The net output process of a system with infinitely many queues

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April 25, 1997

Abstract

We study a system of infinitely many queues with Poisson arrivals and exponential service times. Let the net output process be the difference between the departure process and the arrival process. We impose certain ergodicicity conditions on the underlying Markov chain governing the customer path. These conditions imply the existence of an invariant measure under which the average net output process is positive and proportional to the time. Starting the system with that measure we prove that the net output process is a Poisson process plus a perturbation of order 1. This generalizes a classical theorem (Burke (1956), Kelly (1979)) asserting that the departure process is a Poisson process.

Keywords and phrases. Queuing networks. Zero range process. Burke's theorem. Departure process. Net output process.

AMS-MOS 1991 Classification. 60K35, 60K25, 90B22, 90B15.

Running head. Ouput of infinitely many queues.

1. Introduction. We consider a system of infinitely many queues with arrivals and departures. To describe it let $S = \{-1, 0, 1, \ldots\}$ and $p(x, y)$

be the transition rates of a continuous time Markov chain on S. Consider that at each $x \geq 0$ there is a queue formed by a nonnegative number of indistinguishable customers. If this number is positive, we can think that one of the customers is being served and the others are waiting for service. The service time of queue $x \geq 0$ is exponential with rate $\mu(x) = \sum_y p(x, y)$. Once the customer is served it jumps to queue y with probability $p(x, y)/\mu(x)$. "Queue" -1 is out of the system and it is considered as a queue for notational convenience. We assume that there are infinitely many customers at -1 at all times. This implies that customers are entering from the outside of the system to queue $y \geq 0$ according to a Poisson process of rate $p(-1, y)$. This system is a particular case of the called zero range process with an external source/sink of customers. Under suitable conditions on the rates $p(x, y)$ ((5) below) it can be shown that the process is well defined, at least for a class of initial configurations. The techniques of Andjel (1982) are appropriate to show existence under this condition. We call η_t the resulting process in $I\!N^{I\!N}$ where for $x \geq 0$, $\eta_t(x)$ is the number of customers at queue x by time t.

We are interested in the case when in average there is a positive net flux of customers out of the system. In order to guarantee this we ask the transition matrix $p(x, y)$ to satisfy some conditions. One of them is ergodicity. In other words, we ask —in (6) below— the transition matrix $p(x, y)$ to accept a unique finite invariant measure m on S satisfying $m(-1) = 1$. Furthermore we ask $1 > m(x)$ for all $x \geq 0$. We also assume that there exists a sigma finite measure ρ on S satisfying

$$\rho(y) = \sum_{x \geq -1} \rho(x) p(x,y), \quad y \geq 0, \qquad \rho(-1) = 1$$

such that $1 > \rho(x) > m(x)$, $x \geq 0$. Let ν_ρ be the product measure on Ω whose marginals are geometric with parameter $\rho(x)$. In other words, for any finite set $A \subset I\!N$,

$$\nu_\rho(\eta(x) = k_x, x \in A) = \prod_{x \in A} \rho(x)^{k_x} (1 - \rho(x)) \tag{1}$$

If ρ satisfies the conditions above then ν_ρ is invariant for the system, as a consequence of Proposition 1 below. This was proven by Jackson (1963) for finite networks, by Andjel (1982) for infinite conservative systems and by Ferrari (1986) for a special case of $p(x,y)$ satisfying our conditions.

We assume other mild conditions on the functions $p(x,y)$, $m(x)$ and $\rho(x)$ stated in (8) and (9) below. Let the departure process D_t be the number of customers leaving the system in $[0,t]$ and the arrival process A_t be the number of customers entering the system in $[0,t]$. The departure (respectively arrival) process at time t counts the number of customers jumping to (respectively from) -1 in the interval $[0,t]$. Let the net output process be $X_t = D_t - A_t$. Burke (1956) showed that for one queue in equilibrium with Poisson arrivals and exponential service times, the departure process D_t is a Poisson process. This result was extended to systems with a finite number of queues, see Kelly (1979) and references therein. The extension to infinitely many queues is immediate:

Theorem 1 *Let $p(x,y)$ be the transition rates of a continuous Markov chain on S satisfying (5) to (7) of next section. Let η_t be a system of queues with rates $p(x,y)$ whose generator is given by (10). Assume that the initial distribution ν_ρ is given by (1). Then the departure process D_t is a Poisson process of parameter $\sum_{x\geq 0}\rho(x)p(x,-1)$.*

To prove Theorem 1 one follows Kelly's (1979) idea: construct η_t^*, the reverse process of η_t with respect to the invariant measure ν_ρ and verify that in distribution $D_t = A_t^*$, the number of customers entering the reverse process. By construction A_t^* is a Poisson process with rate $\sum_{x\geq 0}\rho(x)p(x,-1)$.

If $p(-1,x) \equiv 0$, then $X_t = D_t$ is a Poisson process. When $p(-1,y)$ is not identically zero, customers enter into the system producing memory which prevent X_t from being a Poisson process. We then have the following theorem, which is our main result.

Theorem 2 *Let $p(x,y)$ be the transition rates of a continuous Markov chain on S satisfying conditions (5) to (9) of next section. Let η_t be the system of infinitely many queues on $I\!N^{I\!N}$ with rates $p(x,y)$, whose generator is given by (10) of next section. Assume that the initial distribution of η_t is the invariant measure ν_ρ defined in (1). Then the net output process X_t can be expressed as the following sum*

$$X_t = R_t + B_t, \tag{2}$$

where R_t is a Poisson process with rate

$$\sum_{x\geq 0}[\rho(x) - m(x)]p(x,-1) \tag{3}$$

and B_t is a mean zero process such that the distribution of $|B_t|$ decays exponentially, uniformly in t, that is, there are positive constants C and β such that

$$P(|B_t| > x) \leq Ce^{-\beta x},\ \ x,t \geq 0. \tag{4}$$

This result is not trivial only for systems with infinitely many queues. For finite systems there is only one invariant measure and one writes $D_t - A_t = |\eta_0| - |\eta_t|$, where $|\eta_t|$ is the total number of customers in the system by time t, which is of order 1 as in equilibrium η_t has the same distribution as η_0. In this case R_t is the trivial Poisson process of rate 0 and $B_t = |\eta_0| - |\eta_t|$.

We sketch now the proof of Theorem 2 in which we also use the reverse process but in a somewhat more subtle way. The main point is to distinguish between two type of customers: those that enter the system from -1 that we call black customers and the others that we call red customers. We can think that the red customers come from "infinity". The motion of this two species system is based on the assumption that black and red customers behave in the same way in the queues $x \geq 0$. When the server at queue x finishes a service —this happens at rate $\mu(x)$— it chooses uniformly one customer among the ones in its queue. Hence the probability of choosing a black one is the quocient between the number of black customers and the total number of customers in that queue. Then this customer jumps to queue y with probability $p(x,y)/\mu(x)$. The only difference between black and red customers occurs at the boundary: only black customers enter into the system from -1 and they do so to queue y at rate $p(-1,y)$. We call (σ_t,ξ_t) the resulting system, where $\sigma_t(x)$ (respectively $\xi_t(x)$) is the number of black (respectively red) customers at queue x by time t. The net ouput processes of black and red customers are denoted B_t and R_t respectively. By construction $\eta_t = \sigma_t + \xi_t$ coordinatewise and $X_t = B_t + R_t$. This means that if we disregard colors we recover the original system η_t.

Next step is to find the invariant measure ν_2 for (σ_t,ξ_t) and the reverse process $(\sigma_t,\xi_t)^* = (\sigma_t^*,\xi_t^*)$ with respect to ν_2. We simply exibit them and show the assertion. Call R_t^* (respectively B_t^*), the net input process of red (respectively black) customers into the reverse system. In the reverse process the red customers do not leave the system and R_t^* is a Poisson process. On the other hand, by mass conservation B_t^* is equal to $|\sigma_0| - |\sigma_t|$, that has the required properties under our hypothesis. To conclude the proof of Theorem 2 we use the same idea of the proof of Theorem 1. By reversing the process $B_t = B_t^*$ and $R_t = R_t^*$ in distribution.

The proof is quite straigthforward but the path we followed to guess the invariant measure and the reverse process for the two species system was not so direct. We explain briefly our reasoning now. Following De Masi et al. (1988) one guess η_t^*, the reverse process of η_t with respect to ν_ρ. This turns out to be a system with infinitely many queues as η_t but with transition rates given by

$$p^*(x,y) = \frac{p(y,x)\rho(y)}{\rho(x)}$$

(and service rates $\mu^*(x) = \sum_y p^*(x,y) = \mu(x)$). Since $p(x,y)$ accepts a finite invariant measure, each customer of η_t has a drift towards -1 and once it arrives to -1 we can think that it will stay there. Hence in the original process η_t there is a general flux of customers from infinity to the exit at -1. This implies that in the reverse process η_t^*, there is a flux of customers from -1 to infinity. But a fraction of these customers fail to go to infinity and come back to -1. One can construct the reverse process by labeling the customers and keeping track of each customer. Take the labeled reverse process and make a partition of the trajectory space by specifying which are the customers that will eventually leave the system by -1. These are called black* customers and the remained customers are called red* . The customers entering after time zero are specified according to the queue they arrive. If the probability that a customer at queue x be specified black* is given by $m(x)/\rho(x)$, the probability that a chain on S with rates $p^*(x,y)$ starting at x hits eventually -1, then the process conditioned in this way to the future is still Markovian. The transition rates for black* and red* customers are different and given by the corresponding conditioned chains defined in (13) below. ¿From this observation one construct the invariant measure for (σ_t^*, ξ_t^*) by choosing a η^* according to ν_ρ and then labeling each customer at x

independently as black with probability $m(x)/\rho(x)$ and red with probability $1-m(x)/\rho(x)$. Call the resulting measure ν_2. When one looks for the reverse process of (σ_t^*, ξ_t^*) with respect to ν_2, one finds (σ_t, ξ_t). Since the black* and red* transition rates are different, the process $\sigma_t^* + \xi_t^*$ is not Markovian, so it can not be the same as η_t^*. Nevertheless it is true that the law of $\sigma_t^* + \xi_t^*$ starting with ν_2 is the same as the law of η_t^* starting with ν_ρ.

In Section 2 we state the assumptions needed to show Theorem 2 and define the process. In Section 3 we introduce the two species process and show Theorem 2. In Section 4 we give some concluding remarks and extensions.

2. A system of infinitely many queues.

Let $p(x,y)$ be the transition rates of a (continuous time) Markov chain in the state space $S = \{-1, 0, 1, \dots\}$ satisfying

1. Bounded total rates.

$$\sup_y \sum_x p(x,y) < \infty, \ \sup_y \sum_x p(y,x) < \infty; \tag{5}$$

2. The chain is ergodic and the invariant measure of -1 is strictly bigger than the measure of any other $x \geq 0$. In other words, the chain has a unique invariant finite measure m such that

$$m(y) = \sum_{x \geq -1} m(x)p(x,y), \quad y \geq -1, \quad m(-1) = 1 > m(x), \ x \geq 0. \tag{6}$$

3. There is a measure ρ such that

$$\rho(x)\mu(x) = \textstyle\sum_y \rho(y)p(y,x), \ x \geq 0, \tag{7}$$

$$\rho(-1) = 1 > \rho(x) > m(x),\ x \geq 0,$$

where $\mu(x) = \sum_y p(x,y)$;

4. The measures m and ρ satisfy

$$\sum_{x \geq 0} \frac{m(x)}{1 - \rho(x) + m(x)} < \infty; \tag{8}$$

5. we further require

$$\sup_y\ m(y) \sum_x p(y,x)/m(x) < \infty,$$
$$\sup_y (\rho(y) - m(y)) \sum_x p(y,x)/(\rho(x) - m(x)) < \infty. \tag{9}$$

Remark. We see below that Condition (5) is sufficient to show the existence of the process. Condition (6) is necessary in order to have an invariant measure concentrating mass on configurations with a finite number of customers with probability one. Condition (7) guarantees the existence of an invariant measure with infinitely many customers. Condition (8) is necessary and sufficient to get the exponential moment of B_t. Condition (9) is sufficient for the existence of the reverse process mentioned in the introduction. It is automatically satisfied if $p(x,y)$ is of finite range.

Let $(\eta_t)_{t \geq 0}$ be the infinite system of queues indexed by $I\!N = \{0, 1, \ldots\}$ with servicing rates $\mu(x)$ and transition rates $p(x,y)$ given above. The state -1 is regarded as out of the system, from where and to where customers enter and exit the system. The process η_t counts the number of customers at each queue at time t. It takes values in $\Omega = I\!N^{I\!N}$ and its generator is given by

$$\mathbf{L}f(\eta) = \sum_{x,y \geq -1} 1\{\eta(x) > 0\} p(x,y) [f(\eta^{xy}) - f(\eta)], \tag{10}$$

where η is the initial state of the system ($\eta(-1)$ is defined as $+\infty$), η^{xy} is defined by

$$\eta^{xy}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y \\ \eta(x) - 1, & \text{if } z = x \\ \eta(y) + 1, & \text{if } z = y, \end{cases}$$

and f is a local function in Ω. Condition (5) insures the existence of this process. This can be proven as Andjel (1982) does for an infinite system which conserves the number of customers.

Since the customers are indistinguishable, we can assume the following rule for each queue. After a service period is finished, the server chooses the customer that has been served uniformly among the ones in his queue by that time.

2. The two species system.

For the proof of Theorem 1 we introduce the process $(\sigma_t, \xi_t) \in \Omega_2 = \Omega \times \Omega$ as follows. At each queue we have two types of customers: black and red. The number of black (respectively red) customers at queue x at time t is $\sigma_t(x)$ (respectively $\xi_t(x)$). Let the system evolve with the provision that only black customers enter it and they do so at site y with rate $p(-1, y)$. The evolution in the system is the following. Given that a service time finished at queue x, the probability that a black customer has been served is the number of black customers in x over the total number of customers in x. The service corresponds to a red customer with the complementar probability. Then the chosen customer jumps to queue y with probability $p(x,y)/\mu(x)$. Notice that this specification implies that our system is not a classical system of queues with two types of customers. In a classical system the customer is chosen according to some rule (that may be random as our rule) at the beginning of the service time interval. In our system the customer is chosen at the end of this interval. In this way a customer that has arrived after the starting of the service time interval can be the one served by the end of the interval. The generator of this process is given by

$$\mathbf{L}_2 f(\sigma,\xi) = \sum_{x,y\geq -1} p(x,y) \quad \Big\{ \tfrac{\sigma(x)}{\sigma(x)+\xi(x)} [f(\sigma^{xy},\xi) - f(\sigma,\xi)] \\ + \quad \tfrac{\xi(x)}{\sigma(x)+\xi(x)} [f(\sigma,\xi^{xy}) - f(\sigma,\xi)] \Big\},$$

where (σ,ξ) is the initial state of the system (with $\sigma(-1) = +\infty$ and $\xi(-1) = 0$), σ^{xy} and ξ^{xy} are defined analogously as η^{xy} with the exception $\xi^{x,-1}(-1) \equiv 0$. f is a local function in Ω_2 and we adopt the convention that $\infty/\infty = 1$ and $0/0 = 0$. Then we have the following lemma whose proof is straightforward.

Lemma 1 *Let the process (σ_t,ξ_t) have initial configuration (σ,ξ). Then $\sigma_t + \xi_t$ has the same distribution as η_t with initial configuration $\sigma+\xi$. In other words, if f is a local function defined on Ω, then $L_2 f(\sigma+\xi) = Lf(\sigma+\xi)$.*

Define a measure ν_2 on Ω_2 as follows. Let η be chosen from ν_ρ. Let $\alpha(x) = m(x)/\rho(x)$. It is easy to check that $\alpha(x)$ is the probability that a

Markov chain on S with transition rates $p^*(x,y)$ hits eventually -1. For all $x \geq 0$, each η customer in queue x will be called black, with probability $\alpha(x)$ or red, with probability $1-\alpha(x)$, independently of each other and of other queues. We call ν_2 the resulting measure. From Lemma 1 and the construction of ν_2 we have

$$\int d\nu_2(\sigma,\xi) L_2 f(\sigma+\xi) = \int d\nu_\rho(\eta) L f(\eta) \tag{11}$$

Define new Markov transition rates $p^*(x,y)$ on S as follows

$$p^*(x,y) = \frac{p(y,x)\rho(y)}{\rho(x)} \tag{12}$$

and a new system of queues η_t^* in Ω with transition rates $p^*(x,y)$ and service rates $\mu^*(x) = \sum_y p^*(x,y)$. (Notice that $\mu^*(x) = \mu(x),\ x \in I\!N$.)

Like in the system η_t, we distinguish between two kinds of customers in this new system, which we call black* and red*. Consider the Markov rates defined on S as

$$\begin{aligned} p_b^*(x,y) &= p^*(x,y)\alpha(y)/\alpha(x), \\ p_r^*(x,y) &= p^*(x,y)(1-\alpha(y))/(1-\alpha(x)), \\ p_r^*(-1,y) &= p^*(-1,y)(1-\alpha(y)). \end{aligned} \tag{13}$$

If we consider a Markov chain on S with transition rates p^* and condition it to be eventually absorbed at -1 we get a new Markov chain with transition rates given by p_b^*. If we condition to non absorption we get rates p_r^*. Notice that $\sum_y p_b^*(x,y) = \sum_y p_r^*(x,y) = \mu^*(x) = \mu(x)$. We define now a process (σ^*_t, ξ^*_t) with the following evolution. At rate $\mu^*(x)$, a customer is selected in queue x uniformly among the $\sigma^*_t(x) + \xi^*_t(x)$ customers. If it is a black* one, then the customer jumps to y with probability $p^*_b(x,y)/\mu^*(x)$ and if it is a red* one, then the same jump is with probability $p^*_r(x,y)/\mu^*(x)$. The black* customers enter the system to queue y at rate $p^*_b(-1,y)$ and red* ones do it at rate $p^*_r(-1,y)$. Let $\sigma^*_t(x)$ and $\xi^*_t(x)$ respectively count the number of black* and red* customers in queue x at time t. Then, (σ^*_t, ξ^*_t) takes values in Ω_2 and has the following generator.

$$\begin{aligned}\mathbf{L}^*_2 f(\sigma,\xi) = \sum_{x,y\ge -1} \quad & \Big\{ p^*_b(x,y)\tfrac{\sigma(x)}{\sigma(x)+\xi(x)}[f(\sigma^{xy},\xi) - f(\sigma,\xi)] \\ + \quad & p^*_r(x,y)\tfrac{\xi(x)}{\sigma(x)+\xi(x)}[f(\sigma,\xi^{xy}) - f(\sigma,\xi)]\Big\},\end{aligned}$$

where $\frac{\sigma(-1)}{\sigma(-1)+\xi(-1)} = \frac{\xi(-1)}{\sigma(-1)+\xi(-1)} = 1$. Condition (9) insures the existence of this process.

Here is the main ingredient in the proof of Theorem 1.

Proposition 1 *The processes (σ_t, ξ_t) and (σ^*_t, ξ^*_t) are the reverse of one another with respect to ν_2.*

Proof of Theorem 1: From Proposition 1 and (11) it follows that ν_ρ is invariant for the process η_t. From Lemma 1, we have $\eta_t = \sigma_t + \xi_t$ and

$$X_t = B_t + R_t \tag{14}$$

where B_t and R_t are the net output process of black and red customers of the process (σ_t, ξ_t) under initial distribution ν_2. Proposition 1 implies

$$B_t = B^*_t, \quad R_t = R^*_t \tag{15}$$

in distribution, where B_t^* (respectively R_t^*) is the net input process of black* (respectively red*) customers for the reverse system. Since red* customers do not leave the system, R_t^* is a Poisson process of rate given by (3). On the other hand

$$B_t^* = |\sigma_t^*| - |\sigma_0^*|,$$

which has mean zero since σ_t^* and σ_0^* have the same distribution. This and the exponential decay of $|\sigma_0^*|$ completes the proof, the decay being the consequence of the fact that $|\sigma_0^*|$ is the series of independent geometric random variables with summable parameters (it is here that (8) enters).

Proof of Proposition 1: We want to show that

$$\int f\mathbf{L}_2 g d\nu_2 = \int g\mathbf{L}_2^* f d\nu_2, \tag{16}$$

for local functions f and g on Ω_2. The above equality will be established if we prove the following equalities

$$\begin{aligned} &p(x,y)\int \frac{\sigma(x)}{\sigma(x)+\xi(x)} f(\sigma^{xy},\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi) \\ &= p_b^*(y,x)\int \frac{\sigma(y)}{\sigma(y)+\xi(y)} f(\sigma,\xi)g(\sigma^{yx},\xi)d\nu_2(\sigma,\xi) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &p(x,y)\int \frac{\sigma(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi^{xy})g(\sigma,\xi)d\nu_2(\sigma,\xi) \\ &= p_r^*(y,x)\int \frac{\sigma(y)}{\sigma(y)+\xi(y)} f(\sigma,\xi)g(\sigma,\xi^{yx})d\nu_2(\sigma,\xi), \end{aligned} \tag{18}$$

for $x, y \in S$ and

$$
\begin{aligned}
&\textstyle\sum_{x,y} p(x,y) \int 1\{\sigma(x)+\xi(x)>0\} f(\sigma,\xi) g(\sigma,\xi) d\nu_2(\sigma,\xi) \\
&= \textstyle\sum_{x,y} p_b^*(x,y) \displaystyle\int \frac{\sigma(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi) g(\sigma,\xi) d\nu_2(\sigma,\xi) \\
&+ \textstyle\sum_{x,y} p_r^*(x,y) \displaystyle\int \frac{\xi(x)}{\sigma(x)+\xi(x)} f(\sigma,\xi) g(\sigma,\xi) d\nu_2(\sigma,\xi),
\end{aligned} \tag{19}
$$

where the above sums are done in the set $\{x, y : x \text{ or } y \in [-1, \ldots, N]\}$, where N is big enough so that $[0, N]$ contains the supports of f and g. To prove (17) and (18), we can take $f,\ g$ of the type

$$\prod_{v \in A} 1\{\sigma(v) = k_v\} \prod_{w \in B} 1\{\xi(w) = l_w\},$$

where A and B are arbitrary finite subsets of $I\!N$ and k_v and l_w are arbitrary nonnegative integers. Since ν_2 is a product measure, it is then sufficient to consider the cases

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x + 1, \sigma(y) = k_y, \xi(x) = l_x, \xi(y) = l_y\}$$

$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \sigma(y) = k_y + 1, \xi(x) = l_x, \xi(y) = l_y\}$$

and

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x, \sigma(y) = k_y, \xi(x) = l_x + 1, \xi(y) = l_y\}$$

$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \sigma(y) = k_y, \xi(x) = l_x, \xi(y) = l_y + 1\}$$

respectively for (17) and (18), when $x, y \in I\!N$. When $y = -1$, take

$$f(\sigma,\xi) = 1\{\sigma(x) = k_x + 1, \xi(x) = l_x\}$$

$$g(\sigma,\xi) = 1\{\sigma(x) = k_x, \xi(x) = l_x\}$$

and

$$f(\sigma,\xi)=1\{\sigma(x)=k_x,\xi(x)=l_x+1\}$$
$$g(\sigma,\xi)=1\{\sigma(x)=k_x,\xi(x)=l_x\}$$

respectively for (17) and (18). When $x=-1$, (18) is trivially satisfied (both sides vanish), for (17) one should take

$$f(\sigma,\xi)=1\{\sigma(y)=k_y,\xi(y)=l_y\}$$
$$g(\sigma,\xi)=1\{\sigma(y)=k_y+1,\xi(y)=l_y\}.$$

For all these cases, it is a simple calculation to verify (17) and (18). The equality (19) is easily verified for the sums taken in the set $\{0\le x\le N,\ y\ge -1\}$ from the equalities

$$\sum_y p(x,y)=\sum_y p_b^*(x,y)=\sum_y p_r^*(x,y),\ x\ge 0.$$

So it suffices to show that

$$\begin{aligned}&\int f(\sigma,\xi)g(\sigma,\xi)\sum_y p(-1,y)d\nu_2(\sigma,\xi)\\ &+\textstyle\sum_{x>N}\sum_{y\le N}p(x,y)\displaystyle\int 1\{\sigma(x)+\xi(x)>0\}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi) \qquad (20)\\ &=\int fg\sum_y p_b^*(-1,y)d\nu_2+\int fg\sum_y p_r^*(-1,y)d\nu_2\\ &+\textstyle\sum_{x>N}\sum_{y\le N}p_b^*(x,y)\displaystyle\int\frac{\sigma(x)}{\sigma(x)+\xi(x)}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi)\\ &+\textstyle\sum_{x>N}\sum_{y\le N}p_r^*(x,y)\displaystyle\int\frac{\xi(x)}{\sigma(x)+\xi(x)}f(\sigma,\xi)g(\sigma,\xi)d\nu_2(\sigma,\xi). \qquad (21)\end{aligned}$$

The second term in the l.h.s. and the third and fourth terms in the r.h.s. equal respectively

$$\int fg \sum_{x>N} \sum_{y \leq N} \rho(x)p(x,y)d\nu_2,$$
$$\int fg \sum_{x>N} \sum_{y \leq N} \alpha(x)\rho(x)p_b^*(x,y)d\nu_2,$$
$$\int fg \sum_{x>N} \sum_{y \leq N} (1-\alpha(x))\rho(x)p_r^*(x,y)d\nu_2,$$

so (21) can be rewritten (when $\int fgd\nu_2 \neq 0$, otherwise we are done) as

$$\begin{aligned} & \textstyle\sum_y p(-1,y) + \sum_{x>N} \sum_{y \leq N} \rho(x)p(x,y) \\ = \; & \textstyle\sum_y p_b^*(-1,y) + \sum_y p_r^*(-1,y) + \sum_{x>N} \sum_{y \leq N} \rho(y)p(y,x). \end{aligned}$$

So, it suffices to prove the equality

$$\begin{aligned} & \textstyle\sum_{x>N} \sum_{y \leq N} \rho(x)p(x,y) - \sum_{x>N} \sum_{y \leq N} \rho(y)p(y,x) \\ & \qquad = \textstyle\sum_y \rho(y)p(y,-1) - \sum_y p(-1,y), \end{aligned} \tag{22}$$

which we do by rewriting the l.h.s. as

$$\sum_{x \geq -1} \sum_{y \leq N} \rho(x)p(x,y) - \sum_{x \geq -1} \sum_{y \leq N} \rho(y)p(y,x).$$

This equals the r.h.s. of (22) plus

$$\sum_{x \geq -1} \sum_{0 \leq y \leq N} \rho(x)p(x,y) - \sum_{x \geq -1} \sum_{0 \leq y \leq N} \rho(y)p(y,x)$$

and the equality of these two terms follows from (7).

4. Final remarks.

The method of proof of Theorem 1 allows one to show that the departure process from queue y is a Poisson process of parameter $\rho(y)p(y,-1)$. Furthermore the departure processes from different queues are independent (Kelly (1979)). One is tempted to extend the result to the net flux out of queue y. But this is more difficult. It is true that the departure process of red customers from different queues are independent Poisson processes with rate $[\rho(y)-m(y)]p(y,-1)$. But it is hard to control the net ouput process of black customers because they can enter in other queues different from y and the compensation that allows one to prove Theorem 2 does not occur.

One particular case of transition function satisfying (5)-(9) is the nearest neighbor asymmetric random walk

$$p(x,y)=\begin{cases} p, & \text{if } y=x+1\\ q, & \text{if } x\geq 0,\ y=x-1\\ 0, & \text{otherwise,}\end{cases}$$

where $p+q=1$ and $q>p$. In this case

$$m(x)=(p/q)^{x+1},$$

and since $p(x,y)$ is doubly stochastic we can take

$$\rho(x)=m(x)+(1-m(x))\rho,$$

where ρ is any number in the interval $(0,1)$ (Ferrari (1986)). When $p=0$ Theorem 1 applies and X_t is a Poisson process. For nearest neighbor jumps, the system of queues is isomorphic to the simple exclusion process as seen from the leftmost particle. This is mentioned for this model by Spitzer (1970) quoting an oral communication of Kesten who noticed that Theorem 1 holds for this last process. Liggett (1985) (Theorem 4.7) proves Theorem 1 by a direct computation for the exclusion process. Kipnis (1986) makes the relationship between Burke's result and Kesten's observation.

It is possible to prove Theorem 2 for the net flux of customers between two consecutive queues in a doubly infinite system. To be more precise, let ζ_t be the process on $\mathbb{Z}^{\mathbb{N}}$ with transition rates p and q respectively for customer jumps to the right (respectively left) nearest neighbor. Let ν_ρ be the product measure of geometrics of parameter ρ. Let Y_t be the net flux of customers between queues 0 and -1. Then Y_t is a Poisson process of parameter $(q-p)\rho$ plus a perturbation of order one as in Theorem 2. The case $p = 0$ is immediate. The proof for the case $p > q > 0$ is based in a coupling between the semi infinite process η_t and the double infinite process ζ_t via the asymmetric simple exclusion process. This proves a conjecture of Arratia (1983) and will appear in Ferrari and Fontes (1993).

Acknowledgements. We thank Marcos Magalhães for discussions on the background of queuing theory and for indicating us the relevant references. This work is partially supported by FAPESP, Projeto Temático, 90/3918-5 and CNPq.

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