CHEBYSHEV'S RECURSION — SOME ANALYTICAL, COMPUTATIONAL AND APPLIED ASPECTS

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Summary

We consider polynomials orthogonal with respect to some scalar product on spaces of polynomials on the real line and unit circle. A basic problem in the constructive theory of such polynomials is to determine their three-term recurrence relations. Depending on what is known about the measure corresponding to a given scalar product, there are different ways to proceed. If, as is typical in applications, one knows the measure only through its moment information, the appropriate procedure is an algorithm that goes back to Chebyshev. The algorithm in effect implements the nonlinear map from the given moments or modified moments to the desired recurrence coefficients. We study some algebraic and computational structures connected with scalar products on spaces of polynomials on the real line and unit circle and Chebyshev's algorithm. To illustrate up-to-date versions of the Chebyshev algorithm various applied problems are considered.

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Running title: Chebyshev's recursion — some aspects

Contents

1.Introduction

In this article, we suggest numerical procedures based on the consistent application of a number of fundamental propositions of the moment problem and the theory of orthogonal polynomials. We do not seek to compare the efficiency of these procedures with that of other possible procedures in solving the same problems. We were simply interested in the fact that these procedures, based on fundamental theoretical propositions, were efficient to some extent. This efficiency may be suggestive of some unknown potential that the methods of the moment problem may possess, and may serve to clarify its essence.

In 1859 P.L. Chebyshev proposed an effective technique to construct the system of polynomials orthogonal with respect to some measure on the real line by its power moments [2]. The Chebyshev algorithm actually realizes a transition from one system of polynomials with three-term recurrence relations to the other system of polynomials orthogonal with respect to some given measure on $\mathbb{R}^1$. It turned out that the Chebyshev idea is applicable to different problems. Algorithms based on this idea have been used for applications in numerical analysis and in theoretical physics and chemistry with great effect ([6,21]).

In the following we will give a brief survey on the constructive theory of orthogonal polynomials and consider some analytical, computational and applied procedures connected with the scalar product $<,>$ on the space $\{\mathcal{P}_{R^1}\}$ of polynomials on the real line $\mathbb{R}^1$ and the Chebyshev algorithm. The scalar product $<,>$ is assumed to have the property $< P,Q >=< P\overline{Q},1 >$, and it is not necessary definite. We also study the scalar product on the space $\{\mathcal{P}_C\}$ of trigonometric polynomials and the corresponding modifications of the Chebyshev algorithm.

To illustrate up-to-date variants of the Chebyshev technique ((m)-T-recursion and its modifications), we consider some applied problems such as the test of positive definiteness of Hankel and Toeplitz quadratic forms and the calculation of arbitrary-order determinants of Hankel – Hadamard and Toeplitz. The (m)-T-recursion, i.e. the Chebyshev recursion modification, and the classical moment theory [1] enable us to construct effective algorithms for checking positivity of functions of one and many variables on given sets by its moments. We also consider the application of the modified Chebyshev technique to the method of symmetric and Hermitian forms in the theory of the separation of roots of algebraic equations expounded in [13]. To apply the Chebyshev idea to another circle of problems, we consider the moment method in the study of the Schrödinger equation and construct an algorithm based on this technique to estimate the eigenvalue boundaries and the solutions of the Schrödinger equation with a special kind of potential.

Some results of this work have been given in [16,17,19] and have been announced in [11].

2. Scalar products on polynomial space

2.1. *Orthogonal systems of polynomials*

Let $\{\mathcal{P}_{R^1}\}$ be the linear space of polynomials on the real line $\mathbb{R}^1$, $<,>$ denote a scalar product on the space $\{\mathcal{P}_{R^1}\}$, which is real for real polynomials and has the property

$$< P, Q >=< P\overline{Q}, 1 > . \tag{A}$$

The scalar product $<,>$ is not assumed to be definite. It is assumed below that all scalar products considered on the space $\{\mathcal{P}_{R^1}\}$ have the property (A). We also assume that the scalar product $<,>$ is nontrivial, i.e., all the Hankel – Hadamard determinants

$$H_{<,>}(n) = \det \begin{vmatrix} < 1, 1 > & < \lambda, 1 > & \dots & < \lambda^n, 1 > \\ < 1, \lambda > & < \lambda, \lambda > & \dots & < \lambda^n, \lambda > \\ \vdots & \vdots & \ddots & \vdots \\ < 1, \lambda^n > & < \lambda, \lambda^n > & \dots & < \lambda^n, \lambda^n > \end{vmatrix} \neq 0. \tag{2.1}$$

Letting

$$s_k =< \lambda^k, 1 >$$

we obtain the traditional notation for Hankel – Hadamard determinants $H_{<,>}(n)$:

$$H_{<,>}(n) = \det \|s_{i+j}\|_{i,j=0}^n.$$

For the scalar product considered there exists an $<,>$-orthogonal system of polynomials, i.e., a system of polynomials $\{P_n(\lambda)\}_{n=0}^\infty$, such that

1) $degP_n(\lambda) = n$

2) the highest-degree coefficient is positive

3) $< P_n(\lambda), P_m(\lambda) >= 0$, if $n \neq m$.

Two systems of the $<,>$-orthogonal polynomials are in use:

a) the set of monic orthogonal polynomials

$$P_n(\lambda) = \frac{1}{H_{n-1}} \det \begin{vmatrix} <1,1> & <\lambda,1> & \dots & <\lambda^n,1> \\ <1,\lambda> & <\lambda,\lambda> & \dots & <\lambda^n,\lambda> \\ \vdots & \vdots & \ddots & \vdots \\ <1,\lambda^{n-1}> & <\lambda,\lambda^{n-1}> & \dots & <\lambda^n,\lambda^{n-1}> \\ 1 & \lambda & \dots & \lambda^n \end{vmatrix}. \tag{2.2}$$

$$= \lambda^n + \text{lower degree terms}$$

b) in the case of a definite scalar product the orthonormal system of the $<,>$-orthogonal polynomials is of special interest. This system is defined by the conditions $< P_n(\lambda), P_n(\lambda) >= 1, n = 0,\ 1,\ \dots$ and

$$\widetilde{P}_n(\lambda) = \frac{1}{\sqrt{H_{n-1}H_n}} \det \begin{vmatrix} <1,1> & <\lambda,1> & \dots & <\lambda^n,1> \\ <1,\lambda> & <\lambda,\lambda> & \dots & <\lambda^n,\lambda> \\ \vdots & \vdots & \ddots & \vdots \\ <1,\lambda^{n-1}> & <\lambda,\lambda^{n-1}> & \dots & <\lambda^n,\lambda^{n-1}> \\ 1 & \lambda & \dots & \lambda^n \end{vmatrix}. \tag{2.3}$$

The relations (2.2), (2.3) are fundamental in the constructive theory of orthogonal polynomials. However, direct calculations of the orthogonal polynomials by these determinant formulas are difficult and even impossible for large n.

2.2. *Recurrence relations for the $<,>$-orthogonal system of polynomials*

The set of monic $<,>$-orthogonal polynomials $\{P_n(\lambda)\}_{n=0}^{\infty}$ satisfies the well-known three-term recurrence relation

$$P_{n+1}(\lambda) = (\lambda - a_n)P_n(\lambda) - b_n P_{n-1}(\lambda), \tag{2.4}$$

$$P_{-1}(\lambda) \equiv 0, \qquad P_0(\lambda) \equiv 1.$$

Note that in fact b_0 does not take part in the recurrence relation (2.4), but it is advisable to take $b_0 =< 1, 1 >$.

Furthermore, in order to establish this relation, we expand $\lambda P_{n-1}(\lambda)$ as a linear combination of the polynomials $\{P_n(\lambda)\}$:

$$\lambda P_{n-1}(\lambda) = \sum_{k=0}^{n} \alpha_{n,k} P_k(\lambda). \tag{2.5}$$

The $<,>$-orthogonality properties show that

$$< \lambda P_{n-1}(\lambda), P_k(\lambda) >= h_k \alpha_{n,k},$$

where

$$h_k = \frac{H_k}{H_{k-1}}.$$

However

$$< \lambda P_{n-1}(\lambda), P_k(\lambda) >=< P_{n-1}(\lambda), \lambda P_k(\lambda) >,$$

which shows that $\alpha_{n,k}$ is equal to zero if $k < n-2$. On other hand, comparison of highest order terms in (2.5) shows that

$$\alpha_{n,n} = 1, \qquad h_n =< \lambda P_{n-1}(\lambda), P_n(\lambda) >, \tag{2.6}$$

and

$$h_{n-1}\alpha_{n,n-1} =< \lambda P_{n-1}(\lambda), P_{n-1}(\lambda) >,$$

$$h_{n-2}\alpha_{n,n-2} =< \lambda P_{n-1}(\lambda), P_{n-2}(\lambda) >=< \lambda P_{n-2}(\lambda), P_{n-1}(\lambda) > .$$

In view of (2.6), we get

$$h_{n-2}\alpha_{n,n-2} = h_{n-1}.$$

Introducing the notations

$$a_{n-1} = \alpha_{n,n-1}, \qquad b_{n-1} = \alpha_{n,n-2},$$

we get (2.4) with

$$a_n = \frac{< \lambda P_n(\lambda), P_n(\lambda) >}{< P_n(\lambda), P_n(\lambda) >}, \tag{2.7}$$

$$b_n = \frac{< P_n(\lambda), P_n(\lambda) >}{< P_{n-1}(\lambda), P_{n-1}(\lambda) >}. \tag{2.8}$$

>From the relation (2.8) it follows that

$$< P_n(\lambda), P_n(\lambda) >= b_0 b_1 \dots b_n \tag{2.9}$$

and

$$H_n = b_0^{n+1} b_1^n \dots b_{n-1}^2 b_n \tag{2.10}$$

Note that according to (2.10), the two scalar products $<,>_1$ and $<,>_2$ on the polynomial space $\{\mathcal{P}_{R^1}\}$ have the same Hankel – Hadamard determinants if and only if the sets of coefficients $\{b_k\}$ in recurrence relations (2.4) that are corresponding to scalar products are equal.

The following theorem due to Favard holds:

Theorem 2.1. *The scalar product* $<,>$ *on the polynomial space* $\{\mathcal{P}_{R^1}\}$ *is definite*

$$\forall P \in \{\mathcal{P}_{R^1}\} \quad < P, P > \geq 0,$$

if and only if b_k *is positive for all non-negative integer* k.

Proof. If a scalar product is definite, then from (2.8) we have that b_k is positive for all $k = 0,\ 1,\ 2, \ldots$. Next we expand an arbitrary polynomial $P(\lambda)$ as a linear combination of the polynomials $\{P_k(\lambda)\}$

$$P(\lambda) = \sum_{k=0}^{n} \xi_k P_k(\lambda)$$

Then

$$< P(\lambda), P(\lambda) >= \sum_{k=0}^{n} |\xi_k|^2 \prod_{l=0}^{k} b_l$$

and if b_k is positive for all $k \in Z_+$, we obtain

$$< P, P > \geq 0.$$

The proof is finished.

A set of polynomials $\{P_k(\lambda)\}$ is closely connected with the Haar matrix $\|C^r_{j,k}\|$. This matrix is a "multiplication table" for polynomials and can be obtained from the relation

$$P_j(\lambda) P_k(\lambda) = \sum_{|j-k|}^{j+k} C^r_{j,k} P_r(\lambda). \tag{2.11}$$

It is clear, that

$$C^r_{j,k} =< P_j(\lambda) P_k(\lambda) P_r(\lambda), 1 > . \tag{2.12}$$

That is why

$$C^r_{j,k} = C^j_{k,r} = C^k_{r,j}.$$

We note also the relation

$$C^r_{j,k} = \|P_r(\mathcal{J})\|_{j,k}, \tag{2.13}$$

where $\mathcal{J}$ is the Jacobi matrix of the normal system of polynomials.

2.3. *The Chebyshev matrix, modified Chebyshev algorithm, relations between two sets of orthogonal polynomials for different scalar products*

Let $<,>$ denote a scalar product on the space $\{\mathcal{P}_{R^1}\}$, $\{\widehat{P}_k(\lambda)\}$ be a set of the $<,>$-orthogonal monic polynomials. They satisfy a three-term recurrence relation

$$\widehat{P}_{n+1}(\lambda) = (\lambda - \widehat{a}_n)\widehat{P}_n(\lambda) - \widehat{b}_n \widehat{P}_{n-1}(\lambda), \tag{2.14}$$

$$\widehat{P}_{-1}(\lambda) \equiv 0, \qquad \widehat{P}_0(\lambda) \equiv 1.$$

The coefficients $\{\widehat{a}_k, \widehat{b}_k\}$ are not assumed to be known. At the same time we consider a set of monic polynomials $\{P_n(\lambda)\}$, which is given by the system of polynomials satisfying the three-term recurrence relation

$$P_{n+1}(\lambda) = (\lambda - a_n)P_n(\lambda) - b_n P_{n-1}(\lambda),$$

$$P_{-1}(\lambda) \equiv 0, \qquad P_0(\lambda) \equiv 1,$$

where the constants a_k, b_k are known.

The sets of coefficients $\{a_k, b_k\}$ and $\{\widehat{a}_k, \widehat{b}_k\}$ are connected with each other by a remarkable relation that goes back to P.L. Chebyshev [2]. The algorithm presented below is a direct generalization of the original Chebyshev algorithm. In stating the modified Chebyshev algorithm we will in principal follow [6].

Definition. *The matrix $T_{\widehat{P},P}$ of the modified moments, where*

$$\|T_{\widehat{P},P}\|_{k,l} = \sigma_{k,l} \stackrel{\text{def}}{=} < \widehat{P}_k, P_l >, \tag{2.15}$$

will be called a Chebyshev matrix.

If $k > l$, then $\sigma_{k,l} = 0$, i.e., the Chebyshev matrix is triangular. In particular,

$$\sigma_{k+1,k-1} = 0, \qquad \sigma_{k+1,k} = 0.$$

We get from (2.15)

$$\begin{aligned} 0 = \sigma_{k+1,k-1} &= < \widehat{P}_{k+1}, P_{k-1} > = < (\lambda - \widehat{a}_k)\widehat{P}_k - \widehat{b}_k \widehat{P}_{k-1}, P_{k-1} > \\ &= < \lambda \widehat{P}_k, P_{k-1} > - \widehat{b}_k \sigma_{k-1,k-1} = < \widehat{P}_k, \lambda P_{k-1} > - \widehat{b}_k \sigma_{k-1,k-1} \\ &= < \widehat{P}_k, P_k > + a_{k-1} < \widehat{P}_k, P_{k-1} > + b_{k-1} < \widehat{P}_k, P_{k-2} > - \widehat{b}_k \sigma_{k-1,k-1} \\ &= \sigma_{k,k} - \widehat{b}_k \sigma_{k-1,k-1} \end{aligned}$$

Thus

$$\widehat{b}_k = \frac{\sigma_{k,k}}{\sigma_{k-1,k-1}}. \tag{2.16}$$

Next

$$
\begin{aligned}
0 = \sigma_{k+1,k} &=< \widehat{P}_{k+1}, P_k >=< (\lambda - \widehat{a}_k)\widehat{P}_k - \widehat{b}_k \widehat{P}_{k-1}, P_k > \\
&=< \lambda \widehat{P}_k, P_k > -\widehat{a}_k \sigma_{k,k} - \widehat{b}_k \sigma_{k-1,k} =< \widehat{P}_k, \lambda P_k > -\widehat{a}_k \sigma_{k,k} - \widehat{b}_k \sigma_{k-1,k} \\
&=< \widehat{P}_k, P_{k+1} + a_k P_k + b_k P_{k-1} > -\widehat{a}_k \sigma_{k,k} - \widehat{b}_k \sigma_{k-1,k} \\
&= \sigma_{k,k+1} + a_k \sigma_{k,k} - \widehat{a}_k \sigma_{k,k} - \widehat{b}_k \sigma_{k-1,k}
\end{aligned}
$$

Therefore we have

$$
\widehat{a}_k = a_k + \frac{\sigma_{k,k+1}}{\sigma_{k,k}} - \frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}}. \tag{2.17}
$$

As one can see from relations (2.16), (2.17), to calculate the values $\widehat{a}_k$ and $\widehat{b}_k$ for any fixed index k it is enough to know two entries of row $(k-1)$ and row k of the matrix $T_{\widehat{P},P}$ in addition to the known parameters a_k and b_k . If the scalar product $<,>$ on the space $\{\mathcal{P}_{R^1}\}$ and initial data are known, we can sequentially generate all rows of the matrix $T_{\widehat{P},P}$. Relations (2.16) and (2.17) determine the coefficients $\widehat{a}_k$ and $\widehat{b}_k$, which provides the most stable algorithm that we known to construct the system $\{\widehat{P}_k(\lambda)\}$ of the $<,>$-orthogonal polynomials. The quantities $\sigma_{k,l} =< \widehat{P}_k, P_l >$ can also be obtained by the recursion. Namely, on the basis of the property (A) of the scalar product $<,>$, we get

$$
\begin{aligned}
\sigma_{k,l} &=< \widehat{P}_k, P_l >=< (\lambda - \widehat{a}_{k-1})\widehat{P}_{k-1} - \widehat{b}_{k-1}\widehat{P}_{k-2}, P_l > \\
&=< \lambda \widehat{P}_{k-1}, P_l > -\widehat{a}_{k-1} < \widehat{P}_{k-1}, P_l > -\widehat{b}_{k-1} < \widehat{P}_{k-2}, P_l > \\
&=< \widehat{P}_{k-1}, P_{l+1} + a_l P_l + b_l P_{l-1} > -\widehat{a}_{k-1} < \widehat{P}_{k-1}, P_l > -\widehat{b}_{k-1} < \widehat{P}_{k-2}, P_l > \\
&= \sigma_{k-1,l+1} + a_l \sigma_{k-1,l} + b_l \sigma_{k-1,l-1} - \widehat{a}_{k-1}\sigma_{k-1,l} - \widehat{b}_{k-1}\sigma_{k-2,l}.
\end{aligned}
$$

Thus

$$
\sigma_{k,l} = \sigma_{k-1,l+1} + a_l \sigma_{k-1,l} + b_l \sigma_{k-1,l-1} - \widehat{a}_{k-1}\sigma_{k-1,l} - \widehat{b}_{k-1}\sigma_{k-2,l}. \tag{2.18}
$$

The relations

$$
\begin{aligned}
\sigma_{k,l} &= \sigma_{k-1,l+1} - (\widehat{a}_{k-1} - a_l)\sigma_{k-1,l} - \widehat{b}_{k-1}\sigma_{k-2,l} + b_l \sigma_{k-1,l-1}, \\
\widehat{a}_k &= a_k + \frac{\sigma_{k,k+1}}{\sigma_{k,k}} - \frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}}, \\
\widehat{b}_k &= \frac{\sigma_{k,k}}{\sigma_{k-1,k-1}},
\end{aligned} \tag{$(m)-T$}
$$

with obvious initial conditions

$$\sigma_{-1,l} = 0, \qquad \sigma_{0,l} =< \widehat{P}_0, P_l >,$$

$$a_0 = a_0 + \frac{\sigma_{0,1}}{\sigma_{0,0}}, \qquad \widehat{b}_0 = \sigma_{0,0}$$

form the recurrence process, the so-called the modified Chebyshev algorithm. These relations give us an opportunity to organize a computational cycle for the sequential calculation of the $\sigma_{k,l}$ by the recurrence relations (2.18) and then of the $\widehat{a}_k, \widehat{b}_k$ by relations (2.16) and (2.17).

Thus, having the coefficients $\{a_k, b_k\}$ and having the well-conditioned algorithm of constructing the set $\{P_k(\lambda)\}$ we can calculate the first n coefficients $\widehat{a}_k, \widehat{b}_k, k = 0,\ 1, \ldots, n-1$ from the first $2n$ quantities $< 1, P_l >, l = 0,\ 1, \ldots, 2n-1$, the modified moments, using the modified Chebyshev algorithm.

>From (2.16), (2.9) and (2.10) we get

$$< \widehat{P}_k(\lambda), P_k(\lambda) >= \sigma_{k,k}, \tag{2.19}$$

$$H_{<,>}(k) = \prod_{l=0}^{k} \sigma_{l,l}. \tag{2.20}$$

Note that formula (2.20) expresses the Hankel–Hadamard determinant by product of the diagonal entries of the Chebyshev matrix.

The formula connecting the set of monic polynomials satisfying the recurrence relation (2.4) and the monic system of polynomials corresponding to the scalar product $<,>$ is given by the following proposition:

Theorem 2.2. *Let* $P = [P_0(\lambda), P_1(\lambda), \ldots, P_n(\lambda), \ldots]^T$ *and* $\widehat{P} = [\widehat{P}_0(\lambda), \widehat{P}_1(\lambda), \ldots, \widehat{P}_n(\lambda), \ldots]^T$ *be the vectors consisting of the polynomials satisfying the recurrence relations (2.4) and (2.14), respectively, and let* $T_{\widehat{P},P}$ *be the Chebyshev matrix. Then*

$$P = T^T_{\widehat{P},P} \| \mathrm{diag}(\sigma^{-1}_{k,k}) \| \widehat{P}.$$

2.4. *Algebraic definition of the scalar product*

The transition from one scalar product on the polynomial space to another scalar product is necessary in various problems. This transition can be formally described by an algebraic definition of the scalar product on the polynomial space $\{\mathcal{P}_{R^1}\}$.

Let $<,>$ be a scalar product on the polynomial space $\{\mathcal{P}_{R^1}\}$; $\{P_k(\lambda)\}$ be the orthonormal set of polynomials corresponding to the scalar product $<,>$:

$$P_j(\lambda) = \sum_{l=0}^{j} \gamma_{l,j}\lambda^l. \tag{2.21}$$

Then any sequence $\{\alpha\} = \alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ of real numbers defines a new scalar product $<,>_\alpha$ (with the property $< P, Q >_\alpha = < P\overline{Q}, 1 >_\alpha$) on the polynomial space $\{\mathcal{P}_{R^1}\}$. Namely let us define the scalar product as

$$< P, Q >_\alpha = \sum_{j=0}^{m+n} \gamma_j \alpha_j, \tag{2.22}$$

where m is the degree of $P(\lambda)$, n is the degree of $Q(\lambda)$, and γ_j are the coefficients of the polynomial

$$P\overline{Q} = \sum_{j=0}^{m+n} \gamma_j P_j. \tag{2.23}$$

Note that

$$< P_r, 1 >_\alpha = \alpha_r. \tag{2.24}$$

Thus, any scalar product on the polynomial space $\{\mathcal{P}_{R^1}\}$ is defined in a unique way as $<,>_\alpha$-scalar product, if the scalar product $<,>$ on polynomial space $\{\mathcal{P}_{R^1}\}$ and the $<,>$-orthogonal system of polynomials are fixed.

At the same time the initial scalar product $<,>$ can also be represented through the $<,>$-orthogonal system in an algebraic manner by means of the sequence

$$\overset{(-1)}{\alpha_r} = < \widehat{P}_r, 1 > . \tag{2.25}$$

We consider the following *local property* to be very important and fundamental: *to compute the first n elements of the sequence $\{\overset{(-1)}{\alpha_k}\}$ it is sufficient to know first n elements of the sequence $\{\alpha_k\}$* . This follows from relation (2.25) and Theorem 2.2.

The kernel

$$\mathcal{K}_\alpha(j;k) = < P_j, P_k >_\alpha \tag{2.26}$$

is connected with the transition from the scalar product $<,>$ to $<,>_\alpha$. This kernel can be represented through the Haar matrix $\|C^r_{j,k}\|$ of the system $\{P_j\}$ and the sequence $\{\alpha_k\}$ as

$$\mathcal{K}_\alpha(j;k) = \sum_{|j-k|}^{j+k} C^r_{j,k}\alpha_r. \tag{2.27}$$

If $\mathcal{J}$ is the Jacobi matrix of the system $\{P_j(\lambda)\}$, then thanks to (2.13) we have

$$\mathcal{K}_\alpha(j;k) = \| \sum_{|j-k|}^{j+k} \alpha_r P_k(\mathcal{J}) \|_{j,k}. \tag{2.28}$$

The next theorem is self-evident:

Theorem 2.3. *The following properties are equivalent: a) the scalar product $<,>_\alpha$ is defined, i.e.*

$$\forall P \in \{\mathcal{P}_{R^1}\}: \qquad < P, P >_\alpha \geq 0;$$

b) the kernel $\mathcal{K}_\alpha(j;k)$ is positive definite; c) the sequence

$$s_\alpha(n) =< \lambda^n, 1 >_\alpha \tag{2.29}$$

forms a positive definite kernel $s_\alpha(j+k)$; d) the sequence $\{\alpha_k\}$ is a positive modified moment sequence with regard to the system $\{P_j(\lambda)\}$, i.e. there exists a non-negative measure $\mu_\alpha(d\lambda)$ having moments of all orders, such that

$$\alpha_j = \int_{-\infty}^{\infty} P_j(\lambda)\, \mu_\alpha(d\lambda); \tag{2.30}$$

e) the form defined on the dense in $\mathcal{H}_{<,>}$ set of polynomials by means of the relation

$$\alpha(\mathcal{J})\,(P,Q) =< P, Q >_\alpha$$

is positive. The Hilbert space $\mathcal{H}_{<,>}$ is a closed set of polynomials with respect to the norm $\sqrt{< P, P >}$.

Note that

$$\alpha(\mathcal{J})\,(P_j, P_k) = \mathcal{K}_\alpha(j;k). \tag{2.31}$$

If $\alpha(\lambda)$ is a polynomial, then $\alpha(\mathcal{J})$ is an operator on the space $\mathcal{H}_{<,>}$ from a dense set in $\{\mathcal{P}_{R^1}\}$ onto $\{\mathcal{P}_{R^1}\}$:

$$\alpha(\mathcal{J}) \colon P(\lambda) \to \alpha(\lambda) P(\lambda).$$

Theorem 2.4. *Let $<,>_\alpha$ be the "α''-scalar product on the space $\{\mathcal{P}_{R^1}\}$ defined by (2.22)–(2.23) and $< P, P >_\alpha \geq 0$. Then the set Σ_α of measures $\mu_\alpha(d\lambda) \geq 0$, such that*

$$\alpha_j = \int_{-\infty}^{\infty} P_j(\lambda)\, \mu_\alpha(d\lambda),$$

and the set Σ_{s_α} of measures, such that sequence $s_\alpha(n) =< \lambda^n, 1 >_\alpha$ is represented in the form

$$s_\alpha(n) = \int_{-\infty}^{\infty} \lambda^n\, \mu_\alpha(d\lambda), \qquad \mu_\alpha(d\lambda) \geq 0, \tag{2.32}$$

are identical. The sequence $\{s_\alpha(n)\}$ can be obtained from the triangular system of linear equations

$$\sum_{l=0}^{j} \gamma_{l,j} s_\alpha(l) = \alpha_j, \qquad j \in Z_+. \tag{2.33}$$

2.5. *"α''-definition as multiplication by function*

The function

$$\alpha(\gamma) = \sum_{j=0}^{\infty} \alpha_j P_j(\gamma) \tag{2.34}$$

can be formally compared with a sequence $\{\alpha_j\}$ for a fixed sequence of polynomials $\{P_j(\gamma)\}$. If $\varrho(d\lambda)$ is a measure having moments of all orders, the integral

$$\int_{-\infty}^{\infty} P(\lambda)\,\alpha(\lambda)\,\varrho(d\lambda),$$

where $P(\lambda)$ is a polynomial, can be rewritten in the following way:

$$\int\limits_{-\infty}^{\infty} P(\lambda)\,\alpha(\lambda)\,\varrho(d\lambda) = \sum_{j=0}^{\deg P} \gamma_j \alpha_j. \tag{2.35}$$

Here, γ_j are the coefficients of the expansion $P(\lambda)$ as a linear combination of the polynomials $\{P_j(\gamma)\}$.

Theorem 2.5. *Let $<,>_1$ and $<,>_2$ be the scalar products on the polynomial space $\{\mathcal{P}_{R^1}\}$; $\{P_j(\gamma)\}$ be the $<,>_2$-orthogonal system of polynomials;*

$$\alpha_j =< P_j, 1 >_1 \tag{2.36}$$

be the modified moments. Then

$$< P, Q >_1 =< P, Q >_{2,\alpha} , \tag{2.37}$$

and if $<,>_2 =<,>_{\varrho(d\lambda)}$, then

$$< P, Q >_1 =< P, Q >_{\alpha(\lambda)\varrho(d\lambda)} . \tag{2.38}$$

3. Scalar product on the space of trigonometric polynomials

In this section we shall consider polynomials of one independent complex variable z on the unit circle $\mathbb{C}=\{z:|z|=1\}$ which are defined by their recurrence relations

$$u_{n+1}=a_{n+1}zu_n+b_{n+1}v_n,$$

$$v_{n+1}=\overline{b}_{n+1}zu_n+a_{n+1}v_n,\qquad n=0,\ 1,\ 2,\dots$$

and the initial conditions

$$u_0(z)\equiv 1,\qquad v_0(z)\equiv 1.$$

Assuming that

$$a_n=\overline{a}_n,\qquad a_n^2-|b_n|^2\neq 0,\qquad n=1,\ 2,\dots$$

we will develop a technique analogous to the one expounded for the polynomial space $\{\mathcal{P}_{R^1}\}$ on $\mathbb{R}^1$.

Let $\{\mathcal{P}_C\}$ be a space of polynomials with complex coefficients on the unit circle $\mathbb{C}=\{z:|z|=1\}$, i.e. a space of trigonometric polynomials, and let $<,>$ denote a scalar product on the space $\{\mathcal{P}_C\}$ with the property

$$<P,Q>=<P\overline{Q},1>.$$

We shall assume that the Toeplitz determinants are nonzero:

$$\Delta_n=\det\begin{vmatrix} <1,1> & <z,1> & \dots & <z^n,1> \\ \overline{<z,1>} & <1,1> & \dots & <z^{n-1},1> \\ \vdots & \vdots & \ddots & \vdots \\ \overline{<z^n,1,>} & \overline{<z^{n-1},1>} & \dots & <1,1> \end{vmatrix}\neq 0,\qquad n=0,\ 1,\dots.$$

It is well known (see for example [1]) that there exists a $<,>$-orthogonal system of polynomials $\{P_n(z)\}$ such that

1) $deg P_n(z)=n$

2) the highest-degree coefficient is positive

3) $<P_n(z),P_m(z)>=0$, if $n\neq m$.

As in Section 2, we will also consider two sets of $<,>$-orthogonal polynomials: the monic system and the normalized system.

The monic polynomial is characterized by the highest degree coefficient being equal to 1 and in terms of determinants is represented as

$$P_n(z)=\frac{1}{\Delta_{n-1}}\det\begin{vmatrix} <1,1> & <z,1> & \dots & <z^n,1> \\ \overline{<z,1>} & <1,1> & \dots & <z^{n-1},1> \\ \vdots & \vdots & \ddots & \vdots \\ \overline{<z^{n-1},1>} & \overline{<z^{n-2},1>} & \dots & <z,1> \\ 1 & z & \dots & z^n \end{vmatrix}.$$

The n-th degree normalized polynomial $P_n(z)$ is

$$\widetilde{P}_n(z) = \frac{1}{\Delta_n} \det \begin{vmatrix} <1,1> & <z,1> & \dots & <z^n,1> \\ <z,1> & <1,1> & \dots & <z^{n-1},1> \\ \vdots & \vdots & \ddots & \vdots \\ <z^{n-1},1> & <z^{n-2},1> & \dots & <z,1> \\ 1 & z & \dots & z^n \end{vmatrix}.$$

The polynomials $\{P_n(z)\}$ and $\{\widetilde{P}_n(z)\}$ satisfy the recurrence relations for the orthogonal sets of polynomials:

$$P_{n+1}(z) = zP_n(z) + b_{n+1}P_n^*(z), \tag{3.1}$$

$$\widetilde{P}_{n+1}(z) = \widetilde{a}_{n+1} z\widetilde{P}_n(z) + \widetilde{b}_{n+1}\widetilde{P}_n^*(z), \tag{3.2}$$

where $R^*(z) \stackrel{\text{def}}{=} z^k\overline{R}(1/z)$ for any polynomial $R(z)$ of degree k; $\widetilde{a}_k$ and $\widetilde{b}_k$ are some constants for which formal analytic relations in determinant form can be obtained. Note that b_0 does not take part in the recursion (3.1), however, below it will be convenient to define

$$|b_0|^2 = 1- <1,1> .$$

>From recurrence relations (3.1)–(3.2) the following three-term recurrence relations immediately follow:

$$b_{n+1}P_{n+2}(z) = (zb_{n+1} + b_{n+2})P_{n+1}(z) - b_{n+2}(1 - |b_{n+1}|^2)zP_n(z), \tag{3.3}$$

$$\widetilde{b}_{n+1}\widetilde{P}_{n+2}(z) = (z\widetilde{a}_{n+2}b_{n+1} + \widetilde{a}_{n+1}\widetilde{b}_{n+2})\widetilde{P}_{n+1}(z) - \widetilde{b}_{n+2}(\widetilde{a}_{n+1}^2 - |\widetilde{b}_{n+1}|^2)z\widetilde{P}_n(z). \tag{3.4}$$

We will use h_k defined for $k \in Z_+$ by

$$h_k =< P_k(z), P_k(z) >, \qquad k = 0,\ 1, \dots\ .$$

It is easy to check [1] that

$$h_k = \frac{\Delta_k}{\Delta_{k-1}}$$

and

$$h_k = h_{k-1}(1 - |b_k|^2). \tag{3.5}$$

Then, in complete analogy to the real line case,

$$< P_k(z), P_k(z) >= (1 - |b_0|^2)(1 - |b_1|^2)\dots(1 - |b_k|^2) \tag{3.6}$$

and

$$\Delta_k = (1 - |b_0|^2)^k(1 - |b_1|^2)^{k-1}\dots(1 - |b_k|^2). \tag{3.7}$$

The relation (3.6) allows us to normalize the monic orthogonal polynomials.

The trigonometric version of the Favard theorem it as follows:

Theorem 3.1. *The scalar product* $<,>$ *on the polynomial space* $\{\mathcal{P}_C\}$ *is definite:*

$$\forall P \in \{\mathcal{P}_C\}: \qquad < P, P > \geq 0$$

if and only if

$$|b_k|^2 < 1, \qquad k = 0,\ 1,\ 2, \dots \ . \tag{3.8}$$

Omitting details, we now give the trigonometric version of the modified Chebyshev algorithm.

Let $\{P_n(z)\}$ be a set of monic orthogonal polynomials satisfying recurrence relations (3.3), and let $\{\pi_n(z)\}$ be a set of monic $<,>$-orthogonal polynomials with the following recurrence relations

$$\beta_{n+1}\pi_{n+2}(z) = (z\beta_{n+1} + \beta_{n+2})\pi_{n+1}(z) - \beta_{n+2}(1 - |\beta_{n+1}|^2)z\pi_n(z)\ . \tag{3.9}$$

The coefficients $\{\beta_n\}$ are not assumed to be known.

Definition. *The matrix* $T_{\pi,P}$ *of modified moments, where*

$$\|T_{\pi,P}\|_{k,l} = \sigma_{k,l} \stackrel{\text{def}}{=} < \pi_k, P_l >$$

will be called a Chebyshev matrix.

Note that $\sigma_{k,l} = 0$ for all $k > l$, i.e. the Chebyshev matrix is triangular.

The entries of the Chebyshev matrix $T_{\pi,P}$ and the coefficients of the three-term recurrence formulas (3.3) and (3.9) are connected by relations that can also be considered as a recursion process to determine entries of the Chebyshev matrix and unknown elements β_k. Writing it down in a recursive manner, we have:

the initial conditions: $\sigma_{-1,l} = 0, \quad \sigma_{0,l} =< 1, P_l >, \quad l = 0,\ 1, \dots, n,$

$$\beta_0 = 1$$

and for all $k = 1,\ 2, \dots,\ n$

$$\beta_k = \frac{1}{\beta_{k-1}}\Big[-\frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}} + \frac{\overline{b}_k}{\overline{b}_{k-1}} + (1 - |\beta_{k-1}|^2)\frac{\sigma_{k-2,k-1}}{\sigma_{k-1,k-1}} - (1 - |\beta_{k-1}|^2)\frac{\overline{b}_k}{\overline{b}_{k-1}}(1 - |b_{k-1}|^2)\frac{\sigma_{k-2,k-2}}{\sigma_{k-1,k-1}}\Big]$$

$$(m) - T_C$$

$$\sigma_{k,l} = \begin{cases} 0, & l = 0,\ 1, \ldots,\ n; \\ \dfrac{\bar{b}_l}{\bar{b}_{l-1}}\sigma_{k,l-1} + \dfrac{\beta_k}{\beta_{k-1}}\sigma_{k-1,l} - \dfrac{\beta_k}{\beta_{k-1}}\dfrac{\bar{b}_l}{\bar{b}_{l-1}}\sigma_{k-1,l-1} + \sigma_{k-1,l-1} \\ -\dfrac{\bar{b}_l}{\bar{b}_{l-1}}(1-|b_{l-1}|^2)\sigma_{k-1,l-2} - \dfrac{\beta_k}{\beta_{k-1}}(1-|\beta_{k-1}|^2)\sigma_{k-2,l-1} \\ +\dfrac{\beta_k}{\beta_{k-1}}(1-|\beta_{k-1}|^2)\dfrac{\bar{b}_l}{\bar{b}_{l-1}}(1-|b_{l-1}|^2)\sigma_{k-2,l-2}, & l = k,\ k+1, \ldots,\ n \end{cases}$$

In the case $P_k(z) = z^k$, i.e. $b_k = 0$ for $k = 0,\ 1, \ldots,$ the $(m){-}T_C$-recurrence relations are transformed into the well-known Levinson algorithm [14]:

step 1: $\pi_0(z) = 1, \qquad \sigma_{0,0} =< 1, 1 >,$

step 2: for $k = 1,\ 2, \ldots,\ n$

$$\begin{aligned} \beta_k &= -\frac{1}{\sigma_{k-1,k-1}} < z\pi_{k-1}(z), 1 > \\ \sigma_{k,k} &= (1-|\beta_k|^2)\sigma_{k-1,k-1}, \\ \pi_k(z) &= z\pi_{k-1}(z) + \beta_k\pi^*_{k-1}(z). \end{aligned} \tag{3.10}$$

It is obvious by analogy with Section 2 that the relations (3.10) can be considered as an effective way to transform $\{z^k\}$ into $\{\pi_k(z)\}$.

Theorem 3.2. *Let $P = [P_0(z), P_1(z), \ldots, P_n(z), \ldots]^T$ and $\pi = [\pi_0(z), \pi_1(z), \ldots, \pi_n(z), \ldots]^T$ be the vectors consisting of the polynomials satisfying the recurrence relations (3.3) and (3.9) respectively and let $T_{\pi,P}$ be the Chebyshev matrix. Then*

$$P = T^T_{\pi,P}\|diag(\sigma^{-1}_{k,k})\|\pi.$$

We will not consider the questions concerning the algebraic definition of the scalar product on the space $\{\mathcal{P}_C\}$ of trigonometric polynomials, which are entirely analogous to the case of the scalar product on the space $\{\mathcal{P}_{R^1}\}$ of polynomials on the real line $\mathbb{R}^1$.

4. Test of the positive definiteness of the Hankel and Toeplitz quadratic forms

The question of checking the positive definiteness of the Hankel and Toeplitz quadratic forms is not only of theoretical interest, but it is also very important in signal processing problems [15,19] and studying quantum-mechanical systems [7–9,20].

Let $\|s_{j+k}\|_{j,k=0}^{\infty}$ be a Hankel (respectively $\|c_{j-k}\|_{j,k=0}^{\infty}$ be a Toeplitz) matrix of a quadratic form. Theoretically the question about its positive definiteness is solved by finding the signature of a form or, what is equivalent, by finding the signs of the determinants in the sequence of $H_k = \det\|s_{j+m}\|_{j,m=0}^{k}$ $(\Delta_k = \|c_{j-m}\|_{j,m=0}^{k})$, $k = 0,\ 1, \ldots$. However, due to the above reasons the calculation and even the determination of the sign of H_k (Δ_k) is a very difficult problem and it becomes even more difficult in cases where the entries of H_k (Δ_k) depend on some parameter being subject to decision.

On the other hand, the Favard theorem allows one to find the definition index of the scalar product $<,>_s$ ($<,>_c$) on the polynomial space $\{\mathcal{P}_{R^1}\}$ ($\{\mathcal{P}_C\}$) and in this way the question about positive definiteness of the corresponding quadratic form can be solved.

Namely, by relations (2.9) and (3.6)

$$< P_m, P_m >= \prod_{k=0}^{m} \widetilde{b}_k \qquad (\quad < P_m, P_m >= \prod_{k=0}^{m} (1 - |\widetilde{\beta}_k|^2) \quad) ,$$

where by definition $\widetilde{b}_0$ is equal to s_0 (c_0), the remaining $\widetilde{b}_k$ are determined by the modified Chebyshev algorithm with $a_k = b_k = 0$ (accordingly, the Levinson algorithm in the trigonometrical case) and the initial data $\sigma_{0,l} = s_l \quad (\sigma_{0,l} = c_l)$.

Testing the positive definiteness of the quadratic forms is connected with problems of special integral representation of the sequences with a lacuna or a lacuna set in the spectrum of the representative measure.

Below the case of one lacuna in the spectrum of the representative measure is considered. In essence, it is comprehensive.

a) *Stieltjes case*

It is well known [1] that the question of special integral representation of the sequence $\{s_k\}$ with the measure defined on the right-hand half-axis is equivalent to positive definiteness of two quadratic forms with matrices $S = \|s_{j+k}\|_{j,k=0}^{\infty}$ and $S^{(1)} = \|s_{j+k+1}\|_{j,k=0}^{\infty}$. By the modified Chebyshev algorithm it can be reformulated as follows:

Theorem 4.1. *Let* $\{\widehat{b}_k\}$ *and* $\{\widehat{b}_k^{(1)}\}$ *be the parameters calculated by two modified Chebyshev algorithms with the initial conditions* $\sigma_{0,l} = s_l$ *and* $\sigma_{0,l} = s_{l+1}$ *respectively. Then the sequence* $\{s_k\}$ *is a Stieltjes one if and only if* $\widehat{b}_k > 0$ *and* $\widehat{b}_k^{(1)} > 0$ *for all* $k \in Z_+$.

b) *Case of a fixed bounded lacuna* (a, b)

Theorem 4.2. *The sequence* $\{s_k\}$ *can be represented by an integral with a measure defined on the real line* $\mathbb{R}^1$ *with the lacuna* (a, b) $(-\infty < a < b < \infty)$ *if and only if the coefficients* $\widehat{b}_k$ *and* $\widehat{b}_k^{(1)}$ *obtained by the modified Chebyshev algorithms with the initial data*

$$\sigma_{0,l} = s_l \qquad \text{and} \qquad \sigma_{0,l} = s_{l+2} - (a+b)s_{l+1} + abs_l$$

respectively, are positive.

For a lacuna set the question about representation of the sequence $\{s_k\}$ by an integral with a measure having the given system of lacunas can be solved analogously. It is important to note that the necessary modified Chebyshev algorithms can be carried out independently and simultaneously.

c) *The trigonometric case*

Theorem 4.3. *The sequence* $\{c_k\}_{k=0}^{n}$ *can be represented by an integral with a measure defined on the unit circle* $\mathbb{C}$ *with the lacuna* (α, β) $(-\pi < \alpha < \beta < \pi)$ *if and only if the values* $a_k, k = 1, 2, \dots, n-1$, *and* $b_k, k = 1, 2, \dots, n-2$, *obtained by the following Levinson algorithms*

$$P_0(z) = 1, \qquad h_0 = c_0,$$

for $k = 1, 2, \dots, n-1$

$$a_k = \frac{1}{h_{k-1}} \sum_{j=0}^{k-1} c_{j+1} p_{k-1,j}; \qquad h_k = h_{k-1}(1 - |a_k|^2); \qquad P_k(z) = zP_{k-1}(z) - a_k P_{k-1}^*(z),$$

and

$$P'_0(z) = 1, \qquad h'_0 = \mu_0,$$

for $k = 1,\ 2, \dots,\ n-2$

$$b'_k = \frac{1}{h'_{k-1}} \sum_{j=0}^{k-1} \mu_{j+1} p'_{k-1,j}; \qquad h'_k = h'_{k-1}(1 - |b_k|^2); \qquad P'_k(z) = zP'_{k-1}(z) - b_k P'^{*}_{k-1}(z),$$

where

$$\mu_k = 2\cos\frac{\alpha-\beta}{2} c_k - e^{i\frac{\alpha+\beta}{2}} c_{k+1} - e^{-i\frac{\alpha+\beta}{2}} c_{k-1}, \qquad P_k(z) = \sum_{j=0}^{k} p_{k,j} z^j, \quad p_{k,k} = 1.$$

satisfy the conditions

$$|a_k| < 1,\ k = 1,\ 2, \dots, n-1 \qquad \textit{and} \qquad |b_k| < 1,\ k = 1,\ 2, \dots, n-2.$$

Note that the Levinson algorithms for the sequences $\{c_k\}_{k=0}^{n}$ and $\{\mu_k\}_{k=0}^{n-1}$ can be carried out independently and simultaneously.

5. The symmetric and Hermitian forms technique

Below we will point out the ties between the modified Chebyshev algorithm and the symmetric and Hermitian forms technique in the theory of the separation of roots of algebraic equations. As it follows from the title of this section we shall be guided by the classical account of this technique in the well known monograph [13]. This technique is also expounded in [5]. In our opinion there is not much to add to the text of these books. We shall be concerned only with the computational aspects of some propositions given in [5,13], which are closely connected with the modified Chebyshev algorithm.

5.1. *Test for the number of different real roots of a real polynomial. Borchardt's criterion*

Let $f(z)$ be a real polynomial having distinct roots $\alpha_1,\ \alpha_2, \dots,\ \alpha_\tau$ with multiplicities $n_1,\ n_2, \dots, n_\tau$, respectively,

$$\begin{aligned} f(z) &= q_0(z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2}\dots(z-\alpha_\tau)^{n_\tau} \\ &= q_0 z^n - q_1 z^{n-1} + q_2 z^{n-2} - \dots + (-1)^n q_n, \end{aligned}$$

where

$$n_1 + n_2 + \cdots + n_\tau = n.$$

The Newton sums are defined by

$$s_p = \sum_{j=1}^{\tau} n_j \alpha_j^p, \qquad p = 0,\ 1,\ 2, \ldots\ .$$

By a theorem on sums of like power of the roots, the following Newton formulas hold:

$$q_0 s_\lambda - q_1 s_{\lambda-1} + q_2 s_{\lambda-2} - \cdots + (-1)^\lambda \lambda q_\lambda = 0, \qquad \lambda = 1,\ 2, \ldots, n-1, \tag{5.1}$$

and

$$q_0 s_{n+\lambda} - q_1 s_{n+\lambda-1} + q_2 s_{n+\lambda-2} - \cdots + (-1)^n q_n s_\lambda = 0, \qquad \lambda = 0,\ 1,\ 2, \ldots\ . \tag{5.2}$$

The formulas (5.1), (5.2) allow us to calculate the values s_λ in consecutive order using the already known values $s_1,\ s_2, \ldots, s_{\lambda-1}$. Thus, the sequence of the Newton sums $\{s_p\}$ can be calculated in an effective way by means of the recurrence relations (5.1), (5.2) and hence this sequence can be determined by the coefficients of the polynomial $f(z)$. All calculations based on the formulas (5.1), (5.2) should be as accurate as possible, for which special procedures are necessary.

The Borchardt criterion given below is formulated in terms of sign alternation in the sequence of the Hankel–Hadamard determinants H_k, where $H_k = \det \|s_{i+j}\|_{i,j=0}^k, k = 0,\ 1, \ldots$. Namely:

Theorem 5.1. *(Borchardt) The number of distinct real roots of the real polynomial $f(z)$ is equal to the surplus of the number of constancies of sign over the number of sign alternations in the sequence $1,\ H_0,\ H_1, \ldots,$ H_{r-1}, where $H_k = \det \|s_{i+j}\|_{i,j=0}^k$ are the Hankel–Hadamard determinants, s_p are the Newton sums, r is a rank of the Hankel form $S_n(x,x) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k$, n is the degree of the polynomial $f(z)$. Accordingly the number of distinct complex conjugate roots of the real polynomials is equal to the number of the sign alternation in the sequence $1, H_0, H_1, \ldots, H_{r-1}$.*

It has been noted that the direct calculation of the determinants H_k and even the determination of its sign is a difficult computational problem not only for large k. However, the determinants H_k can be calculated by the modified Chebyshev algorithm. For this we suppose that $a_k = b_k = 0, \quad \sigma_{0,l} = s_l$ in the relations of the modified Chebyshev algorithm.

Note that for any pair of complex conjugate roots α, β of the real polynomial $f(z)$ and for all $k \in Z_+$ the value $\alpha^k + \beta^k$ is real and so all the Newton sums s_p are real and the relations $< x^k, x^l > \stackrel{\text{def}}{=} s_{k+l}$ determine the scalar product $<,>$ on the polynomial space $\{\mathcal{P}_{R^1}\}$, which is real for real polynomials and has the property (A).

As it can be seen from the Borchardt criterion, the investigated problem is reduced to study of the definition index of the scalar product introduced on the polynomial space $\{\mathcal{P}_{R^1}\}$.

In terms of the Chebyshev matrix, the Borchardt criterion can be reformulated in the following manner:

Theorem 5.2. *The number of the complex conjugate roots of the real polynomial $f(z)$ is equal to the number of negative elements in the sequence of the diagonal entries of the Chebyshev matrix $\sigma_{0,0}, \sigma_{1,1}, \ldots, \sigma_{r,r}$. Accordingly, the number of distinct real roots of the real polynomial is equal to the surplus of the number of positive elements over the number of negative elements in the sequence of the diagonal entries of the Chebyshev matrix corresponding to the sequence of Newton sums s_p added by unity: $1, \sigma_{0,0}, \sigma_{1,1}, \ldots, \sigma_{r,r}$.*

5.2. *Separation of real roots of the real polynomial*

Theorem 5.3. *(Joachimsthal) The number of distinct real roots of the real polynomial $f(z)$ in the interval (a, b) is equal to the loss of the number of sign alternation when going from a to b in the sequence of polynomials*

$$1, \delta_1(t), \delta_2(t), \ldots, \delta_r(r),$$

where

$$\delta_k(t) = \det \begin{vmatrix} s_0 t - s_1 & s_1 t - s_2 & \ldots & s_{k-1} t - s_k \\ s_1 t - s_2 & s_2 t - s_3 & \ldots & s_k t - s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} t - s_k & s_k t - s_{k+1} & \ldots & s_{2k-2} t - s_{2k-1} \end{vmatrix}$$
$$= \det \begin{vmatrix} s_0 & s_1 & \ldots & s_k \\ s_1 & s_2 & \ldots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k-1} & s_k & \ldots & s_{2k-1} \\ 1 & t & \ldots & t^k \end{vmatrix}, \qquad k = 1, 2, \ldots, r$$

and in this sequence there are no two successive zeros for $t = a$ and $t = b$; s_p are the Newton sums of the polynomial $f(z)$.

Considering the value s_p as $< t^p, 1 >$ we can see that the functions $\delta_k(t), k = 1,\ 2, \ldots,\ r$ are the sequence of orthogonal polynomials with regard to the scalar product $<, >$. Therefore if the determinants H_k (for example, they have been calculated in the above way) and the system of monic $<, >$-orthogonal polynomials are known, then the system $\{\delta_k(t)\}$ is known too (with a collection of the coefficients of the recurrence relation of the system). The presence of elements of the sets $\{\widehat{a}_k; \widehat{b}_k\}$ permits to calculate $\{\delta_k(t)\}$ for any value t in a stable way, and then the Joachimsthal criterion can be applied.

6. Test of function range

6.1 . *Univariate functions; discretization of the estimation problem.*

In this section we will consider the problems connected with the test of the function range by its moments.

First of all, let us give some heuristic considerations. Let $\varrho(d\lambda) \geq 0$ be some measure on $\mathbb{R}^1$ having moments of all orders; $f(\lambda) \in L^2_\varrho$ and $f(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}^1$. Obviously the sequence $\{s_f(n)\}$, where

$$s_f(n) = \int_{-\infty}^{\infty} \lambda^n f(\lambda)\varrho(d\lambda), \tag{6.1}$$

is positive definite. Generally speaking, the inverse proposition, i.e.

$$(\ \{s_f(n)\} \quad \text{is positive definite} \ ) \quad \Longrightarrow \quad (\ f(n) \geq 0 \qquad \varrho - a.e. \ ), \tag{6.2}$$

is not true. This can be easily seen in the case where the measure $\varrho(d\lambda)$ generates an indeterminate moment sequence, and this is also true for a measure $\varrho(d\lambda)$ corresponding to determined moment sequence.

Let the measure $\varrho(d\lambda)$ satisfying (6.2) be chosen and the sequence $\{s_f(n)\}$ be positive definite, i.e. the sequence $\{s_f(n)\}$ generates a definite scalar product on the polynomial space $\{\mathcal{P}_{R^1}\}$. Then $f(n) \geq 0 \quad \varrho - a.e.$. Obviously, this way can be put into practice to find different double-side function ranges.

Definition. *Let $\varrho(d\lambda) \geq 0$ be a measure having moments of all orders. It will be called an U.p.-measure if the moment sequence corresponding to $\varrho(d\lambda)$ is positive definite and implication (6.2) is true.*

As is known, *U.p.*-measures were first considered in [12], but were not given any terminological designation.

We will confine ourselves to two classes of *U.p.*-measures:

1) measures with bounded support

2) measures for which there exists $a > 0$ such that $\int_{-\infty}^{\infty} \exp(a|\lambda|)\varrho(d\lambda) < \infty$.

Let $\varrho(d\lambda)$ be some *U.p.*-measure; $\{P_n(\lambda)\}$ be the system of monic $<,>_{\varrho(d\lambda)}$-orthogonal polynomials. If $\varrho(d\lambda)$ is one of the standard measures, then, as it was shown above, the polynomial set $\{P_n(\lambda)\}$ and, what is the same, the coefficients of the recurrence relation, can be found by means of the modified Chebyshev algorithm.

The monic set of $<,>_{f(\lambda)\varrho(d\lambda)}$-orthogonal polynomials is found by the modified Chebyshev algorithm, where a_k, b_k are the coefficients of the recurrence relation of the $<,>_{\varrho(d\lambda)}$-orthogonal system; the initial data of the recursion are

$$\sigma_{-1,l} = 0, \qquad \sigma_{0,l} = < f(\lambda), P_l(\lambda) >_{\varrho(d\lambda)}.$$

In our case the condition $f(n) \geq 0 \quad \varrho - a.e.$ is equivalent to the definiteness of the scalar product $<,>_{f(\lambda)\varrho(d\lambda)}$ on the polynomial space $\{\mathcal{P}_{R^1}\}$.

Theorem 6.1. *Let $\varrho(d\lambda)$ be some U.p.-measure; $f(\lambda) \in L^2_\varrho$ and let the entries $\sigma_{k,k}$ of the Chebyshev matrix obtained by the modified Chebyshev algorithm with initial conditions*

$$\sigma_{-1,l} = 0, \qquad \sigma_{0,l} = < f(\lambda), P_l(\lambda) >_{\varrho(d\lambda)}.$$

be positive. Then $f(\lambda) \geq 0 \quad \varrho - a.e.$.

Note that the system $\{\widehat{P}_k(\lambda)\}$ of $<,>_{f(\lambda)\varrho(d\lambda)}$-orthogonal polynomials is found by the modified Chebyshev algorithm, which is useful for many problems (see Section 7 below).

The formulated theorem gives the criterion of positivity for the function $f(\lambda)$. At the same time it is often necessary to know whether $f(\lambda)$ has negative values. Obviously, the indicator of negative values is diagonal entries of the Chebyshev matrix and, first of all, the first negative element in the sequence of the coefficients b_k.

Let us illustrate this on examples [1].

Example 1. Let $f(\lambda) = (3 - \lambda^2)^5(\lambda - 0.5)^2 - \varepsilon$.

Obviously, this function takes negative values on some interval $(a, b) \subset [-1, 1]$, where $b - a$ is a function of ε. Therefore the numerical computations by the modified Chebyshev algorithm must give negative values in the diagonal entries of the Chebyshev matrix and in the coefficients $\widehat{b}_k$ of the recurrence relation.

The results of such calculations is given in Table 1, where the index of the first negative coefficients $\widehat{b}_k$ for distinct values ε are given. For this, the set of Chebyshev polynomials of the second type with the weight $\varrho(d\lambda) = \sqrt{1 - \lambda^2}$ was chosen for the initial $<,>_{\varrho(d\lambda)}$-orthogonal system.

Table 1.

ε	1	0.5	0.2	0.1	0.05	0.02	0.01
n	14	22	35	52	74	118	169

[1] All computations reported on were carried out on the PC AT in single precision

Example 2 (lacuna case). Let $f(\lambda) = (\lambda - a)(\lambda - b)$, $\quad (a < b)$. Obviously, in this case the function $f(\lambda)$ takes negative values on the interval (a, b). The minimal value of the function $f(\lambda)$ depends on the size of the interval (a, b) and is equal to $\varepsilon = -\frac{(a-b)^2}{4}$. The results of the calculations for this example are given in Table 2, where n is the index of the first negative coefficient $\widehat{b}_k$ in the corresponding modified Chebyshev algorithm.

Table 2.

a	0.1	0.1	0.1	0.1	0.1	0.1	0.1
b	0.4	0.2	0.15	0.11	0.105	0.101	0.1005
ε	-2.25$\cdot 10^{-2}$	-2.5$\cdot 10^{-3}$	-6.25$\cdot 10^{-4}$	-2.5$\cdot 10^{-5}$	-6.25$\cdot 10^{-6}$	-2.5$\cdot 10^{-7}$	-6.25$\cdot 10^{-8}$
n	8	29	61	311	623	3124	6250

Example 3. $\quad f(\lambda) = \cosh(\lambda - a) - 1 - \varepsilon$.

The results of the calculations by the modified Chebyshev algorithm for $a = 0.1$ are given in Table 3.

Table 3.

ε	0.1	0.05	0.02	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001	0.00005
n	2	3	6	9	13	23	33	74	78	109	154

Systems of orthogonal polynomials corresponding to various *U.p.*-measures can be used in solving problems of function estimation. In practical terms, this may increase the reliability of estimation.

The "shadow" polynomials were also considered for all given examples. The term *shadow polynomial* is meant to indicate the polynomial $\widehat{P}_{n_0}(\lambda)$, where n_0 is the index of the first negative coefficient $\widehat{b}_k$. Note that

$$\int_{-\infty}^{\infty} |\widehat{P}_{N_0}(\lambda)|^2 f(\lambda) \varrho(d\lambda) < 0.$$

It may be expected that in the considered examples the polynomial $\widehat{P}_{n_0}(\lambda)$ has small values on the positivity set of the function $f(\lambda)$ and large values on the negativity set of the function $f(\lambda)$. The computations which have become possible by means of the modified Chebyshev algorithm confirm this qualitative analysis. Let us give the example corresponding to the case $f(\lambda) = (3-\lambda^2)^5(\lambda-0.5)^2-\varepsilon$ for $\varepsilon = 0.05$. We have

$$\widehat{P}_{74}(\lambda) \sim 6 \cdot 10^{-19} \qquad \text{for} \qquad \lambda = 0.5$$

and

$$\widehat{P}_{74}(\lambda) \sim 1 \qquad \text{and} \qquad \lambda \sim 0.3 \qquad \text{and} \qquad \lambda \sim 0.7.$$

6.2. *Case of two variables*

To estimate the ranges of a function as well as a polynomials of two variables, a more deeper mathematical technique is necessary: first of all, the technique of measure disintegration and the Hausdorff moment problem technique. In this paper we will restrict ourselves only to original illustrations.

As it has been shown in [17] the modified Chebyshev algorithm can also be applied to investigate the structure of the scalar product on the polynomial space $\{\mathcal{P}_{R^2}\}$. It permits to construct an algorithm to test the range of a function of two variables. We will consider some basic situations. The extension of this sketch to more common cases is easily made.

Let D be the rectangle $0 \leq \lambda; \mu \leq 1$;

$$P(\lambda,\mu) = \sum_{k,l=0}^{\infty} d_{k,l}\lambda^k\mu^l \tag{6.3}$$

be a smooth function given by the set of real numbers $\{d_{k,l}\}_{k,l=0}^{\infty}$. Let us choose some bounded measure $\varrho(d\lambda) \geq 0$, *supp* $\varrho = [0,1]$. The set of the orthogonal polynomials of the first type $\{P_l(\lambda)\}$ corresponding to the measure $\varrho(d\lambda)$ is assumed to be known. Next, let

$$P(\lambda,\mu) = \sum_{k,l=0}^{\infty} P_l(\lambda)Q_l(\mu), \tag{6.4}$$

$$Q_l(\mu) = < P(\lambda,\mu), P_l(\lambda) >_{\varrho(d\lambda)}, \tag{6.5}$$

and $\|C^r_{j,k}\|$ be the Haar matrix of the system $\{P_l(\lambda)\}$.

Theorem 6.2. *The inequality $P(\lambda,\mu)\mid_D \geq 0$ is true if and only if the kernel*

$$\mathcal{K}_{P,\varrho}(j,k;\mu) = \| \sum_{|j-k|}^{j+k} C^r_{j,k} Q_r(\mu) \| \tag{6.6}$$

is positive definite for $0 \leq \mu \leq 1$.

According to the Silvester criterion, the test of positive definiteness of the kernel $\mathcal{K}_{P,\varrho}(j,k;\mu)$ is reduced to the test of positivity of a known sequence of determinants $d_0(\mu), d_1(\mu), \ldots, d_1(\mu), \ldots$. The kernel (6.6) application for testing the condition $P(\lambda,\mu)\mid_D \geq 0$ is simplified if $P(\lambda,\mu)$ is a polynomial. In this case the kernel $\mathcal{K}_{P,\varrho}(j,k;\mu)$ is a band matrix. In addition the simplification can sometimes be achieved by a special choice of the Haar matrix $\|C^r_{j,k}\|$. However, the troubles concerning with the calculation of the higher-order determinants do not disappear.

6.3. *The Hausdorff differences and non-negativity test*

In Theorem 6.2 we did not use the limited size of the rectangle D [2]. Let us now apply the specificity of the Hausdorff moment problem. From (6.5) and (6.6) we have

$$\mathcal{K}_{P,\varrho}(j,k;\mu) = < P_j(\lambda), P_k(\lambda) >_{P(\lambda,\mu)\varrho(d\lambda)} = \int_0^1 P_j(\lambda)P_k(\lambda)P(\lambda,\mu)\varrho(d\lambda).$$

The one-dimensional moment sequence corresponding to the kernel $\mathcal{K}_{P,\varrho}(j,k;\mu)$ is

$$s_\mu(n) = \int_0^1 P(\lambda,\mu)\varrho(d\lambda). \tag{6.7}$$

Having formed the sequence of the Hausdorff differences

$$\begin{aligned}\Delta_{k,r}s_\mu &= s_\mu(r) - C^1_k s_\mu(r+1) + C^2_k s_\mu(r+2) + \cdots + (-1)^k s_\mu(r+k) \\ &= \int_0^1 \lambda^r (1-\lambda)^k P(\lambda,\mu)\varrho(d\lambda)\end{aligned} \tag{6.8}$$

for the sequence $\{s_\mu(n)\}$ we obtain the following proposition by the well-known Hausdorff criterion.

[2] The reasoning given can be easily modified for cases of non-trivial unbounded regions

Theorem 6.3. *The following conditions are equivalent*

$$\begin{aligned} &1. \quad P(\lambda,\mu)\big|_D \geq 0, \\ &2. \quad \Delta_{k,r}s_\mu \geq 0, \qquad k,r = 0,\ 1,\dots\ . \end{aligned} \tag{6.9}$$

Note that the test of the "two-dimensional" sequence $\Delta_{k,r}s_\mu$ of the functions from one variable can be carried out in the above way through an application of the modified Chebyshev algorithm. The applications of the modified Chebyshev algorithm also facilitate the computation of the integrals (6.8) in any complicated cases. The above constructions are particularly clear if $P(\lambda,\mu)$ is a polynomial. In this case all the functions $\Delta_{k,r}s_\mu$ are polynomials.

Example 4. To illustrate this, let us give the numerical results for the polynomial $Q(\lambda,\mu)$ $-1 \leq \lambda \leq 1, 0 \leq \mu \leq 1$, where

$$Q(\lambda,\mu) = (\lambda - 0.6)^2 + 4(\lambda - 0.6)(\mu - 0.1) + 4.1(\mu - 0.1)^2 + \varepsilon, \qquad \varepsilon = -0.07.$$

The smallest n, such that $\widehat{b}_n < 0$ and $\widehat{b}_{n-1} < 0$, are given in Table 4 for $k,r = 0,\ 1,\dots,\ 10$.

Table 4.

$k \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	35	19	15	13	13	11	11	11	11	11	11
1	19	16	14	14	12	12	12	11	11	11	11
2	15	15	14	14	12	12	12	12	12	11	11
3	14	13	13	13	12	12	12	12	12	12	10
4	12	13	13	13	12	12	12	12	12	12	12
5	12	12	13	13	11	11	12	12	12	12	12
6	12	12	12	12	11	11	11	11	11	11	11
7	12	12	12	11	11	11	11	11	11	11	11
8	11	12	12	11	11	11	11	11	11	11	11
9	11	12	12	11	11	11	11	11	11	11	11
10	11	12	12	11	11	11	11	11	11	11	11

Note: the test of the k-dimensional function given in a hyperrectangle is reduced to the check of non-negativity of a 2^{k-1}-dimensional countable sequence of one-dimensional functions by the procedure given above.

7. Moment method for a Schrödinger equation and the modified Chebyshev algorithm

In recent works [7-9,20] to investigate the Schrödinger equation with a semi-bounded potential, the moment method has been used with great success. The main reason of the successful application of the moment method to this problems is a phenomenon of exponential decrease of the solution of the Schrödinger equation with a given potential. Roughly, the exponential decrease is :

Let

$$-\Delta\psi + (V - E)\psi = 0 \tag{7.1}$$

be the Schrödinger equation with a semi-bounded potential V. Then the values E for which there exist non-trivial solutions ψ, are subdivided into two classes: the continuous spectrum and the discrete spectrum arranged on the left of it. The solutions corresponding to the discrete spectrum decrease exponentially. The rate of decrease depends on the distance between the considered eigenvalues and the continuous spectrum.

If we consider the ground state solution $\psi_{g.s.}(x)$ of the Schrödinger equation corresponding to the smallest energy value $E_{g.}$, then from physical considerations it will follow that $\psi_{g.s.}(x) \geq 0$. The non-negativity of $\psi_{g.s.}(x)$ and the property of exponential decrease allow us to transform the Schrödinger equation (7.1) for $\psi_{g.s.}(x)$ with a linear fractional semi-bounded potential by means of comparison of $\psi_{g.s.}(x)$ and its power moment μ_n sequence into the equivalent recurrence relation

$$\mu_n = M(\widetilde{\mu}_0, \widetilde{\mu}_1, \ldots, \widetilde{\mu}_s, \mu_{n-k}, E), \tag{7.2}$$

where E is a parameter to be determined. Assuming that $E_{g.}$ and the set of missing moments $\{\widetilde{\mu}_{n_1}, \widetilde{\mu}_{n_2}, \ldots, \widetilde{\mu}_{n_k}\}$ are known, we can calculate all moments of the function $\psi_{g.s.}(x)$ by the recurrence relation (7.2) and, hence, taking into account the nature of the decrease of $\psi_{g.s.}(x)$ we can reconstruct $\psi_{g.s.}(x)$ in principle.

However, this cannot be done immediately because the power system is very far from the orthogonal system of polynomials corresponding to the weight $\psi_{g.s.}(x)$.

In this section we limit ourselves to considering the one-dimensional equation (7.1). First of all, let us note that, as it was shown in [7-9,20], the sequential upper and lower energy bounds for the ground state energy $E_{g.}$ are defined by the conditions

$$\det \|\mu_{i+j}(E)\|_{i,j=0}^{n} > 0.$$

Moreover, in the considered model examples it is sufficient to restrict ourselves to the half-axis case.

Next, let

$$\psi(x) \geq 0 \quad (0 \leq x < \infty), \qquad \exists k > 0: \quad \psi(x) \leq Ce^{-kx} \quad (x > R),$$

and all moments

$$m_n = \int_0^\infty x^n \psi(x) dx$$

are known.

Now we introduce the auxiliary weight $w(x) = e^{\gamma x}, \gamma < 2R$ with the known $<,>_w$-orthogonal system of monic polynomials $\{P_j(x)\}$ and with the coefficients $\{a_n; b_n\}$ of the recurrence relations. The set of values

$$\sigma_{0,l} = \sum_{s=0}^{l} \gamma_{s,l} m_l,$$

where

$$P_j(x) = \sum_{l=0}^{j} \gamma_{l,j} x^l,$$

is also known for the function $\psi(x)/w(x)$. Let us take $\{\sigma_{0,l}\}$ for the initial data of the modified Chebyshev algorithm. The result of calculation by this modified Chebyshev algorithm is the set $\{\widehat{P}_j(x)\}$ of monic polynomials orthogonal with respect to the weight $\frac{\psi(x)}{w(x)} w(x)dx$, i.e. the $<,>_{\psi(x)dx}$-orthogonal set of polynomials. The last system of polynomials allows one to compute the following integral

$$I_j(\Omega) = \int_\Omega (\psi(x) - C)\widehat{P}_j(x) w(x) dx, \tag{7.3}$$

where Ω is some set on the half-axis.

Note that $\psi(x)$ generates some *U.p.*-measure and hence the knowledge of the integral (7.3) allows one to determine the range of the function $\psi(x) - C$.

The calculation of $\int_\Omega \widehat{P}_j(x)w(x)dx$ can be accomplished by a quadrature formula corresponding to the weight $w(x)dx$, because the coefficients of recurrence relation for the system $\{\widehat{P}_j(x)\}$ are obtained by the modified Chebyshev algorithm.

For the calculation of the integral $\int_\Omega \psi(x)\widehat{P}_j(x)w(x)dx$ (the function $\psi(x)$ is unknown) the quadrature formula for the weight $w(x)dx$ can be used since the system $\{\widehat{P}_j(x)\}$ is known. It is sufficient to construct a $<,>_\psi$-quadrature formula. It should be taken into account that the set Ω is realized by summing up on notes from this set.

To give an illustrative example, we consider the usual continuum harmonic-oscillator problem

$$-\psi'' + x^2\psi = E\psi. \tag{7.4}$$

Obviously, in this case there exists a unique point $E_{g.}$ of the discrete spectrum. If we take

$$\mu_p = \int_{-\infty}^{\infty} x^p \psi_{g.s.}(x)dx,$$

then the equation (7.4) is transformed into relation for the moments:

$$-p(p-1)\mu_{p-2} + \mu_{p+2} = E\mu_p.$$

Next, for all odd k, $\mu_k = 0$. Therefore, after replacements $p = 2k$ and $\mu_{2k} = m_k$ we have

$$m_{k+1} = Em_k + 2k(2k-1)m_{k-1}. \tag{7.5}$$

Thus, to recover all the moments $\{m_k\}$ corresponding to E in our example, it is sufficient to know only the two first moments m_0, m_1, i.e. these values are the missing moments. Taking $k = 0$ in relation (7.5), we obtain $m_1 = Em_0$. Hence in the considered example we can restrict ourselves only to one moment m_0 and without loss generality, we can take $m_0 = 1$.

If the moment sequence $\{m_k(E)\}_{k=0}^{\infty}$ is known, we can apply the necessary modified Chebyshev algorithms to check the criterion of solvability of the Stieltjes moment problem (Theorem 4.1) for all fixed values E.

Obviously, the chosen E is nearer to $E_{g.} = 1$, the more first members of the sequences $\{\widehat{b}_k\}$ and $\{\widehat{b}_k^{(1)}\}$ obtained by the corresponding modified Chebyshev algorithm are positive. This directly confirms the numerical results of calculations, that are given in Table 5, where n_0 is the index of the first negative element in the sequence $\{\widehat{b}_k\}$ (left-hand side of Table 5) and $\{\widehat{b}_k^{(1)}\}$ (right-hand side of Table 5) corresponding to the chosen E.

Table 5.

E	n_0	E	n_0
$1-10^{-1}$	3	$1+10^{-1}$	4
$1-10^{-2}$	5	$1+10^{-2}$	5
$1-10^{-3}$	6	$1+10^{-3}$	6
$1-10^{-4}$	7	$1+10^{-4}$	8
$1-10^{-5}$	8	$1+10^{-5}$	9
$1-10^{-6}$	9	$1+10^{-6}$	10
$1-10^{-7}$	10	$1+10^{-7}$	11
$1-10^{-8}$	11	$1+10^{-8}$	12
$1-10^{-9}$	13	$1+10^{-9}$	13
$1-10^{-10}$	14	$1+10^{-10}$	14

In our opinion it is instructive that the comparison of the given numerical results with those from paper [10] emphasizes the principal possibilities, efficiency and accessibility of the modified Chebyshev algorithm as a numerical method of calculation (PC AT for modified Chebyshev algorithm and Cray-2 for method given in [10]). Some more contrast is observed when we compare the effect of the modified Chebyshev algorithm to estimate the range of the function $\psi_{g.s.}(x)$.

8. Hilbert spaces of entire functions and the modified Chebyshev algorithm

To describe the solutions of the indeterminate moment problem, the extension problem of Hermitian-positive functions, the solutions of various interpolation problems, the isometrical inclusions of the De Branges spaces of entire functions, it is necessary to construct the entire second-order matrix-function $W(z)$ accomplishing the linear fractional transform of the Nevanlinna function class. To construct such a matrix-function it is desirable to have an effective computational technique.

In our opinion, the modified Chebyshev algorithm is an effective technique for the indeterminate power moment problem. Indeed, accepting the moment sequence as initial data for the modified Chebyshev algorithm with $a_k = b_k = 0$, we obtain the set of $<,>$-orthogonal polynomials $\{P_k(\lambda)\}$ of the first type, and at the same time the entries $\{\widehat{a}_k, \widehat{b}_k\}$ of the Jacobi matrix . This matrix can be used to construct the set of $<,>$-orthogonal polynomials $\{Q_k(\lambda)\}$ of the second type by recursion. After that the approximations of the entries

$$
\begin{aligned}
A(z) &= z\sum_{k=0}^{\infty} Q_k(0)Q_k(z),\\
B(z) &= -1 + \sum_{k=0}^{\infty} Q_k(0)P_k(z),\\
C(z) &= 1 + \sum_{k=0}^{\infty} P_k(0)Q_k(z),\\
D(z) &= z\sum_{k=0}^{\infty} P_k(0)P_k(z).
\end{aligned}
$$

of the matrix-function $W(z)$ can be calculated.

Note that for the undetermined power moment problem the written series converge uniformly on all compact sets in the complex plane.

The approximation of the phase function corresponding to the Hilbert spaces of entire functions [3] can be obtained in the same way. This gives a sequence of canonical approximations for solutions of the infinite moment problem.

If we have a truncated trigonometric moment problem, the matrix-function $W(z)$ corresponding to it can be obtained by the Levinson algorithm (see Section 3).

The procedure of construction of the matrix-function $W(z) = \|W_{jk}(z)\|_{j,k=1}^2$ described above is considerably more stable than a procedure directly based on determinant equations (2.2) and (2.3). However, errors are accumulated in this case too, which necessitates more accurate calculation and impedes advance to high-degree orthogonal polynomials. If the system of $\widetilde{P}_k(\lambda) = \sum \eta_{l,k}\lambda^l$-orthogonal polynomials is used as a starting-point, knowledge of the values of $\sum \eta_{l,k}s(l)$ is necessary. Their direct calculation is a fairly complicated task. It does, however, sometimes become possible to indicate a "close" to $\{P_k\}$; then $\sum \eta_{l,k}s(l)$ will have temperate growth (see Section 2.5).Sometimes, if sush approach is taken, some a priori information can be taken into consideration on measures representing the sequence $s(l)$, e.g. the existence of measures possessing a support $C[0,\infty)$ as its principal property (Stiltjes's case). Another example of indication of a "starting" system of orthogonal polynomials "close" to one which is sought are moment methods of finding the energy of the ground state (see Section 7) and excited states for the Schrödinger equation with semi-bounded potential. Some "starting" systems of orthogonal polynomials are to be found in the theory of spectral functions of string [4].

The construction of the corresponding modified Chebyshev technique for the arbitrary De Branges spaces of entire functions seems to us a very interesting and important problem.

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9. References

1. Achieser N.I. (1965): The Classical Moment Problem and Some related Questions in Analysis. Oliver & Boyd, Edinburgh
2. Chebyshev, P.L. (1859): Sur l'interpolation par la méthode des moindres carrés, Mém.Acad.Imper.Sci. St.Petersbourg. 7, **1**, 15, 1–24
3. De Branges L. (1968): Hilbert spaces of entire functions. Prentice-Hall, Englewood Cliffs, N.J.
4. Dym H., McKean H.P. (1976): Gaussian Processes, Function Theory, and the Inverse Spectral Problem. Academic Press.
5. Gantmaher F.R. (1988): Matrix Theory. Nauka, Moscow
6. Gautschi W. (1982) On generating orthogonal polynomials, SIAM J. Sci. Statist. Comput. **3**, 289–317
7. Handy C.R. (1987): Moment method quantisation of a linear differential equation for $|\psi|^2$, Phys.Rev. A, **36**, 12, 4411–4416
8. Handy C.R., Bessis D. (1985): Rapidly Lower Bounds for Schrödinger Equation ground State Energy, Phys.Rev.Letters, **55**, 9, 931–934
9. Handy C.R., Bessis D., Morley T.D. (1988): Generating quantum energy bounds by the moment method: A linear programming approach. Phys.Rev. A, **37**, 12, 4557–4569
10. Handy C.R., Pei J.Q. (1988): Moment-method analysis of ground state of discretized bozonic systems, Phys.Rev. A., **38**, 7, 3175–3181
11. Korzh S.A., Ovĉarenko I.E., Ugrinovsky R.A. (1991): Chebyshev recursion and its applications. In Proc.16th USSR Conf. on Operator Theory in Functional Spaces, pp.114 (N.Novgorod) (in Russian)
12. Kowalski M.A. (1985): Scalar product on space of polynomials, Acta Math.Hung., **46**, 1–2, 101–109
13. Krein M.G., Naimark M.A. (1936): The method of symmetric and Hermitian forms in the theory of the separation of roots of algebraic equations. English translation in Linear and Multilinear Algebra, 1981, **10**, 265–308
14. Levinson N. (1947): The Wiener rms (root-mean-square) Error Criterion in Filter Design and Prediction, J.Math.Physics. **25**, 261–278.
15. Notik A.I., Knafel' A.I., Turchin V.I., Ugrinovsky V.A., Ugrinovsky R.A. (1990): Spectral analysis via continual maximum entropy thechnique, Radiotechnika & Electronika, **35**, 9, 1904–1912 (in Russian)

16. Ovĉarenko I.E. (1990): Scalar product on polynomial space and positivity, Dokl. AN of the Ukr.SSR, ser.A., 7, 17–21 (in Russian)

17. Ovĉarenko I.E. (1991): Some applications of the Chebyshev recursion, Dokl. AN of the USSR, **319**, 2, 287–291 (in Russian)

18. Sushkevich A.K. (1941): The principles of higher algebra. Gostexizdat (in Russian)

19. Ugrinovsky R.A. (1992): About trigonometric moment problem with lacuna, Levinson algorithm and some applied problem, Ukraine Math.J., **44**, 5, 713–716 (in Russian)

20. Vrscay E.R., Handy C.R. (1989): The perturbed two dimensional oscillator; eigenvalues and infinite-fields limits via continued fractions renormalised perturbation theory and moment methods, J.Phys.A: Math.Gen., **22**, 7, 823–834

21. Wheeler J.C. (1984): Modified moments and continued fraction coefficients for the diatomic linear chain, J.Chem.Phys. **30**, 1, 472–476