BASIC COHOMOLOGY OF ASSOCIATIVE ALGEBRAS

Michel DUBOIS-VIOLETTE and Thierry MASSON

Laboratoire de Physique Théorique[1]
Bâtiment 211, Université Paris XI

F-91405 ORSAY Cedex

E-mail: flad@qcd.th.u-psud.fr and masson@qcd.th.u-psud.fr

April 25, 1997

Abstract

We define a new cohomology for associative algebras which we compute for algebras with units.

L.P.T.H.E.-ORSAY 94/10

alg-geom/9404014

[1]Laboratoire associé au C.N.R.S.

1 Introduction: Definition of the basic cohomology of an associative algebra

Let $\mathcal{A}$ be an associative algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{A}_{Lie}$ be the underlying Lie algebra (with the commutator as Lie bracket). For each integer $n \in \mathbb{N}$, let $C^n(\mathcal{A})$ be the vector space of n-linear forms on $\mathcal{A}$, i.e. $C^n(\mathcal{A}) = (\mathcal{A}^{\otimes^n})^*$. For $\omega \in C^n(\mathcal{A})$ and $\tau \in C^m(\mathcal{A})$ one defines $\omega.\tau \in C^{n+m}(\mathcal{A})$ by:

$$\omega.\tau(A_1, \ldots, A_{n+m}) = \omega(A_1, \ldots, A_n)\tau(A_{n+1}, \ldots, A_{n+m}),\ \forall A_i \in \mathcal{A}.$$

Equipped with this product, $C(\mathcal{A}) = \underset{n}{\oplus} C^n(\mathcal{A})$ becomes an associative graded algebra with unit ($C^0(\mathcal{A}) = \mathbb{K}$). One defines a differential d on $C(\mathcal{A})$ by setting for $\omega \in C^n(\mathcal{A})$, $A_i \in \mathcal{A}$

$$d\omega(A_1, \ldots, A_{n+1}) = \sum_{k=1}^{n} (-1)^k \omega(A_1, \ldots, A_{k-1}, A_k A_{k+1}, A_{k+2}, \ldots, A_{n+1}).$$

Indeed, d is the extension as antiderivation of $C(\mathcal{A})$ of minus the dual of the product of $\mathcal{A}$ and $d^2 = 0$ is then equivalent to the associativity of the product of $\mathcal{A}$. The graded differential algebra $C(\mathcal{A})$ is together with a bimodule $\mathcal{M}$ the basic building blocks of the Hochschild complex giving the Hochschild cohomology with value in $\mathcal{M}$. Here we do not want to introduce bimodules like $\mathcal{M}$. However it is well known, see below, that the cohomology of $C(\mathcal{A})$ is trivial whenever $\mathcal{A}$ has a unit. Nevertheless, there are two classical cohomologies which can be extracted from the differential algebra $C(\mathcal{A})$, namely the Lie algebra cohomology of $\mathcal{A}_{Lie}$ and the cyclic cohomology of $\mathcal{A}$. In fact let $\mathcal{S} : C(\mathcal{A}) \to C(\mathcal{A})$ and $\mathcal{C} : C(\mathcal{A}) \to C(\mathcal{A})$ be defined by

$$(\mathcal{S}\omega)(A_1, \ldots, A_n) = \sum_{\pi \in \mathcal{S}_n} \varepsilon(\pi)\omega(A_{\pi(1)}, \ldots, A_{\pi(n)})$$

and

$$(\mathcal{C}\omega)(\mathcal{A}_\infty,\ldots,\mathcal{A}_\backslash)=\sum_{\gamma\in\mathcal{C}_\backslash}\varepsilon(\gamma)\omega(\mathcal{A}_{\gamma(\infty)},\ldots,\mathcal{A}_{\gamma(\backslash)})$$

for $\omega\in C^n(\mathcal{A})$, $A_k\in\mathcal{A}$ and where $\mathcal{S}_n$ is the group of permutations of $\{1,\ldots,n\}$ and $\mathcal{C}_n$ is the subgroup of cyclic permutations. One has $\mathcal{S}\circ d=\delta\circ\mathcal{S}$ where δ is the Chevalley-Eilenberg differential so $(Im\ \mathcal{S},\delta)$ is a differential algebra whose cohomology is the Lie algebra cohomology $H(\mathcal{A}_{Lie})$ of the Lie algebra $\mathcal{A}_{Lie}$ [3],[6],[5]. On the other hand, see Lemma 3 in [4] part II, one has $\mathcal{C}\circ d=b\circ\mathcal{C}$ where b is the Hochschild differential of $C(\mathcal{A},\mathcal{A}^*)$ so $(Im\,\mathcal{C},b)$ is a complex whose cohomology is the cyclic cohomology $H_\lambda(\mathcal{A})$ of $\mathcal{A}$ up to a shift -1 in degree [4], (it is worth noticing, and this is not accidental, that the same shift occurs in the Loday-Quillen theorem [7]).

We want now to point out that there is another natural non-trivial cohomology which may be extracted from the differential algebra $C(\mathcal{A})$. This cohomology is connected with the existence of a canonical operation, in the sense of H. Cartan [2], [5], of the Lie algebra $\mathcal{A}_{Lie}$ in the graded differential algebra $C(\mathcal{A})$. For $A\in\mathcal{A}=\mathcal{A}_{Lie}$, define $i_A:C^n(\mathcal{A})\to C^{n-1}(\mathcal{A})$ by

$$i_A\omega(A_1,\ldots,A_{n-1})=\sum_{k=0}^{n-1}(-1)^k\omega(A_1,\ldots,A_k,A,A_{k+1},\ldots,A_{n-1})$$

$\forall\omega\in C^n(\mathcal{A})$, $\forall A_i\in\mathcal{A}$, for $n\geq 1$ and $i_AC^0(\mathcal{A})=0$. For each $A\in\mathcal{A}$, i_A is an antiderivation of degree -1 of $C(\mathcal{A})$ and one has, with $L_A=i_Ad+di_A$, $i_Ai_B+i_Bi_A=0$, $[L_A,i_B]=i_{[A,B]}$, $[L_A,L_B]=L_{[A,B]}$ which are the relations which characterize an operation of $\mathcal{A}_{Lie}$ in $C(\mathcal{A})$. Notice that then, for $A\in\mathcal{A}$, the derivation L_A of degree 0 of $C(\mathcal{A})$ is given by

$$L_A\omega(A_1,\ldots,A_n)=\sum_{k=1}^{n}\omega(A_1,\ldots,[A_k,A],\ldots,A_n)$$

for $\omega \in C^n(\mathcal{A})$, $A_i \in \mathcal{A}$. An element $\omega \in C(\mathcal{A})$ is called *horizontal* if $i_A\omega = 0$ for any $A \in \mathcal{A}$, it is called *invariant* if $L_A\omega = 0$ for any $A \in \mathcal{A}$ and it is called *basic* if it is horizontal and invariant, i.e. if $i_A\omega = 0$ and $L_A\omega = 0$ for any $A \in \mathcal{A}$. The set $C_H(\mathcal{A})$ of horizontal elements of $\mathcal{A}$ is a graded subalgebra of $C(\mathcal{A})$ which is stable by the L_A, $A \in \mathcal{A}$. The set $C_I(\mathcal{A})$ of invariant elements of $\mathcal{A}$ and the set $C_B(\mathcal{A})$ of basic elements of $\mathcal{A}$ are two graded differential subalgebras of $C(\mathcal{A})$ ($C_B(\mathcal{A}) \subset C_I(\mathcal{A})$); their cohomologies $H_I(\mathcal{A})$ and $H_B(\mathcal{A})$ are called the invariant cohomology and the basic cohomology of $\mathcal{A}$. As already claimed, if $\mathcal{A}$ has a unit then the cohomology $H(\mathcal{A})$ of $C(\mathcal{A})$ is trivial and it turns out that the same is true for the invariant cohomology ; one has the following proposition.

PROPOSITION 1 *If $\mathcal{A}$ has a unit, then one has $H^n(\mathcal{A}) = 0$, $H^n_I(\mathcal{A}) = 0$ for $n \geq 1$ and $H^0(\mathcal{A}) = H^0_I(\mathcal{A}) = \mathbb{K}$.*

Proof. Let $\mathbb{1}$ be the unit of $\mathcal{A}$ and let us define for $n \geq 1$
$h : C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ by $h\omega(A_1, \ldots, A_{n-1}) = -\omega(\mathbb{1}, A_1, \ldots, A_{n-1})$,
for $\omega \in C^n(\mathcal{A})$ and $A_i \in \mathcal{A}$. One has $(dh + hd)\omega = \omega$ and $(L_A h - hL_A)\omega = 0$ for $\omega \in C^n(\mathcal{A})$ and $A \in \mathcal{A}$. It follows that h is a contracting homotopy for $C^+(\mathcal{A}) = \bigoplus_{n\geq 1} C^n(\mathcal{A})$ and for $C^+_I(\mathcal{A}) = \bigoplus_{n\geq 1} C^n_I(\mathcal{A})$, which proves the result.□

The basic cohomology $H_B(\mathcal{A})$ is however non-trivial. In fact it is already non-trivial for $\mathcal{A} = \mathbb{K}$.

PROPOSITION 2 *The basic cohomology $H_B(\mathbb{K})$ of $\mathbb{K}$ is the free graded commutative algebra with unit generated by an element of degree two; $H^{2k}_B(\mathbb{K}) = \mathbb{K}$, $H^{2k+1}_B(\mathbb{K}) = \not\vdash$ and $H_B(\mathbb{K})$ identifies to the algebra $\mathbb{K}[\mathbb{X}^{\not\vdash}]$ of polynomials in one indeterminate X^2 of degree two, (X^2 being identified to a non-vanishing element of $H^2_B(\mathbb{K})$).*

Proof. $C(\ \mathbb{K}\ )$ can be identified to $\mathbb{K}\ [\mathbb{X}]$ and coincides with $C_I(\ \mathbb{K}\ )$ since $L_1 = 0$. One has $i_1 = 0$ on the elements of even degrees and $i_1 \neq 0$ on the non-vanishing elements of odd degrees.
Therefore $C_B(\ \mathbb{K}\ ) = \underset{\urcorner}{\oplus}\mathbb{C}^{\not\in\urcorner}(\ \mathbb{K}\ ) = \ \mathbb{K}\ [\mathbb{X}^{\not\in}] = \mathbb{H}_{\mathbb{B}}(\mathbb{K})$. $\square$

It is worth noticing here that one has $C_B^1(\mathcal{A}) = 0$ and therefore $H_B^1(\mathcal{A}) = 0$ for any associative $\mathbb{K}$ -algebra $\mathcal{A}$.
In the next section we shall compute $H_B(\mathcal{A})$ for an arbitrary associative $\mathbb{K}$ -algebra $\mathcal{A}$ with unit.

2 Computation of the basic cohomology of unital algebras

In this section, $\mathcal{A}$ is an associative $\mathbb{K}$ -algebra with a unit denoted by $\mathbb{1}$. Let $\mathcal{I}_{\mathcal{S}}^{\setminus}(\mathcal{A}_{\mathcal{L}\rangle\urcorner})$ denote the space of ad*-invariant homogeneous polynomials of degree n on the underlying Lie algebra $\mathcal{A}_{Lie}$ of $\mathcal{A}$. We shall prove the following theorem which generalizes the proposition 2 of §1.

THEOREM 1 *The basic cohomology $H_B(\mathcal{A})$ of $\mathcal{A}$ identifies with the algebra $\mathcal{I}_{\mathcal{S}}(\mathcal{A}_{\mathcal{L}\rangle\urcorner})$ of invariant polynomials on the Lie algebra $\mathcal{A}_{Lie}$ where the degree $2n$ is given to the homogeneous polynomials of degree n, i.e. $H_B^{2n}(\mathcal{A}) \simeq \mathcal{I}_{\mathcal{S}}^{\setminus}(\mathcal{A}_{\mathcal{L}\rangle\urcorner})$ and $H_B^{2n+1}(\mathcal{A}) = 0$. In particular, $H_B(\mathcal{A})$ is commutative and graded commutative.*

In order to prove this theorem, we shall need some constructions used in equivariant cohomology [1]. Let $\mathcal{P}^{m,n}$ denote the space of homogeneous polynomial mappings of degree m of $\mathcal{A}$ in $C^n(\mathcal{A})$. The direct sum $\mathcal{P} = \underset{m,n}{\oplus}\mathcal{P}^{m,n}$ is an associative bigraded algebra in a natural way. One defines the total degree of an element of $\mathcal{P}^{m,n}$ to be $2m + n$; $\mathcal{P}$ is a graded algebra for the

total degree and $C(\mathcal{A}) = \oplus_n \mathcal{P}^{0,n}$ is a graded subalgebra of $\mathcal{P}$. The composition with the differential d of $C(\mathcal{A})$ is a differential, again denoted by d, of the graded algebra $\mathcal{P}$ which extends the differential d of $C(\mathcal{A})$. One has $d\mathcal{P}^{m,n} \subset \mathcal{P}^{m,n+1}$. By using the operation $A \mapsto i_A$, one can define another differential, δ, on $\mathcal{P}$. Namely if $\omega \in \mathcal{P}$ is the polynomial mapping $A \mapsto \omega_A$ of $\mathcal{A}$ in $C(\mathcal{A})$, then $\delta\omega$ is the polynomial mapping $A \mapsto (\delta\omega)_A = i_A\omega_A$ of $\mathcal{A}$ in $C(\mathcal{A})$. One has $\delta\mathcal{P}^{m,n} \subset \mathcal{P}^{m+1,n-1}$ so δ is of total degree $2-1=1$ and the fact that δ is an antiderivation satisfying $\delta^2 = 0$ follows from the fact that, for any $A \in \mathcal{A}$, i_A is an antiderivation of $C(\mathcal{A})$ satisfying $i_A^2 = 0$. Notice that $C_H^n(\mathcal{A})$ is the kernel of $\delta \restriction C^n(\mathcal{A}) = \mathcal{P}^{0,n}$ $(: \mathcal{P}^{0,n} \to \mathcal{P}^{1,n-1})$.

As a vector space, $\mathcal{P}^{m,n}$ can be identified to the subspace of elements of $C^{m+n}(\mathcal{A})$ which are symmetric in their m first arguments: For $\omega \in \mathcal{P}^{m,n}$, $A \mapsto \omega_A$, there is a unique $\xi_\omega \in C^{m+n}(\mathcal{A})$ symmetric in the m first arguments such that

$$\omega_A(A_1, \ldots, A_n) = \xi_\omega(\underbrace{A, \ldots, A}_{m}, A_1, \ldots, A_n),\ \forall A, A_i \in \mathcal{A}.$$

Let $\mathcal{I}^{\Updownarrow,\backslash}$ denote the subspace of $\mathcal{P}^{m,n}$ consisting of the $\omega \in \mathcal{P}^{m,n}$ such that $\xi_\omega \in C_I^{m+n}(\mathcal{A})$, (i.e. such that ξ_ω is invariant). $\mathcal{I} = \oplus\mathcal{I}^{\Updownarrow,\backslash}$ is a graded subalgebra (also a bigraded subalgebra in the obvious sense) of $\mathcal{P}$ which is stable by d and δ and, furthermore, d and δ anticommute on $\mathcal{I}$.

Notice that one has $\mathcal{I}^{\Updownarrow,\prime} = \mathcal{I}_{\mathcal{S}}^{\Updownarrow}(\mathcal{A}_{\mathcal{L}\rangle\rceil})$ and $\mathcal{I}^{\prime,\backslash} = \mathcal{C}_{\mathcal{I}}^{\backslash}(\mathcal{A})$ and that $C_B^n(\mathcal{A})$ is the kernel of $\delta \restriction C_I^n(\mathcal{A}) = \mathcal{I}^{\prime,\backslash}$ $(: \mathcal{I}^{\prime,\backslash} \to \mathcal{I}^{\infty,\backslash -\infty})$. The algebras $\mathcal{P}$ and $\mathcal{I}$ are bigraded and d and δ are bihomogeneous, therefore the d and the δ cohomologies of $\mathcal{P}$ and $\mathcal{I}$ are also bigraded algebras. By using composition with the homotopy h of the proof of proposition 1 and by noticing that $\mathcal{I}$ is stable by this composition, one obtains the following generalization of proposition 1.

PROPOSITION 3 *One has* $H^{m,n}(\mathcal{P},d)=0$, $H^{m,n}(\mathcal{I},\lceil)=\prime$ *for* $n\geq 1$ *and* $H^{m,0}(\mathcal{P},d)=\mathcal{P}^{m,0}$, $H^{m,0}(\mathcal{I},\lceil)=\mathcal{I}^{\Updownarrow,\prime}=\mathcal{I}^{\Updownarrow}_{\mathcal{S}}(\mathcal{A}_{\mathcal{L}\rangle\rceil})$.

Concerning the cohomology of δ one has the following result

PROPOSITION 4 *One has* $H^{m,n}(\mathcal{P},\delta)=0$, $H^{m,n}(\mathcal{I},\delta)=0$ *for* $m\geq 1$ *and* $H^{0,n}(\mathcal{P},\delta)=C^n_H(\mathcal{A})$, $H^{0,n}(\mathcal{I},\delta)=C^n_B(\mathcal{A})$.

Proof. The last part of the proposition ($m=0$) is obvious since one has $H^{0,n}(\mathcal{P},\delta)=\ker(\delta\restriction C^n(\mathcal{A}))$ and $H^{0,n}(\mathcal{I},\delta)=\ker(\delta\restriction C^n_I(\mathcal{A}))$. Therefore from now on, assume that one has $m\geq 1$. Define a linear mapping ℓ of $\mathcal{P}$ in itself with $\ell(\mathcal{P}^{m,n})\subset\mathcal{P}^{m-1,n+1}$ by

$$(\ell\omega)_A(A_1,\ldots,A_{n+1})=\frac{d}{dt}\omega_{A+tA_1}(A_2,\ldots,A_{n+1})|_{t=0}$$

for $\omega\in\mathcal{P}^{m,n}$. One has $(\delta\ell+\ell\delta)\omega=m\omega+\mathcal{H}\omega$ where $\mathcal{H}\omega$ is given by

$$(\mathcal{H}\omega)_A(A_1,\ldots,A_n)=\sum_{p=2}^{n+1}(-1)^p\omega_A(A_2,\ldots,A_{p-1},A_1,A_p,\ldots,A_n),$$

($\omega\in\mathcal{P}^{m,n}$). Notice that if ω is such that $\omega_A(A_1,\ldots,A_n)$ is antisymmetric in $A_1,\ldots,A_n$, then $\mathcal{H}\omega=n\omega$ and therefore ℓ gives an homotopy for such ω. The following lemma, which is a combinatorial statement in the algebra of the permutation group, will lead to an homotopy for the general case. The proof of this lemma (which is probably known) will be given in appendix.

LEMMA 1 *One has on* $\mathcal{P}^{m,n}$, $\prod_{p=0}^{n-2}(\mathcal{H}-p\ id)=\prod_{p=0}^{n-1}(\mathcal{H}-p\ id)=\mathcal{S}$, *where* $\mathcal{S}\omega$ *is given as before (antisymmetrisation) by* $(\mathcal{S}\omega)_A(A_1,\ldots,A_n)=\sum_{\pi\in\mathcal{S}_n}\varepsilon(\pi)\omega_A(A_{\pi(1)},\ldots,A_{\pi(n)})$, *i.e.* $(\mathcal{S}\omega)_A=\mathcal{S}\omega_A$.

Let $\omega \in \mathcal{P}^{m,n}$ with $m \geq 1$ be such that $\delta\omega = 0$. Then $\delta\ell\omega = m\omega + \mathcal{H}\omega$, so one also have $\delta\mathcal{H}\omega = 0$ and, by induction, $\delta\mathcal{H}^p\omega = 0$ for any integer p; i.e. one has $\delta P(\mathcal{H})\omega = 0$ for any polynomial P. Define $\omega_r \in \mathcal{P}^{m,n}$, for $r = 1, 2, \ldots, n$, by $\omega_1 = \omega,\ \omega_2 = \mathcal{H}\omega - (n-2)\omega, \ldots,\ \omega_r = \prod_{p=2}^{r}(\mathcal{H} - (n-p)id)\omega, \ldots,$ $\omega_n = \mathcal{H}(\mathcal{H} - id)\ldots(\mathcal{H} - (n-2)id)\omega$. One has $\delta\ell\omega_r = m\omega_r + \mathcal{H}\omega_r =$ $(m+n-r-1)\omega_r + \omega_{r+1}$, i.e.

$$\omega_r = \delta\ell\left(\frac{\omega_r}{m+n-(r+1)}\right) - \frac{\omega_{r+1}}{m+n-(r+1)},$$

for $r \leq n-1$. This implies that

$$\omega = \delta\ell\left(\sum_{r=1}^{n-1}\frac{(-1)^{r+1}}{\prod_{p=2}^{r+1}(m+n-p)}\omega_r\right) - \frac{(-1)^n}{\prod_{p=2}^{n}(m+n-p)}\omega_n.$$

On the other hand, it follows from the lemma and the previous discussion (antisymmetry) that $\omega_n = \delta\ell(\frac{1}{m+n}\omega_n)$ and therefore one has an homotopy formula, for $\omega \in \mathcal{P}^{m,n}$ with $m \geq 1$ satisfying $\delta\omega = 0$, of the form $\omega = \delta\delta'\omega$ where $\delta' = \ell \circ Q^{m,n}(\mathcal{H})$ and where the polynomial $Q^{m,n}$ is easily computed from the previous formulae. Since ℓ and $\mathcal{H}$ preserve $\mathcal{I}$ this achieves the proof of proposition 4.$\square$

The proof of the theorem 1 will now follow from $H^{m,n}(\mathcal{I}, d) = 0$ for $n \geq 1$, $H^{m,n}(\mathcal{I}, \delta) = 0$ for $m \geq 1$,

$$H^{m,0}(\mathcal{I}, d) = \mathcal{I}_S^m(\mathcal{A}_{Lie}) \text{ and } H^{0,n}(\mathcal{I}, \delta) = C_B^n(\mathcal{A})$$

by a standard spectral sequence argument in the bicomplex $(\mathcal{I}, d, \delta)$.

Let $H(\delta|d)$ denote the δ-cohomology modulo d of $\mathcal{I}$, i.e.

$$H^{m,n}(\delta|d) = Z^{m,n}(\delta|d)/B^{m,n}(\delta|d)$$

where $Z^{m,n}(\delta|d)$ is the space of the $\alpha^{m,n} \in \mathcal{I}^{m,n}$ for which there is an $\alpha^{m+1,n-2} \in \mathcal{I}^{m+1,n-2}$ such that $\delta\alpha^{m,n} + d\alpha^{m+1,n-2} = 0$ and where $B^{m,n}(\delta|d) = \delta\mathcal{I}^{m-1,n+1} + d\mathcal{I}^{m,n-1}$ $(\subset \mathcal{I}^{m,n})$. With these notations, one has the following result.

PROPOSITION 5 *One has the following isomorphisms:*
$H_B^{2p}(\mathcal{A}) \simeq H^{k,2(p-k)-1}(\delta|d) \simeq \mathcal{I}_S^p(\mathcal{A}_{Lie})$ *for* $1 \leq k \leq p-2$,
$H_B^{2p+1}(\mathcal{A}) \simeq H^{k,2(p-k)}(\delta|d) \simeq 0$ *for* $1 \leq k \leq p-1$, $H_B^4(\mathcal{A}) \simeq \mathcal{I}_S^2(\mathcal{A}_{Lie})$, $H_B^3(\mathcal{A}) \simeq 0$ *and* $H_B^2(\mathcal{A}) \simeq \mathcal{I}_S^1(\mathcal{A}_{Lie})$.

Proof. Let $\alpha^{m,n} \in \mathcal{I}^{m,n}$ be a δ-cocycle modulo d, i.e. there is a $\alpha^{m+1,n-2} \in \mathcal{I}^{m+1,n-2}$ such that $\delta\alpha^{m,n} + d\alpha^{m+1,n-2} = 0$. By applying δ, one obtains $\delta d\alpha^{m+1,n-2} = -d\delta\alpha^{m+1,n-2} = 0$, therefore, if $n \geq 4$, there is in view of proposition 3 a $\alpha^{m+2,n-4} \in \mathcal{I}^{m+2,n-4}$ such that $\delta\alpha^{m+1,n-2} + d\alpha^{m+2,n-4} = 0$, which means that $\alpha^{m+1,n-2}$ is also a δ-cocycle modulo d. If $\alpha^{m,n}$ is exact, i.e. if there are $\beta^{m-1,n+1} \in \mathcal{I}^{m-1,n+1}$ and $\beta^{m,n-1} \in \mathcal{I}^{m,n-1}$ such that $\alpha^{m,n} = \delta\beta^{m-1,n+1} + d\beta^{m,n-1}$, then $d(\alpha^{m+1,n-2} - \delta\beta^{m,n-1}) = 0$ which implies, again by proposition 3 (since $n-2 \geq 2 > 0$), that there is a $\beta^{m+1,n-3}$ such that $\alpha^{m+1,n-2} = \delta\beta^{m,n-1} + d\beta^{m+1,n-3}$ i.e. $\alpha^{m+1,n-2}$ is also exact. Therefore there is a well defined linear mapping $\partial : H^{m,n}(\delta|d) \to H^{m+1,n-2}(\delta|d)$ for $n \geq 4$ such that $\partial[\alpha^{m,n}] = [\alpha^{m+1,n-2}]$. Let now $\alpha^{m+1,n-2} \in \mathcal{I}^{m+1,n-2}$ be a δ-cocycle modulo d, i.e. there is $\alpha^{m+2,n-4} \in \mathcal{I}^{m+2,n-4}$ such that $\delta\alpha^{m+1,n-2} + d\alpha^{m+2,n-4} = 0$. By applying d, one obtains $\delta d\alpha^{m+1,n-2} = 0$ which implies, in view of proposition 4, that there is a $\alpha^{m,n} \in \mathcal{I}^{m,n}$ such that $\delta\alpha^{m,n} + d\alpha^{m+1,n-2} = 0$. This means that ∂ is surjective. Assume that $[\alpha^{m+1,n-2}] = 0$ i.e. $\alpha^{m+1,n-2} = \delta\beta^{m,n-1} + d\beta^{m+1,n-3}$ $(\beta \in \mathcal{I})$ then one has $\delta(\alpha^{m,n} - d\beta^{m,n-1}) = 0$ which implies that $[\alpha^{m,n}] = 0$ if $m \geq 1$ or that $\alpha^{0,n} - d\beta^{0,n-1} \in C_B^n(\mathcal{A})$ if $m = 0$, again by proposition 4. *Thus* $\partial : H^{m,n}(\delta|d) \to H^{m+1,n-2}(\delta|d)$ *are*

isomorphisms for $n \geq 4$ *and* $m \geq 1$ *and, for* $m = 0$ $(n \geq 4)$, $\partial : H^{0,n}(\delta|d) \to H^{1,n-2}(\delta|d)$ *is surjective and its kernel is the image of* $C_B^n(\mathcal{A}) = H^{0,n}(\mathcal{I}, \delta)$ *in* $H^{0,n}(\delta|d)$.

On the other hand, if $\alpha^{0,n} \in \mathcal{I}^{0,n}$ is a δ-cocycle modulo d, i.e. $\delta\alpha^{0,n} + d\alpha^{1,n-2} = 0$ then $d\alpha^{0,n} \in C_I^{n+1}(\mathcal{A})$ is a basic cocycle of $\mathcal{A}$ i.e. $d\alpha^{0,n} \in Z_B^{n+1}(\mathcal{A})$ and if $\alpha^{0,n}$ is exact, i.e. $\alpha^{0,n} = d\beta^{0,n}$ with $\beta^{0,n} \in \mathcal{I}^{0,n}$, then $d\alpha^{0,n} = 0$. Therefore, with obvious notations, one has a linear mapping $d^\sharp : H^{0,n}(\delta|d) \to H_B^{n+1}(\mathcal{A})$, $d^\sharp[\alpha^{0,n}] = [d\alpha^{0,n}]$. If $z^{n+1} \in C_B^{n+1}(\mathcal{A})$ is closed i.e. $z^{n+1} \in Z_B^{n+1}(\mathcal{A})$ then, in view of proposition 1, there is a $\alpha^{0,n} \in C_I^n(\mathcal{A}) = \mathcal{I}^{0,n}$ such that $z^{n+1} = d\alpha^{0,n}$; one has $d\delta\alpha^{0,n} = 0$, which implies that $\alpha^{0,n}$ is a δ-cocycle modulo d if $n \geq 2$ (by proposition 3). Thus $d^\sharp$ is surjective for $n \geq 2$ and one obviously has $ker(d^\sharp) =$ image of $C_B^n(\mathcal{A})$ in $H^{0,n}(\delta|d)$. Applying this for $n \geq 4$ and the previous results, one obtains isomorphisms:

$$H_B^{2p}(\mathcal{A}) \simeq H^{k,2(p-k)-1}(\delta|d) \text{ for } 1 \leq k \leq p-2$$

and

$$H_B^{2p+1}(\mathcal{A}) \simeq H^{k,2(p-k)}(\delta|d) \text{ for } 1 \leq k \leq p-1.$$

Thus, to achieve the proof, it remains to show that one has:

(i) $H^{m,2}(\delta|d) = 0$ for $m \geq 1$ and $H^{0,2}(\delta|d) =$ image of $C_B^2(\mathcal{A})$

(ii) $H^{m,3}(\delta|d) \simeq \mathcal{I}_S^{m+2}(\mathcal{A}_{Lie})$ for $m \geq 1$ and $H^{0,3}(\delta|d)/\text{image of } C_B^3(\mathcal{A}) \simeq \mathcal{I}_S^2(\mathcal{A}_{Lie})$

(iii) $H_B^2(\mathcal{A}) \simeq \mathcal{I}_S^1(\mathcal{A}_{Lie})$, (remembering that $C_B^1(\mathcal{A}) = 0$).

Let $\alpha^{m,2} \in \mathcal{I}^{m,2}$ be a δ-cocyle modulo d then, (since $d\alpha^{m+1,0} \equiv 0$), $\alpha^{m,2}$ is a δ-cocycle, i.e. $\delta\alpha^{m,2} = 0$, which implies, by proposition 4, that $\alpha^{m,2} \in \delta\mathcal{I}^{m-1,1}$ for $m \geq 1$ and, for $m = 0$, $\alpha^{0,2} \in C_B^2(\mathcal{A}) = H^{0,2}(\mathcal{I}, \delta)$. This proves (i).

Let $\alpha^{m,3} \in \mathcal{I}^{m,3}$ be a δ-cocycle modulo d, i.e. there is a $\alpha^{m+1,1} \in \mathcal{I}^{m+1,1}$ such that $\delta\alpha^{m,3} + d\alpha^{m+1,1} = 0$. Then one has $\delta\alpha^{m+1,1} = P^{m+2} \in \mathcal{I}_S^{m+2}(\mathcal{A}_{Lie}) = \mathcal{I}^{m+2,0}$. If $\alpha^{m,3} = \delta\beta^{m-1,4} + d\beta^{m,2}$ for $\beta^{m-1,4} \in \mathcal{I}^{m-1,4}$ and $\beta^{m,2} \in \mathcal{I}^{m,2}$, (i.e. if $\alpha^{m,3}$ is exact), one has $d(\alpha^{m+1,1} - \delta\beta^{m,2}) = 0$ which implies, by proposition 3 and by $d\mathcal{I}^{m+1,0} = 0$, that $\alpha^{m+1,1} = \delta\beta^{m,2}$ and therefore $\delta\alpha^{m+1,1} = P^{m+2} = 0$. Thus there is a well defined linear mapping $j : H^{m,3}(\delta|d) \to \mathcal{I}_S^{m+2}(\mathcal{A}_{Lie})$, $(j([\alpha^{m,3}]) = P^{m+2})$. Let P^{m+2} be an arbitrary element of $\mathcal{I}_S^{m+2}(\mathcal{A}_{Lie})$; then, by proposition 4, there is a $\alpha^{m+1,1} \in \mathcal{I}^{m+1,1}$ such that $\delta\alpha^{m+1,1} = P^{m+2}$ and, since $dP^{m+2} = 0$, one has $\delta d\alpha^{m+1,1} = 0$ which implies again by proposition 4 that there is a $\alpha^{m,3}$ such that $\delta\alpha^{m,3} + d\alpha^{m+1,1} = 0$. This shows that j is surjective. If $\delta\alpha^{m+1,1} = 0$, then, by proposition 4, $\alpha^{m+1,1} = \delta\beta^{m,2}$ and therefore $\delta(\alpha^{m,3} - d\beta^{m,2}) = 0$ which implies again by proposition 4 that $\alpha^{m,3} = \delta\beta^{m-1,4} + d\beta^{m,2}$ if $m \geq 1$ and, for $m = 0$, $\alpha^{0,3} - d\beta^{0,2} \in C_B^3(\mathcal{A})$. This proves (ii).
Finally let $z^2 \in C_I^2(\mathcal{A})$ be a basic cocycle, i.e. $dz^2 = 0$ and $\delta z^2 = 0$, then $z^2 = d\alpha^1$ for a unique $\alpha^1 \in C_I^1(\mathcal{A})$ (since $dC^0(\mathcal{A}) = 0$ and by proposition 1). Conversely, if $\alpha^1 \in C_I^1(\mathcal{A})$ then $d\alpha^1$ is basic; therefore $H_B^2(\mathcal{A}) \simeq C_I^1(\mathcal{A})$ since $C_B^1(\mathcal{A}) = 0$. But one has canonically $C_I^1(\mathcal{A}) = \mathcal{I}_S^1(\mathcal{A}_{Lie})$.□

This proves of course Theorem 1, but it is worth noticing that in the above proof there is also a computation of the δ-cohomology modulo d of $\mathcal{I}$.

3 Sketch of another approach: Connection with the Lie algebra cohomology

There is another way to study the basic cohomology of $\mathcal{A}$ which connects it with the Lie algebra cohomology of $\mathcal{A}_{Lie}$: It is to study the spectral sequence

corresponding to the filtration of the differential algebra $C(\mathcal{A})$ associated to the operation i of the Lie algebra $\mathcal{A}_{Lie}$ in the differential algebra $C(\mathcal{A})$, [5]. This filtration $\mathcal{F}$ is defined by

$$\mathcal{F}^p(C^n(\mathcal{A})) = \{\omega \in C^n(\mathcal{A}) | i_{A_1} \dots i_{A_{n-p+1}}(\omega) = 0,\ \forall A_i \in \mathcal{A}\}$$

for $0 \leq p \leq n$ and $\mathcal{F}^p(C(\mathcal{A})) = \bigoplus_{n \geq p} \mathcal{F}^p(C^n(\mathcal{A}))$.
One has

$$\mathcal{F}^0(C(\mathcal{A})) = C(\mathcal{A}),\ \mathcal{F}^p(C(\mathcal{A})) \cdot \mathcal{F}^q(C(\mathcal{A})) \subset \mathcal{F}^{p+q}(C(\mathcal{A}))$$

and

$$d\mathcal{F}^p(C(\mathcal{A})) \subset \mathcal{F}^p(C(\mathcal{A}))$$

i.e. $\mathcal{F}$ is a (decreasing) filtration of graded differential algebra. To such a filtration corresponds a convergent spectral sequence $(E_r, d_r)_{r \in \mathbb{N}}$, where $E_r = \bigoplus_{p,q \in \mathbb{N}} E_r^{p,q}$ is a bigraded algebra and d_r is a homogeneous differential on E_r of bidegree $(r, 1-r)$. The triviality of the cohomology of $C(\mathcal{A})$, (i.e. proposition 1), implies that $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$ and $E_\infty^{0,0} = \mathbb{K}$. The spectral sequence starts with the graded space E_0 associated to the filtration i.e. $E_0^{p,q} = \mathcal{F}^p(C^{p+q}(\mathcal{A}))/\mathcal{F}^{p+1}(C^{p+q}(\mathcal{A}))$ and d_0 is induced by the differential d of $C(\mathcal{A})$. If $\omega \in \mathcal{F}^p(C^{p+q}(\mathcal{A}))$ then $i_{A_1} \dots i_{A_q}\omega$ is in $C_H^p(\mathcal{A})$ and is antisymmetric in $A_1, \dots, A_q$. Therefore $(A_1, \dots, A_q) \mapsto i_{A_1} \dots i_{A_q}\omega$ is a q-cochain of the Lie algebra $\mathcal{A}_{Lie}$ with values in $C_H^p(\mathcal{A})$ for the representation $A \mapsto L_A$ of the Lie algebra $\mathcal{A}_{Lie}$ in $C_H^p(\mathcal{A})$. This defines a linear map of $\mathcal{F}^p(C^{p+q}(\mathcal{A}))$ in the space of q-cochains of $\mathcal{A}_{Lie}$ with values in $C_H^p(\mathcal{A})$. The kernel of this map is, by definition, $\mathcal{F}^{p+1}(C^{p+q}(\mathcal{A}))$. In our case, it is straightforward to show that this map is surjective, i.e. that $E_0^{p,q}$ identifies with the space of q-cochains of the Lie algebra $\mathcal{A}_{Lie}$ with values in the space $C_H^p(\mathcal{A})$ of horizontal elements of $C^p(\mathcal{A})$ and that then, d_0 coincides with the Chevalley-Eilenberg

differential. Thus $E_1 = H(E_0, d_0)$ is the Lie algebra cohomology of $\mathcal{A}_{Lie}$ with value in $C_H(\mathcal{A})$, $E_1^{p,q} = H^q(\mathcal{A}_{Lie}, C_H^p(\mathcal{A}))$. In particular $E_1^{0,*}$ is the ordinary cohomology of $\mathcal{A}_{Lie}$ (i.e. with value in the trivial representation in $\mathbb{K}$) and $E_1^{*,0}$ is the space of invariant elements of $C_H(\mathcal{A})$, i.e. the space $C_B(\mathcal{A})$ of basic elements of $C(\mathcal{A})$, $E_1^{n,0} = C_B^n(\mathcal{A})$. Furthermore, on $E_1^{*,0} = C_B(\mathcal{A})$, d_1 is just the differential d of $C(\mathcal{A})$ restricted to $C_B(\mathcal{A})$. Therefore $E_2^{*,0}$ is the basic cohomology $H_B(\mathcal{A})$ of $\mathcal{A}$, $E_2^{n,0} = H_B^n(\mathcal{A})$. This shows that the spectral sequence connects the basic cohomology of $\mathcal{A}$ to the Lie algebra cohomology of its underlying Lie algebra $\mathcal{A}_{Lie}$. The connection between the Lie algebra cohomology of $\mathcal{A}_{Lie}$ and the ad*-invariant polynomials, i.e. $H_B(\mathcal{A})$ in our case, is well known but an interest of the last approach could be to catch the primitive parts.

Appendix: Proof of Lemma 1

Let $\mathcal{S}_n$ be the group of permutations of $\{1, \ldots, n\}$.
In the algebra of this group, let us define the antisymmetrisation operator

$$\mathcal{S} = \sum_{\pi \in \mathcal{S}_n} \varepsilon(\pi)\pi$$

and the operators

$$\mathcal{H}_{(k)} = \sum_{\substack{\pi \in \mathcal{S}_n \\ \pi^{-1}(k+1) < \ldots < \pi^{-1}(n)}} \varepsilon(\pi)\pi$$

for any $1 \leq k \leq n$, where $\varepsilon(\pi)$ denotes the signature of the permutation π.
Notice that

$$\mathcal{H}_{(n)} = \mathcal{H}_{(n-1)} = \mathcal{S}$$

and one easily shows that

$$\mathcal{H} \equiv \mathcal{H}_{(1)} = \sum_{p=1}^{n} (-1)^{p+1} \gamma_p$$

where γ_p is the permutation $(1, \ldots, p, \ldots, n) \mapsto (2, \ldots, p, 1, p+1, \ldots, n)$.
With these definitions, one has the following result:

LEMMA *For any* $1 \leq k \leq n-1$,

$$\mathcal{H}\mathcal{H}_{(k)} = k\mathcal{H}_{(k)} + \mathcal{H}_{(k+1)}$$

Proof.

$$\begin{aligned}
\mathcal{H}\mathcal{H}_{(k)} &= \left(\sum_{p=1}^{n} (-1)^{p+1} \gamma_p \right) \left(\sum_{\substack{\pi \in \mathcal{S}_n \\ \pi^{-1}(k+1) < \ldots < \pi^{-1}(n)}} \varepsilon(\pi) \pi \right) \\
&= \sum_{p=1}^{k} \sum_{\substack{\pi \in \mathcal{S}_n \\ \pi^{-1}(k+1) < \ldots < \pi^{-1}(n)}} (-1)^{p+1} \varepsilon(\pi) \gamma_p \pi \\
&\quad + \sum_{p=k+1}^{n} \sum_{\substack{\pi \in \mathcal{S}_n \\ \pi^{-1}(k+1) < \ldots < \pi^{-1}(n)}} (-1)^{p+1} \varepsilon(\pi) \gamma_p \pi
\end{aligned}$$

Now, define $\pi' = \gamma_p \pi \in \mathcal{S}_n$; one has $\varepsilon(\pi') = (-1)^{p+1} \varepsilon(\pi)$.
For $p \leq k$, one has $\pi'^{-1}(q) = \pi^{-1}(q)$ for any $k+1 \leq q \leq n$.
So, in the first summation, for a fixed p, the sum over the $\pi \in \mathcal{S}_n$ such that $\pi^{-1}(k+1) < \ldots < \pi^{-1}(n)$ can be replaced by the sum over the $\pi' \in \mathcal{S}_n$ such that $\pi'^{-1}(k+1) < \ldots < \pi'^{-1}(n)$. Thus

$$\sum_{p=1}^{k} \sum_{\substack{\pi \in \mathcal{S}_n \\ \pi^{-1}(k+1) < \ldots < \pi^{-1}(n)}} (-1)^{p+1} \varepsilon(\pi) \gamma_p \pi = \sum_{p=1}^{k} \sum_{\substack{\pi' \in \mathcal{S}_n \\ \pi'^{-1}(k+1) < \ldots < \pi'^{-1}(n)}} \varepsilon(\pi') \pi' = k\mathcal{H}_{(k)}$$

Now, for $p \geq k+1$, one has $\pi'^{-1}(q) = \pi^{-1}(q-1)$ for any $k+2 \leq q \leq p$ and $\pi'^{-1}(q) = \pi^{-1}(q)$ for any $p+1 \leq q \leq n$. So one has only

$$\pi'^{-1}(k+2) < \ldots < \pi'^{-1}(n),$$

and in the second summation the sum over p and π can be replaced by the sum over the $\pi' \in \mathcal{S}_n$ such that $\pi'^{-1}(k+2) < \ldots < \pi'^{-1}(n)$. Thus

$$\sum_{p=k+1}^{n} \sum_{\substack{\pi \in \mathcal{S}_n \\ \pi^{-1}(k+1) < \ldots < \pi^{-1}(n)}} (-1)^{p+1} \varepsilon(\pi) \gamma_p \pi = \mathcal{H}_{(k+1)}. \; \square$$

By induction, this lemma shows that for any $1 \leq k \leq n$

$$\mathcal{H}_{(k)} = \prod_{p=0}^{k-1} (\mathcal{H} - p \; id)$$

where we recall $\mathcal{H} \equiv \mathcal{H}_{(1)}$.

So for $k = n$ and $k = n-1$, one has

$$\mathcal{H}_{(n)} = \prod_{p=0}^{n-1} (\mathcal{H} - p \; id) = \mathcal{S}$$

$$\mathcal{H}_{(n-1)} = \prod_{p=0}^{n-2} (\mathcal{H} - p \; id) = \mathcal{S}$$

Now, notice that the operators $\mathcal{H}$ and $\mathcal{S}$ of Lemma 1 are representations of the operators $\mathcal{H}$ and $\mathcal{S}$ above in the linear space $\mathcal{P}^{m,n}$ (in fact only in $C^n(\mathcal{A})$). This proves Lemma 1.

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