Temperature Correlations of Quantum Spins

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Abstract

We consider isotropic XY model in the transverse magnetic field in one dimension. One can alternatively call this model Heisenberg XXO antiferromagnet. We solve the problem of evaluation of asymptotics of temperature correlations and explain the physical meaning of our result. To do this we represent quantum correlation function as a tau function of a completely integrable differential equation. This is the well-known Ablowitz-Ladik lattice nonlinear differential equation.

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XY model was introduced and studied by E. Lieb, T. Schultz and D. Mattis [1]. It describes interaction of spins 1/2 situated on 1-dimensional periodic lattice. The Hamiltonian of the model is

$$H = -\sum_n \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + h\sigma_n^z\right]. \tag{1}$$

Here σ are Pauli matrices, h is transverse magnetic field, n enumerate the cites of the lattice. At zero temperature the problem of evaluation of asymptotics of correlation functions was solved in [2,3]. Here we consider temperature correlation function

$$g(n,t) = \frac{Tr\{e^{-\frac{H}{T}} \sigma_{n_2}^+(t_2)\sigma_{n_1}^-(t_1)\}}{Tr e^{-\frac{H}{T}}}, \qquad n = n_2 - n_1, \qquad t = t_2 - t_1 \tag{2}$$

for the infinite lattice.

We consider finite temperature $0 < T < \infty$ and moderate magnetic field $0 \leq h < 2$.

We evaluated asymptotics in cases where both space and time separation go to infinity $n \to \infty$, $t \to \infty$, in some direction φ

$$\frac{n}{4t} = \cot\varphi, \qquad 0 \leq \varphi \leq \frac{\pi}{2}. \tag{3}$$

In accordance with our calculations, correlation function $g(n,t)$ decays exponentially in any direction, but the rate of decay depends on the direction. In the space like direction $0 \leq \varphi < \frac{\pi}{4}$ asymptotics is

$$g(n,t) \to C \exp\left\{\frac{n}{2\pi}\int_{-\pi}^{\pi} dp \ln\left|\tanh(\frac{h - 2\cos p}{T})\right|\right\}. \tag{4}$$

In the time like direction $\frac{\pi}{4} < \varphi \leq \frac{\pi}{2}$ asymptotics is different.

$$g(n,t) \to Ct^{(2\nu_+^2 + 2\nu_-^2)} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} dp\, |n - 4t\sin p| \ln\left|\tanh(\frac{h - 2\cos p}{T})\right|\right\}. \tag{5}$$

The values $\nu_\pm$, which define pre-exponent, are

$$\nu_+ = \frac{1}{2\pi}\ln\left|\tanh\left(\frac{h - 2\cos p_0}{T}\right)\right|$$

$$\nu_- = \frac{1}{2\pi}\ln\left|\tanh\left(\frac{h + 2\cos \rho_0}{T}\right)\right| \tag{6}$$

where $\frac{n}{4t} = \sin p_0$. Formula (5) is valid in the whole time like cone, with exception of one direction $h = 2\cos p_0$. Higher asymptotic corrections will modify formulae by factor $(1 + c(t, x))$, (c decays exponentially in the space-like region and as $t^{-1/2}$ in the time-like region). Also, it should be mentioned that the constant factor C in (4) does not depend on the direction φ, but it does depend on φ in (5).

We want to emphasize that for pure time direction $\varphi = \pi/2$ leading factor in asymptotics (exponent in (5)) was first obtained in [10].

To derive these formulae we went through few steps.

The first step: The explicit expression for eigenfunctions of the Hamiltonian (1) (see [1]) was used to represent the correlation function as a determinant of an integral operator (of Fredholm type) [8]. In order to explain we need to introduce notations. Let us consider integral operator $\hat{V}$. Its kernel is equal to

$$V(\lambda\mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{\pi(\lambda - \mu)}. \tag{8}$$

Here λ and μ are complex variables, which go along the circle $|\lambda| = |\mu| = 1$ in a positive direction. Functions $e_\pm$ are

$$e_-(\lambda) = \lambda^{-n/2} \cdot e^{-it(\lambda+1/\lambda)}\sqrt{v(\lambda)} \tag{9}$$

where

$$v(\lambda) = \left\{1 + \exp\left[\frac{2h - 2(\lambda + \frac{1}{\lambda})}{T}\right]\right\}^{-1} \tag{10}$$

and

$$e_+(\lambda) = e_-(\lambda)E(x, t, \lambda). \tag{11}$$

Here E is defined as an integral

$$E(n, t, \lambda) = \frac{1}{\pi} v.p. \int \exp\{2it(\mu + \frac{1}{\mu})\} \cdot \frac{\mu^n d\mu}{\mu - \lambda}. \tag{12}$$

It is convenient to define functions $f_\pm(\lambda)$ as solutions of the following integral equations:

$$(I + \hat{V})f_k = e_k. \tag{13}$$

Here I is an identical operator and $k = \pm$. Next we define potentials $B_{kj}(k, j = \pm)$:

$$B_{kj}(n,t) = \frac{1}{2\pi i}\int f_k(\lambda)e_j(\lambda)\frac{d\lambda}{\lambda}. \tag{14}$$

They depend on space and time variables n, t. These we shall use to define new potentials b_{kj}:

$$\begin{aligned} &b_{--}(n,t) = B_{--}(n,t) \\ &b_{++}(n,t) = B_{++}(n,t) - 2iG(n,t)B_{+-}(n,t) - G(n,t). \end{aligned} \tag{15}$$

Here we used function

$$G(n,t) = \frac{1}{2\pi i}\int \lambda^{n-1}\exp\{2it(\lambda + \frac{1}{\lambda})\}d\lambda. \tag{16}$$

Now all the notation are ready to write determinant formula for the correlation function $g(n,t)$ (see (4)):

$$g(n,t) = e^{-2iht}b_{++}(n,t)\exp\{\sigma(n,t)\}. \tag{17}$$

Here e^{σ} is a determinant of integral operator

$$\exp\{\sigma(n,t)\} = \det(1+\hat{V}). \tag{18}$$

Second step: Formulae (8)-(15) can be used to show that potentials b_{++} and b_{--} satisfy the system of nonlinear differential equations.

$$\begin{aligned} &\frac{i}{2}\frac{\partial}{\partial t}b_{--}(n,t) = (1+4b_{--}(n,t)b_{++}(n,t))(b_{--}(n+1,t)+b_{--}(n-1,t)) \\ &-\frac{i}{2}\frac{\partial}{\partial t}b_{++}(n,t) = (1+4b_{--}(n,t)b_{++}(n,t))(b_{++}(n+1,t)+b_{++}(n-1,t)). \end{aligned} \tag{19}$$

Derivation of these equations is similar to [4,5,7]. Equations (19) are completely integrable differential equations. They were first discovered by Ablowitz and Ladik [9] as an integrable discretization of the Nonlinear equation. Logarithmic derivatives of $\sigma(x,t)$ (see (18)) can be expressed in terms of solutions of the system (19).

$$\begin{aligned} \tfrac{\partial^2\sigma(n,t)}{16\partial t^2} &= 2b_{--}(n,t)b_{++}(n,t) - b_{++}(n-1,t)b_{--}(n+1,t)- \\ &-b_{--}(n-1,t)b_{++}(n+1,t) - 4b_{++}(n,t)b_{--}(n,t)[b_{++}(n-1,t)b_{--}(n+1,t)+ \\ &b_{--}(n-1,t)b_{++}(n+1,t)] \end{aligned} \tag{20}$$

$$\sigma(n+1,t)+\sigma(n-1,t)-2\sigma(n,t)=\ln[1+4b_{--}(n,t)b_{++}(n,t)] \tag{21}$$

$$\frac{\partial}{\partial t}[\sigma(n+1,t)-\sigma(n,t)]=8i[b_{++}(n+1,t)b_{--}(n,t)-b_{++}(n,t)b_{--}(n+1,t)]. \tag{22}$$

This shows that quantum correlation function g (2) can be expressed in terms of solution of the system (19).

The meaning of all these formulae is that correlation function of XY model is the τ function of Ablowitz-Ladik's differential-difference equation. This is similar to quantum Nonlinear Shrodinger equation. In the papers [5,6,7] relation of τ function and quantum correlation functions is explained in more detail.

Third step: In order to evaluate asymptotics one should solve Ablowitz-Ladik's differential equation. Initial data can be extracted from integral representations (8)-(18). We use Riemann-Hilbert problem in order to evaluate asymptotics of solution of equation (19). It is quite similar to the Nonlinear Shrodinger case [6,7].

Let us emphasize that it is quite remarkable that for the XY model (like for Bose gas, [4-7] which is defined by quantum Nonlinear Shrodinger equation) quantum correlation function is the τ function of classical completely integrable differential equation.

In the end let us explain the physical meaning of our asymptotic formula (4). We start from the expression for free energy [1]:

$$f(h)=-h-\frac{T}{2\pi}\int_{-\pi}^{\pi}dp\ln\left(1+\exp[\frac{4\cos p-2h}{T}]\right). \tag{23}$$

We emphasize the dependence on the magnetic field h. The definition of $f(h)$ is standard:

$$Tre^{-\frac{H}{T}}=\exp\{-\frac{L}{T}f(h)\}. \tag{24}$$

Here L is the length of the box. Let us use Jordan-Wigner transformation to transform correlator (2) (in the equal time case):

$$\sigma^+_{n_1}(0)\sigma^-_{n_1}(0)=\psi_{n_2}\exp\left\{i\pi\sum_{k=n_1+1}^{n_2-1}\psi^+_k\psi_k\right\}\psi^+_{n_1},\qquad \psi^+_k\psi_k=\frac{1}{2}(1-\sigma^z_k). \tag{25}$$

Here ψ_k is canonical Fermi field. We can say now that numerator in (2) differs from denominator by replacement of magnetic field $h\to h-i\pi T/2$ on the space interval $[n_1+1,n_2-1]$. This leads us to the following asymptotical expression for correlator $g(n,0)$

$$g(n,0) \to \exp\left\{ Re\frac{n}{T}\left[f(h) - f(h - \frac{i\pi T}{2}) \right] \right\}. \tag{26}$$

The reason why we wrote Re is that $i\pi$ in (25) can be replaced by $-i\pi$. It is remarkable that (26) coincides with the correct answer (4). It is also worth mentioning that to go to exponent in (5) one should replace differential $d(np)$ by expression $|d(np - t\varepsilon(p)|$ where $\varepsilon(p) = -4\cos p + 2h$ is the energy of the quasiparticle of the model.

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