Finite dimensional representations of quantum affine algebras

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Abstract

We give a general construction for finite dimensional representations of $U_q(\hat{\mathcal{G}})$ where $\hat{\mathcal{G}}$ is a non-twisted affine Kac-Moody algebra with no derivation and zero central charge. At $q = 1$ this is trivial because $U(\hat{\mathcal{G}}) = U(\mathcal{G}) \otimes \mathbf{C}(x, x^{-1})$ with $\mathcal{G}$ a finite dimensional Lie algebra. But this fact no longer holds after quantum deformation. In most cases it is necessary to take the direct sum of several irreducible $U_q(\mathcal{G})$-modules to form an irreducible $U_q(\hat{\mathcal{G}})$-module which becomes reducible at $q = 1$. We illustrate our technique by working out explicit examples for $\hat{\mathcal{G}} = \hat{C}_2$ and $\hat{\mathcal{G}} = \hat{G}_2$. These finite dimensional modules determine the multiplet structure of solitons in affine Toda theory.

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1 Introduction

One reason for the importance of quantum algebras $U_q(\mathcal{G})$ in mathematical physics is their relation to the Yang-Baxter equation: each intertwiner (R-matrix) for the tensor product of two finite dimensional representations of a quantum algebra provides a solution to the quantum Yang-Baxter equation [1, 2, 4, 3, 5].

There are at least two areas where it is important to know solutions of the spectral parameter *dependent* Yang-Baxter equation. One are integrable lattice models, where the existence of commuting transfer matrices follows if the Boltzman weights satisfy the spectral parameter dependent Yang-Baxter equation. The other are massive integrable quantum field theories where the spectral parameter dependent Yang-Baxter equation is the consistency condition of the 2-particle factorization of the scattering matrix. The spectral parameter in this case is the rapidity of the particles.

The R-matrices $\check{R}_{ab}$ for $U_q(\mathcal{G})$, where $\mathcal{G}$ is a finite dimensional simple Lie algebra, provide solutions to the Yang-Baxter equation *without* a spectral parameter. Here a, b are the labels of the representation spaces $V_a,\ V_b$, i.e. $\check{R}_{ab}$ is the intertwiner : $V_a \otimes V_b \to V_b \otimes V_a$. The interesting question is: when can a spectral parameter be introduced into $\check{R}_{ab}$ so as to obtain a solution of the spectral parameter *dependent* Yang-Baxter equation. The answer is: whenever $V_a,\ V_b$ carry representations also of the quantum affine algebra $U_q(\hat{\mathcal{G}})$. Here $\hat{\mathcal{G}}$ is the affine algebra $\mathcal{G}\otimes\mathbf{C}(x, x^{-1})$. The parameter x then consistently provides the spectral parameter. [1, 2, 4, 3, 5].

This paper is devoted to studying in which cases this affinization is possible, i.e. which finite dimensional representations spaces of $U_q(\mathcal{G})$ carry representations also of $U_q(\hat{\mathcal{G}})$.

For $\mathcal{G} = A_n$ all representations are affinizable [4] (see also the appendix of [6]). For other algebras this is not the case. Frenkel & Reshetikhin [7] state that "one generally has to enlarge [an irrep] V_λ by adding certain 'smaller' irreducible representations in order to extend the resulting representation to $U_q(\hat{\mathcal{G}})$. An explicit description of this extension is an important open problem." As far as we know, such a description is still lacking.

We begin in section 2 by defining $U_q(\mathcal{G})$ and $U_q(\hat{\mathcal{G}})$. Then in section 3 we give some concrete examples where two irreps of $U_q(\mathcal{G})$ have to be added together to obtain an irrep of $U_q(\hat{\mathcal{G}})$. The first example we choose is the 10-dimensional representation of $U_q(C_2)$ which has to be enlarged by the singlet representation to give an 11-dimensional irrep of $U_q(\hat{C}_2)$. The second one is the 14-dimensional representation of $U_q(G_2)$ which again has to be enlarged by the trivial representation to give a 15-dimensional irrep of $U_q(\hat{G}_2)$. In section

4 we present our general procedure for obtaining irreps of $U_q(\hat{\mathcal{G}})$. Our method is based on the reduction of tensor products of smaller representations. It is therefore very much in the spirit of the fusion procedure used to construct rational [8] and trigonometric [4] R-matrices. The technical device which we will use is the tensor product graph [9]. In section 5 we illustrate our general method again in the cases of $U_q(\hat{C}_2)$ and $U_q(\hat{G}_2)$.

Our physical motivation for this study of finite dimensional representations of quantum affine algebras comes from the desire to gain a better understanding of the solitons in affine Toda quantum field theory. These solitons transform in such representations and we will come back to that point in the discussions in section 6.

2 Definition of quantum algebras

A simple Lie algebra $\mathcal{G}$ is defined through its simple roots $\alpha_i,\ i = 1\dots r$ by the following relations between its Chevalley generators $h_i, e_i, f_i,\ i = 1\dots r$

$$\begin{array}{lll} [h_i, e_j] = (\alpha_i, \alpha_j)e_j, & [h_i, f_j] = -(\alpha_i, \alpha_j)f_j, & \\ [e_i, f_j] = \delta_{ij}h_j, & & \\ (\mathrm{ad}e_i)^{1-a_{ij}}e_j = 0, & (\mathrm{ad}f_i)^{1-a_{ij}}f_j = 0, & (i \neq j) \end{array} \tag{2.1}$$

where $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. The universal enveloping algebra $U(\mathcal{G})$ is the algebra generated freely by the Chevalley generators modulo the relations eq. (2.1). The quantum algebra $U_q(\mathcal{G})$ is a deformation of this [1, 2] where eq. (2.1) is modified to

$$\begin{array}{lll} [h_i, e_j] = (\alpha_i, \alpha_j)e_j, & [h_i, f_j] = -(\alpha_i, \alpha_j)f_j, & \\ [e_i, f_j] = \delta_{ij}[h_j]_q, & & \\ (\mathrm{ad}_q e_i)^{1-a_{ij}}e_j = 0, & (\mathrm{ad}_q f_i)^{1-a_{ij}}f_j = 0, & (i \neq j) \end{array} \tag{2.2}$$

We have introduced the notation

$$[u]_q = \frac{q^u - q^{-u}}{q - q^{-1}}, \tag{2.3}$$

and ad_q is a q-commutator

$$(\mathrm{ad}_q e_i)e_j \equiv [e_i, e_j]_q \equiv e_i e_j - q^{(\alpha_i,\alpha_j)}e_j e_i. \tag{2.4}$$

The most important feature of this deformation is that it is still a Hopf-algebra. The deformed comultiplication is

$$\begin{array}{rcl} \Delta(h_i) & = & h_i \otimes 1 + 1 \otimes h_i, \\ \Delta(e_i) & = & e_i \otimes q^{h_i/2} + q^{-h_i/2} \otimes e_i, \\ \Delta(f_i) & = & f_i \otimes q^{h_i/2} + q^{-h_i/2} \otimes f_i. \end{array} \tag{2.5}$$

The finite dimensional representations of $U_q(\mathcal{G})$ have been studied by Rosso [11] and Lusztig [12]. They found that, for q not a root of unity, the representation theory of $U_q(\mathcal{G})$ is exactly analogous to that of $\mathcal{G}$. Each finite-dimensional irreducible $\mathcal{G}$-module also carries an irrep of $U_q(\mathcal{G})$ and the irreps of $U_q(\mathcal{G})$ are simply deformations of those of $\mathcal{G}$.

Non-twisted affine Lie algebras $\tilde{\mathcal{G}}$, as defined by Kac [10], can be realized as $\tilde{\mathcal{G}} = \mathcal{G} \otimes \mathbf{C}(x, x^{-1}) \oplus \mathbf{C}c \oplus \mathbf{C}d$, where $\mathbf{C}(x, x^{-1})$ is the algebra of Laurent polynomials in x, c is a central charge and d is a derivation. In this paper we are only interested in the algebra $\hat{\mathcal{G}} = \mathcal{G} \otimes \mathbf{C}(x, x^{-1}) \oplus \mathbf{C}c$ obtained from $\tilde{\mathcal{G}}$ by dropping the derivation. The algebra with derivation does not have finite dimensional representations. Following a widespread custom in the literature we will call also the algebra $\hat{\mathcal{G}}$ an affine algebra. From a finite dimensional representation π of $\hat{\mathcal{G}}$ one can easily obtain a loop representation of the algebra with derivation $\tilde{\mathcal{G}}$.

To generate the affine algebra $\hat{\mathcal{G}}$ it is sufficient to add one more pair of raising and lowering operators and one more Cartan subalgebra generator to the Chevalley basis, namely

$$e_0 = f_\psi \otimes x, \qquad f_0 = e_\psi \otimes x^{-1}, \qquad h_0 = (c - h_\psi) \otimes 1, \tag{2.6}$$

where ψ is the highest root of $\mathcal{G}$ and e_ψ, f_ψ are the corresponding raising and lowering operators [10]. The new Chevalley generators again satisfy the relations eq. (2.1), this time with $i, j = 0 \dots r$ and $\alpha_0 = -\psi$. The central charge c will play no role in this paper because it is represented as zero on all finite dimensional modules.

The *quantum* affine algebra $U_q(\hat{\mathcal{G}})$ is defined analogously by the relations eq. (2.2). There is one important difference between $U(\hat{\mathcal{G}})$ and $U_q(\hat{\mathcal{G}})$, i.e. between the classical and the quantum case. Classically e_0 and f_0 are elements of $U(\mathcal{G}) \otimes \mathbf{C}(x, x^{-1})$, see eq. (2.6), and thus

$$U(\hat{\mathcal{G}}) = U(\mathcal{G}) \otimes \mathbf{C}(x, x^{-1}) \oplus \mathbf{C}c. \tag{2.7}$$

In the quantum case however, generically e_0 and f_0 are not elements of $U_q(\mathcal{G}) \otimes \mathbf{C}(x, x^{-1})$, as will be seen in the next section. Thus

$$U_q(\hat{\mathcal{G}}) \neq U_q(\mathcal{G}) \otimes \mathbf{C}(x, x^{-1}) \oplus \mathbf{C}c. \tag{2.8}$$

The only known exceptions to this are $\mathcal{G} = A_n$ [4] (see also the appendix of [6] for details).

Because of eq. (2.7), any $U(\mathcal{G})$-module is also a $U(\hat{\mathcal{G}})$-module on which x and c are represented trivially. This is no longer true in the quantum case. Some representations spaces of $U_q(\mathcal{G})$ may not carry a representation of e_0 and f_0. Obviously those and only those representations spaces which carry a representation of e_0 and f_0 carry a representation of $U_q(\hat{\mathcal{G}})$. It is the aim of this paper to construct such representations.

3 Examples of representations

The easiest way to prove that e_0 is not an element of $U_q(\mathcal{G}) \otimes \mathbf{C}(x, x^{-1})$ in general is to give some simple examples of representation spaces of $U_q(\mathcal{G})$ which do not carry a representation of e_0 and f_0. As we will see, one usually has to take a direct sum of two (or more) irreps of $U_q(\mathcal{G})$ to form an irrep of $U_q(\hat{\mathcal{G}})$.

The first simple example is the 10-dimensional irrep of $U_q(C_2)$. C_2 has two simple roots α_1 and α_2 which satisfy $2(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = -2(\alpha_1,\alpha_2) = -2(\alpha_2,\alpha_1) = 2$. The 10-dimensional representation is the adjoint representation and its weights are the roots $\{2\alpha_1{+}\alpha_2\,,\ \alpha_1{+}\alpha_2\,,\ \alpha_1\,,\ \alpha_2\,,\ 0\,,\ 0\,,\ -\alpha_2\,,\ -\alpha_1\,,\ -\alpha_1-\alpha_2\,,\ -2\alpha_1-\alpha_2\}$. The matrix forms of $h_1,\ h_2,\ e_1$ and e_2 in this representation are

$$\begin{aligned}
h_1 &= e_{11} - e_{33} + e_{44} - e_{77} + e_{88} - e_{10,10} \\
h_2 &= e_{22} + 2e_{33} - e_{44} + e_{77} - 2e_{88} - e_{99} \\
e_1 &= f_1^{\mathrm{t}} = e_{12} + e_{23} + e_{45} + e_{57} + e_{89} + e_{9,10} \\
e_2 &= f_2^{\mathrm{t}} = e_{24} + e_{35} + e_{58} + e_{79} + ([2]_q - 1)^{1/2}(e_{36} + e_{68})
\end{aligned} \tag{3.1}$$

where t stands for transpose and e_{ij} is the matrix with 1 in entry i,j and 0 elsewhere.

One would now like to find two other matrices e_0 and f_0 which satisfy the defining relations eq. (2.2). One can make a general Ansatz and then at first impose all relations except $[e_0, f_0] = [h_0]_q$ and the q-Serre relation involving e_0 and f_0. At this point one finds that e_0 and f_0 are already completely determined up to an overall constant. Unfortunately they do *not* satisfy $[e_0, f_0] = [h_0]_q$ and the q-Serre relations, and this shows that this irrep of $U_q(C_2)$ can not be extended to a representation of $U_q(\hat{C}_2)$.

Next we consider a direct sum of the 10-dimensional irrep with the trivial one-dimensional representation. For this 11-dimensional reducible representation of $U_q(C_2)$, the matrix form for $h_1,\ h_2,\ e_1$ and e_2 looks the same as above. Now it is possible to find matrices e_0 and f_0 satisfying all of the relations eq. (2.2):

$$\begin{aligned}
e_0 = f_0^{\mathrm{t}} \;&=\; e_{51} + e_{72} + e_{94} + e_{10,5} - ([2]_q - 1)^{-1/2}(e_{61} + e_{10,6}) + \\
&\quad + [2]_q^{1/2}\left(\frac{[2]_q - 2}{[2]_q - 1}\right)^{1/2}(e_{10,11} + e_{11,1}).
\end{aligned} \tag{3.2}$$

This representation of $U_q(\hat{C}_2)$ is seen to be irreducible. It becomes reducible at $q = 1$, as can be seen from the coefficient of the last term.

The second example we want to give is the 14-dimensional irrep of $U_q(G_2)$. The simple roots of G_2 are α_1 and α_2 which satisfy $(\alpha_1,\alpha_1) = 3(\alpha_2,\alpha_2) =$

$-2(\alpha_1,\alpha_2) = -2(\alpha_2,\alpha_1) = 6$. The 14-dimensional representation is the adjoint representation with weights equal to the roots $\{2\alpha_1+3\alpha_2\,,\ \alpha_1+3\alpha_2\,,\ \alpha_1+2\alpha_2\,,\ \alpha_1+\alpha_2\,,\ \alpha_1\,,\alpha_2\,,\ 0\,,0\,,\ -\alpha_2\,,\ -\alpha_1\,,\ -\alpha_1-\alpha_2\,,\ -\alpha_1-2\alpha_2\,,\ -\alpha_1-3\alpha_2\,,\ -2\alpha_1-3\alpha_2\}$. The matrix forms of $h_1,\ h_2,\ e_1$ and e_2 for the 14-dimensional irrep of $U_q(G_2)$ are

$$
\begin{aligned}
h_1 &= 3e_{11}-3e_{22}+3e_{44}+6e_{55}-3e_{66}+3e_{99} \\
&\quad -6e_{10,10}-3e_{11,11}+3e_{13,13}-3e_{14,14} \\
h_2 &= 3e_{22}+e_{33}-e_{44}-3e_{55}+2e_{66}-2e_{99} \\
&\quad +3e_{10,10}+e_{11,11}-e_{12,12}-3e_{13,13} \\
e_1 &= f_1^{\rm t} = [3]_q^{1/2}(e_{12}+e_{46}+e_{9,11}+e_{13,14})+\frac{[3]_q}{[2]_q^{1/2}}(e_{58}+e_{8,10}) \\
&\quad +[3]_q^{1/2}\left(\frac{[6]_q}{[3]_q}-\frac{[3]_q}{[2]_q}\right)^{1/2}(e_{57}+e_{7,10}) \\
e_2 &= f_2^{\rm t} = [3]_q^{1/2}(e_{23}+e_{45}+e_{10,11}+e_{12,13})+ \\
&\quad +([3]_q+1)^{1/2}(e_{34}+e_{11,12})+[2]_q^{1/2}(e_{68}+e_{89})
\end{aligned}
\tag{3.3}
$$

Again it can be shown that this irrep of $U_q(G_2)$ can not be extended to a representation of $U_q(\hat{G}_2)$. Next we consider the direct sum of this irrep with the trivial representation of $U_q(G_2)$. Obviously the matrix form of $h_1,\ h_2, e_1$ and e_2 for this 15-dimensional reducible representation are the same as (3.3). It turns out that this reducible representation can be extended to an irrep of $U_q(\hat{G}_2)$. The explicit expressions for e_0 and f_0 are:

$$
\begin{aligned}
e_0 = f_0^{\rm t} &= [3]_q^{1/2}\left(\frac{[6]_q}{[3]_q}-\frac{[3]_q}{[2]_q}\right)^{-1/2}(e_{71}+e_{14,7})+[3]_q^{1/2}(e_{10,2}+e_{11,3}+e_{12,4}+ \\
&\quad +e_{13,5})+\left(\frac{[6]_q^2[2]_q-[3]_q^2[6]_q-[3]_q^2[2]_q}{[6]_q[2]_q-[3]_q^2}\right)^{1/2}(e_{14,15}+e_{15,1})
\end{aligned}
\tag{3.4}
$$

which defines a 15-dimensional irrep of $U_q(\hat{G}_2)$. This irrep becomes reducible only when $q=1$.

4 General construction

Because e_0 does not exist as an element in $U_q(\mathcal{G})\otimes \mathbf{C}(x,x^{-1})$, we will have to construct $\pi(e_0)$ for each representation π separately. Clearly we can not proceed as in the previous section but need a general construction.

Let V_λ be an irreducible finite dimensional $\mathcal{G}$-module and $\pi_\lambda : U_q(\mathcal{G}) \to \mathrm{End}(V_\lambda)$ the representation of $U_q(\mathcal{G})$ which it carries. Assume that on this module it is possible to define $\pi_\lambda(e_0)$ and $\pi_\lambda(f_0)$ and thus make it into an irreducible representation of $U_q(\hat{\mathcal{G}})$. [1] We start with this irrep and want to construct, using it, further irreps of $U_q(\hat{\mathcal{G}})$. To this end we look at the tensor product $V_\lambda \otimes V_\lambda$ which carries the $U_q(\mathcal{G})$-representation

$$\Pi(g) = (\pi_\lambda \otimes \pi_\lambda)\Delta(g), \qquad g \in U_q(\mathcal{G}). \tag{4.1}$$

It is a reducible representation of $U_q(\mathcal{G})$ and it is known that the decomposition into irreps is the same as in the classical case [11, 12]

$$V_\lambda \otimes V_\lambda = \bigoplus_\mu V_\mu. \tag{4.2}$$

We want to see on which of these irreducible modules V_μ or on which direct sums of them we can define irreps of $U_q(\hat{\mathcal{G}})$. It can be checked that the following defines a representation of $U_q(\hat{\mathcal{G}})$ on $V_\lambda \otimes V_\lambda$

$$\begin{aligned}
\Pi^a(g) &= \Pi(g), \qquad g \in U_q(\mathcal{G}), &(4.3)\\
\Pi^a(e_0) &= (\pi_\lambda \otimes \pi_\lambda)(e_0 \otimes q^{h_0/2} + a q^{-h_0/2} \otimes e_0), &(4.4)\\
\Pi^a(f_0) &= (\pi_\lambda \otimes \pi_\lambda)(f_0 \otimes q^{h_0/2} + a^{-1} q^{-h_0/2} \otimes f_0), &(4.5)\\
\Pi^a(h_0) &= (\pi_\lambda \otimes \pi_\lambda)(h_0 \otimes 1 + 1 \otimes h_0), &(4.6)
\end{aligned}$$

for any choice of the parameter $a \in \mathbf{C}$. We will see that for generic value of a the representation Π^a is irreducible but that it becomes reducible for special values and at these values we can define irreducible representations on submodules of $V_\lambda \otimes V_\lambda$.

To visualize the reducibility of the representation Π^a we describe it by a directed graph. A similar graph, called the tensor product graph, was first introduced in [9] and we will rely heavily on ideas from that paper.

Definition 1 *The* **reducibility graph** G^a *associated to the representation* Π^a *of* $U_q(\hat{\mathcal{G}})$ *is a directed graph whose vertices are the irreducible* $\mathcal{G}$*-modules* V_μ *appearing in the decomposition eq. (4.2) of* $V_\lambda \otimes V_\lambda$*. There is an edge directed from a vertex* V_ν *to a vertex* V_μ *iff*

$$P_\mu \Pi^a(e_0) P_\nu \neq 0 \qquad \text{or} \qquad P_\mu \Pi^a(f_0) P_\nu \neq 0, \tag{4.7}$$

where P_μ *is the projector from* $V_\lambda \otimes V_\lambda$ *onto* V_μ*.*

For an example of a reducibility graph see figure 1. According to the definition, an arrow from V_ν to V_μ indicates that $\Pi^a(e_0)$ or $\Pi^a(f_0)$ can bring us

[1] Examples are the "undeformed" representations for $U_q(\mathcal{G})$ with $\mathcal{G} = A_n,\ B_n,\ C_n,\ D_n,\ E_6$ and E_7, which are affinizable, that is e_0 and f_0 exist for those representations, and the minimal representations for $U_q(G_2)$ and $U_q(F_4)$, which are deformed representations but nevertheless are affinizable.

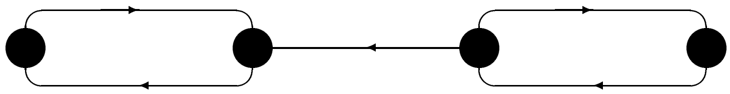

Figure 1: The reducibility graph G^a for the $7 \otimes 7$ of $U_q(G_2)$ at $a = q^2$. The vertices are labeled by the dimension of the corresponding irrep of $U_q(G_2)$.

from the module V_ν to the module V_μ. It implies that a $U_q(\hat{\mathcal{G}})$-submodule of $V_\lambda \otimes V_\lambda$ which contains V_ν also has to contain V_μ. In other words, the $U_q(\hat{\mathcal{G}})$-submodules are described by those subgraphs from which no arrows point outside that subgraph. We formulate this in the next definition and theorem.

Definition 2 *A subgraph G' of a graph G is called*

- **two-way connected** *if for any pair V_μ, V_ν of vertices in G' there exist directed paths from V_μ to V_ν and from V_ν to V_μ.*
- **simply two-way connected** *if it is two-way connected and becomes non-two-way connected if any edge is removed,*
- **closed** *if there is no edge pointing from any vertex in G' to a vertex outside G'.*

Theorem 1 *Every closed subgraph G' of a reducibility graph G^a defines a representation (V', π') of $U_q(\hat{\mathcal{G}})$. The representation space V' is the direct sum of the irreducible $\mathcal{G}$-modules corresponding to the vertices in G'*

$$V' = \bigoplus_{V_\mu \in G'} V_\mu \tag{4.8}$$

Let P' be the projector from $V_\lambda \otimes V_\lambda$ onto V': $P' = \sum_{V_\mu \in G'} P_\mu$. The representation map $\pi' : U_q(\hat{\mathcal{G}}) \to End(V')$ is given by

$$\pi'(g) = P' \Pi^a(g) P' \qquad g \in U_q(\hat{\mathcal{G}}). \tag{4.9}$$

If the subgraph G' is two-way connected then the representation π' is irreducible.

Proof. We only need to show that eq. (4.9) defines a representation using the fact that Π^a does. This becomes trivial with the following observation: Because G' is closed, we know that $\Pi^a(e_0)v' \in V'$ for all $v' \in V'$ and similarly for f_0. Also $\Pi^a(g)v' \in V'$ for all $g \in U_q(\mathcal{G})$ because V' is a sum of $U_q(\mathcal{G})$-modules. Together this gives that $\Pi^a(g)v' \in V'$ for all $g \in U_q(\hat{\mathcal{G}})$. Combining this with $P'v' = v'$ we have that $P'\Pi^a(g)P' = \Pi^a(g)P'$ and therefore the projectors P' can be pulled outside in checking that the relations eq. (2.2) are satisfied. The irreducibility of π' follows immediately from the two-way connectedness of G'.

$\square$

Applying this theorem to the example in figure 1 we see that there is a 15-dimensional irreducible representation of $U_q(\hat{G}_2)$. This reproduces the representation which we found in section 3.

The usefulness of Theorem 1 lies in the fact that the reducibility graph encodes the reducibility of the tensor product and can in most cases be easily constructed using only elementary Lie-algebra representation theory. The facts needed for this purpose are contained in the following lemmas.

Lemma 2 *The reducibility graph G^a is two-way connected for generic values of a. It can be non-two-way connected only if*

$$a = q^{\frac{C(\mu)-C(\nu)}{2}} \tag{4.10}$$

where $C(\lambda) = (\lambda, \lambda + 2\rho)$ is the value of the quadratic Casimir on V_λ.

Proof. Here we can follow [9], who defined a similar graph. For clarity we first consider the classical case $q = 1$. To make the notation simpler we will from now on drop the π_λ and write simply e_i instead of $\pi_\lambda(e_i)$ etc. V_λ was irreducible by definition. This means that by repeatedly acting with the tensor operators $T = \{g \otimes 1 | g \in \mathcal{G}\}$ and $\bar{T} = \{1 \otimes g | g \in \mathcal{G}\}$ we can obtain any vector in $V_\lambda \otimes V_\lambda$ from any other. In particular these tensor operators connect together all irreducible $\mathcal{G}$-modules V_μ contained in $V_\lambda \otimes V_\lambda$. Now $e_0 \otimes 1$ and $1 \otimes e_0$ are just the lowest components of these tensor operators (because at $q = 1$ $e_0 = f_\psi$) and therefore also connect together all modules V_μ. Furthermore $e_0 \otimes 1$ by itself or a linear combination of $e_0 \otimes 1$ with $1 \otimes e_0$ will suffice because of the proportionality

$$P_\mu(e_0 \otimes 1)P_\nu = -P_\mu(1 \otimes e_0)P_\nu \qquad \text{(at } q = 1\text{)}, \tag{4.11}$$

which follows from the fact that P_μ commutes with $e_0 \otimes 1 + 1 \otimes e_0$. This shows for $q = 1$ that $\Pi^a(e_0) = e_0 \otimes 1 + a\, 1 \otimes e_0$ connects all irreps in $V_\lambda \otimes V_\lambda$ unless $a = 1$. At $a = 1$ $P_\mu \Pi^a(e_0) P_\nu$ is always zero according to eq. (4.11). Exactly the same can be said about f_0. Thus at $q = 1$ G^a is two-way connected except at $a = 1$ where it is completely disconnected. This complete disconnectedness at $a = 1$ implies according to Lemma 1 that every irreducible $\mathcal{G}$-module appearing in the tensor product carries a representation of $U(\hat{\mathcal{G}})$, which we observed already in section 2.

The fact that the reducibility graph is two-way connected for generic values of a in the classical case $q = 1$ implies that this is also true in the quantum case $q \neq 1$. This is so because an edge which is present at $q = 1$ can not be absent for $q \neq 1$. Otherwise $P_\mu \Pi^a(e_0) P_\nu$ would not have a smooth limit as $q \to 1$. This proves the first statement of the theorem.

To determine the non-generic values of a at which the reducibility graph may be non-two-way connected we use the quantum analogue of eq. (4.11):

$$\begin{aligned} q^{-C(\mu)/2}\epsilon_\mu P_\mu(e_0 \otimes q^{h_0/2})P_\nu &= q^{-C(\nu)/2}\epsilon_\nu P_\mu(q^{-h_0/2} \otimes e_0)P_\nu, \\ q^{-C(\mu)/2}\epsilon_\mu P_\mu(q^{-h_0/2} \otimes f_0)P_\nu &= q^{-C(\nu)/2}\epsilon_\nu P_\mu(f_0 \otimes q^{h_0/2})P_\nu, \end{aligned} \tag{4.12}$$

where ϵ_μ is the parity of the representation V_μ in $V_\lambda \otimes V_\lambda$. To derive eq. (4.12) consider the $\check{R}$-matrix on $V_\lambda \otimes V_\lambda$. According to Jimbo it is determined by the equations [2]

$$\begin{aligned} &[\check{R}(x), \Pi(\Delta(a))] = 0 \qquad \forall a \in U_q(\mathcal{G}), \qquad (4.13) \\ &\check{R}(x)\left(xe_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes e_0\right) = \left(e_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes xe_0\right)\check{R}(x), \\ &\check{R}(x)\left(x^{-1}f_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes f_0\right) = \left(f_0 \otimes q^{h_0/2} + q^{-h_0/2} \otimes x^{-1}f_0\right)\check{R}(x), \end{aligned}$$

In the limit $x \to \infty$ one obtains the spectral parameter independent $\check{R}$-matrix and eq. (4.13) reduces to

$$[\check{R}, \Pi(\Delta(a))] = 0 \qquad \forall a \in U_q(\mathcal{G}), \tag{4.14}$$

$$\check{R}\left(e_0 \otimes q^{h_0/2}\right) = \left(q^{-h_0/2} \otimes e_0\right)\check{R}, \tag{4.15}$$

$$\check{R}\left(q^{-h_0/2} \otimes f_0\right) = \left(f_0 \otimes q^{h_0/2}\right)\check{R}, \tag{4.16}$$

$\check{R}$ is given by the formula

$$\check{R} = q^{C(\lambda)} \sum_\mu q^{-C(\mu)/2}\epsilon_\mu P_\mu. \tag{4.17}$$

This was proved in the case where $V_\lambda \otimes V_\lambda$ is multiplicity free by Reshetikhin [13] [2] and in the general case by Gould [14]. By inserting eq. (4.17) into eq. (4.15) and eq. (4.16) and multiplying by P_μ from the left and by P_ν from the right we obtain equations eq. (4.12).

Comparison of eq. (4.12) with eq. (4.4) and eq. (4.5) immediately provides the second statement of the theorem, provided $\epsilon_\mu \epsilon_\nu = -1$. To see this later fact we observe that the permutation matrix σ (defined by $\sigma(v \otimes v') = (v' \otimes v)$) satisfies $P_\mu \sigma = \sigma P_\mu = \epsilon_\mu P_\mu$ and thus

$$P_\mu(e_0 \otimes 1)P_\nu = P_\mu \sigma(1 \otimes e_0)\sigma P_\nu = \epsilon_\mu \epsilon_\nu P_\mu(1 \otimes e_0)P_\nu. \tag{4.18}$$

Comparison of this with eq. (4.11) gives $\epsilon_\mu \epsilon_\nu = -1$.

[2] Our $\check{R}$ is the inverse of the R-matrix in this reference.

□

Lemma 3

$$P_\mu \Pi^a(e_0) P_\nu \neq 0 \qquad \Rightarrow \qquad V_\mu \subset V_{adj} \otimes V_\nu, \tag{4.19}$$

$$P_\mu \Pi^a(f_0) P_\nu \neq 0 \qquad \Rightarrow \qquad V_\mu \subset V_{adj} \otimes V_\nu. \tag{4.20}$$

Proof. The proof uses the concept of tensor operators. These are well explained in appendix B of [9]. There it is also shown that $X = q^{-h_0/2} e_0 \otimes 1$ is the lowest component of an adjoint tensor operator[3]. This implies that the vector $X v_\nu$ for $v_\nu \in V_\nu$ must lie in a representation V_μ which is contained in $V_{\text{adj}} \otimes V_\nu$. The same therefore is true for $(e_0 \otimes q^{h_0/2}) v_\nu = \Delta(q^{h_0/2}) X v_\nu$. Similarly also $\bar{X} = 1 \otimes q^{h_0/2} e_0$ is the lowest component of an adjoint tensor operator and thus also $(q^{-h_0/2} \otimes e_0) v_\nu$ must lie in a representation which is contained in $V_{\text{adj}} \otimes V_\nu$. This is therefore also true for $\Pi^a(e_0) v_\nu$ and eq. (4.19) follows. Using similarly that $q^{h_0/2} f_0 \otimes q^{h_0}$ and $q^{-h_0} \otimes q^{h_0/2} f_0$ are the highest components of adjoint tensor operators one shows eq. (4.20).

□

This lemma prompts us to define another directed graph associated to the tensor product $V_\lambda \otimes V_\lambda$.

Definition 3 *The* **tensor product graph** Γ *associated to a tensor product* $V_\lambda \otimes V_\lambda$ *is a directed graph whose vertices are the irreducible* $\mathcal{G}$*-modules appearing in the decomposition eq. (4.2) of* $V_\lambda \otimes V_\lambda$*. There is an edge directed from vertex* V_ν *to a vertex* V_μ *iff*

$$V_\mu \subset V_{adj} \otimes V_\nu. \tag{4.21}$$

Combining Lemma 3 and Lemma 2 we arrive at

Lemma 4 *If the tensor product graph* Γ *is simply two-way connected then it is equal to the reducibility graph* G^a *for generic value of* a.

Proof. According to Lemma 3 the tensor product graph contains every edge that is contained in the reducibility graph. If the tensor product graph is simply two-way connected than the reducibility graph has to contain all its edges, otherwise it could no longer be two-way connected and would violate Lemma 2.

[3] In comparing with reference [9] one should replace q by q^{-1} because [9] uses the opposite coproduct.

□

Lemma 4 is very useful in constructing reducibility graphs because it is easy to construct tensor product graphs. Many worked out examples of undirected tensor product graphs can be found in [9]. To obtain the directed tensor product graph as defined in Definition 3 from the undirected graphs in [9] one has to replace every undirected edge by two directed edges in opposite directions. All known examples are simply two-way connected and so the Lemma 4 applies. We do not yet know exactly how general this simply connectedness of tensor product graphs is. But even when the tensor product graph is multiply connected we still have the following theorem:

Theorem 5 *Let V_λ be an irreducible $\mathcal{G}$-module which carries a representation π_λ of $U_q(\hat{\mathcal{G}})$. Let Γ be the tensor product graph associated with $V_\lambda \otimes V_\lambda$. Let G' be any simply two-way connected subgraph of Γ which can be made closed by deleting just one directed edge from Γ. Let V_ν be the origin and V_μ be the destination of this edge. Let $a = q^{(C(\mu)-C(\nu))/2}$.*

Then G' is a closed two-way connected subgraph of the reducibility graph G^a and carries an irreducible representation of $U_q(\hat{\mathcal{G}})$ as in Theorem 1.

Proof. The proof that G' is a subgraph of the generic reducibility graph is similar to the proof of Lemma 4. The reason why it is a subgraph of G^a for the particular a is that according to Lemma 2 at this a the reducibility graph looses the edge directed from V_ν to V_μ. G' defines an irreducible representation because Theorem 1 applies.

□

Theorem 5 is very easy to apply in practice and we will demonstrate its use in the next section.

5 Specialization to $U_q(\hat{C}_2)$ and $U_q(\hat{G}_2)$

(i) $U_q(C_2)$:
The fundamental 4-dimensional irrep of $U_q(C_2)$ is undeformed and can be extended to an irrep of $U_q(\hat{C}_2)$. We will use Theorem 5 to construct further irreps of $U_q(\hat{C}_2)$ from the tensor product[4]

$$4 \otimes 4 = 10 \bigoplus 5 \bigoplus 1 \tag{5.1}$$

[4] We denote the $U_q(C_2)$-irreps by their dimension.

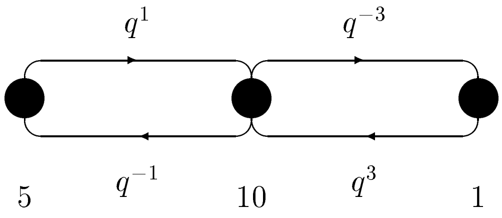

Figure 2: The tensor product graph for the $4 \otimes 4$ of $U_q(C_2)$.

The associated tensor product graph is shown in figure 2. Because it is simply two-way connected it gives also the generic reducibility graph. The numbers associated to the edges in figure 2 are the values of a from Theorem 5, i.e., the values at which the edge disappears from the reducibility graph. They are determined, using eq. (4.10), from $C(1) = 0, C(5) = 4, C(10) = 6$.

We read off from the graph that π^a defines a 5-dimensional irrep of $U_q(\hat{C}_2)$ at $a = q$, a 1-dimensional irrep at $a = q^3$, a $(10 + 5) = 15$-dimensional irrep at $a = q^{-3}$ and a $10 + 1 = 11$-dimensional irrep at $a = q^{-1}$, besides of course the $10+5+1 = 16$-dimensional irrep at generic a. We note in particular that, because the 10-dimensional irrep of $U_q(C_2)$ appears in the middle of the tensor product graph, there is no possibility of having the 10-dimensional irrep in a closed component by itself and thus no irrep of $U_q(\hat{C}_2)$ can be defined on it by itself. The 10 has to be enlarged by adding either the 1, the 5, or both, before it carries a representation of $U_q(\hat{C}_2)$. This reproduces our observation from section 3.

We can also derive the representation matrices eq. (3.1), eq. (3.2) from the general expression eq. (4.9). For this we only need to determine the q-Clebsch-Gordan coefficients of $U_q(C_2)$ for the decomposition of the $4 \otimes 4$. These can easily be calculated by elementary methods. We did this using Mathematica.

Contrary to the 10-dimensional irrep of $U_q(C_2)$, the 5-dimensional irrep can carry an irrep of $U_q(\hat{C}_2)$ by itself. We can repeat the above analysis for the tensor product

$$5 \otimes 5 = 14 \bigoplus 10 \bigoplus 1. \tag{5.2}$$

The associated tensor product graph is shown in figure 3. The truncation values shown above the edges are determined from the Casimir values given earlier and $C(14) = 10$.

We see from the graph that the 14-dimensional irrep of $U_q(C_2)$ can carry an irrep of $U_q(\hat{C}_2)$, but that again the 10-dimensional irrep of $U_q(C_2)$ needs to be extended, either by the 1 or by the 14, again reaffirming our observation from section 3. We may continue the above procedure using the 14-dimensional irrep and get higher irreps of $U_q(\hat{C}_2)$.

(ii) $U_q(G_2)$:

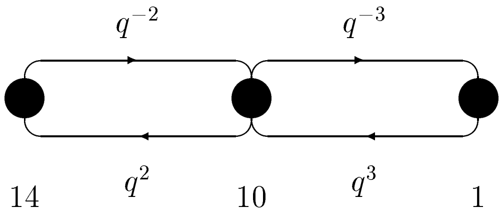

Figure 3: The tensor product graph for the $5 \otimes 5$ of $U_q(C_2)$.

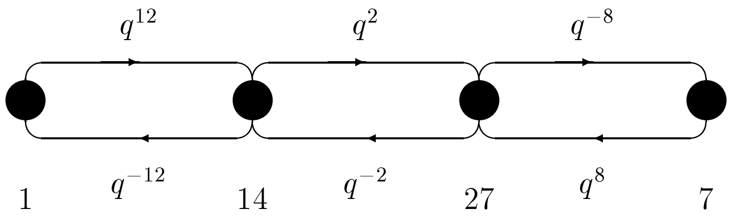

Figure 4: The tensor product graph for the $7 \otimes 7$ of $U_q(G_2)$.

The second example from section 3, the $14+1=15$-dimensional irrep of $U_q(\hat{G}_2)$ can be derived from the tensor product

$$7 \otimes 7 = 27 \bigoplus 14 \bigoplus 7 \bigoplus 1 \tag{5.3}$$

whose associated tensor graph is shown in figure 4.

Again the tensor product graph is simply two-way connected and is therefore equal to the generic reducibility graph. The value of a at which an edge vanishes from the reducibility graph are given in the figure. They were determined from $C(1)=0, C(14)=24, C(27)=28, C(7)=12$. At $a=q^2$ it truncates to the reducibility graph of figure 1, which describes the $14+1=15$-dimensional representation of eq. (3.3), eq. (3.4). Various other irreducible $U_q(\hat{G}_2)$-modules can be read off directly from the other possible truncations of the graph.

6 Discussion

In this paper we have described a practical procedure for constructing finite dimensional representations of quantum affine algebras $U_q(\hat{\mathcal{G}})$. This construction relies on the reduction of tensor product representations. We have introduced the concept of a reducibility graph which encodes the information about which irreducible $U_q(\mathcal{G})$-modules have to be taken together in order to obtain an irreducible $U_q(\hat{\mathcal{G}})$ -module. In practice we exploit the relation of the reducibility graph to another graph, the tensor product graph, which can be constructed by elementary means of classical representation theory.

The construction in the above sections can be extended to the case of the tensor product $V_\lambda \otimes V_{\lambda'}$ with $\lambda \neq \lambda'$. In this case we may still draw the associated tensor product graph, using similar rules to the ones illustrated above; such a graph truncates at some value of a, although the exact form of a is not necessarily given by (4.10) and needs to be determined. Also the construction can be applied to the tensor product of those irreducible $U_q(\hat{\mathcal{G}})$-modules which are reducible as $U_q(\mathcal{G})$-modules. Then the reducibility graph is not two-way connected even for generic a. By using these methods we hope to arrive at a classification of all finite dimensional represenations of $U_q(\hat{\mathcal{G}})$. All those directions and other related aspects are under hard investigations [20].

As mentioned in the introduction, given any two finite dimensional representations of $U_q(\hat{\mathcal{G}})$ one can write down a spectral parameter dependent R-matrix. One method of doing this, applied in [6], is to insert the matrix forms of the generators in the particular representations into the formula for the universal R-matrix. The advantage of this method is that it is totally irrelevant whether the representation is reducible or irreducible, whether the tensor product decomposition is multiplicity-free or with finite multiplicity, (the tensor product decomposition of reducible representation with itself is always with finite multiplicity,) and whether the representations being tensored are the same or different. The disadvantage is, however, that this method requires the explicit form of the universal R-matrix, given in [18], and of e_0 (f_0).

Because of this relation between the existence of a representation of $U_q(\hat{\mathcal{G}})$ on particular $U_q(\mathcal{G})$-modules and the existence of the spectral parameter dependent R-matrices for those modules, our work is related to many works on the construction of R-matrices. In many of these works it has been noticed that often R-matrices can only be constructed on sums of several irreducible $U_q(\mathcal{G})$-modules. Our interpretation of these observations is that only those sums of $U_q(\mathcal{G})$-modules carry representations of $U_q(\hat{\mathcal{G}})$.

The problem of constructing finite dimensional representations also exists for the Yangians $Y(\mathcal{G})$, which give the rational solutions to the Yang-Baxter equation. Already the first paper on the problem [1] by Drinfeld adresses the problem. Drinfeld is able to give a sufficient but not necessary condition for determining wether an irreducible $\mathcal{G}$-modules can carry representations of $Y(\mathcal{G})$. Later he introduced a different realization of Yangians [15] in order to facilitate the construction of finite dimensional representations, but to our knowledge, also in this realization the problem has not yet been completely solved. The relation to our paper lies in the fact that the rational R-matrices of the Yangian $Y(\mathcal{G})$ can be obtained from the trigonometric R-matrices of the quantum affine algebra $U_q(\hat{\mathcal{G}})$ in a limit and therefore all $\mathcal{G}$-modules which we determine to carry representations of $U_q(\hat{\mathcal{G}})$ should also carry representations of $Y(\mathcal{G})$.

Our physical motivation for studying the finite dimensional irreducible representations of quantum affine algebras comes from the study of the solitons in affine Toda quantum field theory. Let us explain briefly:

It is well-known that associated to every affine Lie algebra $\hat{\mathcal{G}}$ there is a 1+1 dimensional affine Toda field theory, denoted $T(\hat{\mathcal{G}})$. It is described by the field equations

$$\Box\vec{\phi} = \frac{\sqrt{-1}}{\beta}\sum_{i=0}^{r} n_i\vec{\alpha}_i\, e^{\sqrt{-1}\beta\,\vec{\alpha}_i\cdot\vec{\phi}} \qquad (6.1)$$

β is the coupling constant and the α_i are the simple roots, $\alpha_0 = -\sum_{i=1}^{r} n_i\alpha_i$. For $\hat{\mathcal{G}} = A_1^{(1)}$ eq.(6.1) specializes to the sine-Gordon (or affine Liouville) equation.

The affine Toda theory $T(\hat{\mathcal{G}})$ posesses symmetry generators e_i , f_i , h_i , $i = 0, 1, \cdots, r$, which generate the quantum affine algebra $U_q(\hat{\mathcal{G}}')$ [16]. Here $\hat{\mathcal{G}}'$ is the dual Lie algebra to $\hat{\mathcal{G}}$, i.e., it is obtained by interchanging the roles of the roots and the coroots. The deformation parameter q is determined by the coupling constant as $q = e^{-\sqrt{-1}\pi/\beta^2}$. The central charge is zero.

The field equations eq. (6.1) have soliton solutions. There exists a very elegant construction of these solitons using the representation theory of the affine Lie algebra $\hat{\mathcal{G}}'$ [17]. The solitons are found to arrange in multiplets given by the fundamental representations of $\mathcal{G}'$ (representations with a fundamental weight as highest weight).

In the quantum theory the classical soliton solutions give rise to particle states and we are interested in the properties of these quantum solitons. Related work for $\hat{\mathcal{G}} = a_n^{(1)}$ has been done by Hollowood [19]. The quantum solitons have to transform in finite dimensional multiplets of the symmetry algebra $U_q(\hat{\mathcal{G}}')$. This paper can be seen as providing some of the necessary mathematical knowledge for extending the elegant group theoretic understanding of the classical solitons to the quantum level. An immediate outcome is that there are often more quantum solitons than the classical solitons filling the fundamental representations. We saw a concrete example: The solitons transforming in the second fundamental representation of $U_q(G_2)$ (the 14-dimensional representation) have to be completed by an additional soliton to make up the 15-dimensional multiplet of $U_q(\hat{G}_2)$ described by eq. (3.3) and eq. (3.4).

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Note added:

Very interesting work on finite dimensional representations of $U_q(\widehat{sl}_n)$ has been done by Chari and Pressley [21,22] using Drinfeld's second realization [15]. Related is also [23] in which Chari and Pressley determine the values of the spectral parameter at which the Yangian R-matrices have poles or zeros. These values should correspond to the values a at which our reducibility graph truncates.

The notion of a tensor product graph, which we borrowed from [9], has also been introduced in [24].

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