QUONS,
AN INTERPOLATION BETWEEN BOSE AND FERMI OSCILLATORS

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Abstract

After a brief mention of Bose and Fermi oscillators and of particles which obey other types of statistics, including intermediate statistics, parastatistics, paronic statistics, anyon statistics and infinite statistics, I discuss the statistics of "quons" (pronounced to rhyme with muons), particles whose annihilation and creation operators obey the q-deformed commutation relation (the quon algebra or q-mutator) which interpolates between fermions and bosons. I emphasize that the operator for interaction with an external source must be an effective Bose operator in all cases. To accomplish this for parabose, parafermi and quon operators, I introduce parabose, parafermi and quon Grassmann numbers, respectively. I also discuss interactions of non-relativistic quons, quantization of quon fields with antiparticles, calculation of vacuum matrix elements of relativistic quon fields, demonstration of the TCP theorem, cluster decomposition, and Wick's theorem for relativistic quon fields, and the failure of local commutativity of observables for relativistic quon fields. I conclude with the bound on the parameter q for electrons due to the Ramberg-Snow experiment.

1 Introduction

I start by reviewing the (Bose) harmonic oscillator. I want to emphasize that the commutation relation,

$$[a_i, a_j^\dagger]_- \equiv a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}, \tag{1}$$

and the vacuum condition which characterizes the Fock representation

$$a_i|0\rangle = 0 \tag{2}$$

suffice to calculate all vacuum matrix elements of polynomials in the annihilation and creation operators. The strategy is to move annihilation operators to the right, picking up terms with a contraction of an annihilation and a creation operator. When the annihilation operator gets to the vacuum on the right, it annihilates it. For example,

$$\begin{aligned}\langle 0|a_{i_1}a_{i_2}\cdots a_{i_n}a^\dagger_{j_m}\cdots a^\dagger_{j_2}a^\dagger_{j_1}|0\rangle &= \delta_{i_n j_m}\langle 0|a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a^\dagger_{j_{m-1}}\cdots a^\dagger_{j_2}a^\dagger_{j_1}|0\rangle \\ &+\langle 0|a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a^\dagger_{j_m}a_{i_n}a^\dagger_{j_{m-1}}\cdots a^\dagger_{j_2}a^\dagger_{j_1}|0\rangle. \end{aligned} \tag{3}$$

Continuing this reduction, it is clear that this vacuum matrix element vanishes, unless the set $\{i_1, i_2, \cdots, i_n\}$ is a permutation of the set $\{j_1, j_2, \cdots, j_m\}$ (this includes $n = m$). In particular, *no relation is needed between two a's or between two $a^\dagger$'s.* As we know, it turns out that

$$[a_i, a_j]_- = 0 = [a^\dagger_i, a^\dagger_j]_-, \tag{4}$$

but these relations are redundant in the Fock representation. Also, only the totally symmetric (one-dimensional) representations of the symmetric (i.e., permutation) group $\mathcal{S}_n$ occur.

To construct observables in the free theory we can use the number operator, n_k, or the transition operator, n_{kl},

$$n_k = n_{kk} = a^\dagger_k a_k, \quad n_{kl} = a^\dagger_k a_l. \tag{5}$$

The commutation relation,

$$[n_{kl}, a^\dagger_m]_- = \delta_{lm} a^\dagger_k, \tag{6}$$

follows from Eq.(1). The number operator has integer eigenvalues,

$$n_k (a^\dagger_k)^{\mathcal{N}}|0\rangle = \mathcal{N}(a^\dagger_k)^{\mathcal{N}}|0\rangle. \tag{7}$$

Using n_k and n_{kl} we can construct the Hamiltonian,

$$H = \sum_k \epsilon_k n_k, \tag{8}$$

and other observables for the free theory. The Hamiltonian obeys

$$[H, a^\dagger_l]_- = \epsilon_l a^\dagger_l. \tag{9}$$

Analogous formulas of higher degree in the a's and $a^\dagger$'s give interaction terms.

I want to pay special attention to couplings to external sources in the quon theory; in preparation for that I write the external Hamiltonian in the Bose case,

$$H_{ext} = \sum_k (j^\star_k a_k + a^\dagger_k j_k), \tag{10}$$

where j_k is a c-number; i.e.,

$$[j_k, a^\dagger_l]_- = [j_k, j^\star_l]_- = 0, \text{ etc.} \tag{11}$$

This satisfies the commutation relation

$$[H_{ext}, a^\dagger_l]_- = j^\star_l. \tag{12}$$

Equations (9) and (12) state that H and H_{ext} are "effective Bose operators" in the context of a free theory with an external source. In particular, Eq. (9) and (12) imply

$$[H, a^\dagger_{l_1} a^\dagger_{l_2} \cdots a^\dagger_{l_n}]_- = \sum_i \epsilon_i a^\dagger_{l_1} a^\dagger_{l_2} \cdots a^\dagger_{l_n} \tag{13}$$

and

$$[H_{ext}, a^\dagger_{l_1} a^\dagger_{l_2} \cdots a^\dagger_{l_n}]_- = \sum_i a^\dagger_{l_1} a^\dagger_{l_2} a^\dagger_{l_{i-1}} j^\star_{l_i} a^\dagger_{l_{i+1}} \cdots a^\dagger_{l_n}, \tag{14}$$

so the energy is additive for a system of free particles. The general definition of an effective Bose operator is that the Hamiltonian density commutes with the field when the points are separated by a large spacelike distance,

$$[\mathcal{H}(\mathbf{x}), \phi(\mathbf{y})]_- \to 0, |\mathbf{x} - \mathbf{y}| \to \infty. \tag{15}$$

This definition holds for all cases, including quons.

Everything I stated for the Bose harmonic oscillator can be repeated for the Fermi oscillator, with obvious modifications. The commutation relation Eq. (1) is replaced by the anticommutation relation;

$$[a_i, a^\dagger_j]_+ \equiv a_i a^\dagger_j + a^\dagger_j a_i = \delta_{ij} \tag{16}$$

that, together with the vacuum condition which characterizes the Fock representation,

$$a_i|0\rangle = 0, \tag{17}$$

again suffices to calculate all vacuum matrix elements of multinomials in the annihilation and creation operators. For example,

$$\begin{aligned}\langle 0|a_{i_1} a_{i_2} \cdots a_{i_n} a^\dagger_{j_m} \cdots a^\dagger_{j_2} a^\dagger_{j_1}|0\rangle = \delta_{i_n j_m} \langle 0|\langle 0|a_{i_1} a_{i_2} \cdots a_{i_{n-1}} a^\dagger_{j_{m-1}} \cdots a^\dagger_{j_2} a^\dagger_{j_1}|0\rangle \\ -\langle 0|a_{i_1} a_{i_2} \cdots a_{i_{n-1}} a^\dagger_{j_m} a_{i_n} a^\dagger_{j_{m-1}} \cdots a^\dagger_{j_2} a^\dagger_{j_1}|0\rangle.\end{aligned} \tag{18}$$

Continuing this reduction, it is clear that this vacuum matrix element vanishes, unless the set $\{i_1, i_2, \cdots, i_n\}$ is a permutation of the set $\{j_1, j_2, \cdots, j_m\}$. In particular, again *no relation is needed between two a's or between two $a^\dagger$'s.* As we know, it turns out that

$$[a_i, a_j]_+ = 0 = [a^\dagger_i, a^\dagger_j]_+, \tag{19}$$

but these relations again are redundant in the Fock representation. Also, as we know, only the totally antisymmetric (one-dimensional) representations of the symmetric group occur.

To construct observables in the free theory we again use the number operator, n_k, or the transition operator, n_{kl},

$$n_k = n_{kk} = a^\dagger_k a_k, \quad n_{kl} = a^\dagger_k a_l. \tag{20}$$

The commutation relation

$$[n_{kl}, a^\dagger_m]_- = \delta_{lm} a^\dagger_k \tag{21}$$

follows from the commutation relation Eq.(16). The number operator again has integer eigenvalues; now, however, the number of particles in a single quantum state can only be zero or one, since Eq.(19), which holds in the Fock representation, implies $a_i^{\dagger 2} = 0$,

$$n_k (a_l^\dagger)^{\mathcal{N}} |0\rangle = \delta_{kl} \mathcal{N} (a_l^\dagger)^{\mathcal{N}} |0\rangle, \mathcal{N} = 0, 1. \tag{22}$$

Using n_k and n_{kl} we again can construct the Hamiltonian,

$$H = \sum_k \epsilon_k n_k \tag{23}$$

and other observables for the free theory. The Hamiltonian obeys

$$[H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{24}$$

Analogous formulas of higher degree in the a's and $a^\dagger$'s give interaction terms.

I again pay special attention to couplings to external sources; the external Hamiltonian in the Fermi case is

$$H_{ext} = \sum_k (f_k^\star a_k + a_k^\dagger f_k), \tag{25}$$

where f_k is an anticommuting (Grassmann) number,

$$[f_k, f_l]_+ = [f_k, f_l^\star]_+ = [f_k, a_l]_+ = [f_k, a_l^\dagger]_+ = 0. \tag{26}$$

The external Hamiltonian satisfies the commutation relation,

$$[H_{ext}, a_l^\dagger]_- = f_l^\star. \tag{27}$$

The commutation relations Eq.(24) and Eq.(27) state that H and H_{ext} are "effective Bose operators" in the context of a free theory with an external source. In particular, Eq.(24) and Eq.(27) imply

$$[H, a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger]_- = \sum_i \epsilon_i a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger \tag{28}$$

and

$$[H_{ext}, a_{l_1}^\dagger a_{l_2}^\dagger \cdots a_{l_n}^\dagger]_- = \sum_i a_{l_1}^\dagger a_{l_2}^\dagger a_{l_{i-1}}^\dagger f_{l_i}^\star a_{l_{i+1}}^\dagger \cdots a_{l_n}^\dagger, \tag{29}$$

so that the energy is additive for a system of free particles.

Notice that Eq.(2,5,6,8,9,13) for the Bose case are *identical* to Eq.(17,20,21,23,24,28) for the Fermi case. Eq.(7,10,12,14) for the Bose case are analogous to Eq.(22,25,27,29) for the Fermi case. Finally, Eq.(1,3,4,11) for the Bose case and Eq.(16,18,19,26) for the Fermi case differ only by minus signs.

2 Generalizations of Bose and Fermi Statistics

As far as I know, the first attempt to go beyond Bose and Fermi statistics was made by G. Gentile [1]. He suggested "intermediate statistics," in which up to n particles can occupy a given quantum state. Clearly Fermi statistics is recovered for $n = 1$ and Bose statistics is recovered in the limit $n \to \infty$. As formulated by Gentile, intermediate statistics is not a proper quantum statistics, because the condition of having at most n particles in a quantum state is not invariant under change of basis.

H.S. Green [2] invented a generalization which is invariant under change of basis. I later dubbed his invention "parastatistics" [3]. Green noticed that the number operator and transition operator, Eq.(5, 20), have the same form for both bosons and fermions, as do the commutation relations between the transition operator and the creation and annihilation operators, Eq.(6, 21). Green generalized the transition operator by writing

$$n_{kl} = (1/2)([a_k^\dagger, a_l]_\pm \mp p\delta_{kl}), \tag{30}$$

where the upper signs are for the generalization of bosons ("parabosons") and the lower signs are for the generalization of fermions ("parafermions"). Since Eq.(30) is trilinear, two conditions the states are necessary to fix the Focklike representation: the usual vacuum condition is

$$a_k|0\rangle = 0; \tag{31}$$

the new condition

$$a_k a_l^\dagger|0\rangle = p\ \delta_{kl}, p \ \text{ integer}, \tag{32}$$

contains the parameter p which is the order of the parastatistics. The Hamiltonian for free particles obeying parastatistics has the same form, in terms of the number operators, as for Bose and Fermi statistics,

$$H = \sum_k \epsilon_k n_k, \text{ where, as usual } [H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{33}$$

For interactions with an external source, we must introduce para-Grassmann numbers which make the interaction Hamiltonian an effective Bose operator. This is in analogy with the cases of external Bose and Fermi sources discussed above. We want

$$[H_{ext}, a_l^\dagger] = c_l^\star. \tag{34}$$

We accomplish this by choosing $H_{ext} = \sum_{kl} H_{kl}^{ext}$, with

$$H_{kl}^{ext} = (1/2)([c_k^\star, a_l]_\pm + [a_k^\dagger, c_l]_\pm), \tag{35}$$

where the para-Grassmann numbers c_k and $c_k^\dagger$ obey

$$[[c_k^\star, c_l]_\pm, c_m^\star]_- = 0, \ [[c_k^\star, a_l]_\pm, a_m^\dagger]_- = 2\delta_{lm}c_k^\star, \text{ etc.}, \tag{36}$$

and the upper (lower) sign is for parabose-Grassmann (parafermi-Grassmann) numbers. The "etc." in Eq.(36) means that when some of the c's or $c^\dagger$'s is replaced by an a or an $a^\dagger$, the relation retains its form, except when the a and $a^\dagger$ can contract, in which case the contraction appears on the right-hand-side.

It is worthwhile to make explicit the fact that in theories with parastatistics states belong to many-dimensional representations of the symmetric group. This contrasts with the cases of Bose and Fermi statistics in which only the one dimensional representations occur.

I emphasize that parastatistics is a perfectly consistent local quantum field theory. The observables, such as the current, are local provided the proper symmetrization or antisymmetrization is used; for example,

$$j^\mu(x) = (1/2)[\bar{\psi}(x), \gamma^\mu \psi(x)]_- \tag{37}$$

for the current of a spin-1/2 field. Further, all norms in a parastatistics theory are positive; there are no negative probabilities. On the other hand, parastatistics of order $p > 1$ gives a gross violation of statistics; for example, for a parafermi theory of order $p > 1$ each quantum state can be occupied p times. A precise experiment is not needed to rule out such a gross violation.

Within the last three years two new approaches to particle statistics (in three or more space dimensions) have been studied in order to provide theories in which the Pauli exclusion principle (i.e., Fermi statistics) and/or Bose statistics can be violated by a small amount. One of these approaches uses deformations [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] of the trilinear commutation relations of H.S. Green [2] and Volkov. [14]. (Deformations of algebras and groups, in particular, quantum groups, are a subject of great interest and activity at present. The extensive literature on this subject can be traced from [15].) The particles, called "parons," which obey this type of statistics have a quantum field theory which is local, but some states of such theories must have negative squared norms (i.e., there are negative probabilities in the theory). The negative squared norms first appear in many-particle states: in the model considered in [5] the first negative norm occurs in the state with Young tableau (3,1). It does not seem that the negative norm states decouple from those with positive squared norms (as, in contrast, the corresponding states do decouple in manifestly covariant quantum electrodynamics). Thus theories with parons seem to have a fatal flaw.

The other approach uses deformations of the bilinear Bose and Fermi commutation relations [16, 17, 18, 19, 20, 21, 23]. The particles which obey this type of statistics, called "quons," have positive-definite squared norms for a range of the deformation parameter, but the observables of such theories fail to have the desired locality properties. This failure raises questions about the validity of relativistic quon theories, but, in contrast to the paron theories, does not cause a problem with non-relativistic quon theories. (As I prove below, the TCP theorem and clustering hold for free relativistic quon theories, so even relativistic quon theories may be interesting.)

Still other approaches to violations of statistics were given in [24, 25, 26]. An interpolation between Bose and Fermi statistics using parastatistics of increasing order was studied in [27]; this also does not give a small violation.

Yet another type of statistics, anyon statistics, has been extensively discussed recently, and applied to the fractional Hall effect and to high-temperature superconductivity. For anyons, the transposition of two particles can give *any* phase,

$$\psi(1,2) = e^{i\phi}\psi(2,1). \tag{38}$$

In the form usually considered, anyons only exist in two space dimensions, and are outside the framework I am considering. I will not discuss them further here; rather I give two relevant references [29, 30].

3 The Quon Algebra

3.1 *The $q = 0$ case*

In their general classification of possible particle statistics, Doplicher, Haag and Roberts [31] included bosons, fermions, parabosons, parafermions and one other case, infinite statistics, in which all representations of the symmetric group could occur, but did not give an algebra which led to this last case. Roger Hegstrom, a chemist at Wake Forest University, suggested averaging the Bose and Fermi commutation relations to get

$$a_k a_l^\dagger = \delta_{kl}, \quad a_k|0\rangle = 0. \tag{39}$$

(Unknown to Hegstrom and me, this algebra had been considered earlier by Cuntz [28].) With Hegstrom's permission, I followed up his idea and showed that this algebra gives an example of infinite statistics. Consider a general scalar product,

$$(a_{k_1}^\dagger \cdots a_{k_n}^\dagger|0\rangle, a_{P^{-1}k_1}^\dagger \cdots a_{P^{-1}k_m}^\dagger|0\rangle). \tag{40}$$

This vanishes unless $n = m$ and P is the identity, and then it equals one. ¿From this it follows that one can choose coefficients $c(P)$ to project into states in each irreducible of S_n and that the norm will be positive,

$$\| \sum_P c(P) a_{P^{-1}k_1}^\dagger \cdots a_{P^{-1}k_n}^\dagger|0\rangle\|^2 > 0; \tag{41}$$

thus every representation of S_n occurs. Note that there is no relation between two a's or two $a^\dagger$'s; as before, the Fock vacuum condition makes such relations unnecessary.

To construct observables, we want a number operator and a transition operator which obey

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger, \quad [n_{kl}, a_m^\dagger]_- = \delta_{lm} a_k^\dagger. \tag{42}$$

Once Eq.(42) holds, the Hamiltonian and other observables can be constructed in the usual way; for example,

$$H = \sum_k \epsilon_k n_k, \quad \text{etc.} \tag{43}$$

The obvious thing is to try

$$n_k = a_k^\dagger a_k. \tag{44}$$

Then

$$[n_k, a_l^\dagger]_- = a_k^\dagger a_k a_l^\dagger - a_l^\dagger a_k^\dagger a_k. \tag{45}$$

The first term in Eq.(45) is $\delta_{kl}a_k^\dagger$ as desired; however the second term is extra and must be canceled. This can be done by adding the term $\sum_t a_t^\dagger a_k^\dagger a_k a_t$ to the term in Eq.(44). This cancels the extra term, but adds a new extra term, which must be canceled by another term. This procedure yields an infinite series for the number operator and for the transition operator,

$$n_{kl} = a_k^\dagger a_l + \sum_t a_t^\dagger a_k^\dagger a_l a_t + \sum_{t_1,t_2} a_{t_2}^\dagger a_{t_1}^\dagger a_k^\dagger a_l a_{t_1} a_{t_2} + \ldots \tag{46}$$

As in the Bose case, this infinite series for the transition or number operator defines an unbounded operator whose domain includes states made by polynomials in the creation operators acting on the vacuum. (As far as I know, this is the first case in which the number operator, Hamiltonian, etc. for a free field are of infinite degree.)

For nonrelativistic theories, the x-space form of the transition operator is

$$\rho_1(\mathbf{x};\mathbf{y}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{y}) + \int d^3z\psi^\dagger(\mathbf{z})\psi^\dagger(\mathbf{x})\psi(\mathbf{y})\psi(\mathbf{z})$$
$$+\int d^3z_1 d^3z_2\psi(\mathbf{z_2})\psi^\dagger(\mathbf{z_1})\psi^\dagger(\mathbf{x})\psi(\mathbf{y})\psi(\mathbf{z_1})\psi(\mathbf{z_2}) + \cdots, \tag{47}$$

which obeys the nonrelativistic locality requirement

$$[\rho_1(\mathbf{x};\mathbf{y}),\psi^\dagger(\mathbf{w})]_- = \delta(\mathbf{y}-\mathbf{w})\psi^\dagger(\mathbf{x}), \quad \text{and} \quad \rho(\mathbf{x};\mathbf{y})|0\rangle = 0. \tag{48}$$

The apparent nonlocality of this formula associated with the space integrals has no physical significance. To support this last statement, consider

$$[Qj_\mu(x), Qj_\nu(y)]_- = 0, \quad x \sim y, \tag{49}$$

where $Q = \int d^e x j^0(x)$. Equation (49) seems to have nonlocality because of the space integral in the Q factors; however, if

$$[j_\mu(x), j_\nu(y)]_- = 0, \quad x \sim y, \tag{50}$$

then Eq.(49) holds, despite the apparent nonlocality. What is relevant is the commutation relation, not the representation in terms of a space integral. (The apparent nonlocality of quantum electrodynamics in the Coulomb gauge is another such example.)

In a similar way,

$$[\rho_2(\mathbf{x},\mathbf{y};\mathbf{y}',\mathbf{x}'),\psi^\dagger(\mathbf{z})]_- = \delta(\mathbf{x}'-\mathbf{z})\psi^\dagger(\mathbf{x})\rho_1(\mathbf{y},\mathbf{y}') + \delta(\mathbf{y}'-\mathbf{z})\psi^\dagger(\mathbf{y})\rho_1(\mathbf{x},\mathbf{x}'). \tag{51}$$

Then the Hamiltonian of a nonrelativistic theory with two-body interactions has the form

$$H = (2m)^{-1}\int d^3x\nabla_x\cdot\nabla_{x'}\rho_1(\mathbf{x},\mathbf{x}')|_{\mathbf{x}=\mathbf{x}'} + \frac{1}{2}\int d^3xd^3yV(|\mathbf{x}-\mathbf{y}|)\rho_2(\mathbf{x},\mathbf{y};bfy,\mathbf{x}). \tag{52}$$

$$[H,\psi^\dagger(\mathbf{z}_1)\ldots\psi^\dagger(\mathbf{z}_n)]_- = -(2m)^{-1}\sum_{j=1}^n\nabla^2_{\mathbf{z}_i} + \sum_{i<j}V(|\mathbf{z}_i-\mathbf{z}_j|)]\psi^\dagger(\mathbf{z}_1)\ldots\psi^\dagger(\mathbf{z}_n)$$
$$+\sum_{j=1}^n\int d^3xV(|\mathbf{x}-\mathbf{z}_j|)\psi^\dagger(\mathbf{z}_1)\cdots\psi^\dagger(\mathbf{z}_n)\rho_1(\mathbf{x},\mathbf{x}'). \tag{53}$$

Since the second term on the right-hand-side of Eq.(53) vanishes when the equation is applied to the vacuum, this equation shows that the usual Schrödinger equation holds for the n-particle system. Thus the usual quantum mechanics is valid, with the sole exception that any permutation symmetry is allowed for the many-particle system. This construction justifies calculating the energy levels of (anomalous) atoms with electrons in states which violate the exclusion principle using the normal Hamiltonian, but allowing anomalous permutation symmetry for the electrons [32].

In general, an arbitrary many-particle state is in a mixture of inequivalent irreducible representations of S_n. If $\mathcal{O}$ is any observable and Ψ is any state, the cross terms between irreducibles in the matrix element $\langle\Psi|\mathcal{O}|\Psi\rangle$ automatically vanish, since observables keep states inside their irreducible representation of S_n.

3.2 *The general quon algebra for* $-1 \leq q \leq 1$.

The quon algebra,

$$a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}, \tag{54}$$

which is a deformation of the Bose and Fermi algebras and interpolates between these algebras as q goes from 1 to -1 on the real axis, shares many qualitative features with the special case of $q = 0$ just discussed. In particular, the quon algebra also allows all representations of S_n. This algebra, supplemented by the vacuum condition

$$a_k|0\rangle = 0, \tag{55}$$

determines a (Fock-like) representation in a linear vector space. For $-1 < q < 1$, the squared norms of all vectors made by limits of polynomials of the creation operators, $a_k^\dagger$, are strictly positive[19, 20, 21]. Among other things, this means that there are $n!$ linearly independent states of n particles with distinct quantum numbers, and all representations of the symmetric group occur. Also, as in the case of $q = 0$, Eqs.(54,55) allow the calculation of the vacuum to vacuum matrix element of any polynomial in the a's and $a^\dagger$'s. As before, no commutation relation between two a's or between two $a^\dagger$'s is needed. Further, in this case, no such rule can be imposed on aa or $a^\dagger a^\dagger$. The relation,

$$a_k a_l - q a_l a_k = 0, \tag{56}$$

between two a's which one might guess in analogy with the Bose and Fermi commutation rules holds only when $q^2 = 1$; and requires that $q = \pm 1$ in Eq.(54); i.e., Eq.(56) can hold only in the Bose and Fermi cases. To see this, interchange k and l in Eq.(56) and put the result back in the initial relation. (Commutation relations between two a's or between two $a^\dagger$'s are also not needed for normal ordering, i.e., to expand a product of a's and $a^\dagger$'s as a sum of terms in which creation operators always stand to the left of annihilation operators. Wick's theorem for quon operators is similar to the usual Wick's theorem; the only difference is that the terms acquire powers of q. I gave the precise algorithm in [33].) As q approaches -1 from above, the more antisymmetric representations become more heavily weighted and at -1 only the antisymmetric representation survives. As q approaches 1 from below, the more symmetric representations become more heavily weighted and at 1 only the symmetric representation survives. Outside the interval $[-1,1]$, the squares of some norms become negative.

Now I discuss the construction of observables both without and with an external source. Without an external source, one again needs a set of number operators n_k such that

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger. \tag{57}$$

Like the $q=0$ case, the expression for n_k or n_{kl} is an infinite series in creation and annihilation operators; unlike the $q=0$ case, the coefficients are complicated. The first two terms are

$$n_{kl} = a_k^\dagger a_l + (1-q^2)^{-1} \sum_t (a_t^\dagger a_k^\dagger - q a_k^\dagger a_t^\dagger)(a_l a_t - q a_t a_l) + \cdots. \tag{58}$$

Here I gave the transition number operator n_{kl} for $k \to l$ since this takes no extra effort. The general formula for the number operator is given in [22] following a conjecture of Zagier [19]. As before, the Hamiltonian is

$$H = \sum_k \epsilon_k n_k, \quad \text{with} \quad [H, a_l^\dagger]_- = \epsilon_l a_l^\dagger. \tag{59}$$

For an external source, it is crucial to insure that H_{ext} is an effective Bose operator. In order to do this, one must choose the external source to be a quon analog of a Grassmann number, i.e., to obey

$$c_k c_l^\star - q c_l^\star c_k = 0; \;\; c_k a_l^\dagger - q a_l^\dagger c_k = 0; \;\; a_k c_l^\star - q c_l^\star a_k = 0. \tag{60}$$

Then H_{ext} must be chosen to obey

$$[H_{ext}, a_l^\dagger]_- = c_l^\star \tag{61}$$

For example, for $q=0$, the first two terms are

$$H_{ext} = \sum_k (c_k^\star a_k + a_k^\dagger c_k) + \sum_k \sum_t a_t^\dagger (c_k^\star a_k + a_k^\dagger c_k) a_t + \cdots \tag{62}$$

For the general case, I give the first two terms of H_{kl}^{ext}, subject to

$$[H_{kl}^{ext}, a_m^\dagger]_- = \delta_{lm} c_k^\star \tag{63}$$

and hermiticity, $(H_{kl}^{ext})^\dagger = H_{lk}^{ext}$,

$$\begin{aligned} H_{kl}^{ext} &= c_k^\star a_l + a_k^\dagger c_l + (1-q^2)^{-1} \sum_t (a_t^\dagger c_k^\star - q c_k^\star a_t^\dagger)(a_l a_t - q a_t a_l) \\ &\quad + \sum_k (1-q^2)^{-1} (a_t^\dagger a_k^\dagger - q a_k^\dagger a_t^\dagger)(c_l a_t - q a_t c_l) + \cdots \end{aligned} \tag{64}$$

If, instead, we incorrectly choose $H_{ext} = \sum_k (j_k^\star a_k + a_k^\dagger j_k)$, where j is a c-number, then the energy of widely separated states is not additive,

$$H_{ext} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_n}^\dagger |0\rangle = [j_{k_1}^\star a_{k_2}^\dagger \cdots a_{k_n}^\dagger + q a_{k_1}^\dagger j_{k_1}^\star \cdots a_{k_n}^\dagger + \cdots q^{n-1} a_{k_1}^\dagger a_{k_2}^\dagger \cdots j_{k_n}^\star]|0\rangle \tag{65}$$

Although this point is transparent for the case of fermions where powers of negative one replace powers of q in Eq.(65), it seems to be less clear in the quon case. Because this point was not recognized, the bound on validity of Bose statistics for photons given in [34] is incorrect.

Again one- and two-body observables can be constructed from $\rho_1(\mathbf{x}, \mathbf{x})$ and $\rho_2(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_2, \mathbf{y}_1)$. The formula for n can be translated into a formula for ρ_1, and at least the first non-trivial term is known for ρ_2. With these, a valid nonrelativistic theory of identical particles with (small) violations of Fermi of Bose statistics can be formulated [35].

The condition that observables must be effective Bose operators leads to conservation of statistics which states that all interactions must involve an even number of fermions or para-fermions and an even number of para particles (except for cases in which p para fields can occur when the order of the parastatistics is p)[36]. I expect that conservation of statistics must also hold for quons and, in particular, that a single quon cannot couple to normal fields [37]. I plan to discuss the conservation of statistics for quons in detail elsewhere. I have discussed the simple case of a single oscillator elsewhere[33], so I will not repeat this discussion here.

To summarize, all irreducible representations of S_n have positive (norm)^2 in this interval. As $q \to \pm 1$ the more symmetric (antisymmetric) irreducibles occur with higher weight. At the endpoints, $q = \pm 1$, only the symmetric (antisymmetric) representation survives.

4 The quon algebra in the presence of antiparticles

The pattern is established by discussing the spin-zero case. Since

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (b_k e^{-ik\cdot x} + d_k^\dagger e^{ik\cdot x}), \tag{66}$$

$$\phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (d_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}), \tag{67}$$

$\omega_k = k^0 = \sqrt{\mathbf{k}^2 + m^2}$, to preserve charge conjugation symmetry one should supplement the commutation relation for the b's and $b^\dagger$'s by

$$d_k d_l^\dagger - q d_l^\dagger d_k = \delta_{kl}, \tag{68}$$

$$d_k b_l^\dagger - q b_l^\dagger d_k = 0, \tag{69}$$

With this choice, the positivity of the norms is preserved in the presence of antiparticles. If, instead, one chooses the x-space relation,

$$\phi(x)\phi^\dagger(y) - q\phi^\dagger(y)\phi(x) = F(x-y) \equiv \mathrm{vev}(lhs) \tag{70}$$

then one finds the usual quon commutation relation for the b's, but

$$d_k d_l^\dagger - q^{-1} d_l^\dagger d_k = \delta_{kl} \tag{71}$$

for the d's. Since Eq.(71) gives positive norms only *outside* $|q| < 1$, this choice is inconsistent. In [39, 40] this last choice has been argued to imply breaking of charge conjugation invariance.

It is amusing to note that the TCP theorem and clustering hold, at least for free quon fields, despite the failure of locality [33].

5 Experiments

In a conference devoted to issues related to harmonic oscillators, it is worthwhile to make some comments about the experimental relevance of the quon oscillator. The quon oscillator provides a parametrization of possible small departures from Fermi or Bose statistics. The simplest way to detect small violations of statistics is to find a state which either Fermi or Bose statistics would not allow. For Fermi (Bose) statistics, this would be a state in which identical particles are not totally antisymmetric (symmetric). The path-breaking high-precision experiment of Ramberg and Snow[41] searches for transitions to a state in which the electrons of the copper atom are not totally antisymmetric. The failure to detect such transitions (above background) leads to the following upper bound on violation of the exclusion principle,

$$\rho_2 = \frac{1}{2}(1-\beta^2)\rho_a + \frac{1}{2}\beta^2\rho_s, \quad \frac{1}{2}\beta^2 \leq 1.7 \times 10^{-26}, \tag{72}$$

ρ_2 is the two-electron density matrix, $\rho_{a(s)}$ is the antisymmetric (symmetric) two-electron density matrix. For two electrons in different states ρ_2 can be expressed in terms of q of the q-mutator as

$$\rho_2 = \frac{1}{2}(1-q)\rho_a + \frac{1}{2}(1+q)\rho_s, \tag{73}$$

so the Ramberg Snow bound is

$$0 \leq (1+q)/2 \leq 1.7 \times 10^{-26}. \tag{74}$$

A high-precision experiment to detect or bound violations of the exclusion principle for electrons in helium is being conducted by D. Kelleher, et al.[42]

I conclude this brief discussion of experimental bounds on small violations of statistics by remarking that there are three types of such experiments: (1) to detect an accumulation of particles in anomalous states, (2) to detect transitions to anomalous states and (3) to detect deviations from the usual statistical properties of many-particle systems. Here and in [8] type (2) experiments are discussed, because they allow detection of single transitions to anomalous states. Type (1) experiments require detection of a small concentration of anomalous states in a macroscopic system; for that reason they are generally less sensitive than type (2) experiments. I have not analyzed type (3) experiments; however it seems likely that they will fail to provide high-precision tests for the same reason that type (1) experiments fail: it will be difficult to detect the modification of the statistical properties of a macroscopic sample due to a small concentration of anomalous states.

6 Summary

The quon oscillator serves as an interpolation between Fermi and Bose statistics. This interpolation preserves positivity of norms and the non-relativistic form of locality, but fails to allow local observables in a relativistic theory. Nonetheless, the TCP theorem and clustering hold in relativistic quon theories. Terms in the Hamiltonian for both self-interacting systems and systems interacting with an external source must be effective Bose operators in order for the additivity of the energy for widely separated subsystems to hold. The quon theory provides a parametrization of possible deviations from Bose or Fermi statistics.

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References

[1] G. Gentile, Nuovo Cimento **17**, 493 (1940).

[2] H.S. Green, Phys. Rev. **90**, 270 (1953).

[3] G.-F. Dell'Antonio, O.W. Greenberg and E.C.G. Sudarshan, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, ed. F. Gürsey (Gordon and Breach, New York, 1964), p403.

[4] A.Yu. Ignatiev and V.A. Kuzmin, Yad. Fys. **46**, 786 (1987) [Sov. J. Nucl.. Phys. **46**,444 (1987).]

[5] O.W. Greenberg and R.N. Mohapatra, Phys. Rev. Lett. **59**, 2507 (1987); erratum, op cit **61**, 1432 (1988).

[6] O.W. Greenberg and R.N. Mohapatra, Phys. Rev. D **39**, 2032 (1989).

[7] A.B. Govorkov, Phys. Lett. A **137**, 7 (1989).

[8] O.W. Greenberg and R.N. Mohapatra, Phys. Rev. Lett. **62**, 712 (1989).

[9] O.W. Greenberg, Nucl. Phys. B (Proc. Suppl.) **6**, 83 (1989).

[10] A.Yu. Ignatiev and V.A. Kuzmin, Pisma Zh. Eksp. Teor. Fiz. **47**, 6 (1988) [JETP Lett. **47**, 4 (1988)].

[11] V.N. Gavrin, A.Yu. Ignatiev and V.A. Kuzmin, Phys. Lett. B **206**, 343 (1988).

[12] L.C. Biedenharn, P. Truini and H. van Dam, J. Phys. A Math. Gen. **22**, L67 (1989).

[13] A.Yu. Ignatiev, Kyoto preprint (1990).

[14] D.V. Volkov, Sov. Phys. JETP **9**, 1107 (1959); **11**, 375 (1960).

[15] Proc. Workshop on Quantum Groups, T. Curtright, D. Fairlie and C. Zachos (eds.), World Scientific (1991).

[16] O.W. Greenberg, Bull. Am. Phys. Soc. **35**, 981 (1990).

[17] O.W. Greenberg, Phys. Rev. Lett. **64,** 705 (1990).

[18] R.N. Mohapatra, Phys. Lett. B **242,** 407 (1990).

[19] D.B. Zagier, Commun. Math. Phys. **147**, 199 (1992).

[20] M. Bozejko and R. Speicher, Commun. Math. Phys. **137**, 519 (1991); and to appear in *Proc. Quantum Probability and Related Topics VII*, New Delhi, 1990).

[21] D.I. Fivel, Phys. Rev. Lett. **65**, 3361 (1990); erratum **69**, 2020 (1992). I have not checked that the erratum repairs the error in the proof of positivity in the article.

[22] S. Stanciu, ibid, **147**, 211 (1992)

[23] O.W. Greenberg, in Proc. Workshop on Quantum Groups, T. Curtright, D. Fairlie and C. Zachos (eds.), World Scientific (1991), p 166.

[24] L.B. Okun, Pisma Zh. Eksp. Teor. Fiz.. **46**, 420 (1987) [JETP Lett. **46**, 529 (1987)]; L.B. Okun, Yad. Fiz. **47**, 1192 (1988) [Sov. J. Nucl. Phys. **47**, 752 (1986)]. L.B. Okun, in *Festival–Festschrift for Val Telegdi*, ed K. Winter (Elsevier Sci. Publishers, 1988), p201. L.B. Okun, Comments on Nucl. and Part. Phys. **19**, 99 (1989).

[25] V. Rahal and A. Campa, Phys. Rev. A **38**, 3728 (1988).

[26] R. Ramanathan, Univ. of Delhi preprint (1990).

[27] O.W. Greenberg and A.M.L. Messiah, J. Math. Phys. **6**, 500 (1965).

[28] J. Cuntz, Commun. Math. Phys. **57,** 173 (1977), discussed a closely related algebra earlier.

[29] J.M. Leinaas and J. Myrheim, Nuovo Cimento **37B**, 1 (1977).

[30] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World, Singapore, 1990).

[31] S. Doplicher, R. Haag and J.E. Roberts, Commun. Math. Phys. **13**, 1 (19069); **15**, 173 (1969); **23**, 199 (1971); **35**, 49 (1974).

[32] G.F.W. Drake, Phys. Rev. A **39**, 897 (1989).

[33] O.W. Greenberg, Phys. Rev. D **43**, 4111 (1991).

[34] D.I. Fivel, Phys. Rev. A **43**, 4913 (1991).

[35] O.W. Greenberg, Physica A **180**, 419 (1992).

[36] O.W. Greenberg and A.M.L. Messiah, Phys. Rev. **138**, B1155, (1965), discuss the selection rules for parafields in detail.

[37] T. Sudbery, Nature **348**, 193 (1990) remarked on the implication of conservation of statistics for the interpretation by Kekez, et al, [38] of experiments searching for gamma rays as setting bounds of violation of the exclusion principle for nucleons.

[38] D. Kekez, A. Ljubicic and B.A. Logan, Nature **348**, 224 (1990).

[39] A.B. Govorkov, Mod. Phys. Lett. A **7**, 2383 (1992).

[40] M.V. Cougo-Pinto, Phys Rev. D **46**, 858 (1992).

[41] E. Ramberg and G.A. Snow, Phys. Lett. B **238**, 438 (1990).

[42] D. Kelleher, National Bureau of Standards (now National Institute of Standards and Technology (NIST))Internal Memo (1988). J.D. Gillaspy of NIST has made a list of experiments relevant to bounds on violations of statistics.

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