# Critical sections of gauge functionals: a symmetry approach

Giuseppe Gaeta
*Centre de Physique Theorique, Ecole Polytechnique*
*F-91128 Palaiseau (France)*
*E-mail: gaeta@orphee.polytechnique.fr*

**Abstract:** We extend the classical theorem of L. Michel on the symmetry breaking patterns in theories described by minimization of a symmetric potential to the case of theories described by minimization of a gauge-symmetric functional.

## Introduction

Some time ago, L. Michel proved a remarkable theorem [1] on geometry of orbit space, and therefore on symmetry patterns in theories described by minimization of an invariant potential (see also [2,3]); indeed, his work was partly motivated by the SU(3) model of particle physics [4].

I aim here at an extension of Michel theorem to gauge theories, i.e. to the case one deals with a functional invariant under a gauge group and looks for critical points of this in the space of sections of a fiber bundle. In this setting one is faced with a number of difficulties, due to non-compactness of the gauge group and infinite dimensionality of the space of sections; indeed, I will not be able to extend Michel theorem in full generality, mainly due to poor understanding of the geometry of gauge orbit space. By a judicious use of the fibered structure (and actually also paying the price of restricting to a dense subspace of the space of sections) one can anyway obtain at least a partial extension of Michel's results to the case of interest here. It turns out that in this way one makes contact with results of Palais [5].

As it already happens with the original work of Michel, which gave origin to the "Equivariant Branching Lemma" [6-8], of great utility in equivariant bifurcation theory and its applications [9-12], the present extension is readily adapted to the bifurcation setting, and fits nicely in the framework of bifurcation theory in the presence of Lie-Point symmetries [13,14] and of "nonlinear" reduction and equivariant branching lemma [15,16] (for an introduction to Lie-Point symmetries, see [17-20]).

Here I only sketch a few (key) steps of this work; a complete version, with full detail, examples, and suggestions of further developements and applications (notably to pattern formation and phase coexistence), will appear elsewhere [21]; related results appear in [16]; see also [20].

## Michel theorem

Let us first shortly recall the setting and statements of Mishel's theorem; the reader is supposed to be familiar with the geometry of group action on manifolds and with the concept of (symmetry) strata; see e.g. [2,3,20,21] for an introduction to these.

Let $M$ be a finite dimensional riemannian manifold (say, embedded in $\mathbf{R}^n$ for definiteness); let $G$ be a compact Lie group, and let $\Lambda = \{\Lambda_g ,\ g \in G\}$ be a representation of $G$ acting on $M$, $\Lambda : M \to M$. We will denote the space of $G$-orbits on $M$, or *orbit space* for short, by $\Omega$; with the present hypotheses for $G$ and $M$ this can be proved to be a semialgebraic manifold [2,3,22].

Let$\mathcal{V}_\Lambda$ be the set of $\mathcal{C}^\infty$ functions $V : M \to \mathbf{R}$ invariant under $\Lambda$, i.e. such that $V(\Lambda_g x) = V(x)\ \forall x \in M$, $\forall g \in G$.

If there is a point $x$ such that $dV(x) = 0$, then necessarily $dV(y) = 0\ \forall y \in \omega(x)$, so that critical points of $V \in \mathcal{V}$ will came in $G$-orbits. An orbit $\omega = \omega(x_c)$ such that $dV(x_c) = 0\ \forall V \in \mathcal{V}$ is called a *critical orbit* for $G$ [1].

To see that these exist, just consider $N = 1$ and $G = Z_2 = \{e, g\}$, with $\Lambda_e : x \to x$, $\Lambda_g : x \to -x$. In other words, every (smooth) even potential has a critical point in the origin. Analogously, for $G = SO(N)$ we have that any rotationally invariant smooth potential has a critical point in the origin. A less trivial example is obtained by considering $M = S^2 \subset R^3$ and $G = SO(2)$ acting in $R^3$ as rotations around the $z$-axis, Any invariant potential will have critical points at North and South poles of the sphere (i.e. at $(0, 0, \pm 1)$). The orbit space is isomorphic to the segment $[-1, 1]$ (this can be thought as the $z$ coordinate of the orbit); all the points in the interior of this belong to the same stratum ($G_x = \{e\}$), and the extrema $x = \pm 1$ form another stratum ($G_x = SO(2)$). An even less trivial example is obtained by considering the adjoint representation of SU(3) (which provided indeed physical motivation for the work of Michel), see [4,9,11].

The theorem of L. Michel [1] tells that: **An orbit is critical for $G$ if and only if it is isolated in its stratum**.

An orbit is, roughly speaking, isolated in its stratum if one can take a neighbourhood of $\omega$ in $\Omega$ which does not contain points of $\Sigma_\omega$ (the stratum of $\omega$ in $\Omega$) other than $\omega$ itself.

In order to talk of neighbourhoods in $\Omega$, one has to provide it with a topology, which will be taken to be the quotient topology: the distance of two orbits will be defined by means of the distance $d(x,y)$ defined in $M$, and will be

$$d(\omega,\omega') = \min\{\ d(x,y)\ ;\ x \in \omega, y \in \omega'\} \tag{1}$$

The minimum of (1), whose existence has to be proven, can be seen also as

$$d(\omega,\omega') = \min\{d(x,\omega')\ ;\ x \in \omega\} \tag{2}$$

where we have introduced the distance of a point from an orbit,

$$d(x,\omega') = \min\{d(x,y)\ ;\ y \in \omega'\} \tag{3}$$

If the point $y \in \omega'$ for which $d(x,y) = d(x,\omega')$ is unique (locally), it is called the retraction of $x$ on $\omega$ and denoted $\rho_{\omega'}(x)$. The function $\rho_\omega(\cdot)$ is an equivariant one:

$$\rho_\omega(\Lambda_g x) = \Lambda_g \rho_\omega(x) \tag{4}$$

and is therefore also called the equivariant retraction [2,3]

Notice that if $G$ is compact, then $\omega, \omega'$ are compact sets, and the minima of (1),(2),(3) do surely exists, and therefore also the equivariant retraction does exist.

We stress that for noncompact group orbits the equivariant retraction could very well not exist, and therefore the concept of an orbit "isolated in its stratum" be ill-defined (that is why we discussed the concept at some lenght). Besides this, the very existence of a stratification is not granted for noncompact groups.

## Gauge theoretical setting

Let us consider a fiber bundle $(E,B,\pi)$ of base space $B$, fiber $\pi^1(x) = M$ and structure group a compact Lie group $G$ acting on $M$ through the representation $\Lambda$ (here we have the same hypotheses made earlier on $G$, $M$, $\Lambda$). The space of smooth sections $\varphi(x)$ of this bundle, $\varphi : B \to E$, $\pi\varphi(x) = x$, will be denoted by $\Phi$.

Consider now the bundle $(\tilde{E}, B, \tilde{\pi})$ of base $B$ and fiber $\pi^{-1}(x) = G$; the space of smooth sections $\gamma(x)$ of this bundle, $\gamma : B \to G$, $\tilde{\pi}\gamma(x) = x$, will be denoted by $\Gamma$ (1). Elements of $\Gamma$ act in a natural way in $\Phi$:

$$(\gamma \circ \varphi)(x) = \Lambda_{\gamma(x)}\varphi(x) \tag{5}$$

Consider now a local functional $L : \Phi \to \mathbf{R}$, with local homogeneous density $\mathcal{L}$:

$$L(\varphi) = \int_B \mathcal{L}[\varphi(x)]dx \tag{6}$$

In particular, we want to consider the case of $L$ invariant under the action of the gauge group (we say that $L$ is a *gauge functional* for short), i.e.

$$L(\gamma \circ \varphi) = L(\varphi) \qquad \forall\gamma \in \Gamma\ ,\ \forall\varphi \in \Phi \tag{7'}$$

(1) $\Gamma$ can also seen as a subgroup of the group of fiber-preserving (or gauge) diffeomorphisms of $E$, $\mathrm{GDiff}(E) = \{f \in \mathrm{Diff}(E)\ /\ f : \pi^{-1}(x) \to \pi^{-1}(x) \quad \forall x \in B\}$. This subgroup is simply given by $\Gamma \simeq \Gamma_E = \{f \in \mathrm{GDiff}(E)\ /\ f_x \equiv f|_{\pi^{-1}(x)} = \Lambda_{\gamma(x)}\ ,\ \gamma \in \Gamma\} \subset \mathrm{GDiff}(E)$ where $f_x$ is the restriction of $f$ to $\pi^{-1}(x)$.

or equivalently

$$\mathcal{L}[\Lambda_{\gamma(x)}\varphi(x)] = \mathcal{L}[\varphi(x)] \qquad \forall\gamma\in\Gamma \ , \ \forall\varphi\in\Phi \tag{7”}$$

Here and in (6) the square brackets mean that $\mathcal{L}$ can depend on derivatives of the field $\varphi$ as well. In this note, anyway, we will consider for the sake of brevity only the case of zero-th order gauge functionals, i.e. $\mathcal{L}$ will depend only on $\varphi(x)$ and not on its derivatives. The general case can be readily recovered by this special one: e.g. for first order theories one considers a field $\Psi$ whose components $\Psi_{0i} = \varphi^i$, $\Psi_{ij} = \partial\varphi^j/\partial x^i$ obey the constraints $\partial\Psi_{0j}(x)/\partial x^i = \Psi_{ij}(x)$ (and similarly for higher derivatives). This amounts to introducing a *contact structure* [23,24,20]; this construction is discussed in full detail in [21] and will not be discussed here.

## A distance in gauge orbit space

One could then try to parallel the standard proof of Michel theorem [1,3,21] for the space $\Phi$, in order to get an analogue of Michel's theorem in this case. A difficulty immediately encountered is that now the space $\Phi$ on which $\Gamma$ acts is infinite dimensional, and therefore not compact; correspondingly, also $\Gamma$ is non-compact(2).

The $\Gamma$-orbit of a section $\varphi\in\Phi$, denoted $\vartheta(\varphi)$, will be defined as

$$\vartheta(\varphi) = \{\varphi'\in\Phi \ / \ \varphi'(x) = \Lambda_{\gamma(x)}\varphi(x) \ , \ \gamma(x)\in\Gamma\}\subset\Phi \tag{8}$$

The orbit space for sections will be denoted $\Theta\equiv\Phi/\Gamma$; the orbit $\vartheta(\varphi)$ will be denoted as $\vartheta_\varphi$ when thought as a point of $\Theta$.

Given a section $\varphi\in\Phi$, we can define its isotropy subgroup $\Gamma_\varphi$ as

$$\Gamma_\varphi = \{\gamma\in\Gamma \ / \ \gamma\cdot\varphi = \varphi\}\subset \mathrm{GDiff}(E) \tag{9}$$

If $\varphi' = \gamma\cdot\varphi$, it is easy to see that $\Gamma_{\varphi'} = \gamma\Gamma_\varphi\gamma^{-1}$. We can therefore [2,3] define a stratification of $\Phi$, at least formally.

The reason for which this is only formal is that in the case of infinite dimensional groups one can have a group conjugated to some of his proper subgroups, so that the order relation could not be well defined. We will assume for the moment that a stratification can be defined, and defer to a later time a discussion of how (and how far) one can actually do it, and of the difficulties it can present.

The standard proof of Michel's theorem [1,3,21] relies mainly on purely geometrical concepts, which are transferred with no harm to the present infinite dimensional setting. The only exception, i.e. obstacle to an infinite dimensional extension, is represented by giving a convenient topology to the orbit space. In our case, anyway, we can take advantage of the fibered structure of the theory, and define a distance between two sections $\varphi,\varphi'\in\Phi$ as

$$d_\Phi(\varphi,\varphi') = \frac{1}{|B|}\int_B d_M\left(\varphi(x),\varphi'(x)\right)dx \tag{10}$$

where $|B| = \int_B 1\cdot dx$, and $d_M(.,.)$ is a distance defined in $M$.

In order to define a distance in $\Theta$, $\delta:\Theta\times\Theta\to\mathbf{R}_+$ (here $\mathbf{R}_+$ is the set of nonnegative reals), we can make use of the distance $d_\Omega$ defined earlier in $\Omega$, i.e. of the equivariant retraction $\rho_\omega$, by

$$\delta(\vartheta_1,\vartheta_2) = \frac{1}{|B|}\int_B d_\Omega\left(\vartheta_1(x),\vartheta_2(x)\right)dx$$

where $\vartheta_i : B\to\Omega$.

(2) Although $G$ is compact.

At this point, we can just repeat the proof of finite dimensional Michel's theorem to obtain its extension to gauge functionals. We will call a $\Gamma$-orbit $\omega(\varphi) \subset \Phi$ a *critical gauge orbit* for $\Gamma$ if for every $\Gamma$-invariant functional $L = \int_B \mathcal{L}(\varphi) dx$, $L : \Phi \to \mathbf{R}$, $\omega(\varphi)$ is a critical orbit for $L$. This means that $\forall \varphi \in \omega(\varphi)$, $\delta L[\varphi] = 0$; or, $\mathcal{L}(\varphi + \epsilon \delta\varphi) = \mathcal{L}(\varphi) + O(\epsilon^2)$.

The reader will notice that we defined a distance in the gauge orbit space $\Theta = \Phi/\Gamma$ without defining an equivariant retraction in $\Phi$; this is not only due to the fact that what we actually need is a distance in $\Theta$, but actually to an impossibility. We will not discuss this here, but just refer, as usual, to [21].

## Stratification of gauge orbit space

Our understanding of the stratification of the gauge orbit space $\Theta$ will be based on the stratification of the orbit space $\Omega = M/G$. This latter one is a classical subject, and we will not discuss it here; see [2,3,22] for a discussion.

We just recall that $\Omega$ is a semialgebraic variety in $\mathbf{R}^k$ (a subset of $\mathbf{R}^k$ defined by equalities and inequalities of polynomials); $\mathbf{R}^k$ can be thought as the space of values assumed by the polynomials $P_1(x), \dots, P_k(x)$ of the Hilbert integrity basis for the $G$-action on $M$.

A semialgebraic variety $\Omega$ in $R^k$ has a natural primary stratification [25], i.e. can be seen as the disjoint union of open manifolds of dimensions from $k$ down to 0 (as an example, a square $S$ is the union of interior points $S_i$ and of border $\partial S$; the latter is the union of points on edges $E_i$ and border of edges $\partial E_i$, which are the vertices $V_i$), so that $\Omega = \cup_{\alpha,i} E_i^{(\alpha)}$, $\dim E_i^{(\alpha)} = \alpha$ where $\cup$ denotes disjoint union, and $E_j^\beta \in \partial E_i^\alpha$, $\beta < \alpha$ The latter relation introduces a partial order in the set $\{E_i^\alpha\}$ of primary strata, i.e. bordering.

It can be proven that the symmetry stratification, i.e. the one based on symmetry of orbits $\omega \in \Omega$ under $G$ follows this primary stratification, in the sense that connected components of symmetry strata correspond to union of primary strata, i.e. $\Sigma_\omega = \cup_{\alpha \in A; i \in I} E_i^\alpha$, for some index sets $A, I$.

¿From this it follow in particular two consequences: first, that only orbits in primary strata of dimension 0 can be isolated in their stratum; second, that nearly all the $\omega \in \Omega$ belong to the maximal dimensional primary stratum, and therefore to the principal stratum (some care should be taken if $\Omega$ is not connected). It can also be seen that more peripheral primary strata have higher symmetry; i.e., the partial ordering given by the bordering relation coincides with the partial ordering given by symmetry relations.

We can now discuss the stratification of the space $\Phi$ and of the $\Gamma$-orbit space $\Theta = \Phi/\Gamma$; from now on, subgroups of $G$ conjugated in $G$ will simply be identified.

We will actually discuss a subclass $\Phi_T \subset \Phi$ of sections of $E$, that of transversal ones (this name will be defined and explained in a moment). Given a section $\sigma(x) : B \to M$, we can consider the set of values it takes in $M$,

$$M_\sigma = \{\sigma(x) \ , \ x \in B\} \subset M \tag{11}$$

and in an obvious way

$$\Omega_\sigma = \{\omega_{\sigma(x)} \ , \ x \in B\} \subseteq \Omega \tag{12}$$

Now, let us consider the primary stratification of $\Omega$,

$$\Omega = \cup . E_i^\alpha \quad , \quad \dim E^\alpha = \alpha \tag{13}$$

introduced above. The primary index $\alpha(\sigma)$ of the section $\sigma$ is the greater $\alpha$ for which there is an $E_i^\alpha$ with

$$\Omega_\sigma \cap E_i^\alpha \neq \{\oslash\} \tag{14}$$

A section is transversal if it meets primary strata of dimension $\beta$ strictly less than its primary index transversally.

We do also define, for the sake of completeness, the set of accessible strata for $\sigma$, $[E]_\sigma$:

$$[E]_\sigma = \{E_i^\alpha \ / \ \Omega_\sigma \cap E_i^\alpha \neq \oslash\} \tag{15}$$

Clearly,

$$E_\sigma \equiv \cup . E_i^\alpha \quad , \quad E_i^\alpha \in [E]_\sigma \tag{16}$$

is a semialgebraic variety; the locus of points $\omega_{\sigma(x)}$, $x \in M$ determines a curve $e(\sigma)$ in $E_\sigma$; it is immediate to see that for a smooth section $\sigma$, $e(\sigma)$ is smooth at points belonging to strata of dimension $\alpha(\sigma)$; it is also immediate to see that for transversal sections, $e(\sigma)$ is singular ($C^0$ but not $C^1$) at points on strata of dimension strictly less than $\alpha(\sigma)$.

It should be remarked that transversality is a structurally stable property, and that transversal sections are dense in $\Phi$; we stress that this follows from the results recalled in the previous section about the geometry of $\Omega$.

This restriction to transverse sections could seem quite misterious, but it is actually related to the above remark, that due to the fact that the gauge group is infinite dimensional, the very concept of stratification is quite delicate. Let us show by a simple example the kind of troubles one is faced with.

Let $B = [0,1]$, $M = \mathbf{R}^2$, $G = SO(2)$; consider two sections given by (with $v_0$ a given unit vector in $M$) $\sigma_i(x) = e^{-1/(x-x_i)^2} v_0$ for $x \leq x_i$; $\sigma_i(x) = 0$ for $x \geq x_i$, $i = 1, 2$. Clearly, the isotropy subgroups $\Gamma_i \subset \Gamma$ of these are $\Gamma_i = \{\gamma \in \Gamma \ / \ \gamma(x) = \{e\} \ \ x \geq x_i\}$, or $\{\gamma : [0, x_i] \to SO(2) \ / \ \gamma(x_i) = 0\}$. Now, $\Gamma_1$ and $\Gamma_2$ are isomorphic but if, say, $x_2 > x_1$, then $\Gamma_1$ is also a proper subgroup of $\Gamma_2$.

It is quite clear that this "patology" does not occurr when restricting to transversal sections. It *seems* to us that this restriction is sufficent for a stratification to be properly defined, but we stress that this is not being proved. We remark anyway this restriction does actually suffice to properly define a stratification in a number of meaningful examples, see [21]. We will formalize our reasonable guess into the

*Assumption:* In the space $\Phi_T \subset \Phi$ of transversal sections, and therefore in $\Theta_T = \Phi_T/\Gamma$, a stratification is properly defined.

Actually, if we require only what we really need for our purposes, it will suffice the

*Weaker assumption:* For the action of $\Gamma$ on the space of transversal sections $\Phi_T$, there is a set (possibly infinite or even continuous) of subgroups $\Gamma_\mu$ which are: *i)* isotropy subgroups for some $\sigma \in \Phi_T$; *ii)* not contained in any other subgroup satisfying i); *iii)* such that no proper subgroup $\Gamma_\mu^{(\alpha)} \subset \Gamma_\mu$ is conjugated to any $\Gamma_{\mu'}$ . In other words, the concept of maximal isotropy subgroup of $\Gamma$ is properly defined.

In the following we will write - for ease of notation - $\Phi$ for $\Phi_T$ and $\Theta$ for $\Theta_T$.

Actually, by looking at examples, one gets easily convinced that an explicit stratification of $\Theta$ is extremely complicate and difficult to describe. We therefore concentrate on the description of most singular strata, i.e. those corresponding to maximal isotropy subgroups(3). These are also the strata on which the extension of Michel's theorem given above can be applied.

(3) It should be stressed that maximal isotropy subgroups correspond to most singular strata, but are not always the only ones corresponding to most singular strata: the hypotheses that they indeed are is known as the maximal isotropy subgroup conjecture, and is now known to be in general not true. A complete discussion of it, including identification of the cases (i.e. of the groups) in which it holds true, has been given recently by Field and Richardson for compact Lie groups [26].

## Maximal isotropy groups in gauge orbit space

Let us consider again the action of $G$ on $F$ by the representation $\Lambda = \{\Lambda_g subgroups(MIS)\mathrm{G}_\mu,\ \mu = 1, \dots, s$, i.e. of subgroups $G_\mu \subseteq G$ such that $\exists z \in M \ / \ \Lambda_g z = z\ \forall g \in G_\mu$, and there is no subgroup $G^*_\mu \subseteq G$ such that $G_\mu \subset G^*_\mu$ and $G^*_\mu$ is an isotropy subgroup. (We stress that the concept of MIS depends on both $M$ and $\Lambda$, for given $G$).

The set $\Omega_\mu = \{\omega \in \Omega \ / \ \Lambda_g z = z\ \forall z \in \omega,\ \forall g \in G_\mu\}$ will correspond to a maximal (i.e. minimal dimensional) stratum.

A section $\sigma$ such that $\sigma(x) \in \Omega_\mu\ \forall x \in B$ (i.e. $\Omega_\sigma \subseteq \Omega_\mu$) will admit as symmetry group

$$\Gamma_\mu = \{\gamma : B \to G_\mu\} \tag{17}$$

The groups $\Gamma_\mu \subset \Gamma$, for $G_\mu$ a MIS of $G$, are MIS of $\Gamma$. In facts, to have $\Gamma_\mu \subset \Gamma'$, $\Gamma'$ must contain $\gamma$'s such that for some $x \in B$, $\gamma(x) \in G \backslash G_\mu$. But we know that every $g$ such that $\exists z \ / \ \Lambda_g z = z$ must belong to some $G_\mu,\ \mu = 1, \dots, s$. Therefore $\gamma \in \Gamma$ can belong to the isotropy group of some section only if

$$\gamma(x) \in \cup_{\mu=1,\dots,s} G_\mu \tag{18}$$

Suppose now that for $x \in B$ $\gamma(x)$ belongs to at least two different $G_\mu$'s, $\gamma(x_i) \in G_i$, $G_i \neq G_j$ for $i \neq j$, and let $B_i = \{x \in B \ / \ \gamma(x) \in G_i\}$. Then necessarily there are points $x \in B_i \cap B_j$; due to smoothness of $\gamma$, in these $\gamma(x) \in G_i \cap G_j$.

Now, the functions $\gamma : B \to G \ / \ \gamma : B_i \to G_i$ can be seen as $n$-ples of functions $\gamma_i$ defined on $B_i$ with values in $G_i$, each of them subject to appropriate boundary conditions: on $B_{ij} \equiv B_i \cap B_j \in \partial B_i$, $\gamma : B_{ij} \to G_i \cap G_j$. Clearly for no distincts $i, j$ one can have $G_i \subset G_i \cap G_j$, for the $G_\mu$ are MIS. This also means that it is not possible to find an isotropy group $\Gamma' \subset \Gamma$ such that $\Gamma_\mu \subset \Gamma'$.

We stress that the above argument shows that all the $\Gamma_\mu$ of the form (1), with $G_\mu$ a MIS of $G$, are maximal isotropy subgroups of $\Gamma$, but in general not all the MIS (even on the set of transverse sections) need to be of the form (1).

In the same way, one can see that given a stratum $\Sigma_\omega \subset \Omega$, with isotropy subgroup $G_0$, the sections $\sigma$ such that

$$\omega_{\sigma(x)} \in \Sigma_\omega \quad \forall x \in B \tag{19}$$

have isotropy subgroup

$$\Gamma_0 = \{\gamma \in \Gamma \ / \ \gamma : B \to G_0\} \tag{20}$$

Let us go back to MIS: if in the stratification of $\Omega$ by $G$ $\Sigma^{(\mu)}_\omega \equiv \Omega_\mu$ is a maximal stratum (i.e. minimal dimensional, corresponding to a MIS), we have just seen that

$$\Phi_\mu = \{\sigma \in \Phi \ / \ \sigma : B \to \Omega_\mu\} \tag{21}$$

form a maximal stratum with isotropy $\Gamma_\mu$. In other words,

$$\Theta_\mu = \Phi_\mu / \Gamma \tag{22}$$

is a maximal stratum in the stratification of $\Theta$ by $\Gamma$. (Remark that actually $\Theta_\mu = \Phi_\mu/(\Gamma/\Gamma_\mu)$, since $\Gamma_\mu$ is the identity on $\Phi_\mu$).

Now, let us assume that $\omega \in \Omega$ is isolated in its stratum; it is immediate that $\vartheta \in \Theta$, where $\vartheta$ is the gauge orbit of sections $\sigma \ / \ \sigma(x) \in \omega\ \forall x \in B$, is isolated in its stratum (recall that $\Theta$ is equipped with the topology induced by the topology of $\Omega$). This gives a constructive way for determining some (not all, in general !) of the critical gauge orbits.

We will summarize our discussion as follows:

**Theorem:** Let $G_1 \subseteq G$ be a MIS for the action of $G$ on $F$ by the representation $\Lambda$, and let $\Omega = M/G$. Then

i) $\Gamma_1 = \{\gamma : B \to G_1\}$ is a MIS of $\Gamma$;

ii) The set $\Theta_1 = \{\vartheta \ / \ \omega \cdot \sigma(x) : B \to \Omega_1 \ \forall x \in B \ \forall \sigma \in \vartheta\}$, where $\Omega_1 = \{\omega \ / \ \Lambda_g z = z \ \forall z \in \omega \ \forall g \in G_1\}$, is a maximal stratum of $\Theta$;

iii) If $\omega_0$ is isolated in its stratum $\Omega_1$, then $\vartheta_0$, the gauge orbit such that any section $\sigma(x) \in \omega_0 \ \forall x \in B$ belongs to $\vartheta_0$, is isolated in its stratum $\Theta_1$;

iv) As a consequence of iii), for every critical orbit $\omega_0 \in \Omega$ there is a critical gauge orbit $\vartheta_0 \in \Theta$.

## Reduction and equivariant branching lemma

A full discussion of reduction and equivariant branching lemma for gauge theories would be out of place here. We will just mention the main results one can obtain in this direction; a detailed discussion is given in [16,21].

*Reduction lemma (variational gauge case):* Let $L[\varphi] = \int_B \mathcal{L}[\varphi(x)]dx$ be a gauge invariant functional with gauge group $\Gamma = \{\gamma : B \to G$. Let the group $G$ admit a Lie subgroup $G_0$ with $W_0 = \mathrm{Fix}(G_0) = \{u \in M \ / \ \Lambda_g u = u \ \forall g \in G_0\}$. Let $\Phi_0 \subset \Phi$ be the set of sections such that $\sigma(x) \in W_0 \subseteq M \quad \forall x \in B$, and let $L_0[\varphi] = \int_B \mathcal{L}_0[\varphi(x)]dx$ be the restriction of $L : \Phi \to \mathbf{R}$ to $\Phi_0$. Then, a section $\sigma \in \Phi_0$ is critical for $\mathcal{L}_0$ if and only if it is critical for $L$.

This is indeed the *"symmetric criticality principle"* of Palais [5], recast in the language usually employed in equivariant bifurcation theory(4).

*Corollary (existence lemma):* Let $L,\ \mathcal{L},\ G,\ G_0,\ W_0,\ M$ be as above, $\mathrm{dim} W_0 = d < \infty$, and let $W_0$ contain a nonempty compact subset $D \subset W_0$ of dimension $d$ such that the vector $\nabla_u \mathcal{L}(x,u)$ points outward of $D$ on $\partial D \ \forall x \in B$. Then there is a local section $\sigma : B \to E$, $\sigma(x) \in M_x$ ($M_x$ is the fiber through $x$) which is entirely contained in $W_0$, i.e. $\sigma(x) \in W_0 \subset M_x \ \forall x \in B$, and which is critical for the functional $L : \Phi \to \mathbf{R}$.

We remark that the section whose existence is ensured by this corollary could happen to be trivial, $\sigma(x) = 0$. If the trivial section is a local maximum for $L$, as it is natural to assume in a bifurcation setting, the same reasoning leads to affirm the existence of a local section $\sigma_0(x) \in W_0$ which is nontrivial and a minimum. Indeed, one could introduce in $\mathcal{L}(x,u)$ a dependence on a control parameter $\ell$ and consider bifurcations of critical sections of $L_\ell[\varphi] = \int_M \mathcal{L}_\ell(x, \varphi(x))dx$; in this setting the above existence lemma becomes an equivariant bifurcation lemma for gauge sections. A complete discussion of this is given in [16].

We stress that the existence of global nonzero sections (and a fortiori of global sections if the fiber is not a linear space, e.g. if the fiber is the group itself) cannot, quite obviously, be granted on purely algebraic terms and would require a topological discussion.

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(4) Palais does actually also consider the case of $G$ acting nontrivially on the base manifold $X$; we have excluded this case from his statement as well as from our discussion just for simplicity, but it does not present new difficulties.

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