DYNAMICAL SYSTEMS AND FACTORS OF FINITE AUTOMATA

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Abstract

We conceive finite automata as dynamical systems on discontinuum and investigate their factors. Factors of finite automata include many well-known simple dynamical systems, e.g. hyperbolic systems and systems with finite attractors. In the quadratic family on the real interval, factors of finite automata include all systems with finite number of periodic points, as well as systems at the band-merging bifurcations. On the other hand the system with a non-chaotic attractor, which occurs at the limit of the period doubling bifurcations (at the edge of chaos), is not of this class. Next we propose as another simplicity criterion for dynamical systems the property of having chaotic limits (every point belongs to a set, whose ω-limit is chaotic). We show that any factor of a finite automaton has chaotic limits but not vice versa.

key words: edge of chaos, discontinuum, renormalization.

1 Introduction

Studies in chaotic dynamics suggest that chaotic dynamical systems are usually rather simple, the most notable example being the shift map. Another class of simple dynamical systems is distinguished by the existence of periodic orbits, which attract all points. Systems with really complex and interesting dynamics are found at the edge of chaos (see Langton [8]). They are characterized by long transient periods, strong dependence on initial conditions and the power of universal computations. In the quadratic family on the real interval, the systems at both period doubling and band merging bifurcations have been shown by Crutchfield and Young [4] to generate simple regular languages. On the other hand the system at their common limit generates a non-regular language.

The feeling that chaotic means simple has been supported by a recent result of Brooks et al. [2]. A standard Devaney's [5] definition reads that a chaotic system is one, which is topologically transitive, whose periodic points are dense, and which is sensitive to initial conditions. Brooks et al. [2] prove that on infinite metric spaces, the first two conditions imply the third. In light of this result it is natural to relax the definition of chaotic system to topological transitivity and density of periodic points. According to this relaxed definition, finite dynamical systems, which are orbits of periodic points, are chaotic too.

One might try to characterize simple systems by the condition, that all points have chaotic ω-limit sets. This would not do, however, since even in the shift system, there are many complex subshifts like Morse sequences. We define therefore a system with chaotic limits as one, whose each point is included in a set, whose ω-limit set is chaotic. This works well in the quadratic family: all systems at the period doubling and band merging bifurcations have chaotic limits, while the system at their common limit has not.

The simplest systems with chaotic limits are the shift map σ on 2^N defined by $\sigma(u)_i = u_{i+1}$, which itself is chaotic, and its inverse σ_0^{-1} defined by $\sigma_0^{-1}(u)_{i+1} = u_i$, $\sigma_0^{-1}(u)_0 = 0$, whose limit is the fixed point 0. Both these systems are instances of finite automata. We conceive a finite automaton as a dynamical system: a finite control connected to infinite input and output tapes. At each time step the automaton reads a symbol from the input tape and writes a symbol to the output tape. The state of the whole system consists of the inner state of the control and the content of both tapes. The shift map is an automaton which works only with the input tape, while its inverse works only with the output tape. The state space of a finite automaton is a discontinuum, i.e. a countable product of finite spaces.

Another simplicity criterion for dynamical systems is based on the fact, that every dynamical system on a compact metric space is a factor of a dynamical system on discontinuum (see Balcar and Simon ([1], Proposition 5.12). In computer science there is a hierarchy ranging from finite automata, stack automata, Turing automata, and cellular automata to neural networks. All these automata can be regarded as dynamical systems on discontinuum. Considering their factors, we obtain a hierarchy of dynamical systems on compact metric spaces. Factors of finite automata are at the bottom of this hierarchy. This works again in the quadratic family: all systems at the period doubling and band merging bifurcations are factors of finite automata, while the system at their common limit is not. We show that every factor of a finite automaton has chaotic limits, while there are systems with chaotic limits, which are not factors of finite automata.

In Crutchfield and Young [4] and Troll [10], the complexity of dynamical systems has been studied via complexity of languages generated by finite partitions. Our concepts of systems with chaotic limits, and factors of finite automata offer an alternative and complementary approach.

2 Systems with chaotic limits

Definition 1 *A dynamical system (X, f) is a continuous map $f : X \to X$ of a compact metric space X to itself. A homomorphism $\varphi : (X, f) \to (Y, g)$ of dynamical systems is a continuous mapping $\varphi : X \to Y$ such that $g\varphi = \varphi f$. We say that (Y, g) is a factor of (X, f), if φ is a factorization (i.e. a surjective map).*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

The n-th iteration $f^n : X \to X$ of f is defined by $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. The orbit $\mathcal{O}(x) = \{f^i(x) : i \geq 0\}$ of x is the set of its iterates. A point $x \in X$ is periodic with period $n > 0$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $0 < i < n$. A fixed point is a periodic point with period 1. A subset $A \subseteq X$ is invariant, if $f(A) \subseteq A$. It is strongly invariant, if $f(A) = A$. The ω-limit set of a set $A \subseteq X$ is defined by $\omega(A) = \bigcap_n \overline{\bigcup_{m>n} f^m(A)}$. It is easy to see that if A is nonempty, then $\omega(A)$ is a nonempty, closed, strongly invariant set, so f restricted to $\omega(A)$ is a dynamical system.

Lemma 1 *Let $\varphi : (X, f) \to (Y, g)$ be a homomorphism, and $A \subseteq X$. Then $\varphi(\omega_f(A)) = \omega_g(\varphi(A))$.*

Proof: If $x \in \omega_f(A)$, then there exist $x_i \in A$ and n_i with $f^{n_i}(x_i) \to x$, so $\varphi(x_i) \in \varphi(A), g^{n_i}(\varphi(x_i)) \to \varphi(x)$, and therefore $\varphi(x) \in \omega_g(\varphi(A))$. If $y \in \omega_g(\varphi(A))$, then there exist $x_k \in A$ and n_k with $g^{n_k}\varphi(x_k) \to y$. There exists a convergent subsequence $f^{n_{k_i}}(x_{k_i}) \to x \in \omega_f(A)$, so $\varphi f^{n_{k_i}}(x_{k_i}) \to \varphi(x)$. It follows $y = \varphi(x)$, and therefore $y \in \varphi(\omega_f(A))$. $\square$

Definition 2 *A dynamical system (X, f) is chaotic, if the set of its periodic points is dense in X and if for any open nonempty sets $U, V \subseteq X$ there exists $k > 0$ such that $f^k(U) \cap V \neq 0$ (topological transitivity). The system (X, f) has chaotic limits, if for each $x \in X$ there exists $A \subseteq X$ such that $x \in A$ and $\omega(A)$ is chaotic.*

In particular a finite system is chaotic iff it is an orbit of a periodic point, and any finite system has chaotic limits. If $\omega(x)$ is a periodic orbit for each $x \in X$, then (X, f) is a system with chaotic limits. Moreover if X or $\omega(X)$ is chaotic, then (X, f) has again chaotic limits.

Proposition 1 *A factor of a chaotic dynamical system is chaotic. A factor of a system with chaotic limits has chaotic limits.*

Proof: Suppose that $\varphi : (X, f) \to (Y, g)$ is a factorization and (X, f) is chaotic. Let $U \subseteq Y$ be a nonempty open set. Then $\varphi^{-1}(U)$ is a nonempty open set and there is a periodic point x in it. It follows that $\varphi(x)$ is a periodic point in U. If $U, V \subseteq Y$ are nonempty open sets, then so are $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$, so there exists $x \in f^k\varphi^{-1}(U) \cap \varphi^{-1}(V)$. It follows $\varphi(x) \in g^k(U) \cap V$, so (Y, g) is chaotic. Suppose now that (X, f) has chaotic limits, and let $y \in Y$. There exists $x \in X$ with $\varphi(x) = y$ and $x \in A \subseteq X$ such that $\omega(A)$ is chaotic. It follows that $y \in \varphi(A)$ and by Lemma 2, $\omega_g(\varphi(A)) = \varphi(\omega_f(A))$ is chaotic. $\square$

3 Finite automata

Definition 3 *Let A, Q, B be finite sets, and $Z = \{... - 1, 0, 1, ...\}$ the set of integers. The discontinuum (over A, Q, B) is the space*

$$X = \{u \in (A \cup Q \cup B)^Z : u_i \in A \text{ for } i > 0,\ u_0 \in Q,\ u_i \in B \text{ for } i < 0\}$$

with metric

$$\varrho(u,v) = 2^{-n} \quad \textit{where} \quad n = \inf\{i \geq 0 : u_i \neq v_i \quad \textit{or} \quad u_{-i} \neq v_{-i}\}$$

Proposition 2 *Let X be a discontinuum. Then X is a compact metric space. For each compact metric space Y there is a factorization $\varphi : X \to Y$. For every dynamical system (Y, g) there exists a dynamical system f on X and a factorization $\varphi : (X, f) \to (Y, g)$.*

See Balcar and Simon [1] for a proof.

Definition 4 *Let A, Q, B be finite sets (input alphabet, set of inner states, and output alphabet), and $f_A : Q \times A \to Q$, $f_B : Q \to B$ input and output functions. A finite automaton over A, Q, B is a dynamical system f on the discontinuum X (over A, Q, B) defined by*

$$f(u)_{-1} = f_B(u_0),\ f(u)_0 = f_A(u_0, u_1),\ f(u)_i = u_{i+1} \ \textit{for} \ \ -1 \neq i \neq 0$$

(X, f) is a finite input automaton (without output tape) if B is a singleton. It is a finite output automaton (without input tape), if A is a singleton.

Thus $\{u_i : i > 0\}$ is the input tape, $\{u_i : i < 0\}$ is the output tape, and $f(...u_{-2}, u_{-1}, u_0, u_1, u_2...) = (...u_{-1}, f_B(u_0), f_A(u_0, u_1), u_2, u_3...)$.

Theorem 1 *Every finite automaton has chaotic limits.*

Proof: Let $w \in X$ and define $Q_0 = \{q \in Q : (\forall m)(\exists n > m)(f^n(w)_0 = q)\}$. Let k be the first integer such that for all $n \geq k$ there is $f^n(w)_0 \in Q_0$. Define $W = \{v \in X : (\forall i \leq k)(v_i = w_i) \ \ \& \ \ (\forall i \geq k)(f^i(v)_0 \in Q_0)\}$. We shall prove that $\omega(W)$ is chaotic. Denote $Y = \{v \in X : (\forall n \geq 0)(f^n(v)_0 \in Q_0)\}$. Then Y is an invariant set and $\omega(W) \subseteq \bigcap\{f^n(Y) : n \geq 0\}$ since $f^k(W) \subseteq Y$. Let $u \in \bigcap\{f^n(Y) : n \geq 0\}$, and pick some $n > 0$. There is $v \in Y$ with $f^n(v) = u$, and there exists $m \geq k$ with $f^m(w)_0 = v_0$. Define $s \in W$ by $s_i = w_i$ for $i \leq m$, $s_i = v_{m+i}$ for $i > m$. Then $\varrho(f^{m+n}(s), u) < 2^{-n}$, so $u \in \omega(W)$. Thus $\omega(W) = \bigcap\{f^n(Y) : n \geq 0\}$. Let $U \subseteq X$ be an open set and $u \in U \cap \omega(W)$. We shall find a periodic point in $U \cap \omega(W)$. There exists n such that the n-th cylinder around u is contained in U: $u \in \{v \in \omega(W) : (\forall : i :\leq n)(v_i = u_i)\} \subseteq U \cap \omega(W)$. There exists $u' \in Y$ such that $u = f^n(u')$. Then there exists $u'' \in Y$ such that $u'_i = u''_i$ for $0 \leq i \leq 2n$ and $f^m(u'')_0 = u''_0$ for some $m > 2n$. Define $s \in X$ by $s_0 = u''_0$, $s_{km+i} = u''_i$, $s_{-km-i} = f^m(u'')_{-i}$

for $1 \le i \le m$, and $k \ge 0$. Then $s \in \omega(W)$ is a periodic point, and $f^n(s)$ is a periodic point in $U \cap \omega(W)$. Thus periodic points are dense in $\omega(W)$. To prove the topological transitivity, let U, V be open sets, $u \in U \cap \omega(W)$, $v \in V \cap \omega(W)$. There is again n such that the n-th cylinders around u and v are contained in U and V respectively. Then there are $u', v' \in Y$ such that $u = f^n(u')$, $v = f^n(v')$. There exists $u'' \in Y$ such that $u'_i = u''_i$ for $0 \le i \le 2n$, and $f^m(u'')_0 = v'_0$ for some $m > 2n$. Define $s \in X$ by $s_i = u''_i$ for $i \le m$ and $s_i = v'_{i-m}$ for $i > m$. Then $f^{m+n}(s) \in f^m(U) \cap V \cap \omega(W)$. $\square$

Corollary 1 *A factor of a finite automaton has chaotic limits.*

Proposition 3 *A subshift of finite type is a factor of a finite input automaton.*

Proof: Let A be a finite alphabet and M a square $A \times A$ matrix with entries from $2 = \{0, 1\}$. Let $Q = A$, $B = \{0\}$, and define f_A so that for each $a, b \in A$ $(\exists c \in A)(f_A(a, c) = b) \Leftrightarrow M(a, b) = 1$. Define $\Sigma_M = \{u \in A^N : (\forall i \ge 0)(M(u_i, u_{i+1}) = 1)\}$, $\sigma : \Sigma_M \to \Sigma_M$ by $\sigma(u)_i = u_{i+1}$, and $\varphi : X \to \Sigma_M$ by $\varphi(u)_i = (f^i(u))_0$. Then $\varphi : (X, f) \to (\Sigma_M, \sigma)$ is a factorization. $\square$

Since a closed hyperbolic set with a local product structure (see Shub [9]) is a factor of a subshift of finite type, it is also a factor of a finite automaton.

Theorem 2 *A dynamical system (Y, g) is a factor of a finite output automaton if and only if $\omega(Y)$ is finite.*

Proof: Let (X, f) be a finite output automaton. Then $f_A : Q \to Q$. Denote $Q_0 \subseteq Q$ the set of periodic points of f_A, and $f_A^{-1} : Q_0 \to Q_0$ the inverse map to f_A. Then $u \in \omega(X)$ iff there exists $q \in Q_0$ such that $u_0 = q$ and $(\forall i > 0)(u_{-i} = f_B f_A^{-i}(q))$. Thus $\omega(X)$ is a finite set. If $\varphi : (X, f) \to (Y, g)$ is a factorization, then $\omega(Y) = \varphi(\omega(X))$ is finite too. Suppose now that $\omega(Y)$ is finite. Since it is strongly invariant, it consists just of periodic points of (Y, g), and there exists a map $g^{-1} : \omega(Y) \to \omega(Y)$ inverse to g. For $p \in \omega(Y)$ denote $t_p > 0$ its period. We shall prove that for every $p \in \omega(Y)$ there exists its closed, g^{t_p}-invariant neighbourhood, which is disjoint with $\omega(Y) - \{p\}$. Suppose by contradiction, that it is not the case. For $A \subseteq X$, $\varepsilon > 0$ denote $B_\varepsilon(A) = \{y \in Y : (\exists z \in A)(\varrho(y, z) < \varepsilon)\}$. For each $k > 0$ the closure of $\bigcup_{n \ge 0} g^{nt_p}(B_{1/k}(p))$ is a closed g^{t_p}-invariant neighbourhood of

p, and therefore meets $\omega(Y) - \{p\}$. It follows that there exists a sequence $y_k \in B_{1/k}(p)$, such that $g^{n_k t_p}(y_k) \in B_{1/k}(\omega(Y) - \{p\})$ for some $n_k > 0$. Let W be a neighbourhood of p disjoint with $\omega(Y) - \{p\}$. There exists an open neighbourhood U of p with $g^{t_p}(U) \subseteq W$, and also an open neighbourhood V of $\omega(Y) - \{p\}$ disjoint with W. For each k let $m_k < n_k$ be the first integer, for which $g^{m_k t_p}(y_k) \notin U$, and therefore $g^{m_k t_p}(y_k) \in Y - U - V$. Since $Y - U - V$ is compact, there is a convergent subsequence $g^{m_{k_i} t_p}(y_{k_i}) \to y \in Y - U - V$, and it follows $y \in \omega(Y)$, which is a contradiction. So we have a closed neighbourhood V_p of p disjoint with $\omega(Y) - \{p\}$, for which $g^{t_p}(V_p) \subseteq V_p$. Let us prove, that $W_p = \{y \in Y : \lim_{k\to\infty} g^{kt_p}(y) = p\}$ is an open set. If $y \in W_p$, there is $k > 0$ for which $g^{kt_p}(y) \in int(V_p)$ and there is a neighbourhood U of y with $g^{kt_p}(U) \subseteq V_p$, so $U \subseteq W_p$. Since $W_p \cap W_q = 0$ for different $p, q \in \omega(Y)$, and $\bigcup\{W_p : p \in \omega(Y)\} = Y$, W_p are also closed. Construct a finite automaton (X, f) with alphabets $A = \{0\}$, $Q = \omega(Y)$, $B = Q \cup \{0\}$ (0 is a symbol not occurring in Q). Define transition and output functions by $f_A(p, 0) = g(p)$, $f_B(p) = g(p)$. For $p \in Q$, $k \geq 0$ define $X_{pk} = \{u \in X : \max\{j \geq 0 : u_{-j} \neq g^{-j}(p)\} = k + 1\}$, $X_{p\infty} = \{u \in X : (\forall j \geq 0)(u_{-j} = g^{-j}(p)\}$. Then $f : X_{pk} \to X_{g(p),k+1}$ is a homeomorphism and has an inverse f^{-1}. Let $W_{p0} = W_p \cap \overline{Y - g(Y)}$. If $W_{p0} \neq 0$ define φ on X_{p0} so that $\varphi(X_{p0}) = W_{p0}$ (X_{p0} is a discontinuum and W_{p0} is closed). If $W_{p0} = 0$, define $\varphi(X_{p0}) = \{p\}$. If $u \in X_{g(p),k+1}$, define $\varphi(u) = g\varphi f^{-1}(u)$ by induction on k. If $u \in X_{p\infty}$, define $\varphi(u) = p$. Then $\varphi : (X, f) \to (Y, g)$ is a factorization. $\square$

A set $A \subseteq X$ is an attractor (see Devaney [5]), if it has a neighbourhood U such that $f(U) \subseteq int(U)$, and $A = \omega(U)$. Thus Theorem 2 says that a system is a factor of a finite output automaton iff all its attractors are finite.

Definition 5 *A sum of dynamical systems (X_0, f_0) and (X_1, f_1) is a dynamical system (X, f), where $X = X_0 \cup X_1$ is the disjoint union, and $f(x) = f_i(x)$ if $x \in X_i$.*

Proposition 4 *The sum of dynamical systems, which are factors of finite automata, is itself a factor of a finite automaton.*

Proof: Let $\varphi_0 : (X_0, f_0) \to (Y_0, g_0)$, $\varphi_1 : (X_1, f_1) \to (Y_1, g_1)$, be factorizations of finite automata. Define a finite automaton (X, f) by $A = A_0 \times A_1$, $Q =$

$Q_0 \cup Q_1$, (disjoint union) $B = B_0 \times B_1$. Pick some $b_0 \in B_0$, $b_1 \in B_1$ and define

$$\begin{array}{lll} f_A(q,(a_0,a_1)) = f_{A_0}(q,a_0), & f_B(q) = (f_{B_0}(q), b_1) & \text{if } q \in Q_0 \\ f_A(q,(a_0,a_1)) = f_{A_1}(q,a_1), & f_B(q) = (b_0, f_{B_1}(q)) & \text{if } q \in Q_1 \end{array}$$

Let (Y,g) be the sum of (Y_0,g_0) and (Y_1,g_1). Define $\varphi : (X,f) \to (Y,g)$ by $\varphi(u) = \varphi_0\pi_0(u)$ if $u_0 \in Q_0$, $\varphi(u) = \varphi_1\pi_1(u)$ if $u_0 \in Q_1$. Here $\pi_i : X \to X_i$ are the projections.

4 Clopen-regular systems

If α is a finite set, denote α^* the set of words over α. For a word $u \in \alpha^*$ denote $: \alpha :\geq 0$ its length and u_i its $i+1$st letter, so $u = u_0 \ldots u_{:u:-1}$. Subsets of α^* are called languages. See Hopcroft and Ullmann [7] for the definition of a regular language.

Definition 6 *Let (X,f) be a dynamical system and α a finite set of subsets of X. Define the language generated by α as*

$$L(f,\alpha) = \{u \in \alpha^* : (\exists x \in X)(\forall i <: u :)(f^i(x) \in u_i)\}$$

The system (X,f) is clopen-regular if $L(f,\alpha)$ is regular for every finite clopen cover α of X.

A subset A of a metric space X is clopen, if it is both closed and open. A set α is a cover of X if its union is X. It is a partition, if it is a cover, whose different sets are disjoint. Let X be discontinuum (over A, Q, B), $w \in X$ and $n > 0$. Then the n-cylinder $\text{œ}w!_n = \{v \in X : (\forall i \in Z)(: i :\leq n \Rightarrow v_i = w_i)\}$ of w is a clopen set (it is a ball with centre w). The set $\alpha_n = \{\text{œ}w!_n : w \in X\}$ of all n-cylinders is a finite clopen partition.

Proposition 5 *Let (X,f) be a dynamical system on discontinuum such that $L(f,\alpha_n)$ is regular for each $n > 0$ (α_n is the partition of n-cylinders). Then (X,f) is clopen-regular.*

Proof: Let β be a finite clopen cover. For each $u \in X$ there exists $n_u > 0$ such that if $u \in B \in \beta$, then $\text{œ}u!_{n_u} \subseteq B$, and if $u \notin B \in \beta$ then $\text{œ}u!_{n_u} \cap B = 0$.

The set {œu!$_{n_u}$ $: u \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover {œu!$_{n_u}$ $: u \in I\}$. Let $n = \max\{n_u : u \in I\}$. Then for each $B \in \beta$ there is a finite set $J_B \subseteq \alpha_n$ such that if $A \in J_B$ then $A \subseteq B$, and if $A \notin J_B$, then $A \cap B = 0$. Since $L(f, \alpha_n)$ is regular, $L(f, \beta)$ is regular too. □

Proposition 6 *A factor of a clopen-regular system is clopen-regular.*

Proof: Let $\varphi : (X, f) \to (Y, g)$ be a factorization, and α a clopen cover of Y. Then $\varphi^{-1}\alpha = \{\varphi^{-1}(A) : A \in \alpha\}$ is a clopen cover of X and $L(f, \varphi^{-1}\alpha) = L(g, \alpha)$. □

Proposition 7 *Any finite automaton is clopen-regular.*

Proof: Let α_n be the clopen partition of n-cylinders. For $A, B \in \alpha_n$ define $M(A, B) = 1$ if $f(A) \cap B \neq 0$ and $M(A, B) = 0$ otherwise. Then $L(f, \alpha_n) = \{u \in \alpha_n^* : (\forall i <: u : -1)(M(u_i, u_{i+1}) = 1)\}$. This is a regular language. □

Corollary 2 *Any factor of a finite automaton is clopen-regular.*

Example 1 *There exists a chaotic dynamical system which is not a factor of any finite automaton.*

Proof: Let $X = \{u \in 2^N : 0^n 1^m 0 \prec u \Rightarrow n \leq m\}$. Here $2 = \{0, 1\}$ and $\prec$ means substring. Thus $u \in X$ if every string of n zeros in u is immediately followed by a string of at least n ones. Then X is a closed, shift-invariant subset of 2^N, and it is easy to see that (X, σ) is chaotic. However, it is not clopen-regular. Consider the partition $\alpha_0 = \{\{u : u_0 = 0\}, \{u : u_0 = 1\}\}$. Then $L(\sigma, \alpha_0) = \{u \in 2^* : 0^n 1^m 0 \prec u \Rightarrow n \leq m\}$ consists just of all substrings of elements of X and is not regular. □

Example 2 *There exists a dynamical system (X, f), such that $\omega(x)$ is a fixed point for all $x \in X$, but (X, f) is not a factor of a finite automaton.*

Proof: Let $M \subseteq N = \{0\}^*$ be a nonregular set of positive integers. Define $X = \{u \in 3^Z : i < j \Rightarrow u_i \leq u_j \ \ \& \ \ 01^m 2 \prec u \Rightarrow m \in M\}$, where $3 = \{0, 1, 2\}$. Thus $u \in X$ if it is nondecreasing, and when it contains a finite number of ones, then their number belongs to M. Again, X is a closed shift-invariant set. Its only fixed points are constant sequences $\overline{0}, \overline{1}, \overline{2}$, and for each $x \in X$, $\omega(x)$ is one of them. However, (X, σ) is not clopen-regular. If α_0 is the partition of the 0-cylinders, then $L(\sigma, \alpha_0) = \{u \in 3^* : i < j \Rightarrow u_i \leq u_j \ \& \ 01^m2 \prec u \Rightarrow m \in M\}$ is again the language of finite substrings of elements of X, which is not regular. $\square$

This example does not contradict Theorem 2, since $\sigma(X) = X$ and therefore $\omega(X) = X$ is infinite. Consider $X_1 = \{u \in X : u_1 \geq 1\}$ Then X_1 is a closed shift-invariant subset of X with only two fixed points. When we factorize them, we get a system with a single fixed point, which is ω-limit of all other points, and still the system is not a factor of a finite automaton. This shows that the Theorem 2 cannot be substantially strengthened.

5 Unimodal systems

Definition 7 *A symmetric real function g defined on $I = \text{œ} - 1, 1!$ is unimodal, if it is continuously differentiable, $g(-1) = g(1) = -1$, $g(0) > -1$ and $g'(x) = 0$ iff $x = 0$. (We speak about unimodal system, if $g(I) \subseteq I$, i.e. if $g(0) \leq 1$.)*

Proposition 8 *Let $g : I \to R$ be a symmetric unimodal function with $g(0) > 0$. Then there is a unique $0 < p < 1$ with $g(-p) = g(p) = p$. Define the renormalization of g as $\mathcal{R}g(x) = -g^2(px)/p$. Then $\mathcal{R}g : I \to R$ is a symmetric unimodal function.*

Proof: Clearly $g'(x) > 0$ for $x < 0$, and $g'(x) < 0$ for $x > 0$. Since $g(x) - x$ is positive for $x = 0$ and negative for $x = 1$, there is a unique fixed point $p > 0$, and $g(-p) = p$. Clearly $\mathcal{R}g(\pm 1) = -g^2(\pm p)/p = -1$. For $x \in (-p, p)$ there is $g(x) > p$, so $g'(g(x)) < 0$. The equation $(g^2)'(x) = g'(g(x))g'(x) = 0$ has in $\text{œ} - p, p!$ unique solution 0, so the equation $(\mathcal{R}g)'(x) = 0$ has in $\text{œ} - 1, 1!$ the unique solution 0. Since g is decreasing for $x > 0$, and $0 < p$, we have $g(0) > p$, and $g^2(0) < p$. It follows $\mathcal{R}g(0) = -g^2(0)/p > -1$. $\square$

Theorem 3 *Let g be a symmetric unimodal system with $g(0) > 0$. If $\mathcal{R}g(0) \leq 1$, and if $\mathcal{R}g$ is a factor of a finite automaton, then g is also a factor of a finite automaton.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\
I_0 \arrow[r, "g^2"] & I_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X'' \arrow[r, "h"] \arrow[d, "\psi"'] & X'' \arrow[d, "\psi"] \\
I \arrow[r, "g"] & I
\end{tikzcd}
$$

Proof: Let p_0 be the positive fixed point of g. There is an increasing sequence $p_k \in (0,1)$ such that $g(-p_k) = g(p_k) = -p_{k-1}$ for $k > 0$. Denote $I_{-\infty} = \{-1\}$, $I_k =$ œ $- p_{-k}, -p_{-k+1}$! for $k < 0$, $I_0 =$ œ $- p_0, p_0$!, $I_k =$ œp_{k-1}, p_k! for $k > 0$, and $I_\infty = \{1\}$. Then $g(I_{-k}) = g(I_k) = I_{-k+1}$ for $k > 0$. Since $\mathcal{R}g(I) \subseteq I$, we have $g^2(I_0) \subseteq I_0$, and $g(I_0) \subseteq I_1$. By the assumption there exists a finite automaton (X, f) and a factorization $\varphi : (X, f) \to (I_0, g^2)$. We construct a finite automaton (X', h). Suppose $A = \{0, ..., m-1\}$ and define $A' = \{0, ..., m\}$, $Q' = \{(q, i) : q \in Q, i \in \{-1, 0, 1, \sharp\}\}$, $B' = B$. For $x \in Q'$ denote $\lfloor x \rfloor$ its first component, and $\lceil x \rceil$ the second, so that $x = (\lfloor x \rfloor, \lceil x \rceil)$. Define

$$
\begin{array}{ll}
X_{-\infty} & = \{u \in X' :\ \lceil u_0 \rceil = -1, (\forall j > 0)(u_j = m)\}, \\
X_k & = \{u \in X' :\ \lceil u_0 \rceil = \mathrm{sgn}(k), \min\{j > 0 : u_j \neq m\} =: k :\},\ \ : k :> 0 \\
X_\infty & = \{u \in X' :\ \lceil u_0 \rceil = 1, (\forall j > 0)(u_j = m)\} \\
X_i & = \{u \in X' :\ \lceil u_0 \rceil = i\} \quad \text{for} \quad i \in \{0, \sharp\}
\end{array}
$$

Define $h_B : Q' \to B'$, $h_A : Q' \times A' \to Q'$ by $h_B(x) = f_B(\lfloor x \rfloor)$,

$$
\begin{array}{lll}
h_A(x, m) & = (\lfloor x \rfloor, -1) & \text{if } \lceil x \rceil \in \{-1, 1\} \\
h_A(x, j) & = (\lfloor x \rfloor, 0) & \text{if } \lceil x \rceil \in \{-1, 1, \sharp\},\ j < m \\
h_A(x, j) & = (f_A(\lfloor x \rfloor, j), \sharp) & \text{if } \lceil x \rceil = 0,\ j < m \\
h_A(x, m) & = h_A(x, m-1) & \text{if } \lceil x \rceil \in \{0, \sharp\}
\end{array}
$$

Then $h(X_{-k}) = h(X_k) = X_{-k+1}$, $h(X_0) \subseteq X_\sharp$, $h(X_\sharp) = X_0$. Denote $X'' = \{u \in X' : (\forall k > 0)(u_k = m \Rightarrow \lceil u_0 \rceil \in \{0, \sharp\}\ \ \&\ \ (\forall 0 < j < k)(u_j = m))\}$. Thus m's are allowed only at the beginning of the input tape and only if $\lceil u_0 \rceil \in \{-1, 1\}$. X'' is a closed and invariant set, and there is a factorization $\chi : (X', h) \to (X'', h)$ defined by
$\chi(u)_k = m - 1$ if $k > 0$, $u_k = m$, and $\lceil u_0 \rceil \in \{0, \sharp\}$ or $(\exists 0 < j < k)(u_j < m)$
$\chi(u)_k = u_k$ otherwise. We construct a projection $\psi : (X'', h) \to (I, g)$. Let $g_0 :$ œ $- 1, g(0)$! $\to$ œ $- 1, 0$!, $g_1 :$ œ $- 1, g(0)$! $\to$ œ$0, 1$! be the two inverse

functions for g. Then define by induction

$$
\begin{array}{lll}
\psi(u)=\varphi(\ldots u_{-4}u_{-2}\lfloor u_0\rfloor u_1u_3\ldots) & \text{if} & \lceil u_0\rceil=0\\
\psi(u)=g_0^k\psi h^k(u) & \text{if} & \lceil u_0\rceil=-1 \text{ and } \lceil h^k(u)_0\rceil=0\\
\psi(u)=g_1\psi h(u) & \text{if} & \lceil u_0\rceil\in\{1,\sharp\}\\
\psi(u)=-1 & \text{if} & u\in X_{-\infty}\\
\psi(u)=1 & \text{if} & u\in X_{\infty}
\end{array}
$$

Then clearly $\psi h(u)=g\psi(u)$ whenever $\lceil u_0\rceil\neq 0$. For $\lceil u_0\rceil=0$ observe that both $\psi h(u)$ and $g\psi(u)$ belong to œ$p,1$!. Since g is decreasing on œ$p,1$! it suffices to show $g\psi h(u)=g^2\psi(u)$. Since $\lceil h(u)_0\rceil\neq 0$, we have $g\psi h(u)=\psi h^2(u)=$
$\psi h(\ldots u_{-2},u_{-1},f_B(\lfloor u_0\rfloor),(f_A(\lfloor u_0\rfloor,u_1),\sharp),u_2,u_3,\ldots)=$
$\psi(\ldots u_{-2},u_{-1},f_B(\lfloor u_0\rfloor),f_Bf_A(\lfloor u_0\rfloor,u_1),(f_A(\lfloor u_0\rfloor,u_1),0),u_3,u_4\ldots)=$
$\varphi(\ldots u_{-4},u_{-2},f_B(\lfloor u_0\rfloor),f_A(\lfloor u_0\rfloor,u_1),u_3,u_5\ldots)=$
$\varphi f(\ldots u_{-4}u_{-2}\lfloor u_0\rfloor u_1u_3\ldots)=g^2\varphi(\ldots u_{-4}u_{-2}\lfloor u_0\rfloor u_1u_3\ldots)=g^2\psi(u)$. $\Box$

Lemma 2 *Let $g: I\to I$ be an increasing continuous function such that $g(x)>x$ iff $x\in(-1,1)$. Then (I,g) is a factor of a finite automaton.*

$$
\begin{tikzcd}
\cdots\ \underline{0}01 \arrow[r, "f"] \arrow[d, "\varphi"'] & \underline{0}1 \arrow[r, "f"] \arrow[d, "\varphi"'] & 0\underline{1} \arrow[r, "f"] \arrow[d, "\varphi"] & 01\underline{1}\ \cdots \arrow[d, "\varphi"]\\
\cdots\ I_{-1} \arrow[r, "g"] & I_0 \arrow[r, "g"] & I_1 \arrow[r, "g"] & I_2\ \cdots
\end{tikzcd}
$$

Proof: For $k\in Z$ denote $I_k=$ œ$g^k(0),g^{k+1}(0)$! (f is a homeomorphism), $I_{-\infty}=\{-1\}$, $I_\infty=\{1\}$. Define $A=Q=B=\{0,1\}$, $f_B(i)=i$, and $f_A(i,j)=\max\{i,j\}$. We have closed sets

$$
\begin{array}{lll}
X_{-\infty} & =\{u\in X:(\forall i\geq 0)(u_i=0)\} & \\
X_k & =\{u\in X:\min\{i\geq 0:u_i=1\}=-k+1\}, & k\leq 0\\
X_k & =\{u\in X:\max\{i\leq 0:u_i=0\}=-k\}, & k>0\\
X_\infty & =\{u\in X:(\forall i\leq 0)(u_i=1)\} &
\end{array}
$$

For $k\in Z$ define a projection $\pi_k: X\to 2^N$ by $\pi_k(u)_i=0$ if $0\leq i<-k$, $\pi_k(u)_i=u_{-k-i-1}$ if $i\geq\max\{0,-k\}$. (Thus $\pi_{-1}(u)=(0,u_{-1},u_{-2}\ldots)$, $\pi_1(u)=(u_{-2}u_{-3}\ldots)$. It is easy to see that $\pi_{k+1}f(u)=\pi_k(u)$ for $u\in X_k$. Since I_0 is compact, there exists a factorization $\psi: 2^N\to I_0$. Define $\varphi(u)=g^k\psi\pi_k(u)$ for $u\in X_k$, $\varphi(u)=-1$ for $u\in X_{-\infty}$, and $\varphi(u)=1$ for $u\in X_\infty$. Then $\varphi:(X,f)\to(I,g)$ is a factorization. $\Box$

Proposition 9 *Let (I, g) be a unimodal system with $g(0) \leq 0$. Then it is a factor of a finite automaton.*

Proof: There is at most one fixed point in $(-1, 0!$. If there is none, we get the result by Theorem 2. Suppose there is a fixed point $-1 < p \leq 0$. Then $I_0 =$ œ $- 1, p!$ and $I_1 =$ œ$p, 0!$ are invariant sets for g. (I_0, g) is a factor of a finite automaton by Lemma 2. (I_1, g) is a factor of a finite automaton by Theorem 2. Their sum is a factor of a finite automaton by Proposition 4. But (œ $- 1, 0!, g)$ is a factor of this sum, so there is a factorization φ : $(X, f) \to$ (œ $- 1, 0!, g)$. Define an automaton (X', f') by $A' = A$, $B' = B$, $Q' = Q \times 2$, $f'_A(q, i) = (f_A(q), 0)$, $f'_B(q, i) = f_B(q)$. Define a factorization φ' : $(X', f) \to (I, g)$ by $\varphi'(u) = \varphi(...u_{-1}, q, u_1...)$ if $u_0 = (q, 0)$, $\varphi'(u) = g_1^{-1}\varphi f(u)$ if $u_0 = (q, 1)$. □

Theorem 4 *Let (I, g) be a symmetric unimodal system with a finite number of periodic points. Then it is a factor of a finite automaton.*

Proof: If $g(0) > 0$, then there exists $\mathcal{R}(g)$ and I is its invariant set (otherwise g would have infinite number of periodic points). If $\mathcal{R}g(0) > 0$, we construct further renormalizations. Since g has only a finite number of periodic points, there exists $k \geq 0$, with $\mathcal{R}^k g(0) \leq 0$. By Proposition 9, $\mathcal{R}^k g$ is a factor of a finite automaton, and by Theorem 3, so is g. □

Proposition 10 *Let (I, g) be a symmetric unimodal system with $g(0) = 1$. Then g is a factor of a finite automaton.*

Proof: Define a closed cover $\alpha = \{$œ$-1, 0!,$ œ$0, 1!\}$, and a mapping $\varphi : 2^N \to I$ by $\varphi(u) = x \in I$ iff $(\forall i \geq 0)(g^i(x) \in \alpha_i\}$. It follows from the results in Collet and Eckmann [3], that φ is a surjective mapping, and $\varphi : (2^N, \sigma) \to (I, g)$ is a factorization. □

Corollary 3 *Let (I, g) be a symmetric unimodal system at a band-merging bifurcation, i.e. $\mathcal{R}^k g(0) = 1$ for some k. Then it is a factor of a finite automaton.*

Example 3 *A symmetric unimodal system at the common limit of period doubling and band merging bifurcations has not chaotic limits.*

Proof: There exists an infinite closed invariant set $Y \subseteq I$, which is topologically transitive but has no periodic point. The set of periodic points $P \subseteq I$ is infinite too, and its closure is $P \cup Y$. Moreover, for any finite $A \subseteq P$ there exists $U \subseteq I$ with $Y \subseteq U$ and $\omega(U) \cap A = 0$ (see Collet and Eckmann [3] and Falconer [6] for more details). Let $y \in Y$ and $y \in A \subseteq I$. If $\omega(A) \subseteq Y$, then it has no periodic point. If $\omega(A) \cap P \neq 0$, then $\omega(A)$ is not topologically transitive. In either case $\omega(A)$ is not chaotic. $\square$

References

[1] B.Balcar, P.Simon: Appendix on general topology. Handbook of Boolean Algebras (J.D.Monk and R.Bonnet, eds.), Elsevier Science Publishers B.V., (1989) 1241.

[2] J.Brooks, G.Cairns, G.Davis, P.Stacey: On Devaney's definition of chaos. The American Mathematical Monthly, 99,4 (1992) 332.

[3] P.Collet, J.P.Eckmann.: Iterated Maps on the Interval as Dynamical Systems. Birkhauser, Basel (1980).

[4] J.P.Crutchfield, K.Young: Computation at the onset of chaos. in: Complexity, Entropy and the Physics of Information, SFI Studies in the Sciences of Complexity, vol VIII (W.H.Zurek, ed.), Addison Wesley (1990) 223.

[5] R.L.Devaney: An Introduction to Chaotic Dynamical Systems. Addison-Wesley, Redwood City (1989).

[6] K.J.Falconer: The Geometry of Fractal Sets. Cambridge University Press, Cambridge (1985).

[7] J.E.Hopcroft, J.D.Ullmann: Introduction to Automata Theory, Languages, and Computation. Addison-Wesley, Menlo Park (1990).

[8] Ch.G.Langton: Life at the edge of chaos. in: Artificial Life II. SFI Studies in the Sciences of Complexity, vol. X (Ch.G.Langton, Ch.Taylor, J.D.Farmer, S.Rasmunsen, eds.) Addison-Wesley, Redwood City (1992).

[9] M.Shub: Global Stability of Dynamical Systems. Springer-Verlag, Berlin (1987).

[10] J.Troll: Formal language characterization of transitions to chaos of truncated horseshoes. Technische Universität Berlin, SFB 288, No 13, 1992.