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On the construction of trigonometric solutions of the Yang-Baxter equation

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Abstract

We describe the construction of trigonometric R-matrices corresponding to the (multiplicity-free) tensor product of two irreducible representations of a quantum algebra $U_q(\mathcal{G})$. Our method is a generalization of the tensor product graph method to the case of two different representations. It yields the decomposition of the R-matrix into projection operators. Many new examples of trigonometric R-matrices (solutions to the spectral parameter dependent Yang-Baxter equation) are constructed using this approach.

1 Introduction

The solutions to the Yang-Baxter equation play a central role in the theory of quantum integrable models [1, 2, 3]. In statistical mechanics they are the Boltzman weights of exactly solvable lattice models [4]. In quantum field theory they give the exact factorizable scattering matrices [5]. For an introduction to the mathematical aspects of the Yang-Baxter equation see e.g. [6].

The Yang-Baxter equation (with additive spectral parameter) has the form

$$R_{12}(\theta)R_{13}(\theta+\theta')R_{23}(\theta') = R_{23}(\theta')R_{13}(\theta'+\theta)R_{12}(\theta). \tag{1.1}$$

The $R_{ab}(\theta)$ are matrices which depend on a spectral parameter θ and which act on the tensor product of two vector spaces V_a and V_b

$$R_{ab}(\theta):\ V_a \otimes V_b \to V_a \otimes V_b. \tag{1.2}$$

The products of R's in eq. (1.1) act on the space $V_1 \otimes V_2 \otimes V_3$.

In this paper we are interested in solutions of eq. (1.1) which are trigonometric functions of the spectral parameter θ and which depend on a further complex parameter q. These solutions arise as the intertwiners of the quantum affine algebras $U_q(\hat{\mathcal{G}})$ [7, 8, 9, 10, 11]. These are deformations of affine Kac-Moody algebras [12]. Associated to any two finite-dimensional irreducible $U_q(\hat{\mathcal{G}})$-modules[1] $V(\lambda)$ and $V(\mu)$ there exists a trigonometric R-matrix $R^{\lambda\mu}(\theta)$. Given three modules, the R-matrices for all pairs of these three modules are a solution of eq. (1.1).

We will describe a method for constructing such R-matrices. The method works for any untwisted quantum affine algebra $U_q(\hat{\mathcal{G}})$ and for all those irreducible finite dimensional $U_q(\hat{\mathcal{G}})$-modules $V(\lambda)$ and $V(\mu)$ which are irreducible also as modules of the associated non-affine quantum algebra $U_q(\mathcal{G})$ and whose tensor product $V(\lambda) \otimes V(\mu)$ is a multiplicity-free direct sum of irreducible $U_q(\mathcal{G})$ modules. The method is a generalization of [14] which treats the special case $V(\lambda) = V(\mu)$.

Let us summarize the previous work on trigonometric R-matrices. Most is known if $V(\lambda) = V(\mu)$ coincides with the defining representation. The corresponding R-matrices have been constructed for $A_n^{(1)}$ [15], $A_2^{(2)}$ [16], $A_2^{(1)}$ [17], and finally for all members of the classical series $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ and $D_n^{(2)}$ [18, 19]. $E_6^{(1)}$ and $E_7^{(1)}$ were treated in [20]. The R-matrices in higher representations can be obtained by applying the fusion procedure [21] to the trigonometric R-matrices. In principle this method yields the R-matrices for the product of any two representations which can be projected out of the multiple tensor product of fundamental representations, using the Hecke algebra in the case of A_n [10, 22] or the Birman-Wenzl-Murakami algebra in the case of B_n, C_n, and D_n [23]. Because of the algebraic

[1] For a construction of the finite-dimensional irreducible representations of $U_q(\hat{\mathcal{G}})$ see [13].

complications this has been done in practice only for the product of the vector with a two-index tensor representation [23].

More practical is the method of [14] which is designed for the multiplicity-free tensor product $V(\lambda)\otimes V(\lambda)$ of some irrep with itself. In [14] the R-matrices in such a special case have been worked out for the symmetric as well as the antisymmetric tensor representations of A_n, the spinor represenations for D_n, the 27-dimensional representation of E_6 and the 54-dimensional representation of E_7.

An alternative general method for the construction of trigonometric R-matrices is to start with the formula for the universal R-matrix of $U_q(\hat{\mathcal{G}})$ [24, 25, 26] and to specialize it to particular representations [27, 28]. This method becomes impractical for larger representations, although it has obvious advantages when multiplicities occur in the tensor product space.

More is known [29, 21, 30, 31, 32, 33, 34, 35, 36, 37] about solutions of the Yang-Baxter equation (1.1) which are rational functions of the spectral parameter θ and which do not depend on an extra parameter q. They arise as the interwiners of tensor products of finite dimensional representations of the Yangians [7, 9]. These rational R-matrices can be obtained from the trigonometric R-matrices, which are the subject of this paper, by taking the limit $q \to 1$.

The paper is organized as follows: In section 2 we review quantum affine algebras and Jimbo equations which arise from them and which uniquely determine trigonometric solutions of the Yang-Baxter equation (1.1). In section 3 we show how to write the general solution to these equations as a sum over twisted projection operators and we give the equations which uniquely determine the coefficients. In many cases it is possible to solve these equations using only classical Lie algebra representation theory and we do this using tensor product graphs in section 4. There we explicitly treat various examples and determine the following R-matrices:

A_n: i) rank a symmetric tensor with rank b symmetric tensor; ii) rank a symmetric tensor with rank b antisymmetric tensor; and iii) rank a antisymmetric tensor with rank b antisymmetric tensor.

B_n: i) spinor times spinor, ii) symmetric traceless tensor of rank a times symmetric traceless tensor of rank b.

C_n: fundamental representation a with fundamental representation b.

D_n: i) spinor times antispinor, ii) symmetric traceless tensor of rank a times symmetric traceless tensor of rank b.

E_6: i) $V(\lambda_1)$ times $V(a\lambda_1)$, ii) $V(\lambda_5)$ times $V(a\lambda_1)$.

F_4: $V(\lambda_4)$ times $V(\lambda_4)$.

We end with a discussion in section 5.

2 Quantum Affine Algebras and the Jimbo Equations

Let $A = (a_{ij})_{0\leq i,j\leq r}$ be a symmetrizable generalized Cartan matrix in the sense of Kac [12] and let $\hat{\mathcal{G}}$ denote the affine Lie algebra associated with the Cartan matrix A, where $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, α_i are the simple roots of $\hat{\mathcal{G}}$ and r is the rank of the corresponding finite-dimensional simple Lie algebra $\mathcal{G}$. We are restricting our attention to the untwisted affine Lie algebras. The quantum affine algebra $U_q(\hat{\mathcal{G}})$ [7, 8, 9] is defined by generators $\{e_i,\ f_i,\ q^{h_i},\ i = 0, 1, ..., r\}$ and relations

$$
\begin{aligned}
&q^h.q^{h'} = q^{h+h'}\,, \qquad h,\ h' = h_i,\ i = 0, 1, ..., r\\
&q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i\,, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i\\
&[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0 \qquad (i \neq j)\\
&\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0 \qquad (i \neq j)
\end{aligned}
\tag{2.1}
$$

where

$$
e_i^{(k)} = \frac{e_i^k}{[k]_{q_i}!}, \quad f_i^{(k)} = \frac{f_i^k}{[k]_{q_i}!}, \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad q_i = q^{(\alpha_i,\alpha_i)/2}, \quad [k]_q! = \prod_{n=1}^{k} [n]_q.
\tag{2.2}
$$

Here we do not include the derivation d as part of $U_q(\hat{\mathcal{G}})$. Furthermore we will set the central charge to zero.

The associated non-affine quantum algebra $U_q(\mathcal{G})$ is generated by $\{e_i,\ f_i,\ q^{h_i},\ i = 1, ..., r\}$ (i.e., without e_0, f_0 and h_0) and the same relations. The algebra $U_q(\hat{\mathcal{G}})$ is a Hopf algebra with coproduct, counit and antipode similar to the case of $U_q(\mathcal{G})$:

$$
\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h\,, \quad h = h_i, \quad i = 0, 1, \cdots, r\\
&\Delta(e_i) = e_i \otimes q^{h_i/2} + q^{-h_i/2} \otimes e_i\\
&\Delta(f_i) = f_i \otimes q^{h_i/2} + q^{-h_i/2} \otimes f_i\\
&S(a) = -q^{h_\rho} a q^{-h_\rho}\,, \quad \forall a \in U_q(\mathcal{G})
\end{aligned}
\tag{2.3}
$$

where ρ is the half-sum of positive roots of $\mathcal{G}$. We have omitted the remaining formulas for the antipode and the counit since they are not required.

Let Δ' be the opposite coproduct: $\Delta' = T\Delta$, where T is the twist map: $T(a\otimes b) = b \otimes a\,,\ \forall a, b \in U_q(\hat{\mathcal{G}})$. Then Δ and Δ' are related by the universal R-matrix R in $U_q(\hat{\mathcal{G}}) \otimes U_q(\hat{\mathcal{G}})$ satisfying, among others, the relations

$$
R\Delta(a) = \Delta'(a)R\,, \qquad \forall a \in U_q(\hat{\mathcal{G}}) \tag{2.4}
$$

$$
(I \otimes \Delta)R = R_{13}R_{12}\,, \qquad (\Delta \otimes I)R = R_{13}R_{23} \tag{2.5}
$$

where if $R = \sum a_i \otimes b_i$ then $R_{12} = \sum a_i \otimes b_i \otimes 1$, $R_{13} = \sum a_i \otimes 1 \otimes b_i$ etc.

For any $x \in \mathbf{C}$, we define an automorphism D_x of $U_q(\hat{\mathcal{G}})$ as [11]

$$D_x(e_i) = x^{\delta_{i0}} e_i \,, \qquad D_x(f_i) = x^{-\delta_{i0}} f_i, \qquad D_x(h_i) = h_i \tag{2.6}$$

and set

$$R(x) = (D_x \otimes I)(R). \tag{2.7}$$

It follows from (2.5) that $R(x)$ solves the QYBE with multiplicative spectral parameter

$$R_{12}(x) R_{13}(xy) R_{23}(y) = R_{23}(y) R_{13}(xy) R_{12}(x). \tag{2.8}$$

This reproduces eq. (1.1) if we set $x = e^{\theta}$, $y = e^{\theta'}$.

Let π_λ , π_μ and π_ν be three arbitrary finite-dimensional irreps of $U_q(\mathcal{G})$ afforded by the irreducible modules $V(\lambda)$, $V(\mu)$ and $V(\nu)$ respectively, where λ , μ and ν are the highest weights of these modules. Assume all π_λ , π_μ and π_ν are affinizable, i.e. they can be extended to finite dimensional irreps of the corresponding $U_q(\hat{\mathcal{G}})$ [2]. Let

$$R^{\lambda\mu}(x) = g_q(x)(\pi_\lambda \otimes \pi_\mu)(R(x)) \tag{2.9}$$

where $g_q(x)$ is a scalar normalization factor (see appendix A). Then $R^{\lambda\mu}(x)$ satisfies the system of linear equations [19] deduced from the intertwining property (2.4)

$$\begin{aligned} & R^{\lambda\mu}(x) \Delta^{\lambda\mu}(a) = \Delta'^{\lambda\mu}(a) R^{\lambda\mu}(x) \qquad \forall a \in U_q(\mathcal{G}), \\ & R^{\lambda\mu}(x) \left(x\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right) \\ & \quad = \left(x\pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2}) + \pi_\lambda(q^{h_0/2}) \otimes \pi_\mu(e_0) \right) R^{\lambda\mu}(x), \end{aligned} \tag{2.10}$$

where $\Delta^{\lambda\mu}(a) \equiv (\pi_\lambda \otimes \pi_\mu)\Delta(a)$. The solution $R^{\lambda\mu}(x)$ to the above equations intertwines the coproduct of $U_q(\hat{\mathcal{G}})$ acting on the $U_q(\hat{\mathcal{G}})$-module $V(\lambda) \otimes V(\mu)$. Because the representation $(\pi_\lambda \otimes \pi_\mu)(D_x \otimes 1)$ is irreducible for generic x, by Schur's lemma the solution is uniquely determined up to the scalar factor $g_q(x)$. $R^{\lambda\mu}(x)$ satisfies the QYBE in $V(\nu) \otimes V(\lambda) \otimes V(\mu)$

$$R^{\nu\lambda}(x) R^{\nu\mu}(xy) R^{\lambda\mu}(y) = R^{\lambda\mu}(y) R^{\nu\mu}(xy) R^{\nu\lambda}(x). \tag{2.11}$$

Now let $P^{\lambda\mu} : V(\lambda) \otimes V(\mu) \to V(\mu) \otimes V(\lambda)$ be the permutation operator such that

$$P^{\lambda\mu}(|f> \otimes |g>) = |g> \otimes |f>, \qquad \forall |f> \in V(\lambda),\ |g> \in V(\mu) \tag{2.12}$$

and denote

$$\check{\mathbf{R}}^{\lambda\mu}(x) = P^{\lambda\mu}\, R^{\lambda\mu}(x)\,, \tag{2.13}$$

[2]For an investigation of this point see [13].

then (2.11) is translated into the following relations:

$$(\check{\mathbf{R}}^{\lambda\mu}(x)\otimes I)(I\otimes\check{\mathbf{R}}^{\nu\mu}(xy))(\check{\mathbf{R}}^{\nu\lambda}(y)\otimes I)=(I\otimes\check{\mathbf{R}}^{\nu\lambda}(y))(\check{\mathbf{R}}^{\nu\mu}(xy)\otimes I)(I\otimes\check{\mathbf{R}}^{\lambda\mu}(x)) \tag{2.14}$$

and eqs.(2.10) can be rewritten as

$$\check{\mathbf{R}}^{\lambda\mu}(x)\Delta^{\lambda\mu}(a)=\Delta^{\mu\lambda}(a)\check{\mathbf{R}}^{\lambda\mu}(x) \qquad \forall a\in U_q(\mathcal{G}), \tag{2.15}$$

$$\begin{aligned}&\check{\mathbf{R}}^{\lambda\mu}(x)\left(x\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2})+\pi_\lambda(q^{-h_0/2})\otimes\pi_\mu(e_0)\right)\\ &\quad=\left(\pi_\mu(e_0)\otimes\pi_\lambda(q^{h_0/2})+x\pi_\mu(q^{-h_0/2})\otimes\pi_\lambda(e_0)\right)\check{\mathbf{R}}^{\lambda\mu}(x),\end{aligned} \tag{2.16}$$

Eqs.(2.15, 2.16) are the so-called Jimbo equations (JEs). In this paper we present a method for solving these equations.

3 General solution to Jimbo equations

Below we assume that $V(\lambda)\otimes V(\mu)$ is a multiplicity-free direct sum of irreducible modules. We choose to normalize the solution so that

$$\check{\mathbf{R}}^{\lambda\mu}(x)\check{\mathbf{R}}^{\mu\lambda}(x^{-1})=I. \tag{3.1}$$

See appendix A for an account of this.

We first focus on the $x=1$ case of the JEs which take the form

$$\check{\mathbf{R}}^{\lambda\mu}(1)\Delta^{\lambda\mu}(a)=\Delta^{\mu\lambda}(a)\check{\mathbf{R}}^{\lambda\mu}(1) \qquad \forall a\in U_q(\mathcal{G}), \tag{3.2}$$

$$\begin{aligned}&\check{\mathbf{R}}^{\lambda\mu}(1)\left(\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2})+\pi_\lambda(q^{-h_0/2})\otimes\pi_\mu(e_0)\right)\\ &\quad=\left(\pi_\mu(e_0)\otimes\pi_\lambda(q^{h_0/2})+\pi_\mu(q^{-h_0/2})\otimes\pi_\lambda(e_0)\right)\check{\mathbf{R}}^{\lambda\mu}(1).\end{aligned} \tag{3.3}$$

In this case we have

$$(R^T)^{\lambda\mu}(1)R^{\lambda\mu}(1)=I=\check{\mathbf{R}}^{\lambda\mu}(1)\check{\mathbf{R}}^{\mu\lambda}(1). \tag{3.4}$$

We decompose the tensor product $V(\lambda)\otimes V(\mu)$ into its irreducible $U_q(\mathcal{G})$-submodules. Due to the similarity between the representation theory of the quantum algebra and the classical algebra $\mathcal{G}$, this decomposition is identical to the classical one [38, 39]

$$V(\lambda)\otimes V(\mu)=\bigoplus_\nu V(\nu) \tag{3.5}$$

where the sum is over the highest weights occuring in the tensor product space (assumed multiplicity-free).

Let $\mathcal{P}_\nu^{\lambda\mu}:\ V(\lambda)\otimes V(\mu)\to V(\nu)$ be the projection operators which satisfy

$$\mathcal{P}_\nu^{\lambda\mu}\,\mathcal{P}_{\nu'}^{\lambda\mu}=\delta_{\nu\nu'}\mathcal{P}_\nu^{\lambda\mu}\,, \qquad \sum_\nu\mathcal{P}_\nu^{\lambda\mu}=I. \tag{3.6}$$

We now define operators $\check{\mathbf{P}}_\nu^{\lambda\mu}$ by

$$\check{\mathbf{P}}_\nu^{\lambda\mu} = \mathcal{P}_\nu^{\mu\lambda}\, \check{\mathbf{R}}^{\lambda\mu}(1) = \check{\mathbf{R}}^{\lambda\mu}(1)\, \mathcal{P}_\nu^{\lambda\mu}. \tag{3.7}$$

Then, by definition

$$\check{\mathbf{R}}^{\lambda\mu}(1) = \sum_\nu \check{\mathbf{P}}_\nu^{\lambda\mu}. \tag{3.8}$$

It is easy to show that

$$\check{\mathbf{P}}_\nu^{\lambda\mu}\, \mathcal{P}_{\nu'}^{\lambda\mu} = \mathcal{P}_{\nu'}^{\mu\lambda}\, \check{\mathbf{P}}_\nu^{\lambda\mu} = \delta_{\nu\nu'} \check{\mathbf{P}}_\nu^{\lambda\mu}, \tag{3.9}$$

and

$$\begin{aligned} \check{\mathbf{P}}_\nu^{\mu\lambda}\, \check{\mathbf{P}}_{\nu'}^{\lambda\mu} &= \mathcal{P}_\nu^{\lambda\mu} \check{\mathbf{R}}^{\mu\lambda}(1) \check{\mathbf{R}}^{\lambda\mu}(1) \mathcal{P}_{\nu'}^{\lambda\mu} \\ &= \mathcal{P}_\nu^{\lambda\mu} \mathcal{P}_{\nu'}^{\lambda\mu} = \delta_{\nu\nu'} \mathcal{P}_\nu^{\lambda\mu} \end{aligned} \tag{3.10}$$

Eqs.(3.9, 3.10) imply that the operators $\check{\mathbf{P}}_\nu^{\lambda\mu}$ are “projection” operators. As can be seen from eq. (3.2), the ”projectors” $\check{\mathbf{P}}_\nu^{\lambda\mu}$ are intertwiners of $U_q(\mathcal{G})$. The general solution of eq. (2.15) is a sum of these elementary intertwiners

$$\check{\mathbf{R}}^{\lambda\mu}(x) = \sum_\nu \rho_\nu(x)\, \check{\mathbf{P}}_\nu^{\lambda\mu} \tag{3.11}$$

where $\rho_\nu(x)$ are some functions of x. According to eq. (3.8) they satisfy

$$\rho_\nu(1) = 1. \tag{3.12}$$

Our task is now to determine these functions so that eq. (3.11) satisfies also Jimbo’s equation eq. (2.16).

We begin by determining $\rho_\nu(x)$ at the special value $x = 0$. At $x = 0$ the R-matrix $\check{\mathbf{R}}^{\lambda\mu}(0)$ reduces to the R-matrix of the quantum algebra $U_q(\mathcal{G})$ in $V(\lambda) \otimes V(\mu)$: $\check{\mathbf{R}}^{\lambda\mu}(0) \equiv \check{\mathbf{R}}^{\lambda\mu}$. We recall the following fact about the universal R-matrix $\bar{R}$ for $U_q(\mathcal{G})$(see also appendix A):

$$\bar{R}^T \bar{R} = (v \otimes v)\Delta(v^{-1})\,, \quad v = uq^{-2h_\rho}\,, \quad \pi_\lambda(v) = q^{-C(\lambda)} \cdot I_{V(\lambda)} \tag{3.13}$$

with $u = \sum_i S(\bar{b}_i)\bar{a}_i$ where $\bar{R} = \sum_i \bar{a}_i \otimes \bar{b}_i$. S is the antipode for $U_q(\mathcal{G})$, $C(\lambda) = (\lambda, \lambda + 2\rho)$ is the eigenvalue of the quadratic Casimir element of $\mathcal{G}$ in an irrep with highest weight λ; ρ is the half-sum of positive roots of $\mathcal{G}$. We now compute $\check{\mathbf{R}}^{\mu\lambda}\check{\mathbf{R}}^{\lambda\mu}$. Using the above equations we have

$$\begin{aligned} \sum_\nu (\rho_\nu(0))^2 \mathcal{P}_\nu^{\lambda\mu} &= \check{\mathbf{R}}^{\mu\lambda}\check{\mathbf{R}}^{\lambda\mu} \equiv P^{\mu\lambda}\bar{R}^{\mu\lambda}P^{\lambda\mu}\bar{R}^{\lambda\mu} \\ &= (\bar{R}^T)^{\lambda\mu}\bar{R}^{\lambda\mu} = (\pi_\lambda \otimes \pi_\mu)(\bar{R}^T\bar{R}) \\ &= \pi_\lambda(v) \otimes \pi_\mu(v)(\pi_\lambda \otimes \pi_\mu)\Delta(v^{-1}) \\ &= \sum_\nu q^{C(\nu)-C(\lambda)-C(\mu)} \mathcal{P}_\nu^{\lambda\mu} \end{aligned} \tag{3.14}$$

where use has been made of $\check{\mathbf{R}}^{\lambda\mu} \equiv P^{\lambda\mu}\bar{R}^{\lambda\mu} \equiv P^{\lambda\mu}(\pi_\lambda \otimes \pi_\mu)(\bar{R})$. It follows immediately that

$$\rho_\nu(0) = \epsilon(\nu) q^{\frac{C(\nu)-C(\lambda)-C(\mu)}{2}} \tag{3.15}$$

where $\epsilon(\nu)$ is the parity of $V(\nu)$ in $V(\lambda)\otimes V(\mu)$ (see appendix B for an explanation). Thus

$$\check{\mathbf{R}}^{\lambda\mu}(0) = \sum_\nu \epsilon(\nu) q^{\frac{C(\nu)-C(\lambda)-C(\mu)}{2}} \check{\mathbf{P}}^{\lambda\mu}_\nu. \tag{3.16}$$

A similar relation was obtained in [40] using a different approach.

Remark: (i) although irrelevant to this paper, it is worth pointing out that in the case of a tensor product decomposition with finite multiplicities a similar spectral decomposition formula for $\check{\mathbf{R}}^{\lambda\mu}(0)$ may be obtained by combining the techniques above and those in [41, 42]; (ii) given the universal R-matrix $\bar{R}$ for $U_q(\mathcal{G})$ (which is known in explicit form for any $\mathcal{G}$ [43]) then $\check{\mathbf{R}}^{\lambda\mu}(0) = \check{\mathbf{R}}^{\lambda\mu}$ can be independently computed by using $\check{\mathbf{R}}^{\lambda\mu}(0) = P^{\lambda\mu}(\pi_\lambda \otimes \pi_\mu)(\bar{R})$. Once $\check{\mathbf{R}}^{\lambda\mu}(0)$ is known through such a independent way, then from the above equation we have

$$\check{\mathbf{P}}^{\lambda\mu}_\nu = \epsilon(\nu) q^{-\frac{C(\nu)-C(\lambda)-C(\mu)}{2}} \check{\mathbf{R}}^{\lambda\mu}(0) \mathcal{P}^{\lambda\mu}_\nu \tag{3.17}$$

which implies that $\check{\mathbf{P}}^{\lambda\mu}_\nu$ (and therefore $\check{\mathbf{R}}^{\lambda\mu}(1)$) are computable this way. $\check{\mathbf{P}}^{\lambda\mu}_\nu$ can also be determined by using pure representation theory of $U_q(\mathcal{G})$. This is explained in an forthcoming paper [47] (for a brief outline of this see appendix C). $\square$

Now we use Jimbo's equation eq. (2.16) to determine the $\rho_\nu(x)$ for general x. We insert (3.11) into (2.16), multiply the resulting equation by $\mathcal{P}^{\mu\lambda}_\nu$ from the left and by $\mathcal{P}^{\lambda\mu}_{\nu'}$ from the right and use the properties of the projectors. We arrive at

$$\begin{aligned}\rho_\nu(x)\check{\mathbf{P}}^{\lambda\mu}_\nu &\left(x\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2})\otimes\pi_\mu(e_0)\right)\mathcal{P}^{\lambda\mu}_{\nu'} \\ &= \rho_{\nu'}(x)\mathcal{P}^{\mu\lambda}_\nu\left(\pi_\mu(e_0)\otimes\pi_\lambda(q^{h_0/2}) + x\pi_\mu(q^{-h_0/2})\otimes\pi_\lambda(e_0)\right)\check{\mathbf{P}}^{\lambda\mu}_{\nu'}.\end{aligned} \tag{3.18}$$

To simplify this we use the following equation, obtained by taking $x=0$ and using eq. (3.15):

$$\begin{aligned}\epsilon(\nu)q^{C(\nu)/2}\check{\mathbf{P}}^{\lambda\mu}_\nu &\left(\pi_\lambda(q^{-h_0/2})\otimes\pi_\mu(e_0)\right)\mathcal{P}^{\lambda\mu}_{\nu'} \\ &= \epsilon(\nu')q^{C(\nu')/2}\mathcal{P}^{\mu\lambda}_\nu\left(\pi_\mu(e_0)\otimes\pi_\lambda(q^{h_0/2})\right)\check{\mathbf{P}}^{\lambda\mu}_{\nu'}.\end{aligned} \tag{3.19}$$

We get a second useful relation by multiplying the above by $\check{\mathbf{P}}^{\mu\lambda}_\nu$ from the left and by $\check{\mathbf{P}}^{\mu\lambda}_{\nu'}$ from the right and then interchanging the labels μ and λ to give

$$\begin{aligned}\epsilon(\nu)q^{-C(\nu)/2}\check{\mathbf{P}}^{\lambda\mu}_\nu &\left(\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2})\right)\mathcal{P}^{\lambda\mu}_{\nu'} \\ &= \epsilon(\nu')q^{-C(\nu')/2}\mathcal{P}^{\mu\lambda}_\nu\left(\pi_\mu(q^{-h_0/2})\otimes\pi_\lambda(e_0)\right)\check{\mathbf{P}}^{\lambda\mu}_{\nu'}.\end{aligned} \tag{3.20}$$

A third important relation follows if we set $x=1$ in eq. (3.18) and use eq. (3.12):

$$\begin{aligned}\check{\mathbf{P}}^{\lambda\mu}_\nu &\left(\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2})\otimes\pi_\mu(e_0)\right)\mathcal{P}^{\lambda\mu}_{\nu'} \\ &= \mathcal{P}^{\mu\lambda}_\nu\left(\pi_\mu(e_0)\otimes\pi_\lambda(q^{h_0/2}) + \pi_\mu(q^{-h_0/2})\otimes\pi_\lambda(e_0)\right)\check{\mathbf{P}}^{\lambda\mu}_{\nu'}.\end{aligned} \tag{3.21}$$

In view of eq.(3.19), eq.(3.20) and eq.(3.21) we obtain from (3.18), for $\nu \neq \nu'$,

$$\rho_\nu(x) \left(x q^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu') q^{C(\nu')/2} \right) \check{\mathbf{P}}_\nu^{\lambda\mu} \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu}$$
$$= \rho_{\nu'}(x) \left(q^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu') x q^{C(\nu')/2} \right) \check{\mathbf{P}}_\nu^{\lambda\mu} \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu} . \quad (3.22)$$

Multiplying by $\check{\mathbf{P}}_\nu^{\mu\lambda}$ from the left we see that, if

$$\mathcal{P}_\nu^{\lambda\mu} \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu} \neq 0 , \qquad \nu \neq \nu' \quad (3.23)$$

the above equation gives rise to a relation between $\rho_\nu(x)$ and $\rho_{\nu'}(x)$

$$\rho_\nu(x) = \rho_{\nu'}(x) \frac{q^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu') x q^{C(\nu')/2}}{x q^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu') q^{C(\nu')/2}} , \qquad \nu \neq \nu' . \quad (3.24)$$

Note that this relation is unchanged under $\nu \leftrightarrow \nu'$, as it must be because also the condition eq. (3.23) has this invariance. It simplifies if we note that $\epsilon(\nu)\epsilon(\nu') = -1$ whenever the condition eq. (3.23) is satisfied. This is explained in appendix B. With the notation

$$\langle a \rangle \equiv \frac{1 - x q^a}{x - q^a} , \quad (3.25)$$

the relation eq. (3.24) then becomes

$$\rho_\nu(x) = \left\langle \frac{C(\nu') - C(\nu)}{2} \right\rangle \rho_{\nu'}(x) , \qquad \nu \neq \nu' . \quad (3.26)$$

4 Tensor Product Graphs and R-matrices

We have a relation eq. (3.26) between the coefficients ρ_ν and $\rho_{\nu'}$ whenever the condition eq. (3.23) is satisfied, i.e., whenever $\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2})$ maps from the module $V(\nu')$ to the module $V(\nu)$. As a graphical aid [14] we introduce the tensor product graph.

Definition 1 *The* **tensor product graph** $G^{\lambda\mu}$ *associated to the tensor product* $V(\lambda) \otimes V(\mu)$ *is a graph whose vertices are the irreducible modules* $V(\nu)$ *appearing in the decomposition eq. (3.5) of* $V(\lambda) \otimes V(\mu)$*. There is an edge between a vertex* $V(\nu)$ *and a vertex* $V(\nu')$ *iff*

$$\mathcal{P}_\nu^{\lambda\mu} \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu} \neq 0. \quad (4.1)$$

For an example of a tensor product graph see figure 1.

If $V(\lambda)$ and $V(\mu)$ are irreducible $U_q(\mathcal{G})$-modules the tensor product graph is always connected, i.e., every node is linked to every other node by a path of edges. This follows from the fact that $\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2})$ is related to the lowest component of an adjoint tensor operator; for details see [14]. This implies that the relations eq. (3.24) are sufficient to determine all the coefficients $\rho_\nu(x)$ uniquely. If the tensor product graph is multiply connected, i.e., if there exists more than two paths between two nodes, then the relations overdetermine the coefficients, i.e., there are consistency conditions [14]. However, because the existence of a solution to JEs is guaranteed by the existence of the universal R-matrix, these consistency conditions will always be satisfied.

The straightforward but tedious and impractical way to determine the tensor product graph is to work out explicitly the left hand side of (4.1). To do so requires a knowledge of the representations $\pi_\lambda(e_0)$ and $\pi_\mu(e_0)$ as well as the projection operators $\mathcal{P}_\nu^{\lambda\mu}$. Although it may be possible to construct these representations [13] and the projectors explicitly, it is more practical to use other approaches.

One is to consider the case of $q = 1$ and to work with the following smaller graph.

Definition 2 *The* **restricted tensor product graph** $G^{(0)\lambda\mu}$ *associated to the tensor product* $V^{(0)}(\lambda) \otimes V^{(0)}(\mu)$, *where* $V^{(0)}(\lambda)$ *etc. denote irreducible* $\mathcal{G}$*-modules, is a graph whose vertices are the irreducible modules* $V^{(0)}(\nu)$ *appearing in the decomposition eq. (3.5) of* $V^{(0)}(\lambda) \otimes V^{(0)}(\mu)$. *There is an edge between a vertex* $V^{(0)}(\nu)$ *and a vertex* $V^{(0)}(\nu')$ *iff*

$$\mathcal{P}_\nu^{(0)\lambda\mu} \left(\pi_\lambda^{(0)}(e_0) \otimes I_\mu\right) \mathcal{P}_{\nu'}^{(0)\lambda\mu} \neq 0. \tag{4.2}$$

where $\mathcal{P}_\nu^{(0)\lambda\mu}$ *are the* $q = 1$ *versions of the projectors* $\mathcal{P}_\nu^{\lambda\mu}$.

Following similar arguments to [14] it is seen that this restricted tensor product graph is in general "smaller" than the tensor product graph, i.e. it may be missing some edges. It is however still connected and thus imposing a relation eq. (3.24) for each of the edges of the restricted tensor product graph will lead to a unique solution. It is thus sufficient to determine the restricted tensor product graph. Unfortunately this too is a difficult task.

Another approach is to work instead with the following larger graph which we will utilize in this paper.

Definition 3 *The* **extended tensor product graph** $\Gamma^{\lambda\mu}$ *associated to the tensor product* $V(\lambda) \otimes V(\mu)$ *is a graph whose vertices are the irreducible modules* $V(\nu)$ *appearing in the decomposition eq. (3.5) of* $V(\lambda) \otimes V(\mu)$. *There is an edge between* *two vertices $V(\nu)$ and $V(\nu')$ iff*

$$V(\nu') \subset V_{adj} \otimes V(\nu) \qquad \textit{and} \quad \epsilon(\nu)\epsilon(\nu') = -1. \tag{4.3}$$

It follows again from the fact that $\pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2})$ is related to the lowest component of an adjoint tensor operator that the condition eq. (4.3) is a necessary condition for eq. (4.1) [14]. This means that every link contained in the tensor product graph is contained also in the extended tensor product graph but the latter may contain more links. Only if the extended tensor product graph is a tree do we know that it is equal to the tensor product graph. If we impose a relation (3.26) on the ρ's for every link in the extended tensor product graph, we may be imposing too many relations and thus may not always find a solution. If however we do find a solution, then this is the unique correct solution which we would have obtained also from the tensor product graph.

The advantage of using the extended tensor product graph is that it can be constructed using only Lie algebra representation theory. We only need to be able to decompose tensor products and to determine the parity of submodules. The decomposition of tensor products can be done for small representations by matching dimensions and Casimir values. In the general case one can use Young tableau techniques or characters.

In the following we will give a number of examples of tensor product graphs. We are interested only in the tensor product graphs for products of those irreducible $U_q(\mathcal{G})$-modules which are affinizable, i.e., which are also modules of the quantum affine algebra $U_q(\hat{\mathcal{G}})$. All so-called miniscule modules are affinizable. These are modules for which all $SU(2)$ submodules are either trivial or two-dimensional. Also those modules are affinizable which correspond to extremal nodes on a tensor product graph of two other affinizable modules [13].

4.1 Examples for $A_n = sl(n+1)$

We introduce the standard orthonormal weight basis $\{\epsilon_i | i = 1, \cdots, n\}$ for A_n. The fundamental weights are

$$\lambda_b = \sum_{i=1}^{b} \epsilon_i \,, \qquad b = 1, 2, \cdots, n \tag{4.4}$$

and the semisum of positive roots is

$$\rho = \sum_{i=1}^{n} (n - 2i + 1)\epsilon_i \,. \tag{4.5}$$

All representations of $U_q(A_n)$ are affinizable and therefore there is an unlimited number of tensor product graphs which give rise to R-matrices. Here we only consider three simple cases: (i) rank a (≥ 1) symmetric tensor (which has the highest weight $a\lambda_1$) with rank b (≥ 1) symmetric tensor; (ii) rank a symmetric tensor with rank b ($1 \leq b \leq n$) antisymmetric tensor (which has the fundamental weight λ_b as its highest weight); and (iii) rank a antisymmetric tensor with rank b antisymmetric tensor ($1 \leq a, b \leq n$). Without loss of generality, we assume $a \geq b$ in the following. We start with case (i). Its tensor product graph is

$$V(a\lambda_1) \otimes V(b\lambda_1) \ = \ \underset{\Lambda_0}{\bullet} \!-\!\!-\!\!-\!\! \underset{\Lambda_1}{\bullet} \!-\!\!- \quad \cdots \quad -\!\!-\!\! \underset{\Lambda_{b-1}}{\bullet} \!-\!\!-\!\!-\!\! \underset{\Lambda_b}{\bullet} \tag{4.6}$$

where

$$\Lambda_c = (a+b-c)\epsilon_1 + c\epsilon_2 = (a+b-2c)\lambda_1 + c\lambda_2, \qquad c = 0, 1, \cdots, b. \tag{4.7}$$

The Casimir takes the following values on the representations appearing in the above graph

$$C(\Lambda_c) = (a+b-c)^2 + c^2 - 2c + (a+b)(n-1). \tag{4.8}$$

The R-matrix is

$$\check{\mathbf{R}}^{a\lambda_1, b\lambda_1}(x) = \rho_{\Lambda_0}(x) \sum_{c=0}^{b} \prod_{i=1}^{c} \langle a+b-2i+2 \rangle \ \check{\mathbf{P}}_{\Lambda_c}^{a\lambda_1, b\lambda_1} \tag{4.9}$$

(our convention here and below is that $\prod_{i=1}^{0}(\ldots) = 1$.). The case of $a = b$ was worked out in [14]. The overall scalar factor $\rho_{\Lambda_0}(x)$ is not determined by the Jimbo equations or by the Yang-Baxter equation but by the normalization condition eq. (3.1). It satisfies eq. (3.12) and eq. (3.15). We will from now on drop such overall scalar factors from the formulas for the R-matrices.

For case (ii) we have the tensor product graph

$$V(a\lambda_1) \otimes V(\lambda_b) \ = \ \underset{\Lambda_1}{\bullet} \!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\! \underset{\Lambda_2}{\bullet} \tag{4.10}$$

where

$$\Lambda_1 = (a+1)\epsilon_1 + \sum_{i=2}^{b} \epsilon_i = a\lambda_1 + \lambda_b, \qquad \Lambda_2 = a\epsilon_1 + \sum_{i=2}^{b+1} \epsilon_i = (a-1)\lambda_1 + \lambda_{b+1}. \tag{4.11}$$

The difference of the Casimirs is

$$\frac{C(\Lambda_1) - C(\Lambda_2)}{2} = a + b \tag{4.12}$$

The R-matrix is

$$\check{\mathbf{R}}^{a\lambda_1, \lambda_b}(x) = \check{\mathbf{P}}_{\Lambda_1}^{a\lambda_1, \lambda_b} + \langle a+b \rangle \ \check{\mathbf{P}}_{\Lambda_2}^{a\lambda_1, \lambda_b} \tag{4.13}$$

Finally for case (iii) the tensor product graph is

$$V(\lambda_a) \otimes V(\lambda_b) \ = \ \underset{\Lambda_0}{\bullet} \!-\!\!-\!\!-\!\! \underset{\Lambda_1}{\bullet} \!-\!\!- \quad \cdots \quad -\!\!-\!\! \underset{\Lambda_{b-1}}{\bullet} \!-\!\!-\!\!-\!\! \underset{\Lambda_{min(b,n-a)}}{\bullet} \tag{4.14}$$

where

$$\Lambda_c = \sum_{i=1}^{c} \epsilon_i + \sum_{i=1}^{a+b-c} \epsilon_i \equiv \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i = \lambda_{a+c} + \lambda_{b-c}, \qquad c = 0, 1, \cdots, min(b, n-a) \tag{4.15}$$

The Casimirs are

$$C(\Lambda_c) = 2c(b-a-1) - 2c^2 + (n+1)(a+b) - a^2 - b^2 \tag{4.16}$$

and the R-matrix is

$$\check{\mathbf{R}}^{\lambda_a,\lambda_b}(x) = \sum_{c=0}^{min(b,n-a)} \prod_{i=1}^{c} \langle 2i + a - b \rangle \; \check{\mathbf{P}}^{\lambda_a,\lambda_b}_{\Lambda_c} . \tag{4.17}$$

Again the case of $a = b$ was worked out in [14]. This R-matrix has been used by Hollowood [44] to construct the soliton S-matrix in $A_n^{(1)}$ Toda theory.

4.2 Examples for $B_n = so(2n+1)$

In the orthonormal weight basis $\{\epsilon_i | i = 1, \ldots, n\}$ for the weight space of B_n the fundamental weights are

$$\lambda_n = \frac{1}{2} \sum_{i=1}^{n} \epsilon_i \; , \qquad \lambda_a = \sum_{i=1}^{a} \epsilon_i \; , \quad a = 1, \ldots, n-1. \tag{4.18}$$

The semisum of positive roots is

$$\rho = \sum_{i=1}^{n} \left(n - i + \frac{1}{2} \right) \epsilon_i . \tag{4.19}$$

We begin with the spinor representation $V(\lambda_n)$ because it is the only miniscule representation of B_n. Its tensor product graph is shown in figure 1.

On the representations appearing in this graph the Casimir takes the values

$$C(2\lambda_n) = n(n+1) \; , \qquad C(\lambda_a) = a(2n+1-a) \; , \quad a = 1, \ldots, n-1. \tag{4.20}$$

Using the graph in figure 1 and these values of the Casimir, we can immediately write down the solution of eq. (3.26). To be explicit we distinguish between the cases of n even and n odd. For n even we find

$$\rho_{\lambda_{2a}}(x) \;\; = \;\; \prod_{i=1}^{\frac{n}{2}-a} \langle 4i - 1 \rangle \; , \qquad a = 0 \ldots , \frac{n}{2} - 1 \, , \tag{4.21}$$

$$\rho_{\lambda_{2a+1}}(x) \;\; = \;\; \prod_{i=1}^{\frac{n}{2}-a} \langle 4i - 3 \rangle \; , \qquad a = 0 \ldots , \frac{n}{2} - 1 \, , \tag{4.22}$$

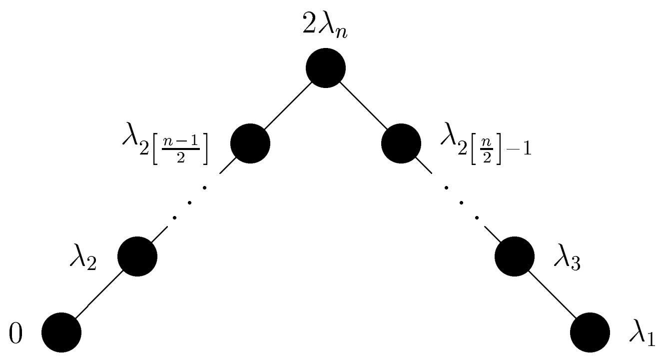


Figure 1: The tensor product graph for the product $V(\lambda_n) \otimes V(\lambda_n)$ of two spinor representations of B_n. $\left[\frac{n}{2}\right]$ means the integer part of $\frac{n}{2}$.

and for n odd

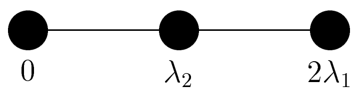
$$\rho_{\lambda_{2a}}(x) = \prod_{i=1}^{\frac{n+1}{2}-a} \langle 4i-3 \rangle \ , \qquad a = 0 \ldots, \frac{n-1}{2} \ , \tag{4.23}$$

$$\rho_{\lambda_{2a+1}}(x) = \prod_{i=1}^{\frac{n-1}{2}-a} \langle 4i-1 \rangle \ , \qquad a = 0 \ldots, \frac{n-1}{2} - 1 \, . \tag{4.24}$$

The R-matrix is

$$\check{\mathbf{R}}^{\lambda_n \lambda_n}(x) = \sum_{a=0}^{n-1} \rho_{\lambda_a}(x) \check{\mathbf{P}}^{\lambda_n \lambda_n}_{\lambda_a} + \check{\mathbf{P}}^{\lambda_n \lambda_n}_{2\lambda_n} \, . \tag{4.25}$$

We also read off from the tensor product graph in figure 1 that the module $V(\lambda_1)$ (i.e., the vector representation) is affinizable, because it sits on an extremal node of the graph. Therefore we can next look at the tensor product graph for the product of two vector representations

$V(\lambda_1) \otimes V(\lambda_1) \ =$ 

(4.26)

From this we see that also the module $V(2\lambda_1)$ (i.e., the symmetric traceless tensor of rank 2) is affinizable and we can look at the tensor product involving it. Continuing in this way we see that all the modules $V(a\lambda_1)$ for any a are affinizable. Figure 2 shows the tensor product graph for the general product $V(a\lambda_1) \otimes V(b\lambda_1)$ $(a \geq b)$. The modules $V((a+b)\lambda_1)$ and $V((a-b)\lambda_1)$ sit on extremal nodes of the graph for $V(a\lambda_1) \otimes V(b\lambda_1)$ and are thus affinizable.

On the modules appearing in figure 2 the Casimir takes the values

$$C(c\lambda_1 + d\lambda_2) = c^2 + 2d^2 + 2cd + (2n-1)c + (4n-4)d. \tag{4.27}$$

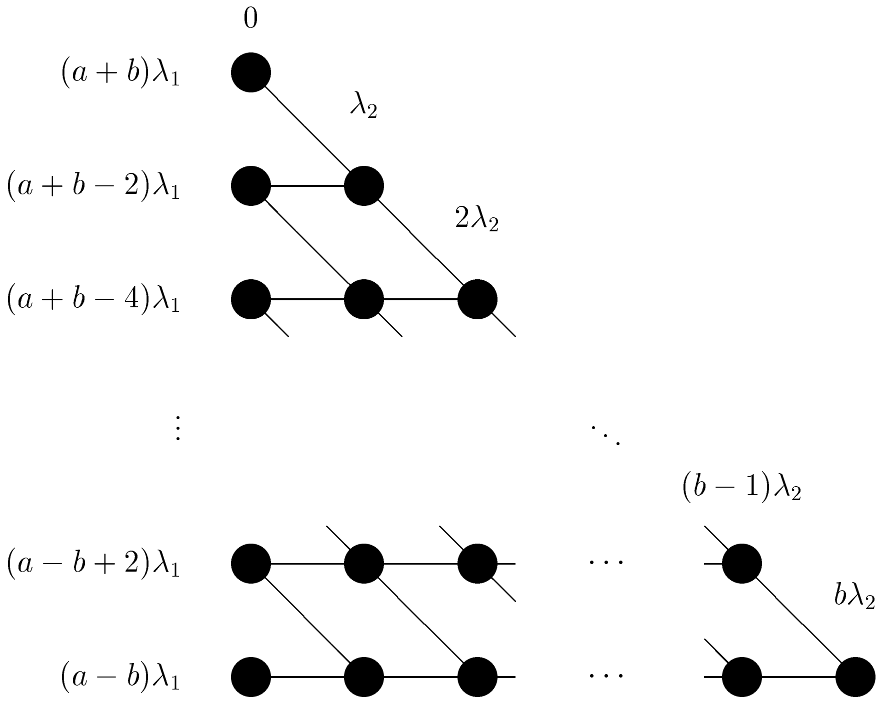


Figure 2: The extended tensor product graph for the product $V(a\lambda_1) \otimes V(b\lambda_1)$ $(a \geq b)$ of symmetric traceless tensors of rank a and rank b. The nodes correspond to modules whose highest weight is the sum of the weight labeling the column and the weight labeling the row.

We need the differences of the values of the Casimir along the edges. Going down diagonally we get

$$\begin{aligned} &C\left((a+b+2-2c)\lambda_1 + (c-d-1)\lambda_2\right) \\ &-C\left((a+b-2c)\lambda_1 + (c-d)\lambda_2\right) = 2(2+a+b-2c). \end{aligned} \tag{4.28}$$

Going from right to left horizontally we get

$$\begin{aligned} &C\left(a+b-2c)\lambda_1 + (c-d+1)\lambda_2\right) \\ &-C\left((a+b-2c)\lambda_1 + (c-d)\lambda_2\right) = 2(2n-1+a+b-2d). \end{aligned} \tag{4.29}$$

The extended tensor product graph in figure 2 is multiply connected. This means that using relation eq. (3.26) one might obtain two different expressions for the coefficients $\rho_\nu(x)$ if one follows two different paths in the tensor product graph. One finds however that the values (4.27) of the Casimir are such that all possible paths give the same result. This can be seen by observing that the right hand side of eq. (4.28) is independent of d and the right hand side of eq. (4.29) is independent of c. If this had not been the case, then it would have been necessary to determine which edges of the extended tensor product graph belong also to the tensor product graph and to use only those.

From figure 2 and the Casimir values we read off that

$$\rho_{(a+b-2c)\lambda_1+(c-d)\lambda_2}(x) = \prod_{i=1}^{c} \langle 2+a+b-2i\rangle \tag{4.30}$$

$$\times \prod_{j=1}^{d} \langle 2n-1+a+b-2j\rangle\,. \tag{4.31}$$

Thus we obtain the R-matrices

$$\check{\mathbf{R}}^{a\lambda_1,b\lambda_1}(x) = \sum_{c=0}^{b}\sum_{d=0}^{c}\prod_{i=1}^{c}\langle 2+a+b-2i\rangle \prod_{j=1}^{d}\langle 2n-1+a+b-2j\rangle \times \check{\mathbf{P}}^{a\lambda_1,b\lambda_1}_{(a+b-2c)\lambda_1+(c-d)\lambda_2}. \tag{4.32}$$

4.3 Examples for $C_n = sp(2n)$

The fundamental weights of C_n are

$$\lambda_a = \sum_{i=1}^{a} \epsilon_i\,, \qquad a = 1,\ldots,n. \tag{4.33}$$

and the semisum of positive roots is

$$\rho = \sum_{i=1}^{n} (n+1-i)\;\epsilon_i. \tag{4.34}$$

The vector representation $V(\lambda_1)$ is miniscule and thus affinizable. Figure 3 shows the extended tensor product graph for the product $V(\lambda_a)\otimes V(\lambda_b)$ of two arbitrary fundamental representations. This graph has been used in [37] to determine the rational R-matrices. By looking at figure 3 we can see that all fundamental representations are affinizable. This is so because $V(\lambda_{a+1})$ appears as an extremal node in $V(\lambda_a)\otimes V(\lambda_1)$ for any a. We will determine the trigonometric R-matrix acting in the product $V(\lambda_a)\otimes V(\lambda_b)$.

On the modules appearing in figure 3 the Casimir takes the values

$$C(\lambda_a+\lambda_b) = (2n+2-a)a + (2n+4-b)b. \tag{4.35}$$

We calculate the differences of the values of the Casimir along the edges. Going down to the right we get

$$C(\lambda_{a+c-d-1}+\lambda_{b-c-d+1}) - C(\lambda_{a+c-d}+\lambda_{b-c-d}) = 2(a-b+2c). \tag{4.36}$$

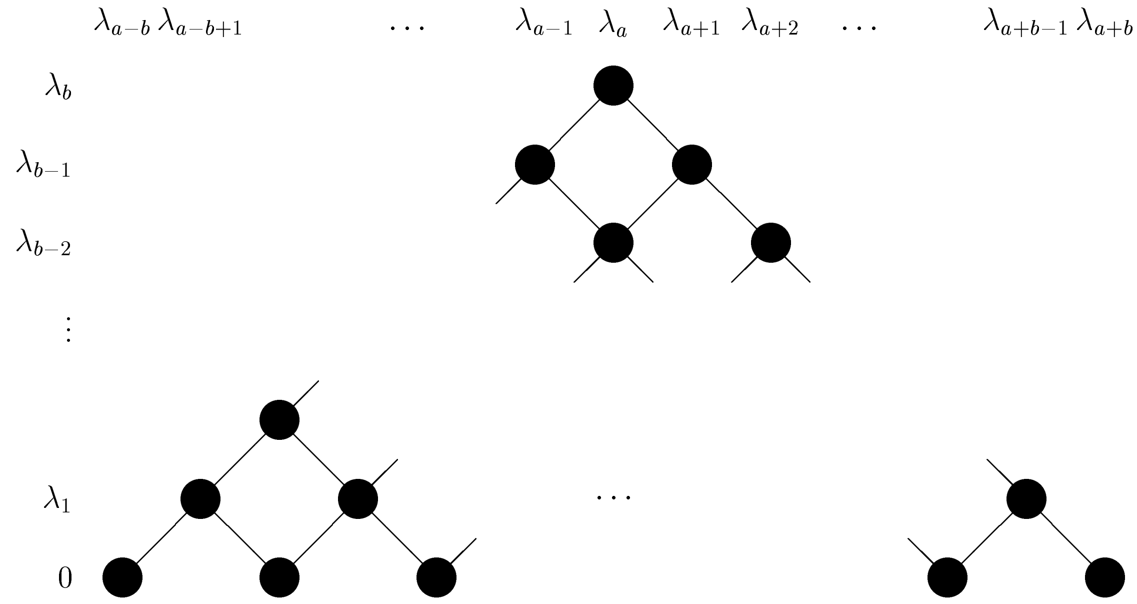


Figure 3: The extended tensor product graph for the product $V(\lambda_a) \otimes V(\lambda_b)$ $(a \geq b)$ of two arbitrary fundamental representations of C_n. The nodes correspond to representations whose highest weight is given by the sum of the weight labeling the column and the weight labeling the row. If $a + b > n$ then the graph extends to the right only up to λ_n.

Going down to the left we get

$$C(\lambda_{a+c-d+1} + \lambda_{b-c-d+1}) - C(\lambda_{a+c-d} + \lambda_{b-c-d}) = 2(2n + 2 - a - b + 2d). \quad (4.37)$$

Note that again the right hand side of eq. (4.36) is independent of d and the right hand side of eq. (4.37) is independent of c. This insures that it is irrelevant, which path through the graph we choose for determining the coefficients ρ using eq. (3.26). We obtain the following R-matrices

$$\check{\mathbf{R}}^{\lambda_a,\lambda_b}(x) = \sum_{c=0}^{min(b,n-a)} \sum_{d=0}^{b-c} \prod_{i=1}^{c} \langle a - b + 2i \rangle \prod_{j=1}^{d} \langle 2n + 2 - a - b + 2j \rangle \times \check{\mathbf{P}}^{\lambda_a,\lambda_b}_{\lambda_{a+c-d}+\lambda_{b-c-d}}. \quad (4.38)$$

These R-matrices, which are important for the construction of the S-matrices for solitons in $D^{(2)}_{n+1}$ Toda theory [45], were first proposed Hollowood [46]. He did however not give a detailed justification and explanation of his formula. It was one of the motivations for the work presented in this paper to derive Hollowood's formula.

4.4 Examples for $D_n = so(2n)$

The fundamental weights of D_n are

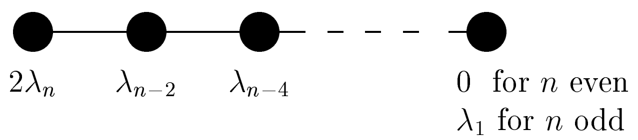
$$\lambda_n = \frac{1}{2}\sum_{i=1}^{n} \epsilon_i \ , \quad \lambda_{n-1} = \lambda_n - \epsilon_n \ , \quad \lambda_a = \sum_{i=1}^{a} \epsilon_i \ , \quad a = 1, \ldots, n-2. \tag{4.39}$$

The semisum of positive roots is

$$\rho = \sum_{i=1}^{n} (n-i)\, \epsilon_i . \tag{4.40}$$

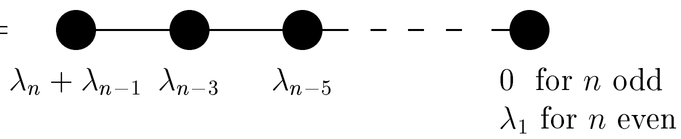
The spinor representations $V(\lambda_n)$ and $V(\lambda_{n-1})$ are miniscule (as well as the vector representation $V(\lambda_1)$). The tensor product graphs for the products of two spinor representations are

$V(\lambda_n) \otimes V(\lambda_n) \ =$



(4.41)

$V(\lambda_n) \otimes V(\lambda_{n-1}) \ =$



(4.42)

The graph for the product $V(\lambda_{n-1}) \otimes V(\lambda_{n-1})$ is obtained from (4.41) by conjugation (i.e., $\lambda_n \leftrightarrow \lambda_{n-1}$). The Casimir values are

$$\begin{aligned} &C(\lambda_a) = a(2n-a) \ , \qquad a = 1, \ldots, n-2 \\ &C(\lambda_n) = n^2 \ , \quad C(\lambda_{n-1}) = n^2 - 1. \end{aligned} \tag{4.43}$$

This leads to the following formulas for the R-matrices

$$\check{\mathbf{R}}^{\lambda_n \lambda_n}(x) \ = \ \check{\mathbf{P}}^{\lambda_n \lambda_n}_{2\lambda_n} + \sum_{a=1}^{\left[\frac{n}{2}\right]} \prod_{i=1}^{a} \langle 4i-2 \rangle \, \check{\mathbf{P}}^{\lambda_n \lambda_n}_{\lambda_{n-2a}} , \tag{4.44}$$

$$\check{\mathbf{R}}^{\lambda_n \lambda_{n-1}}(x) \ = \ \check{\mathbf{P}}^{\lambda_n \lambda_{n-1}}_{\lambda_n + \lambda_{n-1}} + \sum_{a=1}^{\left[\frac{n}{2}\right]} \prod_{i=1}^{a} \langle 4i \rangle \, \check{\mathbf{P}}^{\lambda_n \lambda_{n-1}}_{\lambda_{n-2a}} . \tag{4.45}$$

The R-matrix in eq. (4.44) had already been determined in [14].

The extended tensor product graph for the product of two symmetric traceless tensors of arbitrary rank is identical to the corresponding graph for B_n shown in figure 2. Once again we can show that all the symmetric traceless tensor representations are affinizable, by the same argument as before. So we will determine the corresponding R-matrices. The Casimir takes the values

$$C(c\lambda_1 + d\lambda_2) = c^2 + 2d^2 + 2cd + (2n-2)c + (4n-6)d. \tag{4.46}$$

The differences between the values of the Casimir on connected nodes are, going down along the diagonal

$$\begin{aligned} &C\left((a+b+2-2c)\lambda_1 + (c-d-1)\lambda_2\right) \\ &-C\left((a+b-2c)\lambda_1 + (c-d)\lambda_2\right) = 2(2+a+b-2c), \end{aligned} \tag{4.47}$$

and going left along the horizontal

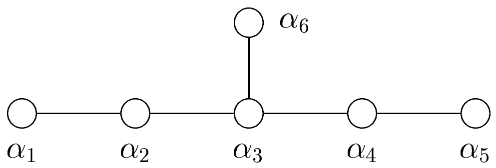
$$\begin{aligned} &C\left(a+b-2c)\lambda_1 + (c-d+1)\lambda_2\right) \\ &-C\left((a+b-2c)\lambda_1 + (c-d)\lambda_2\right) = 2(2n-2+a+b-2d). \end{aligned} \tag{4.48}$$

Again the extended tensor product graph leads to a consistent solution of the relations eq. (3.26) and we obtain the R-matrices

$$\begin{aligned} \check{\mathbf{R}}^{a\lambda_1, b\lambda_1}(x) \quad = \quad & \sum_{c=0}^{b} \sum_{d=0}^{c} \prod_{i=1}^{c} \langle 2+a+b-2i \rangle \prod_{j=1}^{d} \langle 2n-1+a+b-2j \rangle \\ & \times \check{\mathbf{P}}^{a\lambda_1, b\lambda_1}_{(a+b-2c)\lambda_1 + (c-d)\lambda_2}. \end{aligned} \tag{4.49}$$

4.5 Examples for E_6

The Dynkin diagram for E_6 is



(4.50)

We choose the normalization $(\alpha_i, \alpha_i) = 2,\ i = 1, 2, \cdots, 6$. The fundamental weights are

$$\begin{aligned} \lambda_1 &= \frac{1}{3}(4\alpha_1 + 5\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6), \\ \lambda_2 &= \frac{1}{3}(5\alpha_1 + 10\alpha_2 + 12\alpha_3 + 8\alpha_4 + 4\alpha_5 + 6\alpha_6), \\ \lambda_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6), \\ \lambda_4 &= \frac{1}{3}(4\alpha_1 + 8\alpha_2 + 12\alpha_3 + 10\alpha_4 + 5\alpha_5 + 6\alpha_6), \\ \lambda_5 &= \frac{1}{3}(2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6), \\ \lambda_6 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6. \end{aligned} \tag{4.51}$$

The half sum of positive roots is

$$\rho = \sum_{i=1}^{6} \lambda_i = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 15\alpha_4 + 8\alpha_5 + 11\alpha_6. \tag{4.52}$$

The miniscule representations are $V(\lambda_1)$ and $V(\lambda_5)$.

We consider the following tensor product graphs

$$V(\lambda_1)\otimes V(a\lambda_1) \;=\; \underset{(a-1)\lambda_1+\lambda_5}{\bullet} \longrule \underset{(a-1)\lambda_1+\lambda_2}{\bullet} \longrule \underset{(a+1)\lambda_1}{\bullet} \tag{4.53}$$

We see that the modules $V(a\lambda_1)$ are affinizable for any a because $V(a\lambda_1)$ appears as extremal node on the graph for $V(\lambda_1)\otimes V((a-1)\lambda_1)$.

On the modules appearing in (4.53) the Casimir takes the values

$$\begin{aligned}
C((a-1)\lambda_1+\lambda_5) &= \frac{4}{3}(a-1)^2 + \frac{52}{3}(a-1) + \frac{52}{3},\\
C((a-1)\lambda_1+\lambda_2) &= \frac{4}{3}(a-1)^2 + \frac{58}{3}(a-1) + \frac{100}{3},\\
C((a+1)\lambda_1) &= \frac{4}{3}(a+1)^2 + 16(a+1).
\end{aligned} \tag{4.54}$$

We obtain the R-matrices

$$\check{\mathbf{R}}^{\lambda_1,a\lambda_1}(x) = \check{\mathbf{P}}^{\lambda_1,a\lambda_1}_{(a+1)\lambda_1} + \langle a+1\rangle\; \check{\mathbf{P}}^{\lambda_1,a\lambda_1}_{(a-1)\lambda_1+\lambda_2} + \langle a+1\rangle\,\langle a+7\rangle\; \check{\mathbf{P}}^{\lambda_1,a\lambda_1}_{(a-1)\lambda_1+\lambda_5}. \tag{4.55}$$

The case of $a=1$ was also worked in [20, 14].

Next we look at the tensor product graphs

$$V(\lambda_5)\otimes V(a\lambda_1) \;=\; \underset{(a-1)\lambda_1}{\bullet} \longrule \underset{(a-1)\lambda_1+\lambda_6}{\bullet} \longrule \underset{a\lambda_1+\lambda_5}{\bullet} \tag{4.56}$$

The Casimir values are

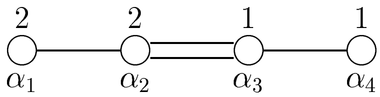
$$\begin{aligned}
C((a-1)\lambda_1) &= \frac{4}{3}(a-1)^2 + 16(a-1),\\
C(a\lambda_1+\lambda_5) &= \frac{4}{3}a^2 + \frac{52}{3}a + \frac{52}{3},\\
C((a-1)\lambda_1+\lambda_6) &= \frac{4}{3}(a-1)^2 + 18(a-1) + 24.
\end{aligned} \tag{4.57}$$

We obtain the R-matrices

$$\check{\mathbf{R}}^{\lambda_5,a\lambda_1}(x) = \check{\mathbf{P}}^{\lambda_5,a\lambda_1}_{a\lambda_1+\lambda_5} + \langle 2(a+1)\rangle\; \check{\mathbf{P}}^{\lambda_5,a\lambda_1}_{(a-1)\lambda_1+\lambda_6} + \langle 2(a+1)\rangle\,\langle a+11\rangle\; \check{\mathbf{P}}^{\lambda_5,a\lambda_1}_{(a-1)\lambda_1}. \tag{4.58}$$

4.6 Example for F_4

The Dynkin diagram for F_4 is



(4.59)

We choose the normalization $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2(\alpha_3, \alpha_3) = 2(\alpha_4, \alpha_4) = 4$ so that $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = (\alpha_2, \alpha_3) = (\alpha_3, \alpha_2) = 2(\alpha_3, \alpha_4) = 2(\alpha_4, \alpha_3) = -2$ with $(\alpha_i, \alpha_j) = 0$ for all other pairs. The fundamental weights are

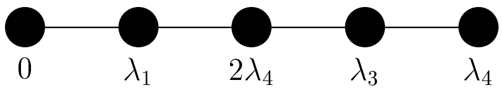
$$\begin{aligned}
\lambda_1 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\,, \\
\lambda_2 &= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4\,, \\
\lambda_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4\,, \\
\lambda_4 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\,.
\end{aligned} \tag{4.60}$$

The half sum of positive roots is

$$\rho = \sum_{i=1}^{4} \lambda_i = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4. \tag{4.61}$$

It can be shown that the minimal representation $V(\lambda_4)$ is affinizable. Thus we consider the following tensor product graph

$V(\lambda_4) \otimes V(\lambda_4) \; =$



(4.62)

On the modules appearing in the above diagram the Casimir has the values

$$C(\lambda_1) = 36, \quad C(2\lambda_4) = 52, \quad C(\lambda_3) = 48, \quad C(\lambda_4) = 24 \tag{4.63}$$

from which we obtain the R-matrix

$$\begin{aligned}
\check{\mathbf{R}}^{\lambda_4\lambda_4}(x) \;=\; & \check{\mathbf{P}}_0^{\lambda_4\lambda_4} + \langle -18\rangle\, \check{\mathbf{P}}_{\lambda_1}^{\lambda_4\lambda_4} + \langle -18\rangle\, \langle -8\rangle\, \check{\mathbf{P}}_{2\lambda_4}^{\lambda_4\lambda_4} \\
& + \langle -18\rangle\, \langle -8\rangle\, \langle 2\rangle\, \check{\mathbf{P}}_{\lambda_3}^{\lambda_4\lambda_4} + \langle -18\rangle\, \langle -8\rangle\, \langle 2\rangle\, \langle 12\rangle\, \check{\mathbf{P}}_{\lambda_4}^{\lambda_4\lambda_4}.
\end{aligned} \tag{4.64}$$

5 Discussion

In this paper we have presented a systematic procedure for obtaining trigonometric solutions $R(x) \in \mathrm{End}(V(\lambda) \otimes V(\mu))$ to the QYBE for a quantum simple Lie algebra $U_q(\mathcal{G})$, where $V(\lambda)$, $V(\mu)$ are irreducible $U_q(\mathcal{G})$-modules which are affinizable and whose tensor product decomposition is multiplicity-free. Our method relies on the extended tensor product graph and certain subgraphs. An important role is played by parities, given by the eigenvalues of the permutation operator for the case $\lambda = \mu$, and which we extended to the case $\lambda \neq \mu$ in the paper. As noted in the introduction spectral dependent R-matrices obtained this way are central to the theory of quantum integrable systems occuring in statistical mechanics, the quantum inverse scattering method and quantum field theory.

Our approach, which is a generalization of that proposed in [14] for the case $\lambda = \mu$, enables the construction of a large number of new R-matrices. As explicit examples we have considered the case where $V(\lambda),\ V(\mu)$ correspond to the symmetric or antisymmetric tensor representations for $\mathcal{G} = A_n$, the spinor irreps or the symmetric traceless tensor irreps for $\mathcal{G} = B_n,\ D_n$, all fundamental irreps for $\mathcal{G} = C_n$ and certain irreps for the exceptional Lie algebras $\mathcal{G} = E_6,\ F_4$.

It should be emphasized that although all finite dimensional irreps of $U_q(\mathcal{G})$ are known to be affinizable for the case $\mathcal{G} = A_n$, this is not true in general for other finite dimensional simple Lie algebras. In this paper we have concentrated on affinizable representations. There are however many more finite-dimensional irreducible representations of $U_q(\hat{\mathcal{G}})$ which are obtained not by affinizing $U_q(\mathcal{G})$ irreducible representations but by affinizing $U_q(\mathcal{G})$ reducible ones. These can be constructed by reduction of tensor product representations [13]. In future work we aim to apply the technique of this paper to obtain the trigonometric R-matrices in these representations. They are important for example for the construction of soliton S-matrices in affine Toda theories.

Finally it is of interest to consider extensions to quantum superalgebras of relevance to supersymmetric quantum integrable systems. Particularly interesting are the type-I quantum superalgebras which admit one parameter families of finite dimensional irreps [48]. The R-matrices arising from these irreps thus automatically contain an extra non-additive parameter [49] corresponding to one parameter families of exactly solvable models.

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A Normalization Convention

We have seen that for the universal R-matrix R for $U_q(\hat{\mathcal{G}})$,

$$R(x) \equiv (D_x \otimes I)(R) \tag{A.1}$$

so that in general

$$R^{\lambda\mu}(x) = g_q(x)(\pi_\lambda \otimes \pi_\mu)(R(x)) \tag{A.2}$$

which follows from the uniqueness of solutions to JEs. Here $g_q(x)$ is an overall scalar function. We thus have in particular

$$\lim_{x\to 0} R^{\lambda\mu}(x) = g_q(0)(\pi_\lambda \otimes \pi_\mu)(\bar{R}) \tag{A.3}$$

for some non-zero $g_q(0)$, with $\bar{R} \equiv R(0)$ the universal R-matrix for $U_q(\mathcal{G})$.

By Schur's lemma, the R-matrix defined above satisfies the unitarity condition

$$(R^T)^{\lambda\mu}(x)R^{\lambda\mu}(x^{-1}) = c_q(x) \cdot I \tag{A.4}$$

where $c_q(x)$ is some overall scalar function, due to the irreducibility of the representation $(\pi_\lambda \otimes \pi_\mu)(D_x \otimes 1)$ for generic x and the fact that the left hand side of above equation intertwines the coproduct of $U_q(\hat{\mathcal{G}})$.

In the context we assume $R^{\lambda\mu}(x)$ normalized so that

$$(R^T)^{\lambda\mu}(x)R^{\lambda\mu}(x^{-1}) = I. \tag{A.5}$$

However this does not uniquely specify $R^{\lambda\mu}(x)$ since it may be multiplied by an arbitrary function $f_q(x)$ such that

$$f_q(x) \cdot f_q(x^{-1}) = 1. \tag{A.6}$$

In particular, choosing

$$f_q(x) = \frac{x + g_q(0)^{-1}}{1 + g_q(0)^{-1}x} \tag{A.7}$$

we obtain an R-matrix $f_q(x)R^{\lambda\mu}(x)$ satisfying both equation (A.5) as well as

$$\lim_{x\to 0} f_q(x)R^{\lambda\mu}(x) = (\pi_\lambda \otimes \pi_\mu)(\bar{R}). \tag{A.8}$$

Hence without loss of generality we may assume $R^{\lambda\mu}(x)$ satisfies (A.5) <u>and</u> the limiting condition

$$\lim_{x\to 0} R^{\lambda\mu}(x) = (\pi_\lambda \otimes \pi_\mu)(\bar{R}) \tag{A.9}$$

as was assumed above in the context.

It is also important to note that the universal R-matrix $\bar{R}$ for $U_q(\mathcal{G})$ is normalized so that

$$\lim_{q\to 1} \bar{R} = I \tag{A.10}$$

and

$$\Delta(\bar{R}^T\bar{R}) = (v \otimes v)\Delta(v^{-1}) \tag{A.11}$$

where v is the canonical Casimir with the eigenvalue $q^{-(\lambda,\lambda+2\rho)}$ on the irreducible $U_q(\mathcal{G})$-module $V(\lambda)$.

B Parity

In this appendix we will clarify the parities of module $V(\nu)$ in $V(\lambda) \otimes V(\mu)$ by finding a parity operator in this case.

From the properties of the "projections" $\check{\mathbf{P}}_{\nu}^{\lambda\mu}$ we note that

$$\check{\mathbf{R}}^{\lambda\mu}(1)\check{\mathbf{P}}_{\nu}^{\mu\lambda} = \check{\mathbf{P}}_{\nu}^{\lambda\mu}\check{\mathbf{R}}^{\mu\lambda}(1) \tag{B.1}$$

from which it follows

$$\check{\mathbf{R}}^{\lambda\mu}(1)\check{\mathbf{R}}^{\mu\lambda}(x) = \check{\mathbf{R}}^{\lambda\mu}(x)\check{\mathbf{R}}^{\mu\lambda}(1). \tag{B.2}$$

We have

$$(R^T)^{\lambda\mu}(1)R^{\lambda\mu}(x) = \check{\mathbf{R}}^{\mu\lambda}(1)\check{\mathbf{R}}^{\lambda\mu}(x) = \sum_{\nu} \rho_{\nu}(x)\mathcal{P}_{\nu}^{\lambda\mu} \tag{B.3}$$

which gives rise to

$$R^{\lambda\mu}(x) = R^{\lambda\mu}(1)\sum_{\nu} \rho_{\nu}(x)\mathcal{P}_{\nu}^{\lambda\mu} \tag{B.4}$$

which gives a direct expression for $R^{\lambda\mu}(x)$ on $V(\lambda)\otimes V(\mu)$. It appears therefore that $R^{\lambda\mu}(1)$ plays a role on $V(\lambda) \otimes V(\mu)$ analogous to the permutation operator when $\lambda = \mu$.

Taking the limit $q \to 1$:

$$R^{\lambda\mu}(0)\Big|_{q=1} \equiv \bar{R}^{\lambda\mu}\Big|_{q=1} = (\pi_{\lambda} \otimes \pi_{\mu})(\bar{R})\Big|_{q=1} = I \tag{B.5}$$

thus by (B.3)

$$(R^T)^{\lambda\mu}(1)\Big|_{q=1} = \sum_{\nu} \epsilon(\nu)\mathcal{P}_{\nu}^{(0)\lambda\mu} \tag{B.6}$$

where $\mathcal{P}_{\nu}^{(0)\lambda\mu} = \mathcal{P}_{\nu}^{\lambda\mu}|_{q=1}$. Since $(R^T)^{\lambda\mu}(1) = R^{\lambda\mu}(1)^{-1}$ it follows that at $q = 1$

$$R^{\lambda\mu}(1) = \sum_{\nu} \epsilon(\nu)\mathcal{P}_{\nu}^{(0)\lambda\mu} \tag{B.7}$$

which squares to the identity:

$$R^{\lambda\mu}(1)^2 = I, \qquad \text{at } q = 1 \tag{B.8}$$

It thus follows that the parities are determined by the eigenvalues of the operator

$$P \equiv R^{\lambda\mu}(1)|_{q=1}, \qquad P^2 = I \tag{B.9}$$

on the space $V(\lambda) \otimes V(\mu)$ at $q = 1$. It remains to give a natural geometric interpretation of this parity operator which, in the case $\lambda = \mu$, concides with the usual permutation operator. This is the problem addressed below.

It follows from (3.19), (3.20) and (3.21) that for $\nu \neq \nu'$

$$\check{\mathbf{P}}_{\nu}^{\lambda\mu}\left(\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2})\right)\mathcal{P}_{\nu'}^{\lambda\mu} = \mathcal{P}_{\nu}^{\mu\lambda}\left(\pi_\mu(e_0)\otimes\pi_\lambda(q^{h_0/2})\right)\check{\mathbf{P}}_{\nu'}^{\lambda\mu}. \tag{B.10}$$

which gives rise to

$$R^{\lambda\mu}(1)\mathcal{P}_{\nu}^{\lambda\mu}\left(\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2})\right)\mathcal{P}_{\nu'}^{\lambda\mu} = \bar{\mathcal{P}}_{\nu}^{\mu\lambda}\left(\pi_\lambda(q^{h_0/2})\otimes\pi_\mu(e_0)\right)\bar{\mathcal{P}}_{\nu'}^{\mu\lambda}R^{\lambda\mu}(1) \tag{B.11}$$

where $\bar{\mathcal{P}}^{\lambda\mu} \equiv P^{\mu\lambda}\mathcal{P}_{\nu}^{\mu\lambda}P^{\lambda\mu}$ is the projection onto submodule $\bar{V}(\nu)$ of $V(\lambda)\otimes V(\mu)$ determined by the opposite coproduct $\Delta^{'\lambda\mu}$ on the tensor product space.

We now consider solely the case $q=1$. In this case the coproduct and the opposite coproduct coincide so the above equation reduces to (omitting π_λ and π_μ below)

$$P\mathcal{P}_{\nu}^{(0)\lambda\mu}\left(e_0\otimes 1\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu} = \mathcal{P}_{\nu}^{(0)\lambda\mu}\left(1\otimes e_0\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu}P\,, \qquad \nu\neq\nu' \tag{B.12}$$

and, since $P^2=1$, we similarily have

$$P\mathcal{P}_{\nu}^{(0)\lambda\mu}\left(1\otimes e_0\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu} = \mathcal{P}_{\nu}^{(0)\lambda\mu}\left(e_0\otimes 1\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu}P\,, \qquad \nu\neq\nu'. \tag{B.13}$$

Moreover, since P is an $\mathcal{G}$-invariant we must have, for $\nu\neq\nu'$

$$\begin{aligned}
P\mathcal{P}_{\nu}^{(0)\lambda\mu}\left(a\otimes 1\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu} &= \mathcal{P}_{\nu}^{(0)\lambda\mu}\left(1\otimes a\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu}P \\
P\mathcal{P}_{\nu}^{(0)\lambda\mu}\left(1\otimes a\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu} &= \mathcal{P}_{\nu}^{(0)\lambda\mu}\left(a\otimes 1\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu}P, \quad \forall a\in\mathcal{G}.
\end{aligned} \tag{B.14}$$

We now observe that $a\otimes 1 - 1\otimes a$ <u>reverses</u> the parity since

$$P\mathcal{P}_{\nu}^{(0)\lambda\mu}\left(a\otimes 1-1\otimes a\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu} = -\mathcal{P}_{\nu}^{(0)\lambda\mu}\left(a\otimes 1-1\otimes a\right)\mathcal{P}_{\nu'}^{(0)\lambda\mu}P\,, \quad \nu\neq\nu'. \tag{B.15}$$

This suggests we define a sequence of $\mathcal{G}$-modules $V_i,\ \bar{V}_i$ recursely according to

$$\begin{aligned}
\bar{V}_i &= \bar{V}_{i-1}\bigoplus V_i, \quad (V_i \text{ the orthocomplement of } \bar{V}_{i-1} \text{ in } \bar{V}_i) \\
\bar{V}_i &= \mathrm{span}\{(a\otimes 1-1\otimes a)v | v\in\bar{V}_{i-1},\ a\in\mathcal{G}\}
\end{aligned} \tag{B.16}$$

starting with $\bar{V}_0 = V_0 = V(\lambda+\mu)$. We then obtain a $\mathcal{G}$-module direct sum decomposition

$$V(\lambda)\otimes V(\mu) = \bigoplus_{i=0}^{k} V_i \tag{B.17}$$

where $\bar{V}_k \equiv V(\lambda)\otimes V(\mu)$ is uniquely determined by

$$(a\otimes 1-1\otimes a)\bar{V}_k \subseteq \bar{V}_k, \quad \forall a\in\mathcal{G} \tag{B.18}$$

We assume the highest component $V(\lambda+\mu)$ with maximal weight vector $v_+^\lambda\otimes v_+^\mu$ has positive parity (by convention). The submodules with positive and negative parities are then given respectively by

$$\begin{aligned}
V_+ &= V_0\bigoplus V_2\bigoplus\cdots \\
V_- &= V_1\bigoplus V_3\bigoplus\cdots
\end{aligned} \tag{B.19}$$

It follows that

$$V(\lambda)\otimes V(\mu)=V_+\bigoplus V_-. \tag{B.20}$$

Thus we have given a method how to determine the parity of $V(\nu)\subset V(\lambda)\otimes V(\mu)$, which, with the help of $R^{\lambda\mu}(1)$, is analogous to the case of $\lambda=\mu$.

We are now to give a necassary condition for eq.(3.23). Given two irreducible modules $V(\nu),\ V(\nu')\subseteq V(\lambda)\otimes V(\mu)$, we have

$$\begin{aligned}\mathcal{P}^{(0)\lambda\mu}_{\nu}(e_0\otimes 1)\mathcal{P}^{(0)\lambda\mu}_{\nu'} &= \frac{1}{2}\mathcal{P}^{(0)\lambda\mu}_{\nu}(e_0\otimes 1+1\otimes e_0)\mathcal{P}^{(0)\lambda\mu}_{\nu'}\\ &\quad+\frac{1}{2}\mathcal{P}^{(0)\lambda\mu}_{\nu}(e_0\otimes 1-1\otimes e_0)\mathcal{P}^{(0)\lambda\mu}_{\nu'}\\ &= \frac{1}{2}\mathcal{P}^{(0)\lambda\mu}_{\nu}(e_0\otimes 1-1\otimes e_0)\mathcal{P}^{(0)\lambda\mu}_{\nu'},\qquad \forall\nu\neq\nu'\end{aligned} \tag{B.21}$$

Thus, for $\nu\neq\nu'$, the left hand side of the above equation can be non-vanishing only if $\nu,\ \nu'$ have opposite parity. Therefore following the similar arguments to [14] we have

Proposition: for any $\nu\neq\nu',\quad \mathcal{P}^{\lambda\mu}_{\nu}\left(\pi_\lambda(e_0)\otimes\pi_\mu(q^{h_0/2})\right)\mathcal{P}^{\lambda\mu}_{\nu'}\neq 0$ only if the parities $\epsilon(\nu),\ \epsilon(\nu')$ associated respectively with $\nu,\ \nu'$ are opposite, i.e. $\epsilon(\nu)\,\epsilon(\nu')=-1$.

C "Projectors"

In this appendix we show that the "projectors" defined in the paper may be determined by using pure representation theory of $U_q(\mathcal{G})$. We only list the results below. The details will be published in a forthcoming paper [47].

Let $\{|e^\nu_\alpha(q)\rangle_{\lambda\otimes\mu}\}$ be a symmetry adapted orthonormal basis for $V(\nu)\subset V(\lambda)\otimes V(\mu)$ under the action defined by the coproduct Δ. Then obviously $\mathcal{P}^{\lambda\mu}_{\nu}$ may expressed as

$$\mathcal{P}^{\lambda\mu}_{\nu}=\sum_\alpha |e^\nu_\alpha(q)\rangle_{\lambda\otimes\mu}\ {}_{\lambda\otimes\mu}\langle e^\nu_\alpha(q)| \tag{C.1}$$

Moreover, it can be shown that $\{|e^\nu_\alpha(q^{-1})\rangle_{\lambda\otimes\mu}\}$ describes a symmetry adapted orthonormal basis under the action defined by the opposite coproduct Δ'.

We define the symmetry adapted basis for $V(\nu)\subset V(\mu)\otimes V(\lambda)$ by

$$|e^\nu_\alpha(q)\rangle_{\mu\otimes\lambda}=\epsilon(\nu)P^{\lambda\mu}\left|e^\nu_\alpha(q^{-1})\right\rangle_{\lambda\otimes\mu}, \tag{C.2}$$

which is easily seen to be orthonormal so that

$${}_{\mu\otimes\lambda}\langle e^\nu_\alpha(q)|=\epsilon(\nu)\ {}_{\lambda\otimes\mu}\left\langle e^\nu_\alpha(q^{-1})\right|P^{\mu\lambda}. \tag{C.3}$$

It can be proven that for real generic $q > 0$, the "projectors" $\check{\mathbf{P}}_\nu^{\lambda\mu}$ may be written as

$$\check{\mathbf{P}}_\nu^{\lambda\mu} = \sum_\alpha |e_\alpha^\nu(q)\rangle_{\mu\otimes\lambda\ \lambda\otimes\mu}\langle e_\alpha^\nu(q)| \tag{C.4}$$

This implies that for real generic $q > 0$ the operators $\check{\mathbf{P}}_\nu^{\lambda\mu}$ defined in section 3 can be computed by pure representation theory (say, CG coefficients) of $U_q(\mathcal{G})$.

By analytic continuation, the above result should extend to all complex q.

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