta_{\lambda y})$ is discontinuous at 0 for these y , so nonregular. For those Dirac states which are regular on the subspace $\mathbf{p} := \big\{ x \in S \;\big|\; B(x,\,\mathcal{C}) = 0 \big\}$ we find that their GNS–representations are Borel, nonregular outside $\mathbf{p}$, and have nonseparable representation space. So when constraints are present, density matrices are not enough for finite dimensional quantum mechanics. Such states have also been studied recently by Acerbi, Morchio and Strocchi [22]. Another kind of nonregular state is the momentum states corresponding to physicist's plane waves, and these were introduced on the CCR algebra by Fannes and Verbeure [24]. Nonregular representations are also central in Narnhofer and Thirring [23], and are included in the analyses by Manuceau, Sirugue, Testard and Verbeure [27] and Goderis, Verbeure and Vets [28].

In the parlance of 2.4(1iii), a Borel representation which has degenerate canonical extension to $C^*_\sigma(S)$ must have a nonseparable representation space.

The Dirac states suggest that what is needed is a structure theory for "partially regular" representations, and indeed, in this case for S finite dimensional, that simply is the structure that if $U \subset S$ is a closed linear subspace, then $C^*_\sigma(U) \subset M(C^*_\sigma(S))$, and a representation can be nondegenerate on $C^*_\sigma(U)$, but degenerate on $C^*_\sigma(S)$ (though not conversely).

(vi) We call a decomposition $\pi = \pi_1 \oplus \pi_2$ as in 2.4(3i) an ideal decomposition.

Def. A nondegenerate representation π on $C^*_\rho(G_d)$ (resp. $\mathcal{E}$) is called *irregular* if $C^*_\rho(G) \subseteq \mathrm{Ker}\,\widetilde{\pi}$ where $\widetilde{\pi}$ is the canonical extension on the same space of π to $\mathcal{E}$. A state $\omega \in \wp(C^*_\rho(G_d))$ is irregular if its GNS–representation π_ω is, ($\wp(\cdot)$ denotes the state space of its argument).

Observe that an irregular representation may still be regular w.r.t. some closed subgroup $H \subset G$. This suggest that we call an irregular representation *absolutely irregular* if it is irregular w.r.t. all closed locally compact subgroups H of G . One example of such, is the GNS–representation of the tracial state above.

Using the canonical extension of representations above, we can define the following canonical (though nonunique) extension of a state $\omega \in \wp(C^*_\rho(G_d))$ to $\mathcal{E}$:

$$\widetilde{\omega}(A) := \big(\Omega_\omega,\, \widetilde{\pi}_\omega(A)\Omega_\omega\big) \qquad \forall\, A \in \mathcal{E}$$

where $(\pi_\omega,\, \Omega_\omega,\, \mathcal{H}_\omega)$ is the GNS–representation for ω . Clearly $\pi_{\widetilde{\omega}}$ is unitarily equivalent to $\widetilde{\pi}_\omega$. Now because $\mathrm{Ker}\,\pi_{\widetilde{\omega}}$ is a maximal two sided ideal in $\mathrm{Ker}\,\widetilde{\omega}$, and $C^*_\rho(G)$ is an ideal of $\mathcal{E}$, we see that a state ω is irregular iff $C^*_\rho(G) \subset \mathrm{Ker}\,\widetilde{\omega}$.

Introduce the following notation:

let $\mathcal{P}^\perp$ (resp. $\wp_{\mathcal{P}}$, $\wp^\perp_{\mathcal{P}}$) denote the set of irregular representations (resp. regular states, irregular states) of $C^*_\rho(G_d)$.

Now Theorem 2.4(3) seems to exhibit asymmetry between the regular and irregular parts of a representation, but it is possible to restore symmetry by enlarging $\mathcal{E}$ as follows. Consider the universal enveloping von Neumann algebra $\mathcal{E}'' \supset \mathcal{E}$. Since $C^*_\rho(G)$ is an ideal of $\mathcal{E}$, there is a central projection $P \in \mathcal{E}'' \cap \mathcal{E}'$ such that $P\mathcal{E}'' = C^*_\rho(G)''$. Define $\widetilde{\mathcal{E}} := C^*(\mathcal{E} \cup \{P\}) \subset \mathcal{E}''$, then any representation on $\mathcal{E}$ extends uniquely on the same space to $\mathcal{E}''$, hence to $\widetilde{\mathcal{E}}$. If $\pi \in \mathrm{Rep}\,\mathcal{E}$ is regular, then $\pi(P) = \mathbb{1}$, and if π is irregular then $\pi(P) = 0$. Now we have the two ideals $\widetilde{\mathcal{E}}_1 := P\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}_2 := (\mathbb{1} - P)\widetilde{\mathcal{E}}$ in $\widetilde{\mathcal{E}}$ such that $AB = 0 = BA$ if $A \in \widetilde{\mathcal{E}}_1$, $B \in \widetilde{\mathcal{E}}_2$, so $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2 = \overline{L(\widetilde{\mathcal{E}}_1 \cup \widetilde{\mathcal{E}}_2)}$. Note that $C^*_\rho(G) \subset \widetilde{\mathcal{E}}_1$, but since $\widetilde{\mathcal{E}}_1$ is unital with unit P , these algebras are not equal. Thus any $\pi \in \mathrm{Rep}\,\widetilde{\mathcal{E}}$ has a unique decomposition $\pi = \pi_1 \oplus \pi_2$ where each $\pi_i \in \mathrm{Rep}\,\widetilde{\mathcal{E}}_i$ and clearly π_1 (resp. π_2) is the regular (resp. irregular) part of π , since $P \in \widetilde{\mathcal{E}}_1$ and $(\mathbb{1} - P) \in \widetilde{\mathcal{E}}_2$. Thus in $\widetilde{\mathcal{E}}$ the characterisation of regular and irregular representations is symmetrical.

For $A = f \in L^1_\rho(G)$, we obtain from 2.4 for the canonical extension:

$$\widetilde{\omega}(f) = \int f(g)\,\omega_1(\delta_g)\,d\mu(g)$$

where $\omega_1(A) := (\Omega_\omega, \widetilde{\pi}_1(A)\,\Omega_\omega)$, and since $\mathcal{E} = \overline{L(C^*_\rho(G) \cup C^*_\rho(G_d))}$, this formula uniquely determines the extension $\widetilde{\omega}$, and $\widetilde{\omega}$ is pure if ω is.

Theorem 2.5. *Let* $\omega \in \wp(C^*_\rho(G_d))$ *with canonical extension* $\widetilde{\omega}$ *to* $\widetilde{\mathcal{E}}$.

(1) *Then there is a unique decomposition* $\widetilde{\omega} = \lambda\omega_1 + (1-\lambda)\omega_2$, $\lambda \in [0, 1]$, *where the state* ω_1 *(resp.* ω_2 *) is regular (resp. irregular), or equivalently,* $\widetilde{\mathcal{E}}_1 \subset \mathrm{Ker}\,\omega_2$ *and* $\widetilde{\mathcal{E}}_2 \subset \mathrm{Ker}\,\omega_1$. *Furthermore, on* $\mathcal{E}$ *the state* ω_1 *is characterised by* $\omega \,\big|\, C^*_\rho(G)$ *being a state and* ω_2 *by* $C^*_\rho(G) \subset \mathrm{Ker}\,\omega_2$. *We also have* $\lambda\omega_1(A) = \widetilde{\omega}(PA)$ *and* $(1-\lambda)\omega_2(A) = \widetilde{\omega}((\mathbb{1} - P)A)$ *for all* $A \in C^*_\rho(G_d)$. *Pure states are either regular or irregular.*

(2) *The canonical extension defines two injections:*

$$\kappa : \wp(C^*_\rho(G_d)) \to \wp(\mathcal{E}) \quad \textit{and} \quad \widetilde{\kappa} : \wp(C^*_\rho(G_d)) \to \wp(\widetilde{\mathcal{E}})$$

which are w–continuous, preserve pure states, and satisfy:*

$$\kappa(\wp_{\mathcal{P}}) = \big\{\, \omega \in \wp(\mathcal{E}) \;\big|\; \omega \;\text{ is a state on }\; C^*_\rho(G) \,\big\} = \wp(C^*_\rho(G))$$

$$\kappa(\wp_{\mathcal{P}}^{\perp}) = \{\, \omega \in \wp(\mathcal{E}) \;\big|\; C^*_\rho(G) \subset \operatorname{Ker}\omega \,\}$$
$$\widetilde{\kappa}(\wp_{\mathcal{P}}) \subset \left\{\, \omega \in \wp(\mathcal{E}) \;\big|\; \widetilde{\mathcal{E}}_2 \subset \operatorname{Ker}\omega \,\right\} = \wp(\widetilde{\mathcal{E}}_1)$$
$$\widetilde{\kappa}(\wp_{\mathcal{P}}^{\perp}) \subset \left\{\, \omega \in \wp(\mathcal{E}) \;\big|\; \widetilde{\mathcal{E}}_1 \subset \operatorname{Ker}\omega \,\right\} = \wp(\widetilde{\mathcal{E}}_2) \ .$$

Remark: Thus a regular state on $C^*_\rho(G_d)$ *is characterised by having a canonical extension which is a state on* $C^*_\rho(G)$ *(equivalently, which vanishes on* $\widetilde{\mathcal{E}}_2$ *).*

Proof: **(1)** The first two statements are direct applications of Dixmier 2.11.7 [8], which states that:

If $\mathcal{I}$ is a closed two–sided ideal of the C*–algebra $\mathcal{A}$ and if f is a positive form on $\mathcal{A}$, then there is a unique decomposition $f = f_1 + f_2$ where f_i are positive forms on $\mathcal{A}$ with $\|f_1\| = \|f_1 \,\big|\, \mathcal{I}\|$ and $f_2(\mathcal{I}) = 0$ and the pair $(\pi_f,\, \xi_f)$ can be identified with $(\pi_{f_1} \oplus \pi_{f_2},\, \xi_{f_1} + \xi_{f_2})$ where $\mathcal{I} \subset \operatorname{Ker}\pi_{f_2}$ and $\pi_{f_1}\,\big|\,\mathcal{I}$ is nondegenerate. Then the property $\|f_1\| = \|f_1 \,\big|\, \mathcal{I}\|$ shows that there is a unique extension of a state on $\mathcal{I}$ to a state on $\mathcal{A}$. The rest follows from this and the fact that $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2$ and $C^*_\rho(G)$ is an ideal of $\mathcal{E}$.

Also recall that $\widetilde{\mathcal{E}}_i \subset \operatorname{Ker}\omega_j$ iff $\widetilde{\mathcal{E}}_i \subset \operatorname{Ker}\pi_{\omega_j}$, $i \neq j$ and $C^*_\rho(G) \subset \operatorname{Ker}\omega_2$ iff $C^*_\rho(G) \subset \operatorname{Ker}\pi_{\omega_2}$ by Dixmier 2.4.10. Let $A \in \widetilde{\mathcal{E}}$, so $A = A_1 + A_2$, $A_i \in \widetilde{\mathcal{E}}_i$, $PA = A_1$, $(\mathbb{1} - P)A = A_2$. Hence
$\widetilde{\omega}(A) = \lambda\omega_1(A) + (1-\lambda)\omega_2(A) = \lambda\omega_1(A_1) + (1-\lambda)\omega_2(A_2)$
and so $\widetilde{\omega}(PA) = \widetilde{\omega}(A_1) = \lambda\omega_1(A_1) = \lambda\omega_1(A)$ and similarly
$\widetilde{\omega}((\mathbb{1} - P)A) = (1-\lambda)\omega_2(A)$.

(2) That κ and $\widetilde{\kappa}$ are injective follows from the uniqueness of the expression for $\widetilde{\omega}$. That it preserves pure states follows from the fact that $\widetilde{\pi}$ is irreducible if π is (cf. 2.4(3)), and the uniqueness of the equivalence class of the GNS–representation of $\widetilde{\omega}$. To see that κ and $\widetilde{\kappa}$ are w*–continuous, let $\{\omega_\beta\}$ be a net in $\wp(C^*_\rho(G_d))$ w*–converging to ω , i.e. $\big|\omega_\beta(A) - \omega(A)\big| \to 0$ as $\beta \to \infty$ for all $A \in C^*_\rho(G_d)$. Then we need to show that $\big|\kappa(\omega_\beta)(A) - \kappa(\omega)(A)\big| \to 0$ as $\beta \to \infty$ for all $A \in \mathcal{E}$. Now $\mathcal{E} = \overline{L(C^*_\rho(G_d) \cup C^*_\rho(G))}$, and since we already have convergence on $C^*_\rho(G_d)$, we only need to show convergence on

$C^*_\rho(G)$. For all $f \in L^1_\rho(G)$:

$$\left|\kappa(\omega_\beta)(f) - \kappa(\omega)(f)\right| = \left| \int f(g) \left(\omega_\beta(\delta_g) - \omega(\delta_g)\right) d\mu(g) \right|$$
$$\leq \int |f(g)| \cdot \left|\omega_\beta(\delta_g) - \omega(\delta_g)\right| d\mu(g) \longrightarrow 0$$

as $\beta \to \infty$, using the Lebesgue dominated convergence theorem together with $|\omega_\beta(\delta_g) - \omega(\delta_g)| \to 0$ as $\beta \to \infty$, and $\left|\omega_\beta(\delta_g) - \omega(\delta_g)\right| \leq \|\delta_g\| + \|\delta_g\| = 2$ which is L^1 for the measure $|f(g)|\,d\mu(g)$.

As for $\widetilde{\kappa}$, observe that $\widetilde{\omega}(P) = \lambda$ (same λ as in part 1), so w*–continuity follows from that on $C^*_\rho(G)$. Any state on $C^*_\rho(G)$ extends uniquely to a state on $\mathcal{E}$, so any $\omega \in \wp(\mathcal{E})$ which is a state on $C^*_\rho(G)$ is uniquely determined by its values on $C^*_\rho(G)$, and this establishes the equality between $\wp(C^*_\rho(G))$ and the set of states on $\mathcal{E}$ restricting to states on $C^*_\rho(G)$ (cf. Dixmier 2.11.8 also). Now from part (1) we have inclusions $\kappa(\wp_{\mathcal{P}}) \subseteq \wp(C^*_\rho(G))$, $\kappa(\wp_{\mathcal{P}}^\perp) \subseteq \left\{\omega \in \wp(\mathcal{E}) \;\middle|\; C^*_\rho(G) \subset \operatorname{Ker}\omega\right\}$, $\widetilde{\kappa}(\wp_{\mathcal{P}}) \subseteq \wp(\widetilde{\mathcal{E}}_1)$, $\widetilde{\kappa}(\wp_{\mathcal{P}}^\perp) \subseteq \wp(\widetilde{\mathcal{E}}_2)$. For the reverse inclusion of the first inclusion, recall that every nondegenerate representation of $C^*_\rho(G)$ is obtained from one on $C^*_\rho(G_d)$ via the relation $\pi(f) = \int f(g)\,\pi(\delta_g)\,d\mu(g)$. In particular, this is true for cyclic representations, and hence every state on $C^*_\rho(G)$ is obtained from one on $C^*_\rho(G_d)$ via the canonical extension. Thus $\kappa(\wp_{\mathcal{P}}) = \wp(C^*_\rho(G))$. For reverse inclusion of the second, let $\omega \in \wp(\mathcal{E})$ with $C^*_\rho(G) \subset \operatorname{Ker}\omega$, then since $\mathcal{E} = \overline{L(C^*_\rho(G) \cup C^*_\rho(G_d))}$, ω is uniquely determined by its values on $C^*_\rho(G_d)$, so ω is in the range of κ . ∎

Next we wish to show that the problem of pointwise norm discontinuous actions on $\overline{\Delta(S,\,B)} = C^*_\sigma(S_d)$ defined from symplectic actions $\beta : \mathcal{G} \to \mathrm{Sp}\,(S,\,B)$ where $\mathcal{G}$ is a nondiscrete topological group, can also be circumvented by working with the algebra $C^*_\sigma(S)$ instead of $\overline{\Delta(S,\,B)}$.

Let $\theta \in \mathrm{Aut}_\rho G := \left\{\theta \in \mathrm{Aut}\,G \;\middle|\; \rho\big(\theta(g),\,\theta(h)\big) = \rho(g,\,h)\;\forall\, g,\, h \in G\right\}$ be a Borel automorphism where G and ρ are as before. Denote the Haar measure of G by μ . Then $\mu \circ \theta$ is again a left invariant measure of

G (on the same σ–algebra), hence by uniqueness of the Haar measure there is a constant $C_\theta \in \mathbb{R}_+$ such that $\mu \circ \theta = C_\theta\,\mu$. Denote $\mathrm{Aut}^B_\rho G := \big\{\theta \in \mathrm{Aut}_\rho G \;\big|\; \theta \;\text{ a Borel automorphism}\big\}$. We endow $\mathrm{Aut}^B_\rho G$ with the topology of pointwise sequential convergence, i.e. a sequence $\{\theta_n\} \subset \mathrm{Aut}^B_\rho G$ converges to $\theta \in \mathrm{Aut}\,G$ if $\theta_n(x) \to \theta(x)$ for each $x \in G$. If one assumes the stronger assumption of joint continuity of ρ , we see that $\mathrm{Aut}^B_\rho G$ is closed in the topology above.

Theorem 2.6. *(i) Each $\theta \in \mathrm{Aut}^B_\rho G$ defines an automorphism $\widetilde{\theta} \in \mathrm{Aut}\,M(C^*_\rho(G))$ which preserves $C^*_\rho(G)$ and $C^*_\rho(G_d)$ (hence $\mathcal{E}$) and is given on $L^1_\rho(G)$ by $(\widetilde{\theta} f)(t) = f(\theta^{-1}t)\cdot C_{\theta^{-1}}$ and on $C^*_\rho(G_d)$ by $\widetilde{\theta}(\delta_t) = \delta_{\theta t}$ $\forall\, t \in G$.*
(ii) Endow $\mathrm{Aut}^B_\rho G$ with the topology above, and give $\mathrm{Aut}\,C^*_\rho(G)$ the pointwise norm topology. Then the map $\widetilde{\ }: \mathrm{Aut}^B_\rho G \to \mathrm{Aut}\,C^*_\rho(G)$ is continuous. (Hence any continuous homomorphism $\beta : \mathcal{G} \to \mathrm{Aut}^B_\rho G$ produces a pointwise norm continuous action $\widetilde{\beta} : \mathcal{G} \to \mathrm{Aut}\,C^*_\rho(G)$).

Proof: (i) First we show that the definition $(\widetilde{\theta} f)(t) := C_{\theta^{-1}} f(\theta^{-1}t)$ produces an automorphism of $L^1_\rho(G)$. It is only necessary to check that $\widetilde{\theta}$ is a homomorphism under the operations of 2.2, as it is obviously linear, invertible and preserves the L^1–norm . Since

$$\begin{aligned}(\mu \circ \theta)(Rx) = \mu\big(\theta(R)\theta(x)\big) &= C_\theta \Delta(\theta(x))\,\mu(R)\\ &= C_\theta \mu(Rx) = C_\theta \Delta(x)\,\mu(R)\end{aligned}$$

for all Borel sets R , we see $\Delta(\theta(x)) = \Delta(x)$. Now

$$\begin{aligned}\big(\widetilde{\theta}(f^*)\big)(x) &= C_{\theta^{-1}} f^*(\theta^{-1}x)\\ &= C_{\theta^{-1}} \Delta(\theta^{-1}x)^{-1}\,\overline{\rho(\theta^{-1}(x),\, \theta^{-1}(x^{-1}))}\;\overline{f(\theta^{-1}(x^{-1}))}\\ &= C_{\theta^{-1}} \delta(x)^{-1}\overline{\rho(x,\, x^{-1})}\;\overline{f(\theta^{-1}(x^{-1}))}\\ &= \Delta(x)^{-1}\,\overline{\rho(x,\, x^{-1})}\;\overline{(\widetilde{\theta} f)(x^{-1})} = (\widetilde{\theta}(f))^*(x) \qquad \text{and}\end{aligned}$$

$$\begin{aligned}\big(\widetilde{\theta}(f * g)\big)(x) &= C_{\theta^{-1}} (f * g)(\theta^{-1}(x))\\ &= C_{\theta^{-1}} \int_G f(y)\, g\big(y^{-1}\theta^{-1}(x)\big)\, \rho(y,\, y^{-1}\theta^{-1}(x))\, d\mu(y)\end{aligned}$$

$$= C_{\theta^{-1}} \int_G f(\theta^{-1}(y))\, g(\theta^{-1}(y^{-1}x))\, \rho(y,\, y^{-1}x)\, d\mu(\theta^{-1}(y))$$

$$= C^2_{\theta^{-1}} \int_G f(\theta^{-1}(y))\, g(\theta^{-1}(y^{-1}x)\, \rho(y,\, y^{-1}x)\, d\mu(y)$$

$$= \int_G (\widetilde{\theta} f)(y)\ (\widetilde{\theta} g)(y^{-1}x)\ \rho(y,\, y^{-1}x)\, d\mu(y) = \big[(\widetilde{\theta} f) * (\widetilde{\theta} g)\big](x)$$

for all $f,\, g \in L^1_\rho(G)$, $x \in G$, $\theta \in \mathrm{Aut}^B_\rho G$. Hence $\widetilde{\theta} \in \mathrm{Aut}\, L^1_\rho(G)$. Then it follows from Dixmier 1.3.7 [8] that $\widetilde{\theta}$ is also continuous in the C*–norm, hence it extends uniquely to an automorphism $\widetilde{\theta} \in \mathrm{Aut}\, C^*_\rho(G)$, and also by the usual arguments to $M(C^*_\rho(G))$. We check that $\widetilde{\theta}$ preserves $C^*_\rho(G_d)$ and that it is given by the formulae in (i) . By 2.3:

$$\widetilde{\theta}(\delta_y * f)(x) = C_{\theta^{-1}}(\delta_y * f)(\theta^{-1}x) = C_{\theta^{-1}}\, f(y^{-1}\theta^{-1}(x))\, \rho(y,\, y^{-1}\theta^{-1}(x))$$

$$= (\widetilde{\theta} f)(\theta(y^{-1})x)\, \rho(\theta(y),\, \theta(y^{-1})x) = \big[\delta_{\theta y} * \widetilde{\theta}(f)\big](x)$$

and hence $\widetilde{\theta}(\delta_y) = \delta_{\theta y}$ $\forall\, y \in G$ which proves that $\widetilde{\theta}$ preserves $C^*_\rho(G_d)$.

(ii) Under the canonical bijection between regular representations $\pi \in \mathcal{P}$ on $C^*_\rho(G_d)$ and nondegenerate representations of $C^*_\rho(G)$, we have $\pi(f) = \int_G f(x)\, \pi(\delta_x)\, d\mu(x)$ for all $f \in L^1_\rho(G)$, and all nondegenerate representations of $L^1_\rho(G)$ are of this form. Let $\{\theta_n\} \subset \mathrm{Aut}^B_\rho G$ be a sequence converging pointwise to $\theta \in \mathrm{Aut}^B_\rho G$, then if π and Ω are the GNS–representation and cyclic vector respectively of a regular state ω ,

$$\big\|\pi(\widetilde{\theta}_n(f) - \widetilde{\theta}(f))\Omega\big\| = \Big\| \int_G \Big[(\widetilde{\theta}_n f)(x) - (\widetilde{\theta} f)(x)\Big]\ \pi(\delta_x)\, d\mu(x)\, \Omega\Big\|$$

$$= \Big\| \int_G \big[C_{\theta_n^{-1}} f(\theta_n^{-1}(x)) - C_{\theta^{-1}} f(\theta^{-1}(x))\big]\ \pi(\delta_x)\, d\mu(x)\, \Omega\Big\|$$

$$= \Big\|\Big[\int_G f(\theta_n^{-1}(x))\, \pi(\delta_x)\, d\mu(\theta_n^{-1}(x)) - \int_G f(\theta^{-1}(x))\, \pi(\delta_x)\, d\mu(\theta^{-1}(x))\Big]\Omega\Big\|$$

$$= \Big\| \int_G \big(\pi(\delta_{\theta_n(x)}) - \pi(\delta_{\theta(x)})\big)\, f(x)\, d\mu(x)\, \Omega\Big\|$$

$$\leq \int_G |f(x)| \cdot \big\|\big(\pi(\delta_{\theta_n(x)}) - \pi(\delta_{\theta(x)})\big)\Omega\big\|\, d\mu(x)$$

$$\leq \|f\|_1 \cdot \sup\big\{\, \big\|(\pi(\delta_{\theta_n(x)}) - \pi(\delta_{\theta(x)}))\Omega\big\|\ \big|\ x \in G\,\big\} \longrightarrow 0$$

as $n \to \infty$ using the strong operator continuity of $\pi(\delta_x)$. Hence

$$\omega\left(\left(\widetilde{\theta}_n(f) - \widetilde{\theta}(f)\right)^*\left(\widetilde{\theta}_n(f) - \widetilde{\theta}(f)\right)\right) \longrightarrow 0 \qquad \text{as} \qquad n \to \infty$$

for all $f \in L^1_\rho(G)$, $\omega \in \wp(C^*_\rho(G))$ using 2.5 (where $\wp$ denotes the state space of its argument). However by Dixmier 2.7.1 [8], this implies that $\|\widetilde{\theta}_n(f) - \widetilde{\theta}(f)\| \to 0$ as $n \to 0$ in the C*–norm of $C^*_\rho(G)$, which completes the proof of (ii) . ∎

Corollary 2.6 If $G = S = \mathbb{R}^n \times \mathbb{R}^n$, endow $\mathrm{Sp}\,(S,\,B)$ with the matrix topology it inherits from its inclusion in $\mathrm{M}^{2n}(\mathbb{R})$, then any continuous homomorphism $\beta : \mathcal{G} \to \mathrm{Sp}\,(S,\,B)$ produces a pointwise norm continuous action $\widetilde{\beta} : \mathcal{G} \to \mathrm{Aut}\, C^*_\sigma(S)$. ∎

Next we wish to extend this structure to deal with the infinite dimensional case, and also to include partially regular representations into the framework.

3. The Infinite Dimensional Case; Regularity of Representations.

If we relax the requirement that $\dim S < \infty$, then S is not locally compact and the structure of the preceding section fails. However, there are many ways of writing the symplectic space S as an inductive limit of finite dimensional subspaces, and for each of these finite dimensional spaces we can still use the preceding theory. To exploit this, we assume that G is written as an inductive limit of closed nondiscrete locally compact subgroups, $G = \varinjlim G_\beta$, $\beta \in \Gamma$ where Γ is a directed set and each G_β is locally compact. This covers the cases where G is S (a separable symplectic space), $\mathrm{Sp}\,(\infty)$, $U(\infty)$, $O(\infty)$. Let $\rho \in Z^2(G,\,\mathbf{T})$ be a two–cocycle such that the restriction of ρ to any G_β is Borel, and on each G_β the maps $g \to \rho(g,\,x)$ are continuous for all x . Consider the monomorphisms of the inductive limit as inclusions. Where different inductive systems are available, we choose the finest nondiscrete one, e.g. for S the inductive system of all its finite dimensional subspaces. Henceforth the choice of inductive system remains fixed.

The aim of this section is to obtain an adequate group C*–algebra for an inductive limit group which is not discrete. More precisely, we want a non–unital C*–algebra $\mathcal{L}$ such that:

(i) its representation space consists of precisely the representations of G which are weak–operator continuous,

(ii) there is a homomorphism $\varphi : C^*_\rho(G_d) \to M(\mathcal{L})$ such that given a representation $\pi \in \mathrm{Rep}\,\mathcal{L}$ with unique extension $\hat{\pi}$ to $M(\mathcal{L})$, then

$$\overline{\pi}(A) := \hat{\pi}(\varphi(A)) \qquad \forall\, A \in C^*_\rho(G_d)$$

defines a regular representation $\overline{\pi}$ on $C^*_\rho(G_d)$, and conversely, every regular representation is obtained this way. (Thus if there are faithful regular representations, φ is injective).

(iii) when G is locally compact, $\mathcal{L} = C^*_\rho(G)$.

We do not address here the existence question of regular ρ–representations for non–locally compact groups.

Recall that we have for G_β the (isometric) imbeddings $C^*_\rho(G_{\beta d}) \subset M(C^*_\rho(G_\beta))$ and $C^*_\rho(G_\delta) \subset M(C^*_\rho(G_\beta))$ if $\delta \leq \beta \in \Gamma$ where $G_{\beta d}$ denotes G_β with the discrete topology. Simplify notation: $\mathcal{N}_\beta := C^*_\rho(G_\beta)$, $\mathcal{N}^d_\beta := C^*_\rho(G_{\beta d})$, $\mathcal{E}_\beta := C^*(\mathcal{N}_\beta \cup \mathcal{N}^d_\beta)$. and we denote the imbeddings above by: $i_{\delta\beta} : \mathcal{N}_\delta \to M(\mathcal{N}_\beta)$, $j_{\delta\beta} : \mathcal{N}^d_\delta \to M(\mathcal{N}_\beta)$ $\quad \forall\, \delta \leq \beta \in \Gamma$. Clearly $\mathcal{N}^d_\delta \subseteq \mathcal{N}^d_\beta$ if $\delta \leq \beta$. Define:

$$\mathcal{A}_\beta := C^*\big(\{\, i_{\delta\beta}(\mathcal{N}_\delta) \;\big|\; \delta \leq \beta \,\}\big) \subset M(\mathcal{N}_\beta)$$
$$\mathcal{M}_\beta := C^*\left(\mathcal{A}_\beta \cup \mathcal{N}^d_\beta\right) \subset M(\mathcal{N}_\beta)$$

Theorem 3.1. *Assume the preceding structures and notation. Then*

(i) *there are injective homomorphisms:*

$$\varphi_{\delta\beta} : \mathcal{A}_\delta \to \mathcal{A}_\beta\ , \qquad \psi_{\delta\beta} : \mathcal{M}_\delta \to \mathcal{M}_\beta \quad \forall\, \delta \leq \beta$$

such that $\psi_{\delta\beta}\big|\mathcal{A}_\delta = \varphi_{\delta\beta}$, $\psi_{\delta\beta}$ *is unital and*

$$\varphi_{\beta\gamma} \circ \varphi_{\delta\beta} = \varphi_{\delta\gamma}\ , \qquad \psi_{\beta\gamma} \circ \psi_{\delta\beta} = \psi_{\delta\gamma} \quad \forall\, \delta \leq \beta \leq \gamma \in \Gamma\,.$$

(ii) *Using the homomorphisms above, the inductive limit C^*–algebra* $\mathcal{M} := \varinjlim \mathcal{M}_\beta$ *exists and contains* $\mathcal{A} := \varinjlim \mathcal{A}_\beta$. *Moreover, if no* G_β *is discrete,* $\mathbb{1} \in \mathcal{M}\backslash\mathcal{A}$.

Proof: (i) Now $\mathcal{A}_\delta \subset \mathcal{M}_\delta \subset M(\mathcal{N}_\delta) \subset \mathcal{N}''_\delta$, and we have the imbedding $i_{\delta\beta} : \mathcal{N}_\delta \to M(\mathcal{N}_\beta) \subset \mathcal{N}''_\beta$, $\delta \leq \beta$ which extends canonically to the multiplier algebra:

$$i_{\delta\beta} : M(\mathcal{N}_\delta) \to M(i_{\delta\beta}(\mathcal{N}_\delta)) \subset [i_{\delta\beta}(\mathcal{N}_\delta)]'' \subset \mathcal{N}''_\beta$$

for $\delta \leq \beta$. Hence if we take $\psi_{\delta\beta} = i_{\delta\beta}\big|\mathcal{M}_\delta$ and $\varphi_{\delta\beta} = \psi_{\delta\beta}\big|\mathcal{A}_\delta$ it is only necessary to show that $i_{\beta\gamma} \circ i_{\delta\beta} = i_{\delta\gamma}$ and $i_{\delta\beta}(\mathcal{M}_\delta) \subset M(\mathcal{N}_\beta)$ to prove 3.1(i). Both of these will be proven if we can show that

$$\big[i_{\beta\gamma}(i_{\delta\beta}(f)h)g\big](x) = \big[i_{\delta\gamma}(f)\, i_{\beta\gamma}(h)\, g\big](x)$$

for all $\delta \leq \beta \leq \gamma$, $x \in G_\gamma$, $g \in L^1(G_\gamma)$, $h \in L^1(G_\beta)$, $f \in L^1(G_\delta)$ to ensure that $i_{\delta\gamma}(\mathcal{A}_\delta) \subset M(\mathcal{N}_\gamma)$ together with an even simpler calculation to show that $i_{\delta\gamma}(\mathcal{M}_\delta) \subset M(\mathcal{N}_\gamma)$. We only do the first calculation:

$$\begin{aligned}\left[i_{\beta\gamma}(i_{\delta\beta}(f)\,h)g\right](x) &= \int_{G_\gamma} [i_{\delta\beta}(f)\,h](y)\,g(y^{-1}x)\,\rho(y,\,y^{-1}x)\,d\mu_\gamma(y) \\ &= \int_{G_\gamma}\int_{G_\delta} f(z)\,h(z^{-1}y)\,\rho(z,\,z^{-1}y)\,d\mu_\delta(z)\ \ g(y^{-1}x)\,\rho(y,\,y^{-1}x)\,d\mu_\gamma(y) \qquad -(*)\end{aligned}$$

$$\begin{aligned}\left[i_{\delta\gamma}(f)\,i_{\beta\gamma}(h)\,g\right](x) &= \int_{G_\delta} f(z)\,[i_{\beta\gamma}(h)\,g](z^{-1}x)\,\rho(z,\,z^{-1}x)\,d\mu_\delta(z) \\ &= \int_{G_\delta} f(z)\int_{G_\gamma} h(y)\,g(y^{-1}z^{-1}x)\,\rho(y,\,y^{-1}z^{-1}x)\,d\mu_\gamma(y)\,\rho(z,\,z^{-1}x)\,d\mu_\delta(z) \\ &= \int_{G_\delta} f(z)\int_{G_\gamma} h(z^{-1}y)\,g(y^{-1}x)\,\rho(z^{-1}y,\,y^{-1}x)\,d\mu_\gamma(y)\,\rho(z,\,z^{-1}x)\,d\mu_\delta(z)\end{aligned}$$

and this is equal to $(*)$ by virtue of Fubini's theorem and the two co-cycle identity: $\rho(a,\,b)\,\rho(ab,\,c) = \rho(a,\,bc)\,\rho(b,\,c)$.

(ii) The existence and inclusion of the inductive limits follow from (i) and Takeda's criterion [12]. The statement about the identity follows from $\mathbb{1} \in \mathcal{N}_\beta^d \quad \forall\,\beta \in \Gamma$ and the lemma 3.1.3 of Blackadar [13] using $\mathbb{1} \notin \mathcal{A}_\beta\ \forall\,\beta$ (since $\mathcal{A}_\beta \subset \operatorname{Ker}\pi$ for any irreducible nonregular representation π of $\mathcal{N}_\beta^d$). ∎

Remarks (i) Clearly $\varinjlim \mathcal{N}_\beta^d = C_\rho^*(G_d) \subset \mathcal{M}$ and hence in the case when $G = S$, we see that $\overline{\Delta(S,\,B)} \subset \mathcal{M}$ if we set $\rho = \sigma = \exp\frac{i}{2}B(\cdot,\cdot)$.

(ii) For the case of the CCRs, the algebra $\mathcal{A}$ was considered by Segal [16] as a CCR field algebra. The algebra $\mathcal{A}$ is not a suitable infinite dimensional analogue of the algebra $C_\rho^*(G)$ of the finite dimensional case, because it has many representations which are not regular. The auxiliary C*–algebra of Schaflitzel [2] is the algebra generated by the identity and the algebras $C_\sigma^*(S_\alpha) \subset \mathcal{M}$ where S_α ranges over all one–dimensional subspaces of S , and it is the one proposed by Kastler [19] as the proper field algebra for the CCRs. More recently, Rieffel has defined a large C*–algebra for the CCRs, containing $\overline{\Delta(S,\,B)}$ [25].

(iii) Since $\{\mathcal{N}_\beta \mid \beta \in \Gamma\}$ generates $\mathcal{A}$, a state or representation of $\mathcal{A}$ is uniquely determined by its restriction to these.

(iv) Since $1\!\!1_{\mathcal{M}} \in C^*_\rho(G_{\beta d}) = \mathcal{N}^d_\beta \quad \forall\, \beta \in \Gamma$, any nondegenerate representation of $\mathcal{M}$ restricts to a nondegenerate representation of $\mathcal{N}^d_\beta$.

(v) The measure algebras $\mathcal{M}(G_\alpha)$ also form an inductive system in the obvious way (if $\alpha < \beta$, imbed $\mu \in \mathcal{M}(G_\alpha)$ in $\mathcal{M}(G_\beta)$ as a measure on G_β with support in G_α), and so one can define the inductive limit algebra $\mathcal{M}_\rho(G) := \varinjlim \mathcal{M}(G_\alpha)$. Kastler [19] used this twisted measure algebra as a "big field algebra," since it is a natural setting in which to examine representation questions for G . In the present case of G being non–locally compact, there is clearly no $L^1_\rho(G)$ algebra in $\mathcal{M}_\rho(G)$, although the discrete algebra $\ell^1_\rho(G) = L^1_\rho(G_d)$ is still in $\mathcal{M}_\rho(G)$, as are all $L^1_\rho(G_\alpha)$, $\alpha \in \Gamma$. As in the locally compact case, given a representation $\pi \in \mathrm{Rep}^0_\rho G$, Borel on each G_α , we can still define a representation $\widetilde{\pi}$ of $\mathcal{M}_\rho(G)$ by

$$\widetilde{\pi}(\nu) = \int_G \pi(g)\, d\nu(g) \qquad \forall\, \nu \in \mathcal{M}(G_\alpha)\,, \;\; \alpha \in \Gamma$$

which restricts to $\pi(g)$ on each $\delta_g \in L^1_\rho(G_d)$. That the representations defined on each $C^*_\rho(G_\alpha)$ by this formula piece together into a well–defined representation for $\mathcal{M}_\rho(G)$, follows from the fact that it is well–defined for each $\mathcal{M}_\alpha$, and then invoking the inductive limit structure of $\mathcal{M}_\rho(G)$. Thus a Borel representation on $C^*_\rho(G_d)$ has an extension $\widetilde{\pi}$ on the same space to $\mathcal{M}_\rho(G)$, hence to $\mathcal{M}$, using the fact that $\mathcal{M}$ is generated by $L^1_\rho(G_\alpha)$, $\alpha \in \Gamma$ and $C^*_\rho(G_d)$. In this non–locally compact framework, we will call this $\widetilde{\pi}$ the *canonical extension* of π . It also extends uniquely to $\mathcal{M}''$.

Next we would like to define a canonical extension for a representation $\pi \in \mathrm{Rep}\, C^*_\rho(G_d)$ which is not Borel on all G_α . Proceed as in Theorem 2.4(3); first define

$$\mathcal{H}_B := \left\{\, \psi \in \mathcal{H}_\pi \;\middle|\; g \in G_\alpha \to \pi(\delta_g)\psi \in \mathcal{H}_\pi \;\text{ is Borel }\; \forall\, \alpha \;\right\}.$$

Since sums and sequential limits of Borel functions are Borel, $\mathcal{H}_B$ is a closed linear space which is clearly preserved by $C^*_\rho(G_d)$. Thus π restricts on $\mathcal{H}_B$ to a Borel representation of $C^*_\rho(G_d)$, hence has a canonical extension on $\mathcal{H}_B$ to $\mathcal{M}$.

On $(\mathcal{H}_B)^\perp$ we set $\widetilde{\pi}(f)\,\psi = 0 \quad \forall f \in L^1_\rho(G_\alpha)$, $\alpha \in \Gamma$, $\psi \in (\mathcal{H}_B)^\perp$. This clearly defines a representation of $\mathcal{M}$ on $\mathcal{H}$, and it is also called *canonical.*

(Note that the restriction of this canonical extension of π to $C^*_\rho(G_\alpha)$ need not be the canonical extension of $\pi \,\big|\, C^*_\rho(G_{\alpha d})$ in Theorem 2.4 (3ii), because $\mathcal{H}_B$ may be smaller here, due to conditions on $\psi \in \mathcal{H}_B$ from G_β , $\beta \neq \alpha$). Using the canonical extension of representations, we also have a canonical extension of a state ω on $C^*_\rho(G_d)$ to $\mathcal{M}$, given by

$$\widetilde{\omega}(A) = (\Omega_\omega,\, \widetilde{\pi}_\omega(A)\Omega_\omega) \quad \forall\, A \in \mathcal{M}.$$

However, since $\mathbb{1} \in C^*_\rho(G_d)$, we know from the Stone–Weierstrass theorem that this extension of ω must be nonunique for some ω .

Theorem 3.2. *Given the hypotheses of theorem 3.1,*

(i) A nondegenerate representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *extends uniquely to a representation* $\widetilde{\pi} : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ *(on the same space).*

(ii) The state ω_0 *of* $C^*_\rho(G_d)$ *defined by*

$$\omega_0(\delta_x) = \begin{cases} 1 & \text{if } x = e \\ 0 & \text{if } x \neq e \end{cases}$$

is absolutely irregular, in the sense that the canonical extension of its GNS–representation vanishes on all $\mathcal{N}_\alpha = C^*_\rho(G_\alpha)$ *,* $\alpha \in \Gamma$ *.*

(iii) $\mathcal{A}$ *is contained in a proper closed two–sided ideal of* $\mathcal{M}$ *.*

Proof: **(i)** Given a nondegenerate representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, denote $\mathcal{H}_\beta := \big[\pi(\mathcal{N}_\beta)\,\mathcal{H}\big]$ for $\beta \in \Gamma$ (we use $[\cdot]$ to denote the closed linear space of its argument). Then $\pi : \mathcal{N}_\beta \to \mathcal{B}(\mathcal{H}_\beta)$ is a nondegenerate representation, hence extends uniquely to a representation $\pi : M(\mathcal{N}_\beta) \to \mathcal{B}(\mathcal{H}_\beta)$ hence to $\pi : \mathcal{N}^d_\beta \to \mathcal{B}(\mathcal{H}_\beta)$. That is, $\pi \,\big|\, \mathcal{N}_\beta$ acting on $\mathcal{H} = [\pi(\mathcal{N}_\beta)\mathcal{H}] \oplus [\pi(\mathcal{N}_\beta)\mathcal{H}]^\perp$ determines a unique (degenerate) representation π of $\mathcal{N}^d_\beta$ on $\mathcal{H}$ with the same essential subspace

$[\pi(\mathcal{N}_\beta)\mathcal{H}]$. Since $\mathcal{N}^d_\beta \subseteq \mathcal{N}^d_\alpha$ if $\beta \leq \alpha$, it is only necessary to check that these representations piece together to produce a representation of $C^*_\rho(G_d)$. Given $\beta \leq \alpha$, $\mathcal{N}_\beta \subset M(\mathcal{N}_\alpha)$ we need to show that the $\widetilde{\pi}$ obtained from $\mathcal{N}_\alpha$ on $\mathcal{N}^d_\alpha$ agrees on $\mathcal{N}^d_\beta \subset \mathcal{N}^d_\alpha$ with the $\widetilde{\widetilde{\pi}}$ obtained on $\mathcal{N}^d_\alpha$ from π on $\mathcal{N}_\beta$. Let $\{u_\gamma\}$ be an approximate identity in $\mathcal{N}_\alpha$, then for all $\psi \in \mathcal{H}$, $A \in \mathcal{N}^d_\alpha$:

$$\widetilde{\pi}(A)\,\psi = \lim_\gamma \pi(Au_\gamma)\,\psi = \lim_\gamma \widetilde{\pi}(A)\,\pi(u_\gamma)\,\psi$$

and likewise, if $\{v_\gamma\}$ is an approximate identity in $\mathcal{N}_\beta$, then for all $A \in \mathcal{N}^d_\beta$, $\psi \in \mathcal{H}$ we have $\widetilde{\widetilde{\pi}}(A)\,\psi = \lim\limits_\gamma \widetilde{\widetilde{\pi}}(A)\pi(v_\gamma)\,\psi$. Moreover $\mathcal{N}_\beta \subset M(\mathcal{N}_\alpha)$ so π extends uniquely on its essential subspace from $\mathcal{N}_\alpha$ to $\mathcal{N}_\beta$. So for $A \in \mathcal{N}^d_\beta$, we find that for all $\psi \in \mathcal{H}$:

$$\begin{aligned}\widetilde{\pi}(A)\,\psi &= \lim_\nu \pi(Au_\nu)\,\psi = \lim_\nu \lim_\gamma \pi(Au_\nu)\,\pi(v_\gamma)\,\psi \\ &= \lim_\nu \lim_\gamma \pi(Au_\nu v_\gamma)\psi = \lim_\nu \lim_\gamma \widetilde{\widetilde{\pi}}(A)\,\pi(u_\nu v_\gamma)\,\psi \\ &= \widetilde{\widetilde{\pi}}(A)\,\psi\,.\end{aligned}$$

Thus π is uniquely determined on $\mathcal{M}_\beta = C^*(\mathcal{A}_\beta \cup \mathcal{N}^d_\beta)$, respects the monomorphisms of the inductive limit, and so determines uniquely a nondegenerate representation $\widetilde{\pi}$ of $\mathcal{M} = \varinjlim \mathcal{M}_\beta$ on $\mathcal{H} = \varinjlim \mathcal{H}_\beta$.

(ii) That the formula defines a state is standard. Given the GNS–representation $(\pi_{\omega_0},\, \Omega_{\omega_0},\, \mathcal{H}_{\omega_0})$ of ω_0 , observe that the set $\left\{\, \pi_{\omega_0}(\delta_x)\Omega_{\omega_0} \;\middle|\; x \in G \,\right\}$ is an orthonormal basis of $\mathcal{H}_{\omega_0}$ (so it is non-separable), and that each basis vector $\psi_x := \pi_{\omega_0}(\delta_x)\Omega_{\omega_0}$ is cyclic for $\pi_{\omega_0}(C^*_\rho(G_d))$. Let $\{y_\gamma\} \subset G$ be any net converging to e . Then for each pair $x,\, z \in G$ we have

$$(\psi_z,\, \pi_{\omega_0}(\delta_{y_\gamma})\psi_x) = 0 \qquad \forall\, y_\gamma \neq zx^{-1}\,,$$

and thus:
$$\lim_\gamma \pi_{\omega_0}(\delta_{y_\gamma})\,\psi_x = 0 \quad \forall\, x \in G\,, \tag{1}$$

using coincidence of weak and strong operator topologies on the unitaries. Now recall that for each G_α we have a decomposition

$$\pi_{\omega_0} \,\big|\, C^*_\rho(G_{\alpha d}) = \pi^\alpha_R \oplus \pi^\alpha_N$$

into G_α–regular and irregular parts. So for each $x \in G$, write $\psi_x = \psi_R^\alpha + \psi_N^\alpha$ according to this decomposition, and we know that for any net $\{y_\gamma\} \subset G_\alpha$ converging to e we must have $\lim\limits_\gamma \pi_{\omega_0}(\delta_{y_\gamma})\,\psi_R^\alpha = \psi_R^\alpha$. Hence

$$\begin{aligned} -\psi_R^\alpha &= \lim_\gamma \pi_{\omega_0}(\delta_{y_\gamma})(\psi_x - \psi_R^\alpha) && \text{(using (1))} \\ &= \lim_\gamma \pi_{\omega_0}(\delta_{y_\gamma})\,\psi_N^\alpha && (2) \end{aligned}$$

(thus also establishing that the last limit exists). However $\pi_{\omega_0}(\delta_{G_\alpha})$ preserves $\mathcal{H}_N^\alpha$, hence we have in (2) an equality of orthogonal vectors, which therefore must be zero. Thus $\psi_R^\alpha = 0$, so ψ_x has no G_α–regular part, i.e. is orthogonal to the space

$$\begin{aligned} \mathcal{H}_R^\alpha &:= \big\{ \psi \in \mathcal{H}_{\omega_0} \;\big|\; g \in G_\alpha \to \pi_{\omega_0}(\delta_g)\,\psi \in \mathcal{H}_{\omega_0} \quad \text{is continuous} \big\} \\ &\supseteq \big\{ \psi \in \mathcal{H}_{\omega_0} \;\big|\; g \in G \to \pi_{\omega_0}(\delta_g)\,\psi \quad \text{is continuous} \big\} \\ &= \bigcap_{\alpha\in\Gamma} \mathcal{H}_R^\alpha = \mathcal{H}_R \;. \end{aligned}$$

Thus the whole basis $\big\{ \psi_x \;\big|\; x \in G \big\}$ is orthogonal to $\mathcal{H}_R^\alpha$, and so $\mathcal{H}_R^\alpha = \{0\}$ for all $\alpha \in \Gamma$, i.e. $\pi_{\omega_0} \big| C_\rho^*(G_{\alpha d})$ is G_α–irregular , so $C_\rho^*(G_\alpha) \subset \mathrm{Ker}\,\widetilde{\pi}_{\omega_0}$ for all α . Since π_{ω_0} is a Borel representation, $\mathcal{H}_B = \mathcal{H}_{\omega_0}$ and so the canonical extension of π_{ω_0} coincides on each $C_\rho^*(G_\alpha)$ with the canonical extension of $\pi_{\omega_0} \big| C_\rho^*(G_{\alpha d})$ defined in Theorem 2.4(3iii). Thus $C_\rho^*(G_\alpha) \subset \mathrm{Ker}\,\widetilde{\pi}_{\omega_0}$ for all $\alpha \in \Gamma$.

(iii) This follows from (ii) since $\mathrm{Ker}\,\widetilde{\pi}_{\omega_0}$ is an ideal. ∎

Remark The ideal generated by $\mathcal{A}$ in $\mathcal{M}$ is $[\mathcal{M}\mathcal{A}\mathcal{M}]$ and so since $\mathbb{1} \in \mathcal{M}$, 3.2(iii) guarantees that $[\mathcal{M}\mathcal{A}\mathcal{M}]$ is proper. In fact, if $G = S$, $\rho = \sigma$ nondegenerate, we know that the CCR–algebra $\overline{\Delta(S,\,B)} = C_\sigma^*(S_d)$ is simple, hence $C_\sigma^*(S_d) \cap [\mathcal{M}\mathcal{A}\mathcal{M}] = \{0\}$.

Def. 3.3 **(i)** A representation $\pi : C_\rho^*(G_d) \to \mathcal{B}(\mathcal{H})$ is called *regular* if for each $\beta \in \Gamma$, the restriction of π to $C_\rho^*(G_{\beta d})$ is regular on its essential subspace. A state ω is regular if its GNS–representation is regular.

(This is equivalent to $\lim_\gamma \omega(\delta_{g_\gamma}) = 1$ for any net $\{g_\gamma\} \subset G$ converging to e).

Recall the inductive limit topology on G , of which the family of its open sets are all those sets $U \subseteq G$ such that $U \cap G_\beta$ is open in G_β for all $\beta \in \Gamma$. Thus for any topological space V , a map $f : G \to V$ is continuous iff f is continuous on each G_β . Thus the regular representations of G are precisely the strong operator continuous representations of G with the inductive limit topology. On the CCR–algebra (with the inductive system of all finite dimensional subspaces of S), this definition of regularity agrees with the usual one. Clearly the definition depends on the choice of inductive system, but as it is assumed to be fixed, usually this will not be explicitly mentioned.

We also wish to examine those closed subgroups $H \subseteq G$ which can be written as an inductive limit of a subsystem of the inductive system for G , so $H = \varinjlim G_\beta$, $\beta \in \Gamma_H \subseteq \Gamma$, and each $\alpha \in \Gamma$ with $\alpha < \beta \in \Gamma_H$ is in Γ_H . These are called *inductive subgroups* and include all G_β and G , but also others. For instance in the case $G = S$, the inductive subgroups can be any closed linear subspace of S . (It was with this in mind that we specified the finest available nondiscrete inductive system for G).

Def. 3.3 (ii) Let $H \subset G$ be an inductive subgroup of G , then we say that a representation π or state ω of $C^*_\rho(G_d)$ or $\mathcal{M}$ is H–regular if it is regular on $C^*_\rho(H_d) \subset C^*_\rho(G_d) \subset \mathcal{M}$.

It is clear that if a state or representation is H–regular , then it is also K–regular for all inductive subgroups $K \subseteq H$. These are the "partially regular" representations and states which we encountered in [10] for the CCRs.

Lemma 3.4. (1) *A representation* $\pi \in \operatorname{Rep} C^*_\rho(G_d)$, *is* G*–regular iff its canonical extension* $\widetilde{\pi}$ *to* $\mathcal{M}$ *is nondegenerate on each* $C^*_\rho(G_\alpha)$.

(2) *If* $\pi \in \operatorname{Rep} C^*_\rho(G_d)$, *is a Borel representation on* G *, and* H *is an inductive subgroup of* G *, then* π *is* H*–regular iff its canonical extension* $\widetilde{\pi}$ *is nondegenerate on* $C^*_\rho(G_\alpha)$ *for each* $\alpha \in \Gamma_H$.

Proof: **(1)** If π is G–regular it is Borel, so $\widetilde{\pi}\,\big|\,C^*_\rho(G_\alpha)$ is the canonical extension of $\pi\,\big|\,C^*_\rho(G_{\alpha d})$ as in Theorem 2.4(3). Since $\mathbb{1}\in C^*_\rho(G_{\alpha d})$, $\pi\,\big|\,C^*_\rho(G_{\alpha d})$ is nondegenerate, so by Theorem 2.4 since π is G_α–regular , $\widetilde{\pi}\,\big|\,C^*_\rho(G_\alpha)$ is nondegenerate, and this is true for all $\alpha\in\Gamma$.

Conversely, if $\widetilde{\pi}$ is nondegenerate on each $C^*_\rho(G_\alpha)$, then it is uniquely determined on each $C^*_\rho(G_{\alpha d})$ and is regular on each G_α , using Theorem 2.4(2). Now since $C^*_\rho(G_d)=\varinjlim C^*_\rho(G_{\alpha d})$, this means that π is uniquely determined on $C^*_\rho(G_d)$ by its values on the collection of algebras $C^*_\rho(G_\alpha)$, $\alpha\in\Gamma$, and it is regular on all G_α , hence on G .

(2) This is obvious, since if π is Borel, then $\widetilde{\pi}\,\big|\,C^*_\rho(G_\alpha)$ is the canonical extension of $\pi\,\big|\,C^*_\rho(G_{\alpha d})$ as in Theorem 2.4(3). ∎

Remark: Any inductive subgroup $H\subset G$ can determine its own "canonical extension" of a given non–Borel representation as follows. For an H–canonical extension, define

$$\mathcal{H}^H_B:=\big\{\,\psi\in\mathcal{H}_\pi\;\big|\;h\in H\longrightarrow\pi(\delta_h)\psi\in\mathcal{H}_\pi\quad\text{is Borel}\,\big\}$$

and proceed as before for $\widetilde{\pi}$, extending from $C^*_\rho(G_d)$ to all $C^*_\rho(G_\beta)$, $\beta\in\Gamma_H$. (Extension to $C^*_\rho(G_\beta)$ for $\beta\in\Gamma\backslash\Gamma_H$ may not be possible on $\mathcal{H}^H_B$). Observe that for $H\subset K\subset G$ we have $\mathcal{H}^K_B\subseteq\mathcal{H}^H_B$. Now part (2) of lemma 3.4 is true for non–Borel representations if we use H–canonical extensions instead.

We now come to the argument for the construction of $\mathcal{L}$. Let $\pi\in\operatorname{Rep}C^*_\rho(G_d)$, and define its regular subspace by:

$$\mathcal{H}_R:=\big\{\,\psi\in\mathcal{H}_\pi\;\big|\;g\to\pi(\delta_g)\psi\quad\text{is continuous from}\quad G\quad\text{to}\quad\mathcal{H}\;\big\}\subset\mathcal{H}_B\,.$$

That $\mathcal{H}_R$ is a linear space, is obvious, that it is closed follows from the fact that

$$\mathcal{H}_R:=\bigcap_{\alpha\in\Gamma}\mathcal{H}^\alpha_R,\qquad\text{where:}$$

$$\mathcal{H}^\alpha_R:=\big\{\,\psi\in\mathcal{H}_\pi\;\big|\;g\in G_\alpha\to\pi(\delta_g)\psi\in\mathcal{H}_\pi\quad\text{is continuous}\,\big\}$$

is the representation space of the G_α–regular part of $\pi \,\big|\, C^*_\rho(G_{\alpha d})$ which we know to be closed. Now all $\pi(\delta_h)$ will preserve $\mathcal{H}_R$ because

$$g \to \pi(\delta_g) \cdot \pi(\delta_h)\psi = \rho(g,\, h)\, \pi(\delta_{gh})\psi$$

is continuous if $\psi \in \mathcal{H}_R$, using the fact that $\rho(g, \cdot)$ is continuous and group multiplication is continuous. Thus both $\pi(C^*_\rho(G_d))$ and $\pi(G)$ will preserve $\mathcal{H}_R$, and so π restricts to a regular representation π_R of $C^*_\rho(G_d)$ on $\mathcal{H}_R$. For the complementary representation, write π_N , so $\pi = \pi_R \oplus \pi_N$. Call π_N the *irregular part* of π . (Note that π_N may still be regular on some proper subgroup of G).

Denote the projection onto $\mathcal{H}_R$ by P^π_R . Now in $\mathcal{M}''$ we define a projection P_R as the projection from the universal representation space $\mathcal{H}_u$ onto

$$P_R\mathcal{H}_u := \oplus \left\{ P^\pi_R \mathcal{H}_\pi \;\middle|\; \pi \in \mathrm{Rep}_c\mathcal{M} \right\}$$

i.e. $P_R = \bigoplus_{\pi\in \mathrm{Rep}_c\mathcal{M}} P^\pi_R$. We need to show that $\pi(\mathcal{M})$ preserves $P^\pi_R\mathcal{H}_\pi$ for all $\pi \in \mathrm{Rep}\,\mathcal{M}$, (at this point we only know that the subalgebra $\pi(C^*_\rho(G_d)) \subset \mathcal{M}$ preserves $\mathcal{H}_R$). It is only necessary to prove that $\pi(L^1_\rho(G_\alpha))$ preserves $P^\pi_R\mathcal{H}_\pi$ for all α and π . Clearly $\pi(C^*_\rho(G_{\alpha d})) \subset \pi(C^*_\rho(G_d))$ preserves $P^\pi_R\mathcal{H}_\pi$. For the ideal decomposition $\pi \,\big|\, \mathcal{E}_\alpha = \pi_1 \oplus \pi_2$ (recall $\pi_2(L^1_\rho(G_\alpha)) = 0$ and $\pi_1(C^*_\rho(G_\alpha))$ is weak operator dense in $\pi_1(\mathcal{E}_\alpha)$), we have that

$$\pi_1(f) = \int_{G_\alpha} f(g)\, \pi_1(\delta_g)\, d\mu_\alpha(g) \qquad \text{for } f \in L^1(G_\alpha)$$

from which it is clear that $\pi_1(C^*_\rho(G_\alpha))$ preserves $\mathcal{H}_{\pi_1} \cap P^\pi_R\mathcal{H}_\pi$. Since $\pi_2(C^*_\rho(G_\alpha)) = 0$, this algebra trivially preserves $\mathcal{H}_{\pi_2} \cap P^\pi_R\mathcal{H}_\pi$, and thus $\pi(C^*_\rho(G_\alpha))$ preserves $P^\pi_R\mathcal{H}_\pi$. Hence $\pi(\mathcal{M})$ preserves $P^\pi_R\mathcal{H}_\pi$ for all representations π . Thus $P^\pi_R \in \mathcal{M}' \cap \mathcal{M}''$, and so $P_R \in \mathcal{M}' \cap \mathcal{M}''$. Define

$$\mathcal{L} := P_R[\mathcal{M}\mathcal{A}\mathcal{M}]\,. \qquad -(\mathbf{3.5})$$

Theorem 3.6. *Given the notation and framework above,* $\mathcal{L}$ *is a closed two–sided ideal of the* C^**–algebra* $\mathcal{E} := C^*\big(\mathcal{L} \cup C^*_\rho(G_d)\big) \subset \mathcal{M}''$ *, hence there is a homomorphism*

$$\varphi : C^*_\rho(G_d) \to M(\mathcal{L})$$

such that in $\mathcal{M}''$, $\varphi(A)L = AL$ *for all* $A \in C^*_\rho(G_d)$, $L \in \mathcal{L}$, *and* φ *is injective when* $C^*_\rho(G_d)$ *has a faithful regular representation.*

Proof: To see that $\mathcal{L}$ is an ideal of $\mathcal{E}$:

$$C^*_\rho(G_d)\mathcal{L} = P_R C^*_\rho(G_d)[\mathcal{M}\mathcal{A}\mathcal{M}] = P_R[\mathcal{M}\mathcal{A}\mathcal{M}] = \mathcal{L}$$

and similarly from the right. Then the homomorphism φ exist by the general properties of multiplier algebras (cf. Pedersen [11]). Let π be a faithful regular representation of $C^*_\rho(G_d)$. Then $\widetilde{\pi}(P_R) = \mathbb{1}$ and $\widetilde{\pi}$ is nondegenerate on $\mathcal{L}$. Hence π is uniquely determined by $\widetilde{\pi}\,\big|\,\mathcal{L}$ using the unique extension of a representation from the ideal $\mathcal{L}$ to $\mathcal{E}$ on the same space. So

$$\begin{aligned} \pi(\delta_g)\psi &= \lim_\lambda \widetilde{\pi}(\delta_g u_\lambda)\psi = \lim_\lambda \widetilde{\pi}(\varphi(\delta_g)u_\lambda)\psi \\ &= \hat{\pi}(\varphi(\delta_g))\psi \qquad\qquad \text{for all } \psi \in \mathcal{H}_\pi \end{aligned}$$

where $\{u_\lambda\}$ is an approximate identity of $\mathcal{L}$ and $\hat{\pi}$ is the unique extension of $\widetilde{\pi}\,\big|\,\mathcal{L}$ to $M(\mathcal{L})$. Thus $\pi(A) = (\hat{\pi}\circ\varphi)(A)$ for all $A \in C^*_\rho(G_d)$. However π is assumed to be faithful, hence so is φ and $\hat{\pi}$. ∎

Remark. Clearly when G has no regular ρ–representations , $\mathcal{L} = \{0\}$. As far as the author knows, the problem of whether there exists a strong–operator continuous ρ–representation $\pi : G \to U(\mathcal{H})$ for a general topological group is unsolved. (In this direction there is a result in a preprint by Pickrell on the nonexistence of a continuous representation of a gauge–type group on a separable Hilbert space). However in the case of the CCR's, where the symplectic space S is a pre–Hilbert space, the symplectic form being the imaginary part of the inner product, Fock–representations exist, are regular, faithful and well–studied, so for this situation $P_S \neq 0$. There are certainly known symplectic spaces $(S,\, B)$ for which the CCR–algebra has neither Fock representations, nor quasi–free representations [20].

Theorem 3.7. (i) *Let* $\pi \in \operatorname{Rep}\mathcal{L}$, *then the unique extension of* π *on the same space to* $C^*_\rho(G_d)$ *defines a regular representation on*

$C^*_\rho(G_d)$.

(ii) Let $\pi \in \mathcal{P}$ ($\equiv$ regular representations of $C^*_\rho(G_d)$), then the canonical extension $\widetilde{\pi}$ of π to $\mathcal{M}$ is nondegenerate on $\mathcal{L}$.

(iii) There is a bijection $t : \mathcal{P} \to \operatorname{Rep}\mathcal{L}$, given by $t(\pi) = \widetilde{\pi}\,\big|\,\mathcal{L}$, preserving irreducibility.

Proof: **(i)** Let $\pi \in \operatorname{Rep}\mathcal{L}$, , then since $\mathcal{L}$ is an ideal of $\mathcal{M}''$, π extends uniquely on the same space to a representation $\hat{\pi}$ of $\mathcal{M}''$. Since $\hat{\pi}$ is nondegenerate on $\mathcal{L}$, checking the definition of $\mathcal{L}$ shows that $\hat{\pi}(P_R) = \mathbb{1}$. This means that $\hat{\pi}$ is regular on $C^*_\rho(G_d)$.

(ii) Let $\pi \in \mathcal{P}$ have canonical extension $\widetilde{\pi}$ to $\mathcal{M}''$. Since π is regular, $\widetilde{\pi}(P_R) = \mathbb{1}$ and so $\widetilde{\pi}$ is nondegenerate on each $C^*_\rho(G_\alpha) \subset \mathcal{A} \subset [\mathcal{MAM}]$ by lemma 3.4, hence it is nondegenerate on $P_R[\mathcal{MAM}] = \mathcal{L}$.

(iii) We show that t is invertible with inverse $t^{-1}(\pi) = \hat{\pi}\,\big|\,C^*_\rho(G_d)$, $\pi \in \operatorname{Rep}\mathcal{L}$, where $\hat{\pi}$ is the unique extension of (i) . Start with $t(\nu) = \widetilde{\nu}\,\big|\,\mathcal{L}$, $\nu \in \mathcal{P}$, then since ν is regular, $\widetilde{\nu}(P_R) = \mathbb{1}$, so $\widetilde{\nu}(P_R) = \mathbb{1}$, so $\widetilde{\nu}(\mathcal{L}) = \widetilde{\nu}([\mathcal{MAM}]) \supset \widetilde{\nu}(C^*_\rho(G_\alpha))$ for all $\alpha \in \Gamma$. By definition of the canonical extension

$$\widetilde{\nu}(f) = \int_{G_\alpha} f(g)\,\nu(\delta_g)\,d\mu_\alpha(g) \qquad \forall\, f \in L^1_\rho(G_\alpha)\;,$$

i.e. the same canonical extension than in the locally compact case on each $C^*_\rho(G_\alpha)$. Now $(t^{-1} \circ t)(\nu) = (\widetilde{\nu}\,|\,\mathcal{L})\hat{}\,\big|\,C^*_\rho(G_d)$. Since ν is regular, by lemma 3.4 $\widetilde{\nu}\,\big|\,C^*_\rho(G_\alpha)$ is nondegenerate for all α . Hence for a fixed α by Theorem 2.4(2) it has a unique extension to $C^*_\rho(G_{\alpha d})$ on the same space, which therefore must agree with $(\widetilde{\nu}\,|\,\mathcal{L})\hat{}\,\Big|\,C^*_\rho(G_{\alpha d})$. However by 2.4(4) we know that these maps are inverses of each other, so $(\widetilde{\nu}\,|\,\mathcal{L})\hat{}\,\Big|\,C^*_\rho(G_{\alpha d}) = \nu\,\big|\,C^*_\rho(G_{\alpha d})$. Since this is true for all $\alpha \in \Gamma$, we find $(t^{-1} \circ t)(\nu) = \nu$. The statement of irreducibility is obvious. ∎

Theorem 3.8. *With notation and assumptions as above, when G is locally compact we have $\mathcal{L} = C^*_\rho(G)$.*

Proof: When the inductive system $\{ G_\beta \mid \beta \in \Gamma \}$ terminates in G, define

$$\mathcal{B} := C^*\left(C^*_\rho(G) \cup C^*_\rho(G_d) \cup \mathcal{L} \cup \{P_R\}\right) \subset \mathcal{M}'' \qquad -(*)$$

Otherwise, observe $\mathcal{M} \cup C^*_\rho(G) \subset M(C^*_\rho(G))$, so define $\mathcal{B}$ by the formula $(*)$, but as a subalgebra of $C^*(\mathcal{M} \cup C^*_\rho(G))'' \supset \mathcal{M}'' \ni P_R$. Using 3.6 and 3.5 as well as $C^*_\rho(G) \subset \mathcal{A} \subset [\mathcal{MAM}]$, we conclude $\mathcal{L}$ is a two–sided closed ideal for $\mathcal{B}$. Now since $\mathcal{M}$ is generated by $C^*_\rho(G_\alpha)$ and $C^*_\rho(G_d)$, it is clear that $C^*_\rho(G)$ is an ideal in $\mathcal{M}$, so we only need to show that $P_R C^*_\rho(G) \subset C^*_\rho(G)$ to establish that $C^*_\rho(G)$ is also an ideal of $\mathcal{B}$. This is obvious since P_R is exactly the projection of $\mathcal{H}_u$ onto the direct sum of the cyclic representations of $C^*_\rho(G)$. So with the imbedding $C^*_\rho(G) \subset \mathcal{M}''$ given by $\pi_u(C^*_\rho(G)) \subset \pi_u(\mathcal{M})''$ we see $P_R \pi_u(C^*_\rho(G)) = \pi_u(C^*_\rho(G))$.
So since both $\mathcal{L}$ and $C^*_\rho(G)$ are ideals of $\mathcal{B}$, we use the result in Pedersen 3.13.8 [11] that any closed ideal of a C*–algebra is the intersection of the primitive ideals containing it. We prove that $\mathcal{L} = C^*_\rho(G)$ by showing $\mathcal{L} \subset \operatorname{Ker}\pi$ iff $C^*_\rho(G) \subset \operatorname{Ker}\pi$ for all $\pi \in \operatorname{Irrd}\mathcal{B}$. Fix $\pi \in \operatorname{Irrd}\mathcal{B}$ and let $\mathcal{L} \subset \operatorname{Ker}\pi$. Consider the ideal decomposition of π w.r.t. $C^*_\rho(G)$. Since π is irreducible, either $C^*_\rho(G) \subset \operatorname{Ker}\pi$ or π is nondegenerate on $C^*_\rho(G)$ (and $\pi(C^*_\rho(G))$ is strong operator dense in $\pi(\mathcal{B})$). In the latter case π has unique extension to $\mathcal{B}$, so by theorem 2.4 must be regular on $C^*_\rho(G_d)$, so $\pi(P_R) = \mathbb{1}$ and

$$\pi(\mathcal{L}) = \pi(P_R[\mathcal{MAM}]) = \pi([\mathcal{MAM}]) \supset \pi(\mathcal{A}) \supset \pi(C^*_\rho(G))$$

so if π is nondegenerate on $C^*_\rho(G)$, this contradicts $\mathcal{L} \subset \operatorname{Ker}\pi$. Thus $C^*_\rho(G) \subset \operatorname{Ker}\pi$.
Conversely, assume $C^*_\rho(G) \subset \operatorname{Ker}\pi$ and consider the ideal decomposition of π w.r.t. $\mathcal{L}$. If π is nondegenerate on $\mathcal{L}$, then by definition of $\mathcal{L}$ we have $\pi(P_R) = \mathbb{1}$, and $\pi \mid \mathcal{L}$ is the canonical extension of $\pi \mid C^*_\rho(G_d)$ by 2.4(3). So since $\pi(C^*_\rho(G)) \subset \pi(\mathcal{L})$, π is also the canonical extension of $\pi \mid C^*_\rho(G_d)$ to $C^*_\rho(G)$. Since it is regular, π is nondegenerate on $C^*_\rho(G)$, and this contradicts $C^*_\rho(G) \subset \operatorname{Ker}\pi$. Thus

$\mathcal{L} \subset \operatorname{Ker} \pi$, and so we have shown that $\mathcal{L} \subset \operatorname{Ker} \pi$ iff $C^*_\rho(G) \subset \operatorname{Ker} \pi$ for all $\pi \in \operatorname{Irrd} \mathcal{B}$. ∎

Thus $\mathcal{L}$ is an adequate infinite dimensional analogue of the twisted group algebra $C^*_\rho(G)$. So in the infinite dimensional case of S , the regular representations form the representation space of a C*–algebra, and this may be useful to obtain direct integral decompositions of regular representations into irreducible regular representations. More general decompositions of regular representations are dealt with in Schaflitzel [2].

Remarks: (i) Clearly for an H–regular representation we can construct the same structure for $C^*_\rho(H_d) \subset C^*_\rho(G_d)$, thus obtaining an algebra $\mathcal{L}_H$ which has as representation space exactly the continuous representations of H . More specifically, let

$$\mathcal{A}_H := \lim_{\beta \in \Gamma_H} \mathcal{A}_\beta = C^*(\{\, \mathcal{N}_\beta \;\big|\; \beta \in \Gamma_H \,\}) \subset \mathcal{A}$$

$$\mathcal{M}_H = \lim_{\beta \in \Gamma_H} \mathcal{M}_\beta = C^*(\{\, \mathcal{N}_\beta \cup \mathcal{N}^d_\beta \;\big|\; \beta \in \Gamma_H \,\}) \subset \mathcal{M}$$

so $$[\mathcal{M}_H \mathcal{A}_H \mathcal{M}_H] \subset [\mathcal{M}\mathcal{A}\mathcal{M}] \,.$$

Now let $P^H_R \in \mathcal{M}''_H \cap \mathcal{M}'_H$ be the projection which maps every representation space $\mathcal{H}_\pi \subset \mathcal{H}_u$ (the universal representation of $\mathcal{M}_H$ used here) onto the space

$$P^H_R \mathcal{H}_\pi := \{\, \psi \in \mathcal{H}_\pi \;\big|\; h \in H \longrightarrow \pi(\delta_h)\psi \in \mathcal{H}_\pi \;\text{ is continuous} \,\}$$

(note that $P^H_R \notin \mathcal{M}'' \cap \mathcal{M}'$). Now define

$$\mathcal{L}_H := P^H_R [\mathcal{M}_H \mathcal{A}_H \mathcal{M}_H] \,.$$

Note that $M(\mathcal{L}_H) \supset C^*_\rho(H_d)$ but it does not contain $C^*_\rho(G_d)$, and furthermore $\mathcal{L}_H \not\subset \mathcal{L}$. Since $P^H_R P_R = P_R$ (using $\mathcal{M}''_H \subset \mathcal{M}''$ to make sense of this), clearly $\mathcal{L}_H \subset M_{\mathcal{M}''}(\mathcal{L})$. Thus we have, just as in the locally compact case that $\mathcal{L}$, $\mathcal{L}_H$ and $C^*_\rho(G_d)$ are all imbedded in a larger C*–algebra $\mathcal{M}''$ such that $\mathcal{L}$ is an ideal for $\mathcal{L}_H$ and $C^*_\rho(G_d)$, i.e. $\mathcal{L}_H \subset M_{\mathcal{M}''}(\mathcal{L}) \supset C^*_\rho(G_d)$ for all inductive subgroups H , and regular representations are obtained from the unique extensions of representations from

$\mathcal{L}$ to $C^*_\rho(G_d)$. Moreover, a representation $\pi \in C^*_\rho(G_d)$ is H–regular iff its H–canonical extension to $\mathcal{M}$ is nondegenerate on $\mathcal{L}_H$. So the structure theory for partially regular representations looks almost like what it is in the locally compact case.

(ii) For the moment we leave open the questions of when $\mathcal{L}$ is simple or separable, and to what extent it depends on the initial inductive system for G .

(iii) At this point we would like to argue that if one requires all physical representations of the CCRs to be regular (this will be when no linear constraints are present, or after the constraints have been imposed), then $\mathcal{L}$ should be the appropriate field algebra for the CCRs rather than $\overline{\Delta(S,\,B)}$.

(iv) The irregular part π_N of a representation $\pi \in \operatorname{Rep} C^*_\rho(G_d)$ must have canonical extension vanishing on $\mathcal{L}$ (clearly since $\mathcal{L}$ consists of products with P_R , and π_N is orthogonal to $P^\pi_R = \pi(P_R)$). This is consistent with the definition of irregularity in the locally compact case, by 3.8. Note that the irregular part π_N may be regular on some inductive subgroup H , in which case π_N can of course be further decomposed into H–regular and irregular parts, but only as a representation of $C^*_\rho(H_d)$, **not** of $C^*_\rho(G_d)$, because $\pi(\delta_g)$ for some $g \in G\backslash H$ may not preserve the H–regular subspace:

$$\mathcal{H}^H_R := \left\{ \psi \in \mathcal{H}_\pi \;\middle|\; h \in H \longrightarrow \pi(\delta_h)\psi \in \mathcal{H}_\pi \text{ is continuous} \right\} .$$

Say an H–regular representation π is *simply* H–regular if it is K–irregular for all inductive subgroups K properly containing H . By the preceding discussion, we cannot in general expect to find a decomposition of a nonregular representation into a direct integral of partially simply regular subrepresentations of $C^*_\rho(G_d)$. However, if one allows more general direct integrals as in Schaflitzel [2], where the subrepresentations are on different subalgebras, such a decomposition may be possible.

4. Automorphisms.

In physics it is frequently the case that actions of the physical transformation groups on the CCR–algebra are weak operator continuous in all regular representations but not pointwise norm continuous on the CCR–algebra. Recall that from the continuous group homomorphism $\gamma : \mathcal{G} \to \mathrm{Sp}\,(S,\,B)$ an action $\alpha : \mathcal{G} \to \mathrm{Aut}\,\overline{\Delta(S,\,B)}$ is defined by $\alpha_g(\delta_x) = \delta_{\gamma_g(x)}$ for all $f \in \mathcal{G}$, $x \in S$. For the finite dimensional case we already have from theorem 2.6 that such a group action defines a pointwise norm continuous action on $C^*_\sigma(S) = \mathcal{L}$ and we aim to prove this for the algebra $\mathcal{L}$ of the infinite dimensional case.

From a continuous group action on G , it is possible to prove that it defines a pointwise norm continuous action on $\mathcal{L}$ via the Ernest–Takesaki–Bichteler [14, 15] realisation of a C*–algebra as admissable operator fields on its representation space, but for concreteness we do a more direct argument below.

Assume the structures of the last section,

$$\{\, G_\beta \;\big|\; \beta \in \Gamma \,\}\,, \qquad G = \varinjlim G_\beta\,, \qquad \rho \in Z^2(G,\,\mathbf{T}) \quad \text{is continuous}$$

and define

$$\theta \in \mathrm{Aut}^0_\rho G \quad \text{iff} \quad \begin{cases} \bullet\ \theta \in \mathrm{Aut}\, G \ \text{ is Borel on each } \ G_\beta \\[2ex] \bullet\ \text{for each } \ \beta \in \Gamma \ \text{ there are } \ \delta\,,\ \gamma \in \Gamma \text{ with} \\ \qquad\qquad \theta(G_\beta) = G_\delta \ \text{ and } \ \theta^{-1}(G_\beta) = G_\gamma \\ \bullet\ \rho\big(\theta(g),\,\theta(h)\big) = \rho(g,\,h) \ \text{ for all } \ g,\,h \in G. \end{cases}$$

so $\mathrm{Aut}^0_\rho G$ are those ρ–preserving automorphisms mapping members of the inductive system to other members of the inductive system, so for each $\theta \in \mathrm{Aut}^0_\rho G$ we obtain a map $\theta : \Gamma \to \Gamma$ by $\theta(G_\beta) =: G_{\theta(\beta)}$ and an abuse of notation.

Theorem 4.1. **(1)** *Each* $\alpha \in \mathrm{Aut}^0_\rho G$ *defines an automorphism of the measure algebra* $\mathcal{M}_\rho(G) = \varinjlim \mathcal{M}(G_\beta)$ *by* $(\alpha \cdot \mu)(D) := \mu(\alpha^{-1}(D))$ *for all* $\mu \in \mathcal{M}(G_\beta)$ *and Borel sets* $D \subset G_{\alpha(\beta)}$.

(2) *The automorphism* $\alpha \in \mathrm{Aut}\,\mathcal{M}_\rho(G)$ *of (1) preserves* $L^1_\rho(G_d)$ *and maps* $L^1_\rho(G_\beta)$ *to* $L^1_\rho(G_{\alpha(\beta)})$ *. Its restriction to* $L^1_\rho(G_d)$ *is given by* $\alpha(\delta_g) = \delta_{\alpha(g)}$ *, which extends to* $C^*_\rho(G_d)$.

Explicitly on $L^1_\rho(G_\beta)$ *we have* $(\alpha\cdot f)(g) = C^\beta_\alpha\, f(\alpha^{-1}(g))$ *where on the Haar measures we have* $\alpha\cdot\mu_\beta = C^\beta_\alpha\,\mu_{\alpha(\beta)}$, $C^\beta_\alpha \in \mathbb{R}_+$.

(3) *Each* $\alpha\in \mathrm{Aut}^0_\rho G$ *defines an automorphism on* $\mathcal{M}$ *which preserves* $\mathcal{A}$. *There is an injective homomorphism* $\Phi : \mathrm{Aut}^0_\rho G \to \mathrm{Aut}\,\mathcal{M}$ *such that any* $\alpha\in\Phi(\mathrm{Aut}^0_\rho G)$ *preserves* $\mathcal{A}$.

(4) *Let* $\pi \in \mathcal{P} \equiv$ *regular representations of* $C^*_\rho(G_d)$. *Then* $\pi\circ\alpha\in\mathcal{P}$ *for all* $\alpha\in\mathrm{Aut}^0_\rho G$.

Proof: **(1)** Clearly the defined $\alpha\cdot\mu$ is a measure on $G_{\alpha(\beta)}$, and $\alpha : \mathcal{M}(G_\beta)\to\mathcal{M}(G_{\alpha(\beta)})$ is an invertible map. Surjectivity is clear, we show that it is a *–isomorphism. Let μ and $\nu\in\mathcal{M}(G_\beta)$, $f\in C_0(G_{\alpha(\beta)})$, so $\alpha(\mu*\nu)$ is defined by

$$\int_{G_{\alpha(\beta)}} f(t)\,d(\alpha(\mu*\nu))(t) = \int_{G_{\alpha(\beta)}} f(t)\,d(\mu*\nu)(\alpha^{-1}(t))$$

$$= \int_{G_\beta} f(\alpha(t))\,d(\mu*\nu)(t) \qquad \text{using isomorphism } \alpha : G_\beta\to G_{\alpha(\beta)}$$

$$= \int\int_{G_\beta} f(\alpha(st))\,\rho(s,\,t)\,d\mu(s)\,d\nu(t)$$

$$= \int\int_{G_{\alpha(\beta)}} f(st)\,\rho(s,\,t)\,d\mu(\alpha^{-1}(s))\,d\nu(\alpha^{-1}(t)) \qquad \text{using } \rho\circ\alpha=\rho$$

$$= \int_{G_{\alpha(\beta)}} f(t)\,d((\alpha\cdot\mu)*(\alpha\cdot\nu))(t)$$

and thus $\alpha(\mu*\nu) = (\alpha\cdot\mu)*(\alpha\cdot\nu)$, so α is a homomorphism $\mathcal{M}(G_\beta)\to\mathcal{M}(G_{\alpha(\beta)})$. To see that it preserves the involution:

$$\int_{G_{\alpha(\beta)}} f(t)\,d(\alpha\cdot\mu)^*(t) = \overline{\int_{G_{\alpha(\beta)}} \overline{f}(t^{-1})\,\rho(t^{-1},\,t)\,d(\alpha\cdot\mu)(t)}$$

$$= \overline{\int_{G_\beta} \overline{f}(\alpha(t)^{-1})\,\rho(\alpha(t)^{-1},\alpha(t))\,d\mu(t)}$$

$$= \overline{\int_{G_\beta} \overline{f}(\alpha(t)^{-1})\,\rho(t^{-1},\,t)\,d\mu(t)} = \int_{G_\beta} f(\alpha(t))\,d\mu^*(t)$$

$$= \int_{G_{\alpha(\beta)}} f(t)\,d(\alpha(\mu^*))(t)$$

hence $\alpha(\mu^*) = (\alpha\cdot\mu)^*$, and so α is a *–isomorphism.

Moreover if $\beta\le\gamma$ then $G_\beta\subseteq G_\gamma$, and so since $\alpha(G_\beta)\subseteq\alpha(G_\gamma)$, we see that $\mathcal{M}(\alpha(G_\beta)) \subseteq \mathcal{M}(\alpha(G_\gamma))$, hence α preserves the partial ordering of the inductive system of $\mathcal{M}_\rho(G)$. Thus, by the universal property of inductive limit algebras (cf. L.1.1 and L.2.1 in [26]) there is a homomorphism from $\mathcal{M}_\rho(G) = \varinjlim \mathcal{M}(G_\beta)$ to $\widetilde{\mathcal{M}}_\rho(G) := \varinjlim \mathcal{M}(\alpha(G_\beta))$, which is an isomorphism since each $\alpha : \mathcal{M}(G_\beta) \to \mathcal{M}(G_{\alpha(\beta)})$ is an isomorphism. Thus $\mathcal{M}_\rho(G) \cong \widetilde{\mathcal{M}}_\rho(G)$ via α . Now there is also an identity isomorphism between these algebras since we can identify each $\mathcal{M}(G_\beta)$ with its copy in both $\mathcal{M}_\rho(G)$ and in $\widetilde{\mathcal{M}}_\rho(G)$. Then with respect to this identification, the isomorphism obtained from α becomes an automorphism, and on the measures of G_β , agrees with the original definition $(\alpha \cdot \mu)(D) = \mu(\alpha^{-1}(D))$ for Borel sets D .

(2) First, the restriction of α to $L^1_\rho(G_d)$ is:

$$(\alpha \cdot \delta_g)(D) = \delta_g\big(\alpha^{-1}(D)\big) = \begin{cases} 1 & \text{if } \alpha(g) \in D, \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{\alpha(g)}(D) \quad \text{for all Borel sets } D.$$

Thus $\alpha \cdot \delta_g = \delta_{\alpha(g)}$, and so α preserves $L^1_\rho(G_d)$.

Next, we show that if $\alpha(G_\beta) = G_{\alpha(\beta)}$, then $\alpha(L^1_\rho(G_\beta)) = L^1_\rho(G_{\alpha(\beta)})$. Since for any Borel set $D \subset G_{\alpha(\beta)} \ni g$ and the Haar measure μ_β of G_β we have

$$(\alpha \cdot \mu_\beta)(gD) = \mu_\beta\big(\alpha^{-1}(g) \cdot \alpha^{-1}(D)\big) = \mu_\beta(\alpha^{-1}(D)) = (\alpha \cdot \mu_\beta)(D)$$

it is clear that α maps Haar measures to Haar measures. Using the uniqueness of Haar measures up to positive multiples, write $\alpha \cdot \mu_\beta = C^\beta_\alpha \mu_{\alpha(\beta)}$ where $C^\beta_\alpha \in \mathbb{R}_+$. Now realise $f \in L^1_\rho(G_\beta)$ as a measure $\nu = f\,\mu_\beta$, so $d(\alpha \cdot \nu)(g) = d\nu(\alpha^{-1}g) = f(\alpha^{-1}(g))\,d\mu_\beta(\alpha^{-1}(g)) = C^\beta_\alpha\, f(\alpha^{-1}(g))\,d\mu_{\alpha(\beta)}(g)$ which is in $L^1_\rho(G_{\alpha(\beta)})$, and so

$$\big(\alpha(f)\big)(g) = C^\beta_\alpha\, f(\alpha^{-1}(g)) \qquad \forall\, g \in G_{\alpha(\beta)},\ f \in L^1_\rho(G_\beta)\ .$$

(3) Since $\alpha \in \operatorname{Aut}\mathcal{M}_\rho(G)$ is norm preserving, it extends uniquely to the C*–enveloping algebras of the L^1_ρ–algebras above, and also to their multiplier algebras. So we obtain $\alpha \in \mathrm{Aut}\, C^*_\rho(G_d)$ and homomorphisms $\alpha : C^*_\rho(G_\beta) \to C^*_\rho(G_{\alpha(\beta)})$ which preserve the inductive limit of these (see argument in proof of (1)), so α defines an automorphism of $\mathcal{M}$ preserving the subalgebras $C^*_\rho(G_d)$ and $\mathcal{A}$ (which is generated by $\left\{ L^1_\rho(G_\beta) \;\middle|\; \beta \in \Gamma \right\}$).

(4) Recall that $\pi \in \mathcal{P}$ iff the canonical extension $\widetilde{\pi}$ to $\mathcal{M}$ is nondegenerate on each $C^*_\rho(G_\beta)$. We first show that $(\widetilde{\pi \circ \alpha}) = \widetilde{\pi} \circ \alpha$ for $\pi \in \mathcal{P}$. Since α is Borel on all G_β and $\pi \in \mathcal{P}$ (henceforth fixed) is Borel on $\mathcal{H}_\pi$, clearly $\pi \circ \alpha$ is Borel on $\mathcal{H}_\pi$, and so $\mathcal{H}_B = \mathcal{H}_\pi$. Now $\widetilde{\pi \circ \alpha}$ is defined on each $L^1_\rho(G_\beta)$ by

$$(\widetilde{\pi \circ \alpha})(f) = \int_{G_\beta} f(g)\,(\pi \circ \alpha)(\delta_g)\, d\mu_\beta(g) = \int_{G_\beta} f(g)\,\pi(\delta_{\alpha(g)})\, d\mu_\beta(g)$$

whereas
$$\begin{aligned}(\widetilde{\pi} \circ \alpha)(f) = \widetilde{\pi}(\alpha(f)) &= \int_{G_{\alpha(\beta)}} \alpha(f)(t)\,\pi(\delta_t)\, d\mu_{\alpha(\beta)}(t) \\ &= \int_{G_{\alpha(\beta)}} C^\beta_\alpha\, f(\alpha^{-1}(t))\,\pi(\delta_t)\, d\mu_{\alpha(\beta)}(t) \\ &= \int_{G_{\alpha(\beta)}} f(\alpha^{-1}(t))\,\pi(\delta_t)\, d\mu_\beta(\alpha^{-1}(t)) \\ &= \int_{G_\beta} f(g)\,\pi(\delta_{\alpha(g)})\, d\mu_\beta(g) = (\widetilde{\pi \circ \alpha})(f)\end{aligned}$$

for all $f \in L^1_\rho(G_\beta)$. Now since π is regular, $\widetilde{\pi}$ is nondegenerate on all $L^1_\rho(G_\beta)$, so using $\alpha(L^1_\rho(G_\beta)) = L^1_\rho(G_{\alpha(\beta)})$ and the invertibility of α we see that $\widetilde{\pi} \circ \alpha = \widetilde{\pi \circ \alpha}$ is nondegenerate on all $L^1_\rho(G_\beta)$. Hence $\widetilde{\pi \circ \alpha} \,\big|\, C^*_\rho(G_d) = \pi \circ \alpha$ is regular. ∎

Corollary 4.2. *Every automorphism* $\alpha \in \mathrm{Aut}^0_\rho\, G$ *defines an automorphism of* $\mathcal{L} = P_R[\mathcal{MAM}]$.

Proof: Since α preserves $\mathcal{M}$ and $\mathcal{A}$, it preserves $[\mathcal{MAM}]$. Now $\alpha \in \mathrm{Aut}\,\mathcal{M}$ extends uniquely to $\mathcal{M}''$, hence to P_R . Since $\widetilde{\pi}(P_R) = \mathbb{1}$ for all $\pi \in \mathcal{P}$ and $\pi \circ \alpha \in \mathcal{P}$ we have $\widetilde{\pi}(\alpha(P_R)) = \mathbb{1}$. Hence, using invertibility of α , $\widetilde{\pi}(P_R) = \mathbb{1}$ iff $\widetilde{\pi}(\alpha(P_R)) = \mathbb{1}$. Thus $\alpha(P_R) = P_R$, recalling that P_R is the projection onto the regular part of $\mathcal{H}_u$. Thus α preserves $\mathcal{L}$. ∎

Next we show that a pointwise continuous action $\theta : \mathcal{G} \to \mathrm{Aut}^0_\rho G$ defines a pointwise continuous action on $\mathcal{L}$.

Theorem 4.3. *Let* $\{\alpha_\nu\} \subset \mathrm{Aut}^0_\rho G$ *be a net converging pointwise to* $\alpha \in \mathrm{Aut}^0_\rho G$ *(i.e.* $\alpha_\nu(x) \to \alpha(x)$ *as* $\nu \to \infty$ *for each* $x \in G$ *).*
Then $\|\alpha_\nu(A) - \alpha(A)\| \to 0$ *as* $\nu \to \infty$ *for each* $A \in \mathcal{L}$ *.*

Proof: Let $\pi \in \mathcal{P}$, so $\big\|\big[\pi(\delta_{\alpha_\nu(x)}) - \pi(\delta_{\alpha(x)})\big]\psi\big\| \to 0$ as $\nu \to \infty$, hence $\big\|\,[\pi(\alpha_\nu(A)) - \pi(\alpha(A))]\,\psi\big\| \to \infty$ for all $A \in C^*_\rho(G_d)$ and $\psi \in \mathcal{H}_\pi$. Now recall that $\widetilde{\pi}$ is defined on $\mathcal{M}_\rho(G)$ by

$$\pi(\mu) = \int_{G_\beta} \pi(\delta_g)\, d\mu(g) \qquad \forall\, \mu \in \mathcal{M}(G_\beta)\,, \qquad \text{so}$$

$$\pi(\alpha\cdot\mu) = \int_{G_{\alpha(\beta)}} \pi(\delta_g)\, d\mu(\alpha^{-1}(g)) = \int_{G_\beta} \pi(\alpha(\delta_g))\, d\mu(g)\,.$$

Then

$$\begin{aligned} &\big\|\big[\pi(\alpha_\nu\cdot\mu) - \pi(\alpha\cdot\mu)\big]\,\psi\big\| \\ &= \Big\|\int_{G_\beta} \pi(\alpha_\nu(\delta_g) - \alpha(\delta_g))\,\psi\, d\mu(g)\Big\| \\ &\le \|\mu\|_1 \cdot \sup\big\{\, \|\pi(\alpha_\nu(\delta_g) - \alpha(\delta_g))\psi\| \;\big|\; g \in G\big\} \longrightarrow 0 \end{aligned}$$

as $\nu \to \infty$ for all $\psi \in \mathcal{H}_\pi$. We also obtain for the corresponding automorphisms α_ν and α on $\mathcal{M}''$ and on $\mathcal{L}$ that for $\pi \in \mathcal{P}$ and $\psi \in \mathcal{H}_\pi$:

$$\|\widetilde{\pi}(\alpha_\nu(A) - \alpha(A))\psi\| \longrightarrow 0 \qquad \text{as} \quad \nu \to \infty$$

for all $A \in \mathcal{M}'' \supset \mathcal{L}$. Now recall that all the representations of $\mathcal{L}$ are obtained from restrictions to $\mathcal{L}$ of canonical extensions of regular representations $\pi \in \mathcal{P}$. Thus

$$\|\pi(\alpha_\nu(A) - \alpha(A))\psi\| \to 0 \qquad \text{as} \quad \nu \to \infty$$

for all $A \in \mathcal{L}$, $\psi \in \mathcal{H}_\pi$, $\pi \in \mathrm{Rep}\,\mathcal{L}$. In particular if (π, ψ) is the GNS–representation and cyclic vector of a state $\omega \in \wp(\mathcal{L})$, we have

$$\omega\Big(\big(\alpha_\nu(A) - \alpha(A)\big)^*\big(\alpha_\nu(A) - \alpha(A)\big)\Big) \to 0 \qquad \text{as} \quad \nu \to \infty$$

and since this is true for all states of $\mathcal{L}$, we find by Dixmier 2.7.1 [8] that $\|\alpha_\nu(A) - \alpha(A)\| \to 0$ in the C*–norm of $\mathcal{L}$. ∎

Corollary 4.4. *Let* $G = S$ *and* $\rho = \sigma$ *with inductive system consisting of all finite dimensional subspaces of* S *. Then* $\mathrm{Sp}(S,\, B) = \mathrm{Aut}^0_\sigma S$ *,* $\mathrm{Sp}(S,\, B)$ *is closed under the pointwise convergent topology, and any continuous homomorphism* $\gamma : \mathcal{G} \to \mathrm{Sp}(S,\, B)$ *defines a pointwise norm continuous action* $\alpha : \mathcal{G} \to \mathrm{Aut}\, \overline{\Delta(S,\, B)}$ *.*

Proof: Let $\{\theta^\nu\} \subset \mathrm{Sp}(S,\, B)$ be a sequence such that $\{\theta^\nu(x)\}$ converges for each $x \in S$, and denote the limit by $\theta(x) \in S$. Then we wish to show that this defines an element $\theta \in \mathrm{Sp}(S,\, B)$. Linearity is obvious, and we only need to show that $B(\theta(x),\, \theta(y)) = B(x,\, y)$ (which will also establish invertiblity) and this follows from the fact that B is (jointly) continuous on $S \times S$ and that $\theta^\nu \in \mathrm{Sp}(S,\, B)$. The rest of the corollary follows directly from 4.3. ∎

Remarks. (i) The requirement that α is in $\mathrm{Aut}^0_\rho G$ should be expected, because $\mathcal{A}$ depends on the inductive system. (This restriction was not visible for the finite dimensional case).

(ii) The possibility now arises of considering actions $\alpha : \mathcal{G} \to \mathrm{Aut}^0_\rho G$ which are only partially continuous, i.e. only continuous on the orbit of a subgroup $H \subset G$. This will be appropriate for Bosonic systems with linear hermitian constraints, where we only need continuity on the physical subalgebra.

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