An extension of the theorem of Milnor and Thurston on the zeta functions of interval maps

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Dedicated to Huzihiro Araki.

Abstract. We consider a piecewise continuous, piecewise monotone interval map and a piecewise constant weight. With these data we associate a weighted kneading matrix which generalizes the Milnor-Thurston matrix. We show that the determinant of this matrix is related to a natural weighted zeta function.

Introduction

Let f be a continuous map from a compact interval, say $[0,1]$, to itself. "Counting" the periodic orbits of f is a basic dynamical problem. A natural assumption is that f is piecewise strictly monotone with a finite number N of monotonicity intervals $[a_{i-1}, a_i]$ where $0 = a_0 < a_1 < a_2 < \ldots < a_{N-1} < a_N = 1$.

Suppose that each iterate $f^{\circ n}$ can have at most one fixed point in each monotonicity interval (this happens for instance when f is piecewise expanding); then it is natural to define the *zeta function*

$$\zeta(t) = \exp \sum_{n \geq 1} \frac{t^n}{n} \#\{x \,:\, f^{\circ n} x = x\}$$

which simply counts periodic points *à la* Artin-Mazur. For more general situations we shall use (see below) a "reduced" zeta function similar to the one introduced by Milnor and Thurston [1988] (this article was widely circulated as a preprint in the late '70s). The main result of Milnor and Thurston relates the zeta function to the determinant of a finite *kneading matrix* (an invariant of the map defined by Milnor and Thurston in the same paper). Using this relation, they are able to show that $\zeta(t)$ is meromorphic in the unit disc and that the first pole occurs at e^{-s} where s is the topological entropy of f.

More recently, *weighted* zeta functions for piecewise monotone interval maps have attracted interest. Here, a weight $g : [0,1] \to \mathbb{C}$ is given, which is supposed either

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constant on each (a_{i-1}, a_i), or of bounded variation on $[0,1]$ (see e.g. Hofbauer-Keller [1984], Baladi-Keller [1990]; for more general g see Keller-Nowicki [1992], Ruelle [1992]). Assuming that the periodic points are isolated, one sets

$$\zeta_g(t) = \exp \sum_{n \geq 1} \frac{t^n}{n} \sum_{x : f^{\circ n} x = x} \prod_{k=0}^{n-1} g(f^{\circ k} x) \, .$$

The analytic properties of $\zeta_g(t)$ are studied with the help of a *transfer operator* acting on a suitable Banach space. (Transfer operators have their origin in statistical mechanics; their study in infinite dimension was initiated by Araki, Ruelle, and Mayer, see for instance Mayer [1980].) Suppose for simplicity that g is real-valued and positive. Under suitable assumptions, one shows that $\zeta_g(t)$ is meromorphic in a disc and that its first pole is the exponential of minus the topological pressure of $\log g$. In some cases the dynamical meaning of the other poles is understood. In particular, when f is topologically mixing, piecewise $\mathcal{C}^2$ and expanding, and when $g = 1/\log|f'|$, the second pole corresponds to a rate of mixing (in this case the first pole occurs at $t = 1$, and the corresponding eigenfunction of the transfer operator gives rise to an absolutely continuous invariant measure for f). The introduction of weights is also useful when studying the unweighted zeta functions of semi-flows obtained by suspending interval maps under non necessarily constant return times.

In this article, we introduce weighted kneading matrices and determinants and extend Milnor and Thurston's main result to the case of weighted zeta functions of the type of $\zeta_g(t)$. (See Section 1 for precise definitions and results.) The weight g is supposed to be constant on each interval (a_{i-1}, a_i) (the case of a weight of bounded variation, where a countable matrix is needed, will be treated in a further work). We also generalise the setting in that we only require the map f to be continuous on the intervals (a_{i-1}, a_i). (The piecewise continuous setting is useful for example in the study of the Lorenz attractor. See e.g. Williams [1979] and Rand [1978] for a definition of kneading series in this case.) We do not need each a_i to be a turning point, i.e., f can be monotone in a neighbourhood of some of the a_i. Finally, our proof, while based on the homotopy argument used by Milnor and Thurston, is somewhat simpler due to the fact that f is allowed to be discontinuous at the a_i (and we can use more elementary transversality arguments).

"Fredholm" matrices analogous to the kneading matrix have been introduced by Mori [1990,1991]. He considers weights which are either locally constant or of bounded variation and uses a renewal equation to relate the determinant of the Fredholm matrix to the corresponding transfer operator and zeta function. However, the relationship with the Milnor-Thurston theory is not made explicit, and the method of proof is completely different from ours.

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1. Definitions and statement of results

Let $a_0 < a_1 < \ldots < a_N$, and $f_i : [a_{i-1}, a_i] \to (a_0, a_N)$ be strictly monotone and continuous maps for $i = 1, \ldots, N$. We write $f = (f_1, \ldots, f_N)$. By abuse of language we shall also denote by f the "multivalued map" $[a_0, a_N] \to [a_0, a_N]$ whose graph is the union of the graphs of the f_i. We let $\epsilon_i = \pm 1$ depending on whether f_i is increasing or decreasing. We also let $z_i \in \mathbb{C}$ and write $Z = (z_1, \ldots, z_N)$. We define functions ϵ, z on $[a_0, a_N]$ such that they have the constant values ϵ_i, z_i on (a_{i-1}, a_i) and 0 on $\{a_0, a_1, \ldots, a_N\}$.

We define the *address* of $x \in [a_0, a_N]$ to be the vector

$$\vec{\alpha}(x) = (\mathrm{sgn}\,(x - a_1), \ldots, \mathrm{sgn}\,(x - a_{N-1})) \in \mathbb{Z}^{N-1}\,.$$

The *invariant coordinate* of x is the $(N-1)$-tuple of formal power series

$$\vec{\theta}(x, Z) = \vec{\theta}_f(x, Z) = \sum_{n=0}^{\infty} \Big(\prod_{k=0}^{n-1} (\epsilon z)(f^{\circ k} x)\Big) \vec{\alpha}(f^{\circ n} x) \in \mathbb{Z}[[z_1, \ldots, z_N]]^{N-1}\,.$$

Note that $\vec{\theta}$ is single valued because if $f^{\circ k} x \in \{a_0, \ldots, a_N\}$ for some $k < n$ then $(\epsilon z)(f^{\circ k} x) = 0$.

Writing $\phi(a\pm) = \lim \phi(x)$ when $x \downarrow a$ or $x \uparrow a$, we let

$$\begin{aligned} \vec{K}_i(Z) &= \frac{1}{2}\Big[\vec{\theta}(a_i+, Z) - \vec{\theta}(a_i-, Z)\Big] \\ &= (K_{i,1}(Z), \ldots, K_{i,N-1}(Z))\,, \end{aligned}$$

for $i = 1, \ldots, N-1$. The $(N-1) \times (N-1)$ matrix $\big[K_{ij}(Z)\big]$ is the *kneading matrix*, and the determinant

$$\Delta(Z) = \Delta_f(Z) = \det\big[K_{ij}(Z)\big] \in \mathbb{Q}[[z_1, \ldots, z_n]]$$

is the *kneading determinant*. Since $K_{ij}(Z) = \delta_{ij} +$ higher order, we have $\Delta(Z) = 1+$higher order. Note that if we set $z_1 = \ldots = z_N = t$, and if we assume that $\epsilon_i = -\epsilon_{i-1}$ for each $1 \le i \le N$, then we recover a simple modification of the kneading determinant of Milnor and Thurston [1988] (see also Preston [1989] for another presentation).

We denote by $\mathrm{Fix}\, f^{\circ m}$ the set of fixed points of $f^{\circ m}$ which have an orbit disjoint from $\{a_0, \ldots, a_N\}$. We assume that for each m the set $\mathrm{Fix}\, f^{\circ m}$ is finite. For $x \in \mathrm{Fix}\, f^{\circ m}$, if the graph of $f^{\circ m}$ does not cross the diagonal at x we set

$$L(x, f^{\circ m}) = 0\,.$$

If the graph of $f^{\circ m}$ crosses the diagonal, we may define

$$L(x, f^{\circ m}) = \lim_{y \to x} \frac{\mathrm{sgn}\,(f^{\circ m} y - y)}{\mathrm{sgn}\,(x - y)}\,,$$

($L(x, f^{\circ m})$ is a Lefschetz index for x) and

$$\nu(x, f^{\circ m}) = -L(x, f^{\circ m}) \cdot \prod_{k=0}^{m-1} \epsilon(f^{\circ k}x) \, .$$

(If $f^{\circ m}$ is decreasing at $x \in \mathrm{Fix}\, f^{\circ m}$ then the assumption that $\#\mathrm{Fix}\, f^{\circ m} < \infty$ for all m implies that x is either an attracting or repelling fixed point for $f^{\circ m}$. In all cases where the graph of $f^{\circ m}$ crosses the diagonal at x, the periodic point x is hence either attracting or repelling, and $\nu(x, f^{\circ m}) = -1$ if and only if $f^{\circ m}$ is increasing and attracting at x.)

We extend now the set $\mathrm{Fix}\, f^{\circ m}$ to a set $\mathrm{Fix}{\star} f^{\circ m}$ obtained by adding some symbols $x*$ where $x \in (a_0, a_N)$ and $*$ is $+$ or $-$:

$$\begin{aligned} \mathrm{Fix}{\star} f^{\circ m} = \mathrm{Fix}\, f^{\circ m} \bigcup \{ x* \, : \, & f^{\circ m}(x*) = x \, , \\ & f^{\circ k}(x*) = a_i \text{ for some } k, i \text{ and } \prod_{s=0}^{m-1} \epsilon(f^{\circ s}(x*)) = +1 \} \, . \end{aligned}$$

If $x* \in \mathrm{Fix}{\star} f^{\circ m} \setminus \mathrm{Fix}\, f^{\circ m}$, we let

$$L(x*, f^{\circ m}) = \begin{cases} 0 & \text{if } x* \text{ is (one-sided) repelling} \\ 1 & \text{if } x* \text{ is (one-sided) attracting} \end{cases}$$

and $\nu(x*, f^{\circ m}) = -L(x*, f^{\circ m})$.

We define now the "reduced" zeta function

$$\zeta(Z) = \zeta_f(Z) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\xi \in \mathrm{Fix}\star f^{\circ m}} \nu(\xi, f^{\circ m}) \cdot \prod_{k=0}^{m-1} z(f^{\circ k}\xi) \, .$$

If $f^{\circ m}(a_i \pm) \neq a_i$ for all i and all $m \geq 1$, we may replace $\mathrm{Fix}{\star} f^{\circ m}$ by $\mathrm{Fix}\, f^{\circ m}$ in this formula.

Denote by Per the set of periodic orbits of f which do not contain any of the a_i, and by Per$\star$ the extended set of periodic orbits formed of elements of $\cup_m \mathrm{Fix}{\star} f^{\circ m}$. If $\gamma \in \mathrm{Per}\star$ is of period $p(\gamma)$ (i.e., $p(\gamma) = \min\{m \geq 1 \, : \, \gamma \subset \mathrm{Fix}{\star} f^{\circ m}\}$) and contains the element ξ_γ we let

$$Z(\gamma) = \prod_{k=0}^{p(\gamma)-1} z(f^{\circ k}\xi_\gamma) \, .$$

With this notation, we may write

$$\zeta(Z) = \prod_{\gamma \in \mathrm{Per}\star} F(\gamma)$$

where

$$F(\gamma) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \nu(\xi_\gamma, f^{\circ m \cdot p(\gamma)}) Z(\gamma)^m \,.$$

Let $\gamma \in \mathrm{Per}$, with $p(\gamma) = p$, and $x_\gamma \in \gamma$, then

(0) $F(\gamma) = 1$ if the graph of $f^{\circ p}$ does not cross the diagonal at x_γ;
(1) $F(\gamma) = \frac{1}{1-Z(\gamma)}$ if $f^{\circ p}$ is increasing at x_γ and γ repelling;
(2) $F(\gamma) = 1 - Z(\gamma)$ if $f^{\circ p}$ is increasing at x_γ and γ attracting;
(3) $F(\gamma) = 1 + Z(\gamma)$ if $f^{\circ p}$ is decreasing at x_γ and γ attracting;
(4) $F(\gamma) = \frac{1}{1-Z(\gamma)}$ if $f^{\circ p}$ is decreasing at x_γ and γ repelling.

Let $\gamma \in \mathrm{Per}\star \setminus \mathrm{Per}$, then

(5) $F(\gamma) = 1$ if γ is one-sided repelling;
(6) $F(\gamma) = 1 - Z(\gamma)$ if γ is one-sided attracting.

It follows that $\zeta(Z) \in \mathbb{Z}[[z_1, \ldots, z_n]]$. Notice that the cases (1) and (4) where γ is repelling give the same expression for $F(\gamma)$. If all periodic points are repelling, and if $f^{\circ m} a_i{*} \neq a_i$, for all $m \geq 1$ and $1 \leq i \leq N-1$, then we recover the usual (weighted) one-dimensional zeta function (see e.g. Hofbauer-Keller [1984]). We can now state our main results.

1.1. Theorem. *With the above notation and assumptions we have*

$$\zeta(Z)\Delta(Z) = 1 - \frac{1}{2}(\epsilon_1 z_1 + \epsilon_N z_N) \,.$$

1.2. Corollary. *If all periodic points are repelling we have*

$$\zeta(Z) = \prod_{\gamma \in \mathrm{Per}} \frac{1}{1 - Z(\gamma)} = \frac{1 - \frac{1}{2}(\epsilon_1 z_1 + \epsilon_N z_N)}{\Delta(Z)} \,.$$

For each fixed choice of $a_0, \ldots, a_N$ and $\epsilon_1, \ldots, \epsilon_N$ we shall prove Theorem 1.1 by a "homotopy argument" very similar to the one used by Milnor and Thurston [1988, §11]. This means that we check the theorem for some particular choice of f and we then compare the changes in $1/\zeta$ and Δ for a suitable one-parameter family of f's, showing that they are both multiplied by the same factor at each bifurcation (see the crucial Lemma 2.4).

2. Proofs

Let $N \geq 1$, $a_0, \ldots, a_N \in \mathbb{R}$ (with $a_0 < a_1 < \ldots < a_N$) and $\epsilon_1, \ldots, \epsilon_N \in \{-1, +1\}$ be fixed. As outlined in Section 1, we start by checking Theorem 1.1 on a simple example:

2.1. Lemma. *Suppose that $f_i^0[a_{i-1}, a_i] \subset (a_0, a_1)$ for $i = 1, \ldots, N$ and that f_1^0 is differentiable with $|(f_1^0)'| < 1$. Then*

$$\zeta_{f^0}(Z)\Delta_{f^0}(Z) = 1 - \frac{1}{2}(\epsilon_1 z_1 + \epsilon_N z_N) \,.$$

(In particular, the f_i^0 may be taken to be polynomials with derivatives vanishing at a_{i-1}, a_i.)

Our assumptions imply that the set of periodic points of f^0 consists of a single fixed point $x \in (a_0, a_1)$, and x is of type (2) if $\epsilon_1 = +1$, (3) if $\epsilon_1 = -1$. Therefore, we have in either case

$$\zeta_{f^0}(Z) = 1 - \epsilon_1 z_1 \, .$$

We also have

$$\vec{\theta}(a_i+, Z) = (\ldots, +1, +1, -1, \ldots) + \epsilon_{i+1} z_{i+1} \cdot \sum_{n=1}^{\infty} (\epsilon_1 z_1)^{n-1} \cdot (-1, \ldots, -1)$$

$$\vec{\theta}(a_i-, Z) = (\ldots, +1, -1, -1, \ldots) + \epsilon_i z_i \cdot \sum_{n=1}^{\infty} (\epsilon_1 z_1)^{n-1} \cdot (-1, \ldots, -1) \, ,$$

and

$$\begin{aligned} \vec{K}_i(Z) &= (0, \ldots, 1, \ldots, 0) \\ &\quad + \frac{1}{2}(\epsilon_{i+1} z_{i+1} - \epsilon_i z_i) \cdot \frac{1}{1 - \epsilon_1 z_1} \cdot (-1, \ldots, -1) \, . \end{aligned}$$

Writing $\alpha_i = \frac{1}{2}(\epsilon_{i+1} z_{i+1} - \epsilon_i z_i) \cdot \frac{1}{1-\epsilon_1 z_1}$ gives $K_{ij}(Z) = \delta_{ij} - \alpha_i$, hence $\Delta_{f^0}(Z) = \det(1 - T)$, where the matrix $T_{ij} = \alpha_i$ is of rank one. Hence

$$\begin{aligned} \Delta_{f^0}(Z) &= \sum_{n=0}^{N-1} (-1)^n \mathrm{Tr} \wedge^n T = 1 - \mathrm{Tr}\, T \\ &= 1 - \sum_i \alpha_i \\ &= 1 + \frac{1}{2}(\epsilon_1 z_1 - \epsilon_N z_N) \cdot \frac{1}{1 - \epsilon_1 z_1} \\ &= [1 - \frac{1}{2}(\epsilon_1 z_z + \epsilon_N z_N)] \cdot \frac{1}{1 - \epsilon_1 z_1} \, , \end{aligned}$$

and finally

$$\zeta_{f^0}(Z) \Delta_{f^0}(Z) = 1 - \frac{1}{2}(\epsilon_1 z_1 + \epsilon_N z_N) \, . \quad \square$$

It is convenient to use the C^1 topology. For $r = 0$ or 1, let $P^r = P^r(N, (a_i), (\epsilon_i))$ denote the set of N-tuples $f = (f_1, \ldots, f_N)$ where each $f_i : [a_{i-1}, a_i] \to (a_0, a_N)$ is $\mathcal{C}^r$ and strictly monotone increasing or monotone decreasing according to whether ϵ_i equals $+1$ or -1. If $r = 1$ we further impose that $f_i'(a_{i-1}) = f_i'(a_i) = 0$ (vanishing of the derivative at interval endpoints). The d_r distance between f and g in P^r is given by the sum of the $\mathcal{C}^r$ distances between the f_i's and g_i's. As already mentioned, we view f and its iterates $f^{\circ m}$ as multivalued maps $[a_0, a_N] \to (a_0, a_N)$. For the definition of ζ_f, we have required that the sets $\mathrm{Fix}\, f^{\circ m}$ be finite, but this condition will be lifted after Lemma 2.2. Let P_M^r consist of those $f \in P^r$ such that $f^{\circ m}(a_i *) \neq a_i$ whenever $1 \leq m \leq M$, $1 \leq i \leq N - 1$ and $* = \pm$.

2.2. Lemma. *P^r_M is an open subset of P^r. If $\mathcal{I}_{M+1}$ is the ideal of elements of order $\geq M+1$ in $\mathbb{Q}[[z_1,\ldots,z_N]]$, the map*

$$f \mapsto \zeta_f \quad (mod\, \mathcal{I}_{M+1})$$

defined on the set $\{f \in P^0_M \,:\, \#Fix\, f^{\circ m} < \infty\,,\, \forall m \geq 1\}$ extends to a locally constant map

$$P^0_M \to \mathbb{Z}[[z_1,\ldots,z_N]]/(\mathcal{I}_{M+1} \cap \mathbb{Z}[[z_1,\ldots,z_N]])\,.$$

P^r_M is defined by a finite number of open conditions, hence is open in P^r. For general $f \in P^0$, we may define $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ to be:

-1 if the left end of the graph of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is $<$ the diagonal and the right end $>$ the diagonal,
$+1$ if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is increasing and the left end of the graph is $\geq$ the diagonal and the right end is $\leq$ the diagonal,
$+1$ if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is decreasing and the left end of the graph is $>$ the diagonal and the right end is $<$ the diagonal,
0 in all other cases (in particular when the domain of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is empty or reduced to a point).

Note that if $f \in P^0_M$ then neither the left nor the right end of the graph of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ can intersect the diagonal for $1 \leq m \leq M$.

When $\mathrm{Fix}\, f^{\circ m}$ is finite we have thus

$$\sum_{\xi \in \mathrm{Fix}\star f^{\circ m}} \nu(\xi, f^{\circ m}) = \sum_{\ell_1,\ldots,\ell_m} \sum_{\xi \in \mathrm{Fix}\star f_{\ell_m} \circ \cdots \circ f_{\ell_1}} \nu(\xi, f_{\ell_m} \circ \cdots \circ f_{\ell_1})$$
$$= - \sum_{\ell_1,\ldots,\ell_m} (\epsilon_{\ell_1} \cdots \epsilon_{\ell_m}) L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})\,.$$

Note that the $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ are locally constant on P^0_M, and that

$$\{f \in P^0_M \,:\, \mathrm{Fix}\, f^{\circ m} \text{ is finite for all } m \geq 1\}$$

is dense in P^0_M [approximate by N-tuples of non-affine polynomials]. Note also that the map $\xi \mapsto \prod_{k=0}^{m-1} z(f^{\circ k}\xi)$ is constant on $\mathrm{Fix}\star f_{\ell_m} \circ \cdots \circ f_{\ell_1}$. Therefore the lemma results from the definition of ζ. $\square$

The formula given above yields a natural definition of ζ_f (mod $\mathcal{I}_{M+1}$) for general $f \in P^0$, and we shall use this definition to prove Theorem 1.1. (Note that our definition of Δ did not use the condition that the sets $\mathrm{Fix}\, f^{\circ m}$ be finite.)

2.3. Lemma. *Let $M \geq 1$ and $\tilde{f}$ be a sufficiently small perturbation of f in P^0 such that the "shrinking condition"*

$$range\ \tilde{f}_{\ell_m} \circ \cdots \circ \tilde{f}_{\ell_1} \subset range\ f_{\ell_m} \circ \cdots \circ f_{\ell_1}$$

holds when $m \leq M$. Then

$$\zeta_{\tilde{f}}(Z) = \zeta_f(Z) \quad (mod\, \mathcal{I}_{M+1})$$
$$\Delta_{\tilde{f}}(Z) = \Delta_f(Z) \quad (mod\, \mathcal{I}_{M+1})\,.$$

Furthermore, if we have "strict shrinking"

$$range\ \tilde{f}_{\ell_m} \circ \cdots \circ \tilde{f}_{\ell_1} \subset\ interior\ range\ f_{\ell_m} \circ \cdots \circ f_{\ell_1}$$

for $m \leq M$, then $\tilde{f}^{\circ m}(a_i \pm) \notin \{a_1, \ldots, a_{N-1}\}$ for all $1 \leq i \leq N$ and $m \leq M$, in particular $\tilde{f} \in P^0_M$.

Our definition of $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ shows that this quantity does not change if the range of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is shrunk by a small amount. Hence, $\zeta_{\tilde{f}}(Z) = \zeta_f(Z)$ (mod $\mathcal{I}_{M+1}$). When $\tilde{f}$ tends to f, then

$$\tilde{f}^{\circ m}(a_i \pm) \to f^{\circ m}(a_i \pm)$$

and the shrinking condition implies that the limit is reached on the same side as the limit

$$f^{\circ m}(x) \to f^{\circ m}(a_i \pm)$$

when $\pm(x - a_i) \downarrow 0$. Therefore the $\vec{\theta}(a_i \pm, Z)$ coincide (mod $\mathcal{I}_{M+1}$) for $\tilde{f}$ and f and the same holds for the $\vec{K}_i$ so that $\Delta_{\tilde{f}}(Z) = \Delta_f(Z)$ (mod $\mathcal{I}_{M+1}$).

The last statement of the lemma follows from the fact that the numbers

$$|\tilde{f}^{\circ m}(a_i \pm) - f^{\circ m}(a_i \pm)|$$

are in an arbitrarily small interval $(0, \delta)$. $\square$

Proof of Theorem 1.1. Given f we may take $\tilde{f}$ to be arbitrarily close to f in P^0, such that $\tilde{f}_i$ and f_i differ only in $[a_{i-1}, a_i + \delta]$, $[a_i - \delta, a_i]$ and $\tilde{f}_i[a_{i-1}, a_i] \subset f_i(a_{i-1}, a_i)$. This will imply the strict shrinking condition of Lemma 2.3 for some fixed $M \geq 1$ if $\delta = \delta(M)$ is sufficiently small.

Let now $f^1 \in P^1_M$ be an N-tuple of polynomials sufficiently close to $\tilde{f} \in P^0_M$; then the invariant coordinates of the a_i* are the same for f^1 and $\tilde{f}$ up to terms of order $> M$. Therefore

$$\zeta_f = \zeta_{f^1} \quad (\text{mod}\ \mathcal{I}_{M+1})$$
$$\Delta_f = \Delta_{f^1} \quad (\text{mod}\ \mathcal{I}_{M+1})$$

and the proof of the theorem reduces to showing that

$$\zeta_{f^1}\, \Delta_{f^1} = 1 - \frac{1}{2}(\epsilon_1 z_1 + \epsilon_N z_N) \quad (\text{mod}\ \mathcal{I}_{M+1})\,.$$

Using f^0 as defined in Lemma 2.1, let $f^\lambda = (1 - \lambda) f^0 + \lambda f^1 \in P^1$. By definition, $f^\lambda = (f^\lambda_1, \ldots, f^\lambda_N)$ is an N-tuple of polynomials, none of which is affine [f^λ_i is non constant with derivative vanishing at a_{i-1} and a_i]. Therefore, $\mathrm{Fix}\,(f^\lambda)^{\circ m}$ is finite for all m. Note that the maps $(x,\lambda) \to f_i^\lambda(x)$, being also polynomials, are naturally defined on $\mathbb{R}^2$. Every composition $f_{\ell_m}^\lambda \circ \cdots \circ f_{\ell_1}^\lambda$ is hence allowed and defined on $\mathbb{R}$. Since $f_{\ell_m}^\lambda \circ \cdots \circ f_{\ell_1}^\lambda(a_i) - a_j$ is a polynomial in λ, of degree D (depending on $\ell_1, \ldots, \ell_m, a_i$), which we may take $\neq 0$ at $\lambda = 0, 1$, there is a finite set Λ of values of λ outside of which

$$(f^\lambda)^{\circ m}(a_i*) \neq a_j \quad \text{if } 1 \leq m \leq M\,, 1 \leq i \leq N-1, 1 \leq j \leq N-1\,.$$

Therefore $\zeta_{f^\lambda}\Delta_{f^\lambda}$ remains constant $(\mathrm{mod}\,\mathcal{I}_{M+1})$ for λ in each interval of $[0,1] \setminus \Lambda$. (In fact, both ζ_{f^λ} and Δ_{f^λ} remain constant individually; the zeta function because of Lemma 2.2 and the determinant because each $\vec{\theta}_{f^\lambda}(a_i\pm, Z)$ is constant.) There remains to show that $1/\zeta_{f^\lambda}$ and Δ_{f^λ} are multiplied by the same factor $(\mathrm{mod}\,\mathcal{I}_{M+1})$ when λ crosses a point of Λ. The changes of sign of the $(f^\lambda)^{\circ m}(a_i*) - a_j$ when λ crosses an element of Λ may be complicated. We shall make them simpler by modifying the family (f^λ) near each $\lambda \in \Lambda$ to obtain a family (g^λ) with nonlinear (but $\mathcal{C}^\infty$) dependence on λ. If the maps $(x,\lambda) \mapsto f_i^\lambda(x)$ are approximated in the $\mathcal{C}^\infty$ topology by maps $(x,\lambda) \mapsto g_i^\lambda(x)$, then $\lambda \mapsto f_{\ell_m}^\lambda \circ \ldots \circ f_{\ell_1}^\lambda(a_i) - a_j$, and $\lambda \mapsto g_{\ell_m}^\lambda \circ \cdots \circ g_{\ell_1}^\lambda(a_i) - a_j$ are $\mathcal{C}^\infty$ close and therefore $g_{\ell_m}^\lambda \circ \cdots \circ g_{\ell_1}^\lambda(a_i) - a_j$ can have at most D zeros and $(g^\lambda)^m(a_i*) - a_j$ at most D' zeros for some fixed D' uniform in the choice of (g^λ). The uniformity of this bound (when $m \leq M$) will be used in a moment.

For a given $\lambda_0 \in \Lambda$ we construct an oriented graph Γ as follows. Let $(f^{\lambda_0})^{\circ m}(a_i*) = a_j$ with $1 \leq m \leq M$, and suppose that $(f^{\lambda_0})^{\circ k}(a_i*) \notin \{a_1, \ldots, a_{N-1}\}$ for $k = 1, \ldots, m-1$; we define a sign $\pm$ by

$$* \prod_{k=0}^{m-1} \epsilon(f^{\circ k}(a_i*)) = \pm 1$$

and place an arrow $(a_i*) \Rightarrow (a_j\pm)$. Note that there may be *simple loops* $(a_i*) \Rightarrow (a_i*)$. An arrow starting from $(a_{i-1}+)$ (respectively (a_i-)) may be removed from the graph corresponding to λ_0 by a $\mathcal{C}^\infty$ small change of f_i^λ near (a_{i-1}, λ_0) (respectively (a_i, λ_0)) while the other arrows are left unchanged. Repeating this operation, we can arrange that the graph corresponding to λ_0 consists of a single arrow (which may be a simple loop). We have thus replaced the family (f^λ) by a new family $(\tilde{f}^\lambda)$, the set Λ by a new finite set $\tilde{\Lambda}$, and the new graph $\tilde{\Gamma}$ corresponding to λ_0 has only one arrow. By a $\mathcal{C}^\infty$ small change of $\tilde{f}^\lambda$ near $\lambda = \lambda_0$ we may assume that $\mathrm{Fix}\,(\tilde{f}^{\lambda_0})^{\circ m}$ is finite for $1 \leq m \leq M$, and that the derivative of $(\tilde{f}^{\lambda_0})^{\circ m}$ at $x \in \mathrm{Fix}\,(\tilde{f}^{\lambda_0})^{\circ m}$ is not equal to 1 (i.e., the periodic points of period $\leq M$ for $\tilde{f}^{\lambda_0}$ are not neutral). Note that the families (f^λ), $(\tilde{f}^\lambda)$ coincide outside of a small neighbourhood of λ_0; to obtain $\tilde{\Lambda}$ from Λ we have replaced λ_0 by a finite set $\{\lambda_0, \lambda_0', \ldots\}$.

We now start again the above process with a new element $\tilde{\lambda}_0$ of $\tilde{\Lambda}$ (being careful to leave $\tilde{f}^{\lambda_0}$ unchanged). Since the cardinality of the sets Λ, $\tilde{\Lambda}$, ..., is uniformly bounded, after a finite number of steps the family (f^λ) is replaced by (g^λ) with the following properties:

(a) $g^\lambda \in P^1$, the map $(x,\lambda) \mapsto g^\lambda(x)$ is $\mathcal{C}^\infty$, $g^0 = f^0$, $g^1 = f^1$;

(b) for λ outside of a finite set G, $(g^\lambda)^{\circ m}(a_i*) \neq a_j$, if $1 \leq m \leq M$, $1 \leq i \leq N-1$, $1 \leq j \leq N-1$;

(c) if $\lambda \in G$, there is a single (a_i*) and a single j such that $(g^\lambda)^{\circ p}(a_i*) = a_j$, for some $p \in \{1, \ldots, M\}$;

(d) if $\lambda \in G$, $\mathrm{Fix}\,(g^\lambda)^{\circ m}$ is finite for $m \leq M$ and the points $x \in \mathrm{Fix}\,(g^\lambda)^{\circ m}$ are not neutral.

To prove the theorem it suffices now (under the assumptions (a),(b),(c),(d)) to show that $1/\zeta_{g^\lambda}$ and Δ_{g^λ} are multiplied by the same factor (mod $\mathcal{I}_{M+1}$) when λ crosses a point in G. This is done in the following lemma. $\square$

2.4. Lemma. *Let $M \geq 1$ and let $f \in P^1$ be such that the sets $Fix\, f^{\circ m}$ are finite for $1 \leq m \leq M$ and such that there is a single (a_i*) and a single j for which*

$$f^{\circ p}(a_i*) = a_j$$

with $1 \leq p \leq M$. If $j = i$, we take the smallest permissible p and write

$$\hat{\epsilon} = \prod_{k=0}^{p-1} \epsilon(f^{\circ k}(a_i*)) \quad , \quad Z(\gamma) = \prod_{k=0}^{p-1} z(f^{\circ k}(a_i*))\,.$$

We assume that the f-periodic points with period at most M are not neutral. Then, given $g, h \in P^1_M$ sufficiently close to f and such that $g^{\circ p}(a_i*) > a_j$, $h^{\circ p}(a_i*) < a_j$ we have (mod $\mathcal{I}_{M+1}$)

$$\begin{array}{lll} \zeta_g = \zeta_h\,, & \Delta_g = \Delta_h & \text{if } j \neq i \\ \zeta_g = \zeta_h \cdot (1 - \hat{\epsilon} Z(\gamma))^{*1}\,, & \Delta_g = \Delta_h/(1 - \hat{\epsilon} Z(\gamma))^{*1} & \text{if } j = i\,. \end{array}$$

Figure: *Graph of $f^{\circ p}$:* (Lemma 2.4, case $i = j$) The graph of $g^{\circ p}$ (resp. $h^{\circ p}$) is obtained by pushing the graph of $f^{\circ p}$ upwards (resp. downwards).

We first discuss the easy proof of the formulas for the zeta function. If $j \neq i$, ζ (mod $\mathcal{I}_{M+1}$) is locally constant at f (Lemma 2.2), hence $\zeta_g = \zeta_h$.

Let $j = i$. If $* = +$, the orbit $(a_i+, f(a_i+), \ldots, f^{\circ p-1}(a_i+))$ for f turns into an attracting period p orbit γ for g, absent for h (see the figure). If $* = -$ the attracting orbit corresponding to $(a_i-, f(a_i-), \ldots, f^{\circ p-1}(a_i-))$ occurs for h and is absent for g. In both cases, the orbit γ is of type (2) if $\hat{\epsilon} = 1$, of type (3) if $\hat{\epsilon} = -1$. Apart from the new attracting orbit γ just described, f, g, h have corresponding orbits γ' with the same weight $Z(\gamma')$ up to order $\geq M+1$ if they are close enough with respect to the distance d_1. Therefore $\zeta_g = \zeta_h \cdot (1 - \hat{\epsilon} Z(\gamma))^{*1}$ as announced.

Let us now consider the changes for Δ. The "jump" at f of the invariant coordinate is (mod $\mathcal{I}_{M+1}$)

$$\begin{aligned}
&\delta\vec{\theta}(a_i*, Z) \\
&\quad := \vec{\theta}_g(a_i*, Z) - \vec{\theta}_h(a_i*, Z) \\
&\quad = \sum_{n=0}^{\infty} [\prod_{k=0}^{n-1} (\epsilon z)(g^{\circ k}(a_i*))\vec{\alpha}(g^{\circ n}(a_i*)) - \prod_{k=0}^{n-1} (\epsilon z)(h^{\circ k}(a_i*))\vec{\alpha}(h^{\circ n}(a_i*))] \\
&\quad = \prod_{s=0}^{p-1} (\epsilon z)(f^{\circ s}(a_i*)) \\
&\qquad \cdot \sum_{\ell=0}^{\infty} [\prod_{k=0}^{\ell-1} (\epsilon z)(g^{\circ p+k}(a_i*))\vec{\alpha}(g^{\circ p+\ell}(a_i*)) - \prod_{k=0}^{\ell-1} (\epsilon z)(h^{\circ p+k}(a_i*))\vec{\alpha}(h^{\circ p+\ell}(a_i*))] \,.
\end{aligned}$$

We shall denote by Φ a function which is constant on each interval (a_{i-1}, a_i) like $\epsilon, z, \vec{\alpha}$. If $j \neq i$, we have $\Phi(g^{\circ p+k}(a_i*)) = \Phi(f^{\circ k}(a_j+))$, $\Phi(h^{\circ p+k}(a_i*)) = \Phi(f^{\circ k}(a_j-))$ when g and h are sufficiently close to f and $p + k \leq M$, hence

$$\delta\vec{\theta}(a_i*, Z) = \prod_{s=0}^{p-1} (\epsilon z)(f^{\circ s}(a_i*)) \cdot \left[\vec{\theta}_f(a_j+, Z) - \vec{\theta}_f(a_j-, Z)\right],$$

which is proportional to $\vec{K}_j(Z)$, hence $\Delta_g = \Delta_h$.

If $j = i$ we may write

$$\prod_{s=0}^{p-1} (\epsilon z)(f^{\circ s}(a_i*)) = \hat{\epsilon} \cdot Z(\gamma)\,,$$

and we shall take for simplicity $* = +$. In what follows we shall always assume g and h sufficiently close to f in P^1.

When $p + k \leq M$, we have

$$\Phi(g^{\circ p+k}(a_i+)) = \Phi(g^{\circ k}(a_i+))\,.$$

[Using the figure, this is readily verified when k is a multiple of p and then in general.] Also, when $p + k \leq M$

$$\Phi(h^{\circ p+k}(a_i+)) = \Phi(f^{\circ k}(a_i-)) = \Phi(g^{\circ k}(a_i-))$$

hence (mod $\mathcal{I}_{M+1}$)

$$\begin{aligned}\delta\vec{\theta}(a_i+,Z) &:= \vec{\theta}_g(a_i+,Z) - \vec{\theta}_h(a_i+,Z)\\ &= \hat{\epsilon}Z(\gamma)\cdot\left[\vec{\theta}_g(a_i+,Z) - \vec{\theta}_g(a_i-,Z)\right]\\ &= \hat{\epsilon}Z(\gamma)\cdot 2\vec{K}_i^{(g)}(Z)\,,\end{aligned}$$

where $\vec{K}_i^{(g)}$ denotes $\vec{K}_i$ computed with g instead of f, so that

$$\begin{aligned}\delta\Delta(Z) &= \Delta_g(Z) - \Delta_h(Z)\\ &= \hat{\epsilon}Z(\gamma)\cdot\Delta_g(Z)\,,\end{aligned}$$

hence

$$(1-\hat{\epsilon}Z(\gamma))\Delta_g = \Delta_h$$

and finally

$$\Delta_g = \Delta_h/(1-\hat{\epsilon}Z(\gamma))\,.$$

The proofs for $* = -$ would be similar. □

Remarks on Lemma 2.4.

(1) The assumptions that $\operatorname{Fix} f^{\circ m}$ is finite for $1 \le m \le M$ and that no periodic point of period at most M is neutral have been inserted for convenience. By using the definition of the zeta function with the $L(f_{\ell_m}\circ\cdots\circ f_{\ell_1})$ given after Lemma 2.2, one could do without them.

(2) By making use of Lemma 11.5 in Milnor-Thurston [1988], we may prove a version of Lemma 2.4 for $M=\infty$. If one wishes to construct a homotopy satisfying the assumptions of Lemma 2.4 in the case $M=\infty$, the method used in the proof of Theorem 1.1 can be replaced by the more abstract arguments of Milnor-Thurston [1988, Lemmas 11.7, 11.8].

(3) Except to prove the case $i=j$ when $M=\infty$ (which we do not require in our proof of Theorem 1.1), where the above-mentioned Lemma 11.5 in Milnor-Thurston [1988] is crucial, we do not really need the $\mathcal{C}^1$ topology (the hypothesis that the periodic orbits $f^{\circ p}(a_i*) = a_i$ are one-sided attracting is a topological one). (This does not contradict the remark on page 533 of Milnor-Thurston [1988], which is precisely concerned with the case $i=j$ and $M=\infty$.)

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