Nonlinear Lie symmetries in bifurcation theory

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Abstract

We examine the presence of general (nonlinear) time-independent Lie point symmetries in dynamical systems, and especially in bifurcation problems. A crucial result is that center manifolds are invariant under these symmetries: this fact, which may be also useful for explicitly finding the center manifold, implies that Lie point symmetries are inherited by the "reduced" bifurcation equation (a result which extends a known property of *linear* symmetries). An interesting situation occurs when a nonlinear symmetry of the original equation results in a linear one (e.g. a rotation - typically related to a Hopf bifurcation) of the reduced problem. We provide a class of explicit examples admitting nonlinear symmetries, which clearly illustrate all these points.

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Bifurcation theory is by now one of the most useful and used tools in attacking nonlinear problems; here we are in particular concerned with equivariant bifurcation theory, i.e. bifurcation in the presence of some symmetry properties [1-7]. In general, all the developments of equivariant bifurcation theory, and its fundamental theorems, deal with *linear* symmetries (or, to be more precise, with linear group actions). Quite recently, the notion of Lie-point (LP) symmetry received diffuse attention [8-12], not only thanks to the theoretical appeal of LP symmetries, but also to their power in the study of concrete nonlinear equations, often in coincidence with symbolic manipulation computer programs [see, e.g., 13].

It is therefore very natural to try to build an equivariant theory for general LP symmetries: in fact, some results in this direction have been already obtained, showing that some typical results of linear equivariant bifurcation theory (e.g. the Sattinger theorem on the symmetries of the bifurcation equations obtained via Lyapunov-Schmidt projection [2,4], and the equivariant bifurcation lemma [14,3]) can be suitably extended to nonlinear LP symmetries [15,16]. Other applications can be found in [8-12, 17-18] and Ref. therein.

We shall provide, in the second part of this paper, a class of explicit examples of dynamical systems exhibiting nonlinear LP symmetries.

Let us recall that the generic dynamical problem

$$\dot{u}_i = F_i(\lambda, u) \qquad u = u(t); \qquad u \in R^n; \qquad i = 1, \dots, n \tag{1}$$

(where F_i are smooth, e.g. C^∞, functions, possibly also depending on one or more real parameters λ) admits a time-independent LP symmetry generated by the C^∞ vector field operator (here and in the following, ∂_{u_i} stands for $\dfrac{\partial}{\partial u_i}$, etc.)

$$\eta = \sum_i \theta_i(\lambda, u)\ \partial_{u_i} \tag{2}$$

if and only if [8,17]

$$\big[\eta, \eta_F\big] = 0 \tag{3}$$

where η_F is the operator (generating the time evolution of the dynamical flow)

$$\eta_F = \sum_i F_i\ \partial_{u_i} \tag{4}$$

Given the generic dynamical problem (1), we may state the following main result.

Theorem. *Let the linearization of the problem* (1) *around the equilibrium point* $u = u_0$ *(and* $\lambda = \lambda_0$*; we can choose* $u_0 = 0, \lambda_0 = 0$*) possess some eigenvalues with vanishing real part, the remaining eigenvalues having strictly negative real part. Assume here that the (attractive) center manifold* $M = W_c$ *is uniquely* *defined. If the problem* (1) *admits a time-independent LP symmetry generated by an operator* η (2) *defined in a neighbourhood of the point* u_0*, then the center manifold is left invariant by this symmetry, and the equations obtained by restriction to the center manifold* M *admit the LP symmetry* $\eta|_M$*.*

In short, this follows from (3) and from the fact that LP symmetries transform solutions into neighbouring ones; if $\bar{u}(t)$ is a solution belonging to the center manifold $M = W_c$, this implies that η transforms it into a solution $u'(t)$ belonging to a neighbourhood U of M; but the only possibility to have a solution $u'(t) \in U$, for all $t \in R$, is precisely that $u'(t) \in M$ [19-23]. In the case of bifurcation problems, the above argument can be extended to a "neighbourhood of criticality", i.e. in an interval about λ_0, by a standard procedure [24, 19-23]. Let us remark also that the nontrivial part of the above result can be written in the form $\eta : M \to TM$.

Notice that our approach is essentially "local", being based on Lie *algebraic* method, and on local properties of manifolds as well: then, our arguments do not involve properties, e.g. the compactness, of the full symmetry group action.

There are some important aspects of this result. First, one has that the center manifold (CM) W_c is an invariant manifold under the LP symmetry η (notice that this is a peculiar property of center manifolds, and more in general of transversally hyperbolic manifolds: in fact, a generic manifold invariant under the LP symmetry is in general *not* invariant under the dynamical flow).

For sake of simplicity, we have stated our result under the assumption that the CM is unique; it is well known, however, that in general this condition is not met: we refer to [19-23] for detailed and careful discussions ensuring the existence of CM and concerning its properties. In the case of nonunique CM, our result above says immediately that a LP symmetry may trasform a CM into another CM; but we can also prove that, given a LP symmetry η, it is possible to choose in general one CM which is invariant under this η, and/or also to modify the symmetry η in such a way that the given CM turns out to be symmetry invariant. Let us recall, incidentally, that the nonuniqueness of the CM does not affect practical calculations based, as usually happens, on power series expansions of the CM [20-23].

The property of the CM of being invariant under the LP symmetry may be useful even for explicitly finding the CM itself: indeed, the expression describing W_c must be a function of the quantities $k_\alpha = k_\alpha(u)$ which are left invariant by the symmetry η, i.e.

$$\eta \cdot k_\alpha = 0 \qquad (5)$$

(cf. also the method of introducing "adapted coordinates" [11], and/or Poincaré - Birkhoff normal forms [25,26]).

Another general and important point of the above theorem concerns the symmetry properties of the "reduced" bifurcation problem, i.e. of the equations restricted to the variables v spanning the center eigenspace E_c. Precisely, putting $u \equiv (v,z)$, where $v \in E_c$, $z \in E_c^\perp$, we can write locally the CM W_c as identified by $z = \gamma(\lambda, v)$ for some function γ; then the reduced equations [20-23] obtained from the original problem (1) are

$$\dot{v} = \widetilde{F}(\lambda, v) \equiv F\big(\lambda, v, \gamma(\lambda, v)\big) \tag{6}$$

Now, if the LP symmetry generator η (2) is written as

$$\eta = \varphi(\lambda, v, z)\partial_v + \psi(\lambda, v, z)\partial_z \tag{7}$$

we then have

$$\eta|_M = \varphi\big(\lambda, v, \gamma(\lambda, v)\big)\partial_v + \psi\big(\lambda, v, \gamma(\lambda, v)\big)\partial_z \tag{8}$$

and the condition to have $\eta|_M : M \to TM$ is simply $\delta z = (\partial\gamma/\partial v)\delta v$, i.e.

$$\psi\big(\lambda, v, \gamma(\lambda, v)\big) = \frac{\partial \gamma}{\partial v}\varphi\big(\lambda, v, \gamma(\lambda, v)\big) \tag{9}$$

Therefore, the symmetry $\widetilde{\eta}$ inherited by the reduced bifurcation equation (6) is given by

$$\widetilde{\eta} = \widetilde{\varphi}(\lambda, v)\partial_v \equiv \varphi\big(\lambda, v, \gamma(\lambda, v)\big)\partial_v \tag{10}$$

This result extends to nonlinear LP symmetries a well known property of linear ones [1,23].

Now, it may also happen that general nonlinear symmetries of the original problem result in *linear* symmetries of the reduced bifurcation equation (6). This result clearly cannot be predicted by standard (linear) equivariant bifurcation theory; in particular, if the original equations (1) have *no* linear symmetry, then linear equivariant bifurcation theory is not able to give any symmetry constraint on the form of functions $\widetilde{F}(\lambda, v)$, i.e. on the form of the reduced bifurcation equations.

Let us provide now a family of dynamical problems admitting a nonlinear LP symmetry; they will be useful also to illustrate all the above discussion. Let us write, for notational convenience,

$$u \equiv (x, y, z_1, \dots, z_{n-2}) \ \in R^n \tag{11}$$

Let

$$w_a = w_a(y, z_a) \qquad\qquad (a = 1, \dots, k = n-2) \tag{12}$$

be arbitrary smooth functions, satisfying the conditions

$$\frac{\partial w_a}{\partial z_a} \neq 0 \qquad \text{(no sum over } a) \tag{13}$$

and put

$$\chi_a = \chi_a(y, z_a) = \frac{\partial_y w_a}{\partial_{z_a} w_a} \qquad \text{(no sum over } a) \tag{14}$$

Consider systems of the following form

$$\begin{aligned}
\dot{x} &= x\ f - y\ g \\
\dot{y} &= y\ f + x\ g \\
\dot{z}_1 &= \ -y\ f\ \chi_1 - x\ g\ \chi_1 + \frac{h_1}{\partial_{z_1} w_1} \\
&\vdots \\
\dot{z}_k &= \ -y\ f\ \chi_k - x\ g\ \chi_k + \frac{h_k}{\partial_{z_k} w_k}
\end{aligned} \tag{15}$$

where $f, g, h_1, \ldots, h_k$ are n arbitrary smooth functions of the quantities

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad w_1, \ldots, w_k \tag{16}$$

(and possibly also of one or more real parameters λ, to include the case of bifurcation problems). Then, using (3), it is easy (even if tedious) to verify that the above system (15) admits the LP symmetry generated by

$$\eta = y\partial_x - x\partial_y + x\chi_1\partial_{z_1} + \ldots + x\chi_k\partial_{z_k} \tag{17}$$

In particular (with $n = 3$, the extension to $n > 3$ is straightforward; we write here z, w, etc. instead of z_1, w_1, etc.), if $w = w(y, z)$ and $h = h(r^2, w)$ in (15) satisfy

$$\lim_{y,z\to 0} \partial_y w = 0 \qquad \text{and} \qquad \lim_{y,z\to 0} \frac{h(r^2, w)}{z\ \partial_z w} = \ell < 0 \tag{18}$$

then the origin $u = u_0 = 0$ is an equilibrium point for the problem (15), with an attractive 2-dimensional center manifold W_c tangent at $u_0 = 0$ to the center eigenspace E_c ($z = 0$). If the functions $f(\lambda, r^2, w)$, $g(\lambda, r^2, w)$ (with $\lambda \in R$) satisfy also

$$f(0,0,0) = 0, \quad \partial_\lambda f(0,0,0) \neq 0, \quad g(0,0,0) \neq 0 \tag{19}$$

then the standard conditions for the appearance of Hopf bifurcation are met.

The invariance under the symmetry (17) implies that the CM must be a function of the quantities $k_\alpha(u)$ defined in (5), which are now just r^2 and w_a. E.g., let us choose in (12,15), with $n = 3$,

$$w = z\ \mathrm{e}^{-y}$$

which gives the nonlinear LP symmetry

$$\eta = y\partial_x - x\partial_y - xz\partial_z \ ,$$

and also choose

$$f = h = -w + r^2$$

leaving arbitrary g. Then, it is easy to explicitly obtain the CM, which is given by the surface

$$z = \mathrm{e}^y r^2 \ .$$

Also, more in general, if in (15), again with $n = 3$, one has $f = h$ $\big($independently of the choice of $w = w(y,z)\big)$, then the equation

$$h(r^2, w) = 0 \tag{20}$$

gives the implicit expression of the center manifold. Another simple situation occurs if in (15) there is some $w = w_0 = \text{const}$ such that $h(r^2, w_0) = 0$; then, for any choice of f, g, the equation

$$w(y,z) = w_0 \tag{21}$$

gives the CM, as easily verified by direct calculation.

The possibility that the reduced bifurcation equations (6) exhibit a linear symmetry, even in the absence of linear symmetries of the original equations, is well illustrated by the following simple problem. Let us choose in (15) $w = z - y^2$, $h = -w$, $f = \lambda - r^2$, with arbitrary $g(\lambda, r^2, w)$ $\big($satisfying in particular all conditions (18,19)$\big)$: the resulting problem

$$\begin{aligned} \dot{x} &= x \ (\lambda - r^2) - y \ g(\lambda, r^2, w) \\ \dot{y} &= y \ (\lambda - r^2) + x \ g(\lambda, r^2, w) \\ \dot{z} &= - \ z + y^2\big(1 + 2(\lambda - r^2)\big) + 2xy \ g(\lambda, r^2, w) \end{aligned} \tag{22}$$

admits the LP symmetry generated by

$$\eta = y\partial_x - x\partial_y - 2xy\partial_z \tag{23}$$

It is easy to verify that $z = y^2$ is the CM $\big($in agreement with (21)$\big)$, and that the reduced equations for the variables $v \equiv (x, y)$ exhibit a (linear) rotation symmetry generated by

$$\eta_0 = y\partial_x - x\partial_y \ .$$

Notice that a stable Hopf bifurcation for (22) appears at $\lambda_0 = 0$, described by

$$r^2 = x^2 + y^2 = \lambda, \qquad z = y^2$$

with frequency $\omega(\lambda) = g(\lambda, \lambda, 0)$.

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