The diffusive phase of a model of self-interacting walks

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Abstract

We consider simple random walk on $\mathbf{Z}^d$ perturbed by a factor $\exp[\beta T^{-p} J_T]$, where T is the length of the walk and $J_T = \sum_{0 \le i < j \le T} \delta_{\omega(i),\omega(j)}$. For $p = 1$ and dimensions $d \ge 2$, we prove that this walk behaves diffusively for all $-\infty < \beta < \beta_0$, with $\beta_0 > 0$. For $d > 2$ the diffusion constant is equal to 1, but for $d = 2$ it is renormalized. For $d = 1$ and $p = 3/2$, we prove diffusion for all real β (positive or negative). For $d > 2$ the scaling limit is Brownian motion, but for $d \le 2$ it is the Edwards model (with the "wrong" sign of the coupling when $\beta > 0$) which governs the limiting behaviour; the latter arises since for $p = \frac{4-d}{2}$, $T^{-p} J_T$ is the discrete self-intersection local time. This establishes existence of a diffusive phase for this model. Existence of a collapsed phase for a very closely related model has been proven in work of Bolthausen and Schmock.

1 Introduction

1.1 The model

We consider homogeneous simple random walks on $\mathbf{Z}^d$ taking nearest neighbour steps with equal probabilities $\frac{1}{2d}$. Given a T-step simple random walk ω beginning at the origin, let

$$J_T \equiv J_T(\omega) = \sum_{0 \le i < j \le T} \delta_{\omega(i),\omega(j)}. \tag{1.1}$$

Define

$$c_T = E \exp[\beta T^{-p} J_T], \tag{1.2}$$

where the expectation is with respect to simple random walk beginning at 0, and $p \geq 0$ and $\beta \in \mathbf{R}$ are parameters. We define a new measure on T-step simple random walks, by assigning to a walk ω the probability

$$\frac{1}{c_T} \frac{1}{(2d)^T} \exp[\beta T^{-p} J_T(\omega)]. \tag{1.3}$$

For $\beta = 0$ this new measure is just the simple random walk. For $\beta > 0$ it defines a model of self-attracting walks, since self-intersections are encouraged by the exponential factor. Similarly, for $\beta < 0$, this is a model of self-repelling walks. The factor T^{-p} diminishes the strength of the self-interaction for long walks, and for p fixed, β provides a measure of the strength of the interaction. For $p = 0$ and $\beta < 0$ this is the Domb–Joyce model of weakly self-avoiding walks, which in the limit $\beta \to -\infty$ gives the usual strictly self-avoiding walk.

We are interested in the phenomenon of a collapse transition, in which for fixed p and d there is a transition from diffusive behaviour to collapsed behaviour when $\beta > 0$ is increased. The order parameter for the transition is the diffusion constant $D(\beta)$, which is defined in terms of the mean-square displacement

$$\langle |\omega(T)|^2 \rangle_\beta = \frac{E\left(|\omega(T)|^2 \exp[\beta T^{-p} J_T]\right)}{E\left(\exp[\beta T^{-p} J_T]\right)} \tag{1.4}$$

by

$$D(\beta) = \lim_{T \to \infty} \frac{1}{T} \langle |\omega(T)|^2 \rangle_\beta. \tag{1.5}$$

The diffusive phase corresponds to $0 < D(\beta) < \infty$, while the collapsed phase is signalled by $D(\beta) = 0$. In fact, typically in the collapsed phase the mean-square displacement remains uniformly bounded as $T \to \infty$. For simple random walk we have $D(0) = 1$, and it is to be expected that for fixed p the diffusion constant should be a nonincreasing function of β, since increasing the encouragement for self-intersections should not increase the typical distance travelled by the walk.

In order to observe a transition at positive β, it is necessary to include a factor T^{-p} with $p \geq 1$ in the interaction, since for $0 \leq p < 1$ the walk is collapsed for all d when $\beta > 0$ [22, 23]. For $d \geq 2$, we shall see that the correct power for observing a transition is $p = 1$. Note that given a transition for $p = 1$ there can be no transition for other values of p, since $p < 1$ always yields collapse, while larger values of p essentially correspond to the diffusive behaviour observed for $p = 1$ and $\beta = 0$.

The study of the large-T limit of the partition function c_T is is also related to the problem of taking the continuum limit of the discrete Edwards model (with the "wrong" sign of the coupling when $\beta > 0$). To see this, we recall that the partition function of the Edwards model is formally given by the following expression, in which traditionally one is interested in the repulsive case $\beta < 0$:

$$E \exp\left[\beta \int_{0\le s<t\le 1} \delta(X_s - X_t) ds\, dt\right]. \tag{1.6}$$

Here E denotes expectation with respect to the Wiener measure on paths in $\mathbf{R}^d$, and δ is the Dirac delta function. For $d \ge 2$ it is necessary to perform a renormalization of the interaction in order to make sense of this formal expression, and this is by now well understood for $\beta < 0$ and $d \le 3$ [3, 9, 19, 29, 31, 32, 33, 34]. For $d = 4$, understanding (1.6) is equivalent to understanding the long-time behaviour of the Domb–Joyce model (see (1.7) below); this has not yet been fully treated [1, 12, 16]. It is perhaps less well appreciated that for dimensions $d \le 2$ the partition function (1.6) can also be given a meaning for *positive* β, for all $\beta > 0$ when $d = 1$ and for small $\beta > 0$ for $d = 2$. We learned this fact from Le Gall [21], and will address some related issues in Section 2.2. Consequently the partition function (1.6) is an entire analytic function of β for $d = 1$, and is analytic in a neighbourhood of $\beta = 0$ for $d = 2$ (when conventionally renormalized).

A discrete space-time version of (1.6) can be obtained by replacing the continuous time interval $[0, 1]$ by the discrete time interval $\{0, 1, \ldots, T\}$, replacing the Brownian expectation by the simple random walk expectation, replacing X_s by $T^{-1/2}\omega(\lfloor sT\rfloor)$, replacing the two time integrals by Riemann sums, and replacing the Dirac delta function by a suitably rescaled Kronecker delta. This leads to the discrete partition function

$$E \exp[\beta T^{(d-4)/2} J_T], \tag{1.7}$$

where now the expectation is with respect to simple random walk starting from the origin. This is just the partition function c_T, with $p = (4 - d)/2$. Thus the $T \to \infty$ limit for this value of p corresponds to the continuum limit of the Edwards model. Note that this is the problem of studying the *ultraviolet* limit for the Edwards model, as the continuum time interval is the finite interval $[0, 1]$. The *infrared* problem for the Edwards model is connected with studying the behaviour when the interval $[0, 1]$ is replaced by a long interval $[0, L]$ with $L \to \infty$, and is not addressed here.

Before stating our results precisely, we remark that the model we analyze is not the standard model for the collapse of long chain polymers. The standard model of polymer collapse involves a self-repellence due to the excluded volume effect that no two monomers can occupy the same region of space, together with a nearest-neighbour attraction due to temperature or solvent effects. In other words to each T-step simple random walk ω there is associated a factor

$$\exp\left[-\lambda_1 \sum_{0\le i<j\le T} \delta_{\omega(i),\omega(j)} + \lambda_2 \sum_{0\le i<j\le T} \delta_{|\omega(i)-\omega(j)|,1}\right], \tag{1.8}$$

with λ_1 and λ_2 both positive. It has been argued, and observed experimentally, that when λ_2 is increased with fixed λ_1, there is a collapse transition. Recent references include [10, 11]. The combination of attraction and repulsion makes this model difficult to treat rigorously; our model avoids this difficulty.

1.2 The results

In this section we formulate the main results proved in this paper. We will be interested in the scaling limit of $X_T(t) = T^{-1/2}\omega(\lfloor t \rfloor T)$ as $T \to \infty$, where the probability of ω is given by $c_T^{-1}(2d)^{-T}\exp[\beta T^{-p} J_T(\omega)]$.

We introduce the renormalized partition function

$$c_T^{ren} = E\exp[\beta T^{-p}(J_T - E(J_T))]. \tag{1.9}$$

Let

$$\beta_0 = \sup\{\beta : \sup_T c_T^{ren} < \infty\}; \tag{1.10}$$

this depends on d and p. We shall see that $\beta_0 > 0$ for $d \geq 2$, $p = 1$, while $\beta_0 = +\infty$ for $d = 1$, $p = \frac{3}{2}$.

We define the simple random walk Green function

$$G(x) = \sum_{T=0}^{\infty} p_T(x), \tag{1.11}$$

where $p_T(x)$ denotes the transition probability for simple random walk to go from 0 to x in T steps. The Green function is finite for $d > 2$ but diverges for $d \leq 2$.

Theorem 1.1 *Let $d > 2$ and $p = 1$. Then $\beta_0 > 0$, and the following statements hold for $-\infty < \beta < \beta_0$. The diffusion constant $D(\beta)$ is equal to 1, and the process X_T converges in distribution to Brownian motion. The partition function satisfies*

$$\lim_{T\to\infty} c_T = \exp[\beta[G(0) - 1]], \qquad \lim_{T\to\infty} c_T^{ren} = 1. \tag{1.12}$$

The Green function occuring in the limiting partition function in the above theorem is divergent when $d = 2$, and this is a symptom of the need for renormalization in two dimensions. For $d = 2$ we have the following result. Let $\underline{\gamma}$ denote the renormalized self-intersection local time for planar Brownian motion on the time interval $[0, 1]$. This gives rigorous meaning [31, 19] to the expression

$$\underline{\gamma} \text{ ``$=$'' } \int_{0\leq s<t\leq 1} \delta(B_t - B_s)ds\,dt - E\left[\int_{0\leq s<t\leq 1} \delta(B_t - B_s)ds\,dt\right], \tag{1.13}$$

which is only formal since both terms on the right side are infinite. Writing dW for Wiener measure, let

$$d\nu_{2,\beta} = \frac{e^{\beta\underline{\gamma}}dW}{\int e^{\beta\underline{\gamma}}dW}. \tag{1.14}$$

Varadhan [31] has proved that the denominator in the above is finite for $\beta < 0$, and Le Gall [21] has extended this to sufficiently small positive β ($\beta < 4\prod_{j=1}^{\infty}(1 - 2^{-j})$ is small enough). We will show in Section 3.3.2 that it is finite if $\beta < \beta_0$.

Theorem 1.2 *Let $d = 2$ and $p = 1$. Then $\beta_0 > 0$, and for $-\infty < \beta < \beta_0$, the following statements hold. The limit defining the diffusion constant exists and equals*

$$D(\beta) = \int B(1)^2 d\nu_{2,\beta}. \tag{1.15}$$

Moreover, D is a strictly decreasing function of $\beta \in [0, \beta_0)$. The law of the process $X_T(t)$ converges in distribution to $\nu_{2,\beta}$. In addition, as $T \to \infty$,

$$c_T \sim [Ee^{\beta\underline{\gamma}}]T^{\beta/\pi}. \tag{1.16}$$

In fact, we prove the above theorem by taking limits separately for the renormalized partition function and the corresponding quantity weighted with a factor $|\omega(T)|^2$, with the ratio giving the mean-square displacement. For more details, see Section 3.3.

Bolthausen and Schmock [5] have studied a closely related model, with $p = 1$ but with a continuous-time random walk with exponential holding times. It is expected that the continuous-time model will behave in a qualitatively similar fashion to the discrete-time model considered here. Define

$$\beta_c = \inf\{\beta : \lim_{T\to\infty} c_T^{1/T} > 0\}. \tag{1.17}$$

Clearly $\beta_0 \leq \beta_c$, so for $d \geq 2$ it is the case that $\beta_c > 0$ (see also Section 1.4). For the continuous-time model with $d \geq 2$ and β exceeding the continuous-time analogue of β_c, Bolthausen and Schmock prove that the mean-square displacement is uniformly bounded in T and hence $D(\beta) = 0$. Together with our results, this suggests that for $d > 2$ there may be a jump discontinuity in the graph of D at some critical value of β, whereas for $d = 2$ the transition may be continuous. The natural questions therefore arise as to whether or not $\beta_0 = \beta_c$, and whether $D(\beta)$ goes continuously to 0. These questions remain open.

For $d = 1$ and $p = 1$, Bolthausen and Schmock [5] have proved that the mean-square displacement of the continuous-time model is uniformly bounded in T for all positive β, and hence $D(\beta) = 0$ for all $\beta > 0$. This indicates that the power $p = 1$ is too small to observe a collapse transition, at positive β, when $d = 1$. On the other hand the existence of the "wrong-sign" Edwards model for $d = 1$ suggests that for $p = 3/2$ there should be diffusive behaviour for all $\beta > 0$. The next theorem shows that this is indeed the case. For its statement we define

$$d\nu_{1,\beta} = \frac{e^{\beta\gamma} dW}{\int e^{\beta\gamma} dW}, \tag{1.18}$$

where now dW denotes the one-dimensional Wiener measure, and

$$\gamma = \int_{0\leq s<t\leq 1} \delta(B_t - B_s) ds\, dt \tag{1.19}$$

denotes the one-dimensional self-intersection local time. The denominator of (1.18) is finite for all $\beta \in \mathbf{R}$ [21].

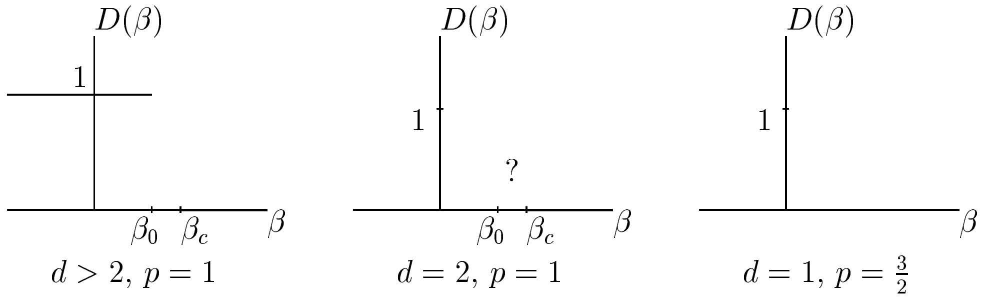

Figure 1: The diffusion constant.

Theorem 1.3 *Let $d = 1$ and $p = \frac{3}{2}$. For any $-\infty < \beta < \infty$, the limit defining the diffusion constant exists, and*

$$D(\beta) = \int B(1)^2 d\nu_{1,\beta}. \tag{1.20}$$

The diffusion constant is strictly decreasing for $\beta \in [0, \infty)$. The law of the process $X_T(t)$ converges in distribution to $\nu_{1,\beta}$.

The behaviour of the diffusion constant is summarized in Figure 1. It has not been established whether $\beta_0 = \beta_c$. Continuity at β_0 for $d = 2$ has not been established, nor has monotonicity for $d = 1$ or 2 when $\beta < 0$. It has not been established whether $D(\beta) \to 0$ as $\beta \to \infty$ for $d = 1$, or whether $D(\beta) \to \infty$ as $\beta \to -\infty$ for $d = 1$ and 2. Concavity or convexity properties have also not been established for $d = 1$ or 2. The curves illustrated for $d = 1$ and $d = 2$ are thus only schematic.

Theorem 1.3 suggests that the transition for $d = 1$ is quite different than for $d \geq 2$, with the transition from diffusive to collapsed behaviour taking place more gradually, as p is varied rather than as β is varied. We expect that, for $\beta > 0$, the behaviour for $p \in (1, \frac{3}{2})$ interpolates between collapse for $p = 1$ and diffusion for $p = \frac{3}{2}$. Such behaviour may be characterized by a continuous and monotone decreasing critical exponent $\nu(p)$ such that

$$\langle |\omega(T)|^2 \rangle \sim \text{Const.}(\beta) T^{2\nu(p)}, \tag{1.21}$$

with $\nu(1) = 0$ and $\nu(\frac{3}{2}) = 1/2$, but (1.21) has not been established.

A complementary problem, also unsolved, is to understand the behaviour of the $d = 1$ model with *repulsive* ($\beta < 0$) interaction with $p = 1$. The behaviour is ballistic for all $\beta < 0$ when $p = 0$ [15], and diffusive for all $\beta < 0$ when $p = \frac{3}{2}$ by Theorem 1.3, and presumably for $p \in (0, \frac{3}{2})$ the behaviour of the walk interpolates between these two extremes as in (1.21). Related issues for the model in which the energy function $T^{-p} J_T$ is replaced by $\sum_{0 \leq i < j \leq T} |j - i|^{-p} \delta_{\omega(i), \omega(j)}$ are discussed in [13, 17].

A different type of attractive walk model was studied in [35], in which the weight of a T-step walk ω is given by

$$\exp\left[\beta \sum_{j=0}^{T} I[\omega(j) = \omega(i) \text{ for some } i < j]\right]. \tag{1.22}$$

For this model, $T^{-1/3}\omega(T)$ converges to a continuous random variable as $T \to \infty$. The interaction here is modified in an unimportant way if we subtract 1 from the indicator function, and this gives the weight

$$\exp\left[-\beta \, \#\{\text{sites visited by } \omega\}\right]. \tag{1.23}$$

This model is a discrete version of the Wiener sausage problem and has been studied in [4, 6], where it is partially proved that the relevant scaling is $T^{-1/(d+2)}\omega(T)$ in all dimensions, and for all $\beta > 0$. Related issues for Brownian motion are treated in [27, 30].

The remainder of the paper is organized as follows. In Section 1.3 we discuss the relation of our results to invariance principles for self-intersection local times. In Section 1.4 we describe a heuristic argument that for $p = 1$ there is a collapse transition at positive β for $d \geq 2$ but not for $d < 2$. In Section 2 we prove uniform bounds on the renormalized partition function. Sections 3.1 and 3.2 contain some preliminaries to the proofs of our main results. The proofs are then completed in Section 3.3, apart from the monotonicity of the diffusion constant for $d = 1$ and 2, which is treated in Section 4.

1.3 Self-intersection local times

This section briefly discusses the relation between our results and invariance principles for self-intersection local times. For a random variable X, we use the notation

$$\underline{X} = X - EX. \tag{1.24}$$

Consider first $d = 2$, and let

$$\gamma_T = \frac{1}{T} \sum_{0 \leq s < t \leq T} \delta_{\omega(s),\omega(t)}. \tag{1.25}$$

It was proven in [26] that $\underline{\gamma_T}$ converges in distribution to $\underline{\gamma}$, for random walks obeying a periodicity condition not satisfied by the simple random walk. We use ideas from [26] in this paper, and as a byproduct obtain a proof of this convergence in distribution for the simple random walk (see Proposition 3.5). We will prove Theorem 1.2 by combining the convergence in distribution of $\underline{\gamma_T}$ to $\underline{\gamma}$ with existence of a uniform exponential moment for $\underline{\gamma_T}$.

For $d = 1$, the discrete self-intersection local time is given by

$$\gamma_T = \frac{1}{T^{3/2}} \sum_{0 \leq s < t \leq T} \delta_{\omega(s),\omega(t)}. \tag{1.26}$$

No renormalization is required for $d = 1$, and it is known that γ_T converges in distribution to its continuum counterpart γ [7, 8, 24]. Combined with existence of a uniform exponential moment for γ_T, this will lead to a proof of Theorem 1.3. The proof we shall give of convergence in distribution of the renormalized self-intersection local time for $d = 2$ applies also to $d = 1$, and thereby provides an alternate approach to some results of [7, 8, 24]. Note that use of $p = 1$ rather than $p = \frac{3}{2}$ in the definition of γ_T would lead to a discrete random variable with an infinite limit, and this much stronger interaction leads to collapse for all positive β.

For dimensions $d > 2$ we define

$$\gamma_T = \frac{1}{T} \sum_{0 \le s < t \le T} \delta_{\omega(s),\omega(t)}. \tag{1.27}$$

This use of $p = 1$ defines a random variable which is smaller than for the choice $p = \frac{4-d}{2}$ corresponding to the discrete self-intersection local time. With $p = 1$ no renormalization is required, and we will prove in Section 3.1 that γ_T converges in L^2 to the constant $G(0) - 1$. Combined with the existence of an exponential moment for γ_T (see Section 2), this will allow for a proof of Theorem 1.1. To our knowledge, it remains an open problem to construct the (repulsive) Edwards model for $d = 3$ as a suitably renormalized continuum limit of the discrete model with $p = \frac{1}{2}$.

The following lemma shows that the effect of the renormalization of γ_T by the subtraction of $E\gamma_T$ is a significant effect when $d = 2$, but not when $d \neq 2$. In particular, for $d \neq 2$ the value of β_0 would be unchanged if the renormalized partition function c_T^{ren} were replaced by c_T in (1.10), but this is not true for $d = 2$. The proof of the lemma makes use of the notation

$$(f * g)_T(x) = \sum_y \sum_{s=0}^{T} f_s(y) g_{T-s}(x - y) \tag{1.28}$$

for convolution in time and space.

Lemma 1.4 *As $T \to \infty$,*

$$E\gamma_T = \begin{cases} G(0) - 1 + O(T^{-\epsilon}) & (d > 2) \\ \frac{1}{\pi} \log T + O(1) & (d = 2) \\ \frac{2}{3}\sqrt{\frac{2}{\pi}} + O(T^{-1/2}) & (d = 1) \end{cases} \tag{1.29}$$

for any $\epsilon < \min\{\frac{d-2}{2}, 1\}$.

Proof. Let $q_t(x) = p_t(0)\delta_{x,0}[1 - \delta_{t,0}]$ and $a = \max\{1, \frac{4-d}{2}\}$. By definition of γ_T and the Markov property,

$$E\gamma_T = \frac{1}{T^a} \sum_x (p * q * p)_T(x) = \frac{1}{T^a} \sum_{s_2=2}^{T} \sum_{s_1=0}^{T-s_2} p_{s_2}(0) = \frac{1}{T^a} \sum_{s_2=2}^{T} (T - s_2 + 1) p_{s_2}(0). \tag{1.30}$$

For $d > 2$ the right side is given by

$$G(0) - 1 - \sum_{s_2=T+1}^{\infty} p_{s_2}(0) - \frac{1}{T}\sum_{s_2=0}^{T}(s_2-1)p_{s_2}(0) - \frac{1}{T}, \tag{1.31}$$

and the desired result then follows from the fact that $p_s(0) = O(s^{-d/2})$. For $d = 1$ or 2, the local central limit theorem states that $p_{2n}(0) = (\pi n)^{-d/2} + O(n^{-1-d/2})$ (see e.g. [18]). With (1.30), this gives the desired result. For example, when $d = 1$ the leading term is the Riemann sum for $\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^1 ds\,(1-s)s^{-1/2}$; the factor $\frac{1}{2}$ arises because only walks with an even number of steps can contribute to $p_{s_2}(0)$. □

1.4 A variational problem

For $p = 1$, we can write the partition function (1.2) in terms of the local time

$$L_y(T) = \frac{1}{T}\sum_{i=0}^{T}\delta_{y,\omega(i)}, \tag{1.32}$$

which is the proportion of time spent by the walk ω at the site y. We also define

$$\tau_y(T) = TL_y(T) = \sum_{i=0}^{T}\delta_{y,\omega(i)} \tag{1.33}$$

to be the number of visits of the walk ω to the site y, up to time T. In terms of $L_y(T)$,

$$T^{-1}J_T = \frac{1}{2}T^{-1}\left[\sum_{i,j=0}^{T}\delta_{\omega(i),\omega(j)} - T\right] = \frac{1}{2}T\sum_y L_y^2(T) - \frac{1}{2}. \tag{1.34}$$

Hence we can rewrite the partition function as

$$c_T = e^{-\beta/2}E\exp\left[\frac{1}{2}\beta T\sum_y L_y^2(T)\right]. \tag{1.35}$$

The following subadditivity argument guarantees existence of the limit $\lim_{T\to\infty} c_T^{1/T}$.

Lemma 1.5 *For $p = 1$ and $\beta \ge 0$, and for any d, the limit $b = \lim_{T\to\infty} T^{-1}\log c_T$ exists and is finite and nonnegative.*

Proof. Define $\tau_y[a,b] = \sum_{i=a}^{b}\delta_{y,\omega(i)}$. By definition,

$$(\tau_y[0,T])^2 \le (\tau_y[0,S] + \tau_y[S,T])^2 = (\tau_y[0,S])^2 + (\tau_y[S,T])^2 + 2\tau_y[0,S]\tau_y[S,T]. \tag{1.36}$$

Estimating the last term on the right side using $2ab \le \alpha a^2 + \alpha^{-1}b^2$, with $\alpha = (T-S)/S$, gives

$$T^{-1}(\tau_y[0,T])^2 \le S^{-1}(\tau_y[0,S])^2 + (T-S)^{-1}(\tau_y[S,T])^2. \tag{1.37}$$

Using (1.33), (1.35) and the Markov property, it follows that $\log(e^{\beta/2}c_T)$ is subadditive. A standard lemma about subadditive sequences then implies that $\lim_{T\to\infty} T^{-1}\log c_T = \inf_{T\geq 1} T^{-1}\log(e^{\beta/2}c_T)$, which gives the result of the lemma. The limit is nonnegative for $\beta \geq 0$, since $c_T \geq 1$. □

Keeping $p = 1$ fixed and restricting $\beta > 0$, we now present a heuristic argument which predicts that there is a collapse transition for $d \geq 2$ but not for $d = 1$, for this value of p. In fact, this argument provides the basis for the proof by Bolthausen and Schmock that there is a collapsed phase. For simplicity we consider the continuous-time model. In view of (1.35) we may as well consider the partition function

$$\tilde{c}_T = \tilde{E}\exp[\beta T \sum_y L_y^2(\omega)], \tag{1.38}$$

where $\tilde{E}_T$ denotes expectation with respect to the continuous-time simple random walk up to time T and L_y is interpreted appropriately.

The Donsker–Varadhan theory of large deviations suggests that

$$\lim_{T\to\infty}\frac{1}{T}\log\tilde{c}_T = \sup\{\beta||\phi||_4^4 - \frac{1}{2}||\nabla\phi||_2^2 : ||\phi||_2 = 1\} \equiv b. \tag{1.39}$$

The gradient is the finite difference gradient associated to the lattice. In general the supremum may or may not be attained. It is clear that for β sufficiently large, and in any dimension, the solution b of the variational problem is strictly positive. In this case it can be shown that the supremum is attained by an exponentially decaying function. For $d = 1$, it is the case for all $\beta > 0$ that $b > 0$ (this can be seen from a scaling argument), and here too the supremum is attained by an exponentially decaying function. On the other hand for $d \geq 2$ there is a Sobolev inequality

$$||\phi||_4^4 \leq C||\phi||_2^2\,||\nabla^2\phi||_2^2, \tag{1.40}$$

and hence $b = 0$ for sufficiently small β.

If ϕ^2 realizes the supremum, then so does any translate. In the collapsed phase it is expected that the law for the process is a mixture of ergodic components, i.e., the process breaks translation invariance by choosing where to collapse, and if we restrict to a component, then $\phi^2(y) = \lim_{T\to\infty} L_y(T)$ a.s.. Exponential decay of the optimal ϕ^2, and hence $b > 0$, thus corresponds to collapse. This type of result is proved by Bolthausen and Schmock. A difficulty in applying the Donsker–Varadhan theory is that the state space here is all of $\mathbf{Z}^d$ and is therefore not compact. This is overcome by making use of the fact that in the collapsed phase the state space is nearly compact, since the walk spends the bulk of its time in a compact subset of $\mathbf{Z}^d$.

Thus, $b > 0$ corresponds to a localized local time, or a confined phase for the walk. On the other hand, $b = 0$ is interpreted as corresponding to the supremum being approximated by a sequence of increasingly more constant (zero) ϕ's, and hence to the local time approaching a constant (zero) function. This is interpreted as extended behaviour for the walk.

The above discussion leads one to expect that there will be a collapse transition, at positive β, for $d \geq 2$ but not for $d = 1$, when $p = 1$.

2 Exponential moments

In this section we begin by showing that γ_T (and hence $\underline{\gamma_T}$) has uniform exponential moments of all orders for $d = 1$ and of small orders for $d > 2$. We then show that, for $d = 2$, $\underline{\gamma_T}$ has uniform exponential moments of small orders.

2.1 Dimensions $d \neq 2$

Recalling the definition of convolution in (1.28), we write f^{*n} for the convolution of n factors of f.

Theorem 2.1 *For $d > 2$, there is a $\beta_0 = \beta_0(d) > 0$ such that for all $-\infty < \beta < \beta_0$,*

$$\sup_T Ee^{\beta\gamma_T} < \infty.$$

For $d = 1$, the above inequality holds for all real β.

Proof. For $\beta \leq 0$, $Ee^{\beta\gamma_T} \leq 1$, so assume $\beta > 0$. Let $d \neq 2$, and let $a = \max\{1, \frac{4-d}{2}\}$. In view of (1.33) and (1.34), it suffices to obtain the uniform bound of the theorem with γ_T replaced by $T^{-a}\sum_y \tau_y^2(T)$. By Jensen's inequality, since $T^{-1}\sum_y \tau_y = 1$, we have

$$\begin{aligned} \exp[\beta T^{-a}\sum_y \tau_y^2] &\leq \frac{1}{T}\sum_y \tau_y \exp[T^{1-a}\beta\tau_y] \\ &= \frac{1}{T}\sum_y \sum_{n=0}^{\infty} \frac{\beta^n}{n!}\frac{1}{T^{(a-1)n}}\tau_y^{n+1}. \end{aligned} \tag{2.1}$$

Therefore

$$E\exp\left[\beta T^{-a}\sum_y \tau_y^2\right] \leq \frac{1}{T}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\frac{1}{T^{(a-1)n}}\sum_y E\tau_y^{n+1}. \tag{2.2}$$

The factor τ_y^{n+1} amounts to a requirement that ω visit the site y at each of $n+1$ times, and ordering these times gives rise to a factor $(n+1)!$. Let $p_t(x)$ denote the probability that simple random walk goes from 0 to x in t steps, and let $q_t(x) = p_t(0)\delta_{x,0}[1-\delta_{t,0}]$. Then we have, using $(p * p)_t(x) = (t+1)p_t(x)$, that

$$\begin{aligned} E\exp\left[\beta T^{-a}\sum_y \tau_y^2\right] &\leq \frac{1}{T}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\frac{1}{T^{(a-1)n}}\sum_x (n+1)!(p * q^{*n} * p)_T(x) \\ &= \frac{1}{T}\sum_{n=0}^{\infty}\beta^n\frac{1}{T^{(a-1)n}}(n+1)\sum_{t=0}^{T}(q^{*n})_t(0)\sum_x (p*p)_{T-t}(x) \\ &= \frac{1}{T}\sum_{n=0}^{\infty}\beta^n\frac{1}{T^{(a-1)n}}(n+1)\sum_{t=2}^{T}(q^{*n})_t(0)(T-t+1) \\ &\leq \sum_{n=0}^{\infty}\beta^n(n+1)\frac{1}{T^{(a-1)n}}\sum_{t=2}^{T}(q^{*n})_t(0). \end{aligned} \tag{2.3}$$

When $d > 2$, so that $a = 1$, a uniform bound on the right side for small β then follows from the fact that

$$\sum_{t=0}^{T} (q^{*n})_t(0) \le G(0)^n. \tag{2.4}$$

For $d = 1$, so that $a = \frac{3}{2}$, we have

$$E \exp\left[\beta T^{-3/2} \sum_y \tau_y^2\right] \le \sum_{n=0}^{\infty} \beta^n (n+1) T^{-n/2} \sum_{t=2}^{T} (q^{*n})_t(0).$$

For $d = 1$, there is a constant c such that $|q_t| \le c/\sqrt{1+t}$. Using this fact, together with the $r = \frac{1}{2}$ version of the inequality (2.6) of Lemma 2.2 below, gives

$$\sum_{t=2}^{T} (q^{*n})_t(0) \le \frac{(cT)^{n/2}}{\sqrt{n!}}.$$

Combining these two inequalities completes the proof. $\square$

Lemma 2.2 *Let $T > 0$, $t_0 = 0$ and $-\infty < r < 1$. There is a constant c (depending on r) such that for all integers $p \ge 0$,*

$$\int_{0 \le t_1 \le \ldots \le t_p \le T} d^p t \prod_{j=1}^{p} \frac{1}{(t_j - t_{j-1})^r} = \frac{[\Gamma(1-r)]^p}{\Gamma(p(1-r)+1)} T^{p(1-r)} \tag{2.5}$$

$$\sum_{0 \le t_1 \le \ldots \le t_p \le T} \prod_{j=1}^{p} \frac{1}{(1 + t_j - t_{j-1})^r} \le \frac{(cT)^{p(1-r)}}{(p!)^{1-r}}. \tag{2.6}$$

Proof. The inequality follows from the identity by Stirling's formula and the fact that

$$\sum_{t=t_{j-1}}^{t_j} \frac{1}{(1+t)^r} \le c \int_{t_{j-1}}^{t_j} \frac{dt}{t^r},$$

so it suffices to prove the identity. Let

$$I_p(T) = \int_{0 \le t_1 \le \ldots \le t_p \le T} d^p t \prod_{j=1}^{p} \frac{1}{(t_j - t_{j-1})^r}.$$

Then $I_p(T) = T^{p(1-r)} I_p(1)$ and therefore

$$\int_0^{\infty} dT\, I_p(T) e^{-T} = \Gamma(p(1-r)+1) I_p(1).$$

On the other hand, by definition

$$\int_0^{\infty} dT\, I_p(T) e^{-T} = \left(\int_0^{\infty} \frac{dt\, e^{-t}}{t^r}\right)^p = [\Gamma(1-r)]^p.$$

Therefore

$$I_p(1) = \frac{[\Gamma(1-r)]^p}{\Gamma(p(1-r)+1)}.$$

The identity then follows from $I_p(T) = T^{p(1-r)} I_p(1)$. □

The above method can also be used to show that the partition function of the continuum Edwards model in $d = 1$ is an entire function of the coupling constant, and provides a slightly different approach to that of [21].

2.2 Dimension $d = 2$

For $d = 2$ the situation is complicated by the need for renormalization: since the Edwards model requires renormalization for a finite partition function in two dimensions, it is to be expected that the discrete partition function must also be renormalized if it is to be uniformly bounded in T. The primary aim of this section is to prove a uniform bound on the renormalized partition function. However, part of the discussion applies also to dimensions $1 \leq d < 4$, and is not restricted to $d = 2$.

For $d < 3$, the renormalized self-intersection local time of planar Brownian motion, $B(s)$, $s \in [0, 1]$, can be defined using a dyadic decomposition introduced by Westwater [32]. The following discussion reviews some results of Le Gall in this regard [21, 19, 20]. We will subsequently prove that his methods give analogous results for random walk, with uniformity in T. We begin by defining

$$\mathcal{T} = \{(s,t) \in \mathbf{R}^2 : 0 \leq s < t \leq 1\} \tag{2.7}$$

and

$$A_k^n = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right) \times \left(\frac{2k-1}{2^n}, \frac{2k}{2^n}\right], \tag{2.8}$$

so that the sets

$$\mathcal{T}_N = \cup_{n=1}^{N} \cup_{k=1}^{2^{n-1}} A_k^n \tag{2.9}$$

increase with N and their union is $\mathcal{T}$. For A any finite union of these sets A_k^n and f a function on $\mathbf{R}^d$, the random variable

$$\gamma(A) = \lim_{f \to \delta} \gamma_f(A), \quad \gamma_f(A) = \int\int_A f(B(s) - B(t)) \tag{2.10}$$

is defined as the limit in L^2 as f tends weakly to the Dirac δ function. The existence of this limit is an easy result because the A_k^n are designed so that

$$\int\int_{A_k^n} ds\, dt\, f(B(s) - B(t)) \stackrel{(d)}{=} \int_0^{2^{-n}} \int_0^{2^{-n}} ds\, dt\, f(B_1(s) - B_2(t)), \tag{2.11}$$

where $B_1,\ B_2$ are independent Brownian motions starting at the origin. Existence of the corresponding random walk limit will be proven in Lemma 3.2 below.

Given a random variable X, we write

$$\underline{X} = X - EX. \tag{2.12}$$

Then the renormalized self-intersection local time is defined[1] by

$$\underline{\gamma}(\mathcal{T}) = \lim_{N\to\infty} \underline{\gamma}(\mathcal{T}_N) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \underline{\gamma}(A_k^n). \tag{2.13}$$

Existence of the limit in L^2 follows easily, using independence of the $\gamma(A_k^n)$ for n fixed and Brownian motion scaling. The analogous argument for random walk is given below in Lemma 2.4. Furthermore, Le Gall [21] has shown that there exists $\beta_1 > 0$ such that for all $\beta < \beta_1$

$$Ee^{\beta\underline{\gamma}(\mathcal{T})} < \infty. \tag{2.14}$$

This existence of an exponential moment for $\underline{\gamma}(\mathcal{T})$ is more delicate than the corresponding results of Section 2.1. For negative β, (2.14) was already proved by Varadhan [31].

Our goal now is to prove versions of (2.13) and (2.14) for random walk, uniformly in T. In Section 3.2, we will also prove a uniform random walk version of (2.10). First we introduce several definitions. Define

$$\mathcal{T}_T = \{(i,j) \in \mathbf{Z}^2 : 0 \leq i < j \leq T\}, \quad \mathcal{A}_T = \{(i,j) \in \mathbf{Z}^2 : 0 \leq i, j \leq T\}. \tag{2.15}$$

and for $A \subset \mathcal{A}_T$, let

$$\alpha_T(A) = \frac{1}{T^{(4-d)/2}} \sum_{(i,j)\in A} \delta_{\omega(i),\omega'(j)}, \tag{2.16}$$

where ω, ω' are two independent T-step simple random walks which both begin at the origin. Given a function h on $\mathbf{Z}^d$, we also define

$$\alpha_{T,h}(A) = \frac{1}{T^{(4-d)/2}} \sum_{(i,j)\in A} h(\omega(i) - \omega'(j)). \tag{2.17}$$

To adapt the proofs of (2.13) and (2.14) for Brownian motion to random walk, we will make use of the following moment estimate on α_T, which for two-dimensional Gaussian processes is proven in (2.15) of [25]. For Brownian motion in $d = 2$ and $d = 3$ it is proven in Lemma 2 of [21] and in Lemma 2.2 of [20], respectively. Our estimates will be limited to $d < 4$; this is a natural limitation since the Brownian motion intersection local time has an infinite first moment for $d \geq 4$. We will make use of the Fourier transform, defined for a function f on $\mathbf{Z}^d$ by

$$\hat{f}(k) = \sum_x f(x)e^{ik\cdot x} \quad (k \in [-\pi,\pi]^d). \tag{2.18}$$

[1]The subtraction of the expectation cannot be commuted with the limit because $E\gamma(\mathcal{T})$ is divergent; the use of the underline notation on the left side of (2.13) is thus a little misleading.

Lemma 2.3 *Let $1 \leq d < 4$. There is a constant C (depending on d) such that for all T and for all $p = 1, 2, 3, \ldots$,*

$$E[(\alpha_T(\mathcal{A}_T))^p] \leq C^p (p!)^{d/2}. \tag{2.19}$$

Proof. Our proof is a variation on a technique of [25]. We first note that since

$$\alpha_T(\mathcal{A}_T) = \frac{1}{T^{(4-d)/2}} \sum_{(i,j)\in\mathcal{A}_T} \delta_{\omega(i),\omega'(j)} = \frac{1}{T^{(4-d)/2}} \sum_x \tau_x(T)\tau'_x(T), \tag{2.20}$$

it suffices to show that

$$E\left[\left(\sum_x \tau_x(T)\tau'_x(T)\right)^p\right] \leq (CT^{(4-d)/2})^p (p!)^{d/2}. \tag{2.21}$$

By the Plancherel theorem,

$$\begin{aligned} E\left[\left(\sum_x \tau_x(T)\tau'_x(T)\right)^p\right] &= \sum_{x_1,\ldots,x_p} \left(E[\tau_{x_1}\cdots\tau_{x_p}]\right)^2 \\ &= (2\pi)^{-dp} \int d^{dp}k \, \left|E[\hat{\tau}_{k_1}\cdots\hat{\tau}_{k_p}]\right|^2 \end{aligned} \tag{2.22}$$

where the integration is over $[-\pi, \pi]^d$ for each of the p variables k. The Fourier transform is given by

$$\hat{\tau}_k = \sum_x e^{ik\cdot x}\tau_x = \sum_{t=0}^{T} e^{ik\cdot\omega(t)}. \tag{2.23}$$

Therefore

$$E[\hat{\tau}_{k_1}\cdots\hat{\tau}_{k_p}] = \sum_{t_1,\ldots,t_p} E\left[\exp[i\sum_{j=1}^p k_j\cdot\omega(t_j)]\right]. \tag{2.24}$$

For a term in the sum for which $t_1 \leq t_2 \leq \ldots \leq t_p$, we rewrite the exponent using

$$\sum_{j=1}^p k_j\cdot\omega(t_j) = \sum_{j=1}^p K_j\cdot(\omega(t_j) - \omega(t_{j-1})), \tag{2.25}$$

where $K_j = k_p + k_{p-1} + \ldots + k_j$ and $t_0 = 0$. Let

$$\hat{D}(k) = \frac{1}{d}\sum_{j=1}^d \cos k_j. \tag{2.26}$$

Since the exponential of the sum (2.25) factors into independent random variables and

$$E\left[\exp[ik\cdot\omega(t)]\right] = \hat{D}^t(k), \tag{2.27}$$

we obtain

$$E\left[\exp[i\sum_{j=1}^p k_j\cdot\omega(t_j)]\right] = \prod_{j=1}^p \hat{D}^{t_j-t_{j-1}}(K_j). \tag{2.28}$$

Now consider a general term in the sum for which $t_1, \ldots, t_p$ are not necessarily in increasing order. Let π be the unique permutation of $\{1, 2, \ldots, p\}$ for which $t_{\pi(1)} \leq t_{\pi(2)} \leq \ldots \leq t_{\pi(p)}$ and whenever $t_{\pi(i)} = t_{\pi(j)}$ for some $i < j$ then $\pi(i) < \pi(j)$. For $j = 1, 2, \ldots, p$ we let

$$\begin{aligned} s_j &= t_{\pi(j)} - t_{\pi(j-1)} \\ K_j &= k_{\pi(p)} + k_{\pi(p-1)} + \ldots + k_{\pi(j)}. \end{aligned}$$

Then

$$E[\hat{\tau}_{k_1} \cdots \hat{\tau}_{k_p}] = \sum_{t_1,\ldots,t_p} \prod_{j=1}^{p} \hat{D}^{s_j}(K_j). \tag{2.29}$$

Therefore

$$\begin{aligned} E\left[\left(\sum_x \tau_x(T)\tau'_x(T)\right)^p\right] &= \sum_{t_1,\ldots,t_p} \sum_{t'_1,\ldots,t'_p} (2\pi)^{-dp} \int d^{dp}k \prod_{j=1}^{p} \left[\hat{D}^{s_j}(K_j)\hat{D}^{s'_j}(-K'_j)\right] \\ &\leq \left[\sum_{t_1,\ldots,t_p} \left[(2\pi)^{-dp} \int d^{dp}k \prod_{j=1}^{p} \hat{D}^{2s_j}(K_j)\right]^{1/2}\right]^2, \end{aligned} \tag{2.30}$$

by the Cauchy–Schwarz inequality. We can perform the k integrals in the order $k_{\pi(1)}, \ldots, k_{\pi(p)}$ by using translation invariance over the torus $[-\pi, \pi]^d$. For this, we observe that

$$p_s(0) = (2\pi)^{-d} \int d^dk\, \hat{D}^s(k) \tag{2.31}$$

is the probability of return to the origin after s steps, and obtain

$$\begin{aligned} E\left[\left(\sum_x \tau_x(T)\tau'_x(T)\right)^p\right] &\leq \left[\sum_{t_1,\ldots,t_p} \prod_{j=1}^{p} \sqrt{p_{2s_j}(0)}\right]^2 \\ &\leq (p!)^2 \left[\sum_{0 \leq t_1 \leq \ldots \leq t_p \leq T} \prod_{j=1}^{p} \sqrt{p_{2t_j - 2t_{j-1}}(0)}\right]^2. \end{aligned}$$

Using the fact that $0 \leq p_t(0) \leq c(1+t)^{-d/2}$ for some constant c and applying Lemma 2.2 then gives (2.21) and completes the proof. It is at this last step that we use the hypothesis $d < 4$. $\square$

Let $d \leq 2$. The $p = 2$ estimate of Lemma 2.3 allows us readily to obtain a uniform version of (2.13) for random walk, for future reference. We first define

$$\mathcal{T}_{T,N} = \cup_{n=1}^{N} \cup_{k=1}^{2^{n-1}} TA_k^n \cap \mathbf{Z}^2, \tag{2.32}$$

$$\gamma_T(A) = \frac{1}{T^{(4-d)/2}} \sum_{(i,j)\in A} \delta_{\omega(i),\omega(j)}, \quad A \subset \mathcal{A}_T, \tag{2.33}$$

and

$$\underline{\gamma}(n,k) = \underline{\gamma_T}(TA_k^n \cap \mathbf{Z}^2). \tag{2.34}$$

Lemma 2.4 *For $d \leq 2$,*

$$\lim_{N\to\infty} \sup_T E\left(\underline{\gamma_T}(\mathcal{T}_T) - \underline{\gamma_T}(\mathcal{T}_{T,N})\right)^2 = 0. \tag{2.35}$$

Proof. For simplicity, we assume first that $T = 2^M$ for some integer M. We write $\|A\|_2 = \sqrt{EA^2}$. The random variables $\underline{\gamma}(n,k)$ are independent for $k = 1, 2, \ldots, 2^{n-1}$, and each has the distribution of $2^{-n(4-d)/2}\underline{\alpha_{2^{-n}T}}(\mathcal{A}_{2^{-n}T})$. Using Lemma 2.3, we have

$$\|\sum_{k=1}^{2^{n-1}} \underline{\gamma}(n,k)\|_2 = 2^{(n-1)/2}\|\underline{\gamma}(n,1)\|_2 = 2^{-1/2}2^{-n(3-d)/2}\|\underline{\alpha_{2^{-n}T}}(\mathcal{A}_{2^{-n}T})\|_2 \leq c2^{-n(3-d)/2}. \tag{2.36}$$

Hence the series is summable for $d < 3$, and since its tail bounds the expectation in (2.35), this proves the lemma for the special case of dyadic T.

The proof for the general case is almost identical. The difference is that a set $TA_k^n \cap \mathbf{Z}^2$ need not be a square, but could also be a rectangle whose side lengths differ only by 1. This small change poses no essential difficulties. For example, a moment of an asymmetric version of α, in which $\mathcal{A}_T$ is replaced by a rectangle, can be bounded above by the moment of the symmetric α on the square obtained by enlarging one side of the rectangle by 1. $\square$

Note that it was actually required that $d < 3$, rather than $d \leq 2$, in the proof of Lemma 2.4. The fact that the lemma fails for $d \geq 3$ is a symptom of the fact that, for $d \geq 3$ and $p = \frac{4-d}{2}$, renormalization beyond subtraction of $E\gamma_T$ is required.

By Lemma 2.3 and the fact that $E[\underline{\alpha_T}(\mathcal{A}_T)] = 0$, for $d \leq 2$ there is a constant C_2 and a $\lambda_0 > 0$, both independent of T, such that

$$E[\exp[\lambda\underline{\alpha_T}(\mathcal{A}_T)]] \leq 1 + C_2\lambda^2 \leq e^{C_2\lambda^2}, \quad (0 \leq \lambda \leq 2\lambda_0). \tag{2.37}$$

The fact that $\underline{\alpha_T}(\mathcal{A}_T)$ has an exponential moment will be the key fact in proving a uniform version of (2.14) for random walk. We now prove such a bound on the renormalized partition function by adapting Le Gall's proof for the analogous continuum problem [21].

Theorem 2.5 *Let $d \leq 2$. There is a constant K such that for $\beta > 0$ sufficiently small and for all T, $E[\exp[\beta\underline{\gamma_T}(\mathcal{T}_T)]] \leq K$.*

Proof. Assume that $T = 2^M$ for some integer M; the general case can be treated similarly. By (2.15), (2.32) and (2.34),

$$\underline{\gamma_T}(\mathcal{T}_T) = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \underline{\gamma}(n,k). \tag{2.38}$$

Note that $TA_k^n \cap \mathbf{Z}^2 = \phi$ for $n > M$, so the sum is finite for each T. Let $\lambda \in (0, \lambda_0)$ (with λ_0 given by (2.37)), and define

$$\beta_N = 2\lambda\prod_{j=2}^{N}(1 - 2^{-j/2}) \quad (N \geq 2), \qquad \beta_1 = 2\lambda. \tag{2.39}$$

The sequence β_N decreases to a limit $\beta_\infty = \beta_\infty(\lambda) > 0$. Let

$$\zeta_T(N) = E\left[\exp\left(\beta_N \sum_{n=1}^{N} \sum_{k=1}^{2^{n-1}} \underline{\gamma}(n,k)\right)\right]. \tag{2.40}$$

Separating the contribution due to $n = N$, and applying Hölder's inequality (with $p = 2^{N/2}/(2^{N/2}-1)$ and $q = 2^{N/2}$), we obtain

$$\zeta_T(N) \leq \zeta_T(N-1)^{1-2^{-N/2}} \left[E\exp\left(2^{N/2}\beta_N \sum_{k=1}^{2^{N-1}} \underline{\gamma}(N,k)\right)\right]^{2^{-N/2}}. \tag{2.41}$$

The random variables $\underline{\gamma}(N,k)$ are independent for $k = 1, 2, \ldots, 2^{N-1}$, and moreover each has the distribution of $2^{-N}\underline{\alpha_{T2^{-N}}}(\mathcal{A}_{T2^{-N}})$. Hence, by (2.37) and the fact that $\beta_N \leq 2\lambda_0$,

$$\left[E\exp\left(2^{N/2}\beta_N \sum_{k=1}^{2^{N-1}} \underline{\gamma}(N,k)\right)\right]^{2^{-N/2}} \leq \exp[2C_2\lambda_0^2 2^{-N/2}]. \tag{2.42}$$

By induction, this gives

$$\zeta_T(N) \leq \zeta_T(1)\exp\left[2C_2\lambda_0^2 \sum_{j=2}^{N} 2^{-j/2}\right] \leq \zeta_T(1)\exp\left[\frac{2C_2\lambda_0^2}{2-\sqrt{2}}\right], \tag{2.43}$$

where

$$\zeta_T(1) = E[e^{\beta_1 \underline{\gamma}(1,1)}] = E\exp[2\lambda \underline{\alpha_{T/2}}(\mathcal{A}_{T/2})] \tag{2.44}$$

is uniformly bounded by (2.37). Applying Fatou's Lemma to take the limit $N \to \infty$ in (2.43), and using the fact that $\lambda \in (0, \lambda_0)$ was arbitrary, this gives a uniform bound on $E\exp[\beta\underline{\gamma_T}(\mathcal{T})]$ for $\beta \in (0, \beta_\infty(\lambda_0))$. □

3 The scaling limit

3.1 Preliminary for $d > 2$

Throughout this section we fix $p = 1$ and the dimension $d > 2$, and as usual we set

$$\gamma_T = \frac{1}{T} \sum_{0 \leq i < j \leq T} \delta_{\omega(i),\omega(j)} \tag{3.1}$$

and

$$\underline{\gamma_T} = \gamma_T - E\gamma_T. \tag{3.2}$$

The combination of the following proposition with existence of an exponential moment will allow us to prove Theorem 1.1.

Proposition 3.1 *In any dimension $d > 2$,*

$$E\left[\left(\underline{\gamma_T}\right)^2\right] = O(T^{-\theta}) \quad \textit{as } T \to \infty, \tag{3.3}$$

for some $\theta > 0$.

Proof. By Lemma 1.4, we have

$$E[\underline{\gamma_T}]^2 = E[\gamma_T]^2 - (G(0) - 1)^2 + O(T^{-\epsilon}). \tag{3.4}$$

By definition, the first term on the right side is given by

$$E[\gamma_T]^2 = \frac{1}{T^2} \sum_{0 \le s_1 < t_1 \le T} \sum_{0 \le s_2 < t_2 \le T} E\delta_{\omega(s_1),\omega(t_1)}\delta_{\omega(s_2),\omega(t_2)}. \tag{3.5}$$

To evaluate the limiting behaviour of the right side, we consider separately the contributions to the double sum arising from the following three cases.
Case (a): the intersection of the two intervals $[s_1, t_1]$, $[s_2, t_2]$ is nonempty but does not contain either of the intervals,
Case (b): one of these two intervals is a subset of the other,
Case (c): the intervals $[s_1, t_1]$ and $[s_2, t_2]$ do not intersect.

We will see that the first two cases correspond to error terms, while the third term gives the main term and will cancel the subtracted term in (3.4).
Case (a): Suppose that $0 \le s_1 < s_2 \le t_1 < t_2 \le T$, or that the inequality with subscripts 1 and 2 interchanged holds. Taking both possibilities into account, the contribution to (3.5) due to this case is

$$2T^{-2} \sum_{0 \le i_1 < i_2 \le i_3 < i_4 \le T} \sum_{x,y} p_{j_1}(x)p_{j_2}(y-x)p_{j_3}(y-x)p_{j_4}(y-x) \tag{3.6}$$

where $j_1 = i_1$, $j_2 = i_2 - i_1$, $j_3 = i_3 - i_2$, and $j_4 = i_4 - i_3$. This is at most

$$2T^{-2} \sum_{\substack{j_1, j_2, j_3, j_4 \\ j_1+j_2+j_3+j_4 \le T}} \sum_{x,y} p_{j_1}(x)p_{j_2}(y-x)p_{j_3}(y-x)p_{j_4}(y-x). \tag{3.7}$$

By symmetry we rewrite this as less than 3! times the same sum but with the additional constraint $j_2 \le j_3 \le j_4$. Then we use $p_j(x) \le O(j^{-d/2})$ for $j = j_3, j_4$ and $\sum_v p_j(v) = 1$ for $j = j_1, j_2$. In this way we find that the contribution of this case is at most

$$T^{-2} \sum_{0 \le j_1 \le T;\, 0 \le j_2 \le j_3 \le j_4 \le T} O(j_3^{-d/2} j_4^{-d/2}) \le O(T^{-\theta}), \tag{3.8}$$

with $\theta > 0$.
Case (b): Suppose $0 \le s_1 \le s_2 < t_2 \le t_1 \le T$, or that the inequality with subscripts 1 and 2 interchanged holds. With the same definitions for j_i, we wish to estimate

$$2T^{-2} \sum_{\substack{j_1, j_2, j_3, j_4 \\ j_1+j_2+j_3+j_4 \le T}} \sum_{x,y} p_{j_1}(x)p_{j_2}(y-x)p_{j_3}(0)p_{j_4}(y-x). \tag{3.9}$$

By symmetry we rewrite this as less than 2! times the same sum but with the additional constraint $j_2 \leq j_4$. Then we use $p_j(x) \leq O(j^{-d/2})$ for $j = j_3, j_4$ and $\sum_v p_j(v) = 1$ for $t = j_1, j_2$. Therefore the contribution of this case is, for some $\theta > 0$, at most

$$T^{-2} \sum_{0 \leq j_1 \leq T;\, 0 \leq j_2 \leq j_4 \leq T;\, 0 \leq j_3 \leq T} O(j_3^{-d/2} j_4^{-d/2}) \leq O(T^{-\theta}). \tag{3.10}$$

Case (c): Suppose now that $0 \leq s_1 < t_1 < s_2 < t_2 \leq T$, or that the inequality with subscripts 1 and 2 interchanged holds. The contribution due to this case is

$$2T^{-2} \sum_{\substack{j_1 \geq 0; j_3 \geq 1; j_2, j_4 \geq 2 \\ 0 \leq j_1 + j_2 + j_3 + j_4 \leq T}} \sum_{x,y} p_{j_1}(x) p_{j_2}(0) p_{j_3}(y - x) p_{j_4}(0). \tag{3.11}$$

Performing the sum over x, y, and then the sum over j_1 and j_3, and letting

$$S(j_2, j_4, T) = \frac{(T - j_2 - j_4)(T - j_2 - j_4 + 1)}{2}, \tag{3.12}$$

this is equal to

$$2T^{-2} \sum_{j_2=2}^{T} \sum_{j_4=2}^{T-j_2} p_{j_2}(0) p_{j_4}(0) S(j_2, j_4, T). \tag{3.13}$$

Using the fact that $p_j(0) = O(j^{-d/2})$, it is not hard to see that the dominant contribution arises from the $T^2/2$ term in S, and that the above is equal to

$$\left(\sum_{j=2}^{\infty} p_j(0) \right)^2 + O(T^{-\theta}) = (G(0) - 1)^2 + O(T^{-\theta}). \tag{3.14}$$

This gives a cancellation in (3.4), and completes the proof. $\square$

3.2 Preliminary for $d \leq 2$

We define an approximate Kronecker delta function by

$$\delta_{x;\epsilon} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} d^d k \, \exp\left[ik \cdot x - \epsilon k^2 \right]. \tag{3.15}$$

The following lemma proves a uniform version of (2.10) for random walk, and is closely related to Lemma 1 of [26]. Its proof is complicated in a minor way by the fact that the simple random walk does not satisfy the strong aperiodicity hypothesis of [26], so that it is not the case that $|\hat{D}(k)| = 1$ only when each component k_j is an integer multiple of 2π (see [28], page 75, P8). Our results could be extended to more general symmetric random walks with second moments by an appropriate generalization of this lemma.

Lemma 3.2 *Let $1 \le d < 4$ and $h_{\epsilon T}(x) = \delta_{x;\epsilon T} - \delta_x$. Then*

$$\lim_{\epsilon\to 0} E\left[\alpha_{T,h_{\epsilon T}}(\mathcal{A}_T)\right]^2 = \lim_{\epsilon\to 0} E\left[\alpha_{T,\delta_{\epsilon T}}(\mathcal{A}_T) - \alpha_T(\mathcal{A}_T)\right]^2 = 0, \tag{3.16}$$

where the limit is uniform in T and $\alpha_{T,f}$ is given by (2.17).

Proof. In the following, k_1, k_2 are each d-component vectors, d^dk_1, d^dk_2 are each Lebesgue measures on $\mathbf{R}^d$ and $I(k_1, k_2)$ is the indicator function of the set $\{k_1, k_2 \in [-\pi, \pi]^d\}$. Going over to the Fourier transform (see (2.18)), we have

$$E\left[\alpha_{T,h_{\epsilon T}}(\mathcal{A}_T)\right]^2 = \frac{1}{T^{4-d}} \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} |\hat{h}_{\epsilon T}(k_1)\hat{h}_{\epsilon T}(k_2)| I(k_1, k_2) \left| \sum_{0\le s_1, s_2\le T} E e^{ik_1\cdot\omega(s_1) + ik_2\cdot\omega(s_2)} \right|^2.$$

Let

$$F_T(k_1, k_2) = \left| \sum_{0\le s_1\le s_2\le T} E e^{ik_1\cdot\omega(s_1)} e^{ik_2\cdot\omega(s_2)} \right| I(k_1, k_2). \tag{3.17}$$

Then, using $\hat{h}_{\epsilon T}(k) = e^{-\epsilon T k^2} - 1 = \hat{h}_\epsilon(\sqrt{T}k)$, we have

$$E\left[\alpha_{T,h_{\epsilon T}}(\mathcal{A}_T)\right]^2 \le \frac{4}{T^{4-d}} \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} |\hat{h}_\epsilon(\sqrt{T}k_1)\hat{h}_\epsilon(\sqrt{T}k_2)| F_T^2(k_1, k_2). \tag{3.18}$$

Let $\vec{\pi}$ denote the vector in $\mathbf{R}^d$ whose components are all equal to π. As $T \to \infty$, the integrand of (3.18) can become singular at the points $(k_1, k_2) = (0, 0)$, $(0, \vec{\pi})$, $(\vec{\pi}, 0)$, $(\vec{\pi}, \vec{\pi})$. We break the domain of integration into subsets which each contain one of these points, and a complementary set where the integrand remains bounded. As before, we write $\hat{D}(k) = d^{-1}\sum_{j=1}^d \cos k_j$.

Beginning with the subset that contains $(0, 0)$, we fix a small constant $a > 0$ so that $|p| < a$ (Euclidean distance) implies that

$$0 < \hat{D}(p) = 1 - \frac{1}{d}\sum_{j=1}^d (1 - \cos p_j) \le 1 - cp^2, \tag{3.19}$$

for some fixed $c > 0$. Let $p_1 = k_1 + k_2$, $p_2 = k_2$ and let $I_1(k_1, k_2)$ be the indicator function of the set $|p_1| < a$ and $|p_2| < a$. Note that

$$E\left[e^{ik_1\cdot\omega(s_1)} e^{ik_2\cdot\omega(s_2)}\right] = \hat{D}^{s_1}(p_1)\hat{D}^{s_2-s_1}(p_2). \tag{3.20}$$

Since $\hat{D}(p_1)$ and $\hat{D}(p_2)$ are positive and bounded by 1,

$$\begin{aligned} F_T(k_1, k_2)I_1(k_1, k_2) &= \sum_{0\le s_1\le s_2\le T} \hat{D}^{s_1}(p_1)\hat{D}^{s_2-s_1}(p_2)I_1(k_1, k_2) \\ &\le \sum_{0\le s_1\le T} \sum_{s_2: s_1\le s_2\le T+s_1} \hat{D}^{s_1}(p_1)\hat{D}^{s_2-s_1}(p_2)I_1(k_1, k_2) \\ &\le \min\{T+1, \frac{1}{1-\hat{D}(p_1)}\} \min\{T+1, \frac{1}{1-\hat{D}(p_2)}\} \\ &\le b_T(p_1)b_T(p_2), \end{aligned} \tag{3.21}$$

where

$$b_T(p) = 2\min\{T, \frac{1}{cp^2}\}. \tag{3.22}$$

Using this estimate and scaling $k \to k/\sqrt{T}$ we have

$$\begin{aligned}\frac{1}{T^{4-d}} \int_{[-\pi,\pi]^d} d^d k_1 \int_{[-\pi,\pi]^d} d^d k_2 \, |\hat{h}_\epsilon(\sqrt{T} k_1) \hat{h}_\epsilon(\sqrt{T} k_2)| F_T^2(k_1, k_2) I_1(k_1, k_2) \\ \leq \int_{R^d} d^d k_1 \int_{R^d} d^d k_2 \, |\hat{h}_\epsilon(k_1)| |\hat{h}_\epsilon(k_2)| b_1^2(p_1) b_1^2(p_2).\end{aligned} \tag{3.23}$$

The right side approaches 0 as $\epsilon \to 0$, by dominated convergence, if $d < 4$. Therefore the contribution from $I_1(k_1, k_2)$ to the right hand side of (3.18) tends to zero uniformly in T.

Next we consider the possible singularity at $(0, \vec{\pi})$. Let $I_2(k_1, k_2)$ be the indicator function of the set $|p_1| < a$ and $|p_2 - \vec{\pi}| < a$. This time, since $\hat{D}(p_1)$ is positive and $|\sum \hat{D}(p_2)^s| \leq 1$ because $\hat{D}(p_2)$ is negative and $|\hat{D}(p_2)| \leq 1$, we use

$$\begin{aligned} F_T(k_1, k_2) I_2(k_1, k_2) &= \left| \sum_{0 \leq s_1 \leq s_2 \leq T} \hat{D}^{s_1}(p_1) \hat{D}^{s_2 - s_1}(p_2) \right| I_2(k_1, k_2) \\ &\leq \sum_{0 \leq s_1 \leq T} \hat{D}^{s_1}(p_1) I_2(k_1, k_2) \\ &\leq b_T(p_1) I_2(k_1, k_2). \end{aligned} \tag{3.24}$$

Therefore

$$\begin{aligned} &\frac{1}{T^{4-d}} \int d^d k_1 \int d^d k_2 \, |\hat{h}_\epsilon(\sqrt{T} k_1) \hat{h}_\epsilon(\sqrt{T} k_2)| F_T^2(k_1, k_2) I_2(k_1, k_2) \\ &\quad \leq \frac{1}{T^{4-d}} \int d^d k_1 \int d^d k_2 \, |\hat{h}_\epsilon(\sqrt{T} k_1) \hat{h}_\epsilon(\sqrt{T} k_2)| b_T^2(p_1) I_2(k_1, k_2) \\ &\quad \leq \frac{1}{T^2} \int_{|p_1| < c\sqrt{T}} d^d p_1 \int_{|p_2 - \sqrt{T}\vec{\pi}| < c\sqrt{T}} d^d p_2 \, |\hat{h}_\epsilon(p_1 - p_2) \hat{h}_\epsilon(p_2)| b_1^2(p_1) \\ &\quad \leq \int_{R^d} d^d p_1 \int_{R^d} d^d p_2 \, |\hat{h}_\epsilon(p_1 - p_2) \hat{h}_\epsilon(p_2)| b_1^2(p_1) \left(\frac{\text{Const.}}{1 + p_2^2} \right)^2, \end{aligned}$$

where in the last step we used the fact that $T^{-1} \leq \text{Const.}(1 + p_2^2)^{-1}$ when $|p_2 - \sqrt{T}\vec{\pi}| < a\sqrt{T}$. The limit, as $\epsilon \to 0$, is zero uniformly in T for $d < 4$. A similar argument applies for $(\vec{\pi}, 0)$. Therefore these contributions to the right hand side of (3.18) tend to zero uniformly in T.

Now consider $(\vec{\pi}, \vec{\pi})$. Let $I_4(k_1, k_2)$ be the indicator function of the set $|p_1 - \vec{\pi}| < a$ and $|p_2 - \vec{\pi}| < a$. Then, using $|\sum \hat{D}(p_2)^s| \leq 1$ as in the previous case,

$$\begin{aligned} F_T(k_1, k_2) I_4(k_1, k_2) &\leq \sum_{0 \leq s_1 \leq T} |\hat{D}^{s_1}(p_1)| I_4(k_1, k_2) \\ &\leq b_T(p_1 - \vec{\pi}), \end{aligned} \tag{3.25}$$

because $|\hat{D}^{s_1}(p_1)| = \hat{D}^{s_1}(p_1 - \vec{\pi})$. Again for $d < 4$ this leads to an integral that tends to zero as $\epsilon \to 0$ uniformly in T.

Finally we have the set complementary to all the potential singularities. Let $I_5(k_1, k_2)$ be the indicator function of the set where $|p_i| \geq a$ and $|p_i - \vec{\pi}| \geq a$ for $i = 1, 2$. Then $|\hat{D}(p_i)|$ is bounded away from 1, and hence

$$\begin{aligned} F_T(k_1,k_2) I_5(k_1,k_2) &\leq \sum_{0 \leq s_1 < \infty} \sum_{0 \leq s_2 < \infty} |\hat{D}^{s_1}(p_1)\hat{D}^{s_2}(p_2)| I_5(k_1,k_2) \\ &\leq \text{Const.} \end{aligned} \tag{3.26}$$

As above,

$$\begin{aligned} &\frac{1}{T^{4-d}} \int d^d k_1 \int d^d k_2 \, |\hat{h}_\epsilon(\sqrt{T}k_1)\hat{h}_\epsilon(\sqrt{T}k_2)| F_T^2(k_1,k_2) I_5(k_1,k_2) \\ &\qquad \leq \frac{\text{Const}}{T^4} \int_{[-\sqrt{T}\pi,\sqrt{T}\pi]^d} d^d k_1 \int_{[-\sqrt{T}\pi,\sqrt{T}\pi]^d} d^d k_2 \, |\hat{h}_\epsilon(k_1)\hat{h}_\epsilon(k_2)| \\ &\qquad \leq \text{Const} \int_{R^d} d^d k_1 \int_{R^d} d^d k_2 \, |\hat{h}_\epsilon(k_1)\hat{h}_\epsilon(k_2)| (1+k_1^2)^{-2}(1+k_2^2)^{-2} \end{aligned}$$

Therefore, for $d < 4$, this contribution to the right hand side of (3.18) tends to zero as $\epsilon \to 0$, uniformly in T, by dominated convergence. $\square$

3.3 Convergence

Throughout this section, given a walk $\omega(i)$ and $t \in [0,1]$, we define a piecewise constant function $X_T(t) = T^{-1/2}\omega(\lfloor tT \rfloor])$. We begin by considering the case of high dimensions.

3.3.1 Dimensions $d > 2$

In this section, we complete the proof of Theorem 1.1, making use of Theorem 2.1 and Proposition 3.1.

Proposition 3.3 *Let $d > 2$ and $-\infty < \beta < \infty$. For $t_m \in [0,1]$, $n = 0, 1, 2, \ldots$, and $k_m \in \mathbf{R}^d$,*

$$\lim_{T\to\infty} E \prod_{m=1}^{n} \left[e^{ik_m \cdot X_T(t_m)} \right] e^{i\beta \underline{\gamma_T}(\mathcal{T}_T)} = E \prod_{m=1}^{n} e^{ik_m \cdot B(t_m)}. \tag{3.27}$$

Hence, the vector $(X_T(t_1), \ldots, X_T(t_n), \underline{\gamma_T}(\mathcal{T}_T))$ converges in distribution to the vector $(B(t_1), \ldots, B(t_n), 0)$.

Proof. We set

$$Q(X_T) = \prod_{m=1}^{n} e^{ik_m \cdot X_T(t_m)} \tag{3.28}$$

and we write Q for $Q(X_T)$ and q for $Q(B)$. Then it suffices to show that

$$EQe^{i\beta\underline{\gamma_T}} - Eq \to 0 \tag{3.29}$$

as $T \to \infty$. The above difference can be written as

$$EQ\left(e^{i\beta\underline{\gamma_T}} - 1\right) + EQ - Eq. \tag{3.30}$$

By Donsker's theorem, $EQ - Eq \to 0$ as $T \to \infty$. Using the inequality $|Q(e^{i\beta\underline{\gamma_T}} - 1)| \leq |\beta\underline{\gamma_T}|$, the first term is bounded above by $|\beta|E|\underline{\gamma_T}| \leq |\beta|[E|\underline{\gamma_T}|^2]^{1/2}$, which approaches zero as $T \to \infty$ by Proposition 3.1.

The last statement of the proposition then follows immediately from the fact that convergence of characteristic functions implies convergence in distribution on $\mathbf{R}^N$; see Theorem 7.6 of [2]. □

The following corollary yields the results of Theorem 1.1.

Corollary 3.4 *Let $d > 2$ and $-\infty < \beta < \beta_0$. For $t_m \in [0,1]$, $i_m \in \{1, \ldots, d\}$, and $n = 0, 1, 2, \ldots$,*

$$\lim_{T\to\infty} E \prod_{m=1}^{n} \left[X_T^{(i_m)}(t_m)\right] e^{\beta\underline{\gamma_T}(\mathcal{T}_T)} = E \prod_{m=1}^{n} \left[B^{(i_m)}(t_m)\right], \tag{3.31}$$

where the superscript (i_m) denotes a component of X_T or B. Moreover, the process X_T converges in distribution to Brownian motion.

Proof. By Proposition 3.3, the expectation of any bounded continuous function of $(X_T(t_1), \ldots, X_T(t_n), \underline{\gamma_T}(\mathcal{T}_T))$ converges to the corresponding continuum expectation. The function appearing in the corollary is not bounded, but it is uniformly integrable. To see this, it is sufficient (see page 32 of [2]) to show that there is an $\epsilon > 0$ such that

$$\sup_T E \left| \prod_{m=1}^{n} \left[X_T^{(i_m)}(t_m)\right] e^{\beta\underline{\gamma_T}(\mathcal{T}_T)} \right|^{1+\epsilon} < \infty. \tag{3.32}$$

But by Hölder's inequality

$$E \left| \prod_{m=1}^{n} \left[X_T^{(i_m)}(t_m)\right] e^{\beta\underline{\gamma_T}(\mathcal{T}_T)} \right|^{1+\epsilon} \leq \left\| \prod_{m=1}^{n} \left[X_T^{(i_m)}(t_m)\right]^{1+\epsilon} \right\|_q \left\| e^{\beta(1+\epsilon)\underline{\gamma_T}(\mathcal{T}_T)} \right\|_p, \tag{3.33}$$

and by Theorem 2.1 the right side is bounded uniformly in T provided p is chosen sufficiently close to 1 and ϵ is chosen sufficiently small that $\beta(1+\epsilon)p < \beta_0$. It then follows from Proposition 3.3, together with Corollary 1 to Theorem 5.1 and Theorem 5.4 of [2], that (3.31) holds.

It is then immediate that the renormalized partition function converges to 1 and that the diffusion constant is equal to 1. The fact that $c_T \to \exp[\beta(G(0) - 1)]$ as $T \to \infty$ then follows from Lemma 1.4.

To complete the proof, it remains to show that X_T is tight. For this, it is sufficient to prove that for any $0 \leq t_1 < t_2 < t_3 \leq 1$ and for some $a > 1/2$ and constant K,

$$\frac{1}{c_T^{ren}} E\left[|X_T(t_2) - X_T(t_1)|^{2a} |X_T(t_3) - X_T(t_2)|^{2a} e^{\beta\underline{\gamma_T}}\right] \leq K|t_2 - t_1|^a |t_3 - t_2|^a \tag{3.34}$$

(see Theorem 8.4 and pages 87-89 of [2]). The normalizing partition function on the left side is asymptotically 1 and can be ignored. Applying Hölder's inequality to separate the exponential interaction factor from the displacement factors, as above, gives (3.34) for any $a \geq 0$. $\square$

3.3.2 Dimensions $d \leq 2$

In this section, we complete the proofs of Theorems 1.2 and 1.3.

Proposition 3.5 *Let $d = 1$ or 2 and $p = \frac{4-d}{2}$. For any $-\infty < \beta < \infty$, $t_m \in [0,1]$, $n = 0, 1, \dots$, and $k_m \in \mathbf{R}^d$,*

$$\lim_{T\to\infty} E \prod_{m=1}^{n} \left[e^{ik_m \cdot X_T(t_m)} \right] e^{i\beta \underline{\gamma_T}(\mathcal{T}_T)} = E \prod_{m=1}^{n} \left[e^{ik_m \cdot B(t_m)} \right] e^{i\beta \underline{\gamma}(\mathcal{T})}. \tag{3.35}$$

Hence, the vector $(X_T(t_1), \dots, X_T(t_n), \underline{\gamma_T}(\mathcal{T}_T))$ converges in distribution to the vector $(B(t_1), \dots, B(t_n), \underline{\gamma}(\mathcal{T}))$.

Proof. We want to apply Donsker's theorem, which says that the expectation of any functional of X_T which is bounded and continuous in the Skohorod topology, converges to its natural continuum limit as $T \to \infty$. However, unlike the simpler situation encountered in Section 3.3.1, we cannot apply it directly because the renormalized self-intersection local time is not a continuous functional. We will use the results of Sections 2.2 and 3.2 to introduce cutoffs and reduce the problem to one involving a bounded continuous functional.

We set

$$Q(X_T) = \prod_{m=1}^{n} \left[e^{ik_m \cdot X_T(t_m)} \right] \tag{3.36}$$

and we write Q for $Q(X_T)$ and q for $Q(B)$. Recalling the definitions of $\mathcal{T}_N$ and $\mathcal{T}_{T,N}$ from (2.9) and (2.32), we write

$$\begin{aligned} EQe^{i\beta \underline{\gamma_T}(\mathcal{T}_T)} - Eqe^{i\beta \underline{\gamma}(\mathcal{T})} &= EQ\left(e^{i\beta \underline{\gamma_T}(\mathcal{T}_T)} - e^{i\beta \underline{\gamma_T}(\mathcal{T}_{T,N})}\right) + EQe^{i\beta \underline{\gamma}(\mathcal{T}_{T,N})} - Eqe^{i\beta \underline{\gamma}(\mathcal{T}_N)} \\ &\quad + Eq\left(e^{i\beta \underline{\gamma}(\mathcal{T}_N)} - e^{i\beta \underline{\gamma}(\mathcal{T})}\right). \end{aligned} \tag{3.37}$$

The first term on the right side can be estimated using the inequality

$$|Q||e^{iA} - e^{iB}| \leq |A - B|, \tag{3.38}$$

together with $E|A - B| \leq \|A - B\|_2$ and Lemma 2.4; its contribution can be made as small as desired by taking N large independent of T. By (2.13) the same assertion holds for the last term on the right side. Thus it suffices to show that for fixed (large) N,

$$\lim_{T\to\infty} E\left[Qe^{i\beta \underline{\gamma_T}(\mathcal{T}_{T,N})}\right] = E\left[qe^{i\beta \underline{\gamma}(\mathcal{T}_N)}\right]. \tag{3.39}$$

We define $\gamma_{T,\epsilon}(\mathcal{T}_{T,N})$ by replacing δ_x in (2.33) by the approximate Kronecker delta

$$\delta_{x;\epsilon T} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} d^d k \, \exp\left[ik \cdot x - \epsilon T k^2\right], \tag{3.40}$$

and define $\gamma_\epsilon(\mathcal{T}_N) = \gamma_{\delta_\epsilon}(\mathcal{T}_N)$ as in (2.10), where

$$\delta_\epsilon(x) = \frac{1}{(2\pi)^d} \int_{R^d} d^d k \, \exp\left[ik \cdot x - \epsilon k^2\right]. \tag{3.41}$$

Adding and subtracting as in the previous paragraph, and using (2.10) and Lemma 3.2, we see that it is sufficient to prove that for fixed (large) N and (small) ϵ,

$$\lim_{T\to\infty} EQe^{i\beta \underline{\gamma_{T,\epsilon}}(\mathcal{T}_{T,N})} = Eqe^{i\beta \underline{\gamma_\epsilon}(\mathcal{T}_N)}. \tag{3.42}$$

By definition,

$$\begin{aligned}
\gamma_{T,\epsilon}(\mathcal{T}_{T,N}) &= \frac{1}{T^{2-d/2}} \sum_{(i,j)\in\mathcal{T}_{T,N}} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} d^d k \, e^{ik\cdot(\omega(i)-\omega(j))} e^{-\epsilon T k^2} \\
&= \frac{1}{T^2} \sum_{(i,j)\in\mathcal{T}_{T,N}} \frac{1}{(2\pi)^d} \int_{[-\sqrt{T}\pi,\sqrt{T}\pi]^d} d^d k \, e^{ik\cdot(X_T(\lfloor\frac{i}{T}\rfloor)-X_T(\lfloor\frac{j}{T}\rfloor))} e^{-\epsilon k^2} \\
&= \frac{1}{T^2} \sum_{(i,j)\in\mathcal{T}_{T,N}} \delta_\epsilon(X_T(\lfloor i/T\rfloor) - X_T(\lfloor i/T\rfloor)) + O(e^{-C_\epsilon T}) \\
&= \int\int_{\mathcal{T}_N} ds\, dt\, \delta_\epsilon(X_T(s) - X_T(t)) + O(T^{-1}) + O(e^{-C_\epsilon T})
\end{aligned} \tag{3.43}$$

for some $C_\epsilon > 0$. The error terms arise from the difference in the ranges of the k integrals in $\delta_\epsilon(x)$ and $\delta_{x;\epsilon}$, and boundary effects in replacing the sum by the integral (note that the terms in the sum are constant on squares of side T), and are uniform in ω. Let

$$F(X_T) = Q(X_T) \exp\left[i\beta \int\int_{\mathcal{T}_N} ds\, dt\, \delta_\epsilon(X_T(s) - X_T(t))\right]. \tag{3.44}$$

It suffices to show that $E[F(X_T)]$ converges to $E[F(B)]$, where B is Brownian motion, But this follows directly from Donsker's theorem [2] because $X_T \mapsto F(X_T)$ is bounded and continuous in the Skorohod topology. □

The following corollary yields the results of Theorems 1.2 and 1.3, apart from the monotonicity of the diffusion constant, which is deferred to Section 4.

Corollary 3.6 *Let $d = 1$ or 2 and $p = \frac{4-d}{2}$, and let β_0 be given by (1.10). For any $-\infty < \beta < \beta_0$, $t_m \in [0,1]$, and $n = 0, 1, \ldots$,*

$$\lim_{T\to\infty} E \prod_{m=1}^{n} [X_T(t_m)]\, e^{\beta \underline{\gamma_T}(\mathcal{T}_T)} = E \prod_{m=1}^{n} [B(t_m)]\, e^{\beta \underline{\gamma}(\mathcal{T})}. \tag{3.45}$$

In particular, for $d = 2$, $Ee^{\beta\underline{\gamma}(\mathcal{T})} < \infty$ for $\beta < \beta_0$. Moreover, X_T converges in distribution to the process with law $d\nu_{d,\beta}$.

Proof. We apply uniform integrability as in Corollary 3.4, using Theorems 2.1 and 2.5 respectively for $d = 1, 2$. $\square$

As a consequence of the corollary for $d = 2$, since $E_T[\gamma_T(\mathcal{T}_T)] \sim \frac{1}{\pi} \log T$ by Lemma 1.4, for $-\infty < \beta < \beta_0$ the unrenormalized partition function satisfies the asymptotic relation

$$c_T = \left[E_T e^{\beta \underline{\gamma_T}(\mathcal{T}_T)}\right] e^{\beta E_T[\gamma_T(\mathcal{T}_T)]} \sim \left[E e^{\beta \underline{\gamma}}\right] T^{\beta/\pi}. \tag{3.46}$$

4 The diffusion constant

For $d > 2$ and $p = 1$, the diffusion constant is equal to 1 for all $-\infty < \beta < \beta_0$, while for $d \le 2$ and $p = \frac{4-d}{2}$, we have

$$D(\beta) = \frac{E[B(1)^2 e^{\beta \underline{\gamma}}]}{E e^{\beta \underline{\gamma}}}. \tag{4.1}$$

For the rest of this section, we consider only $d \le 2$.

The following elementary calculation shows that the diffusion constant is *strictly* decreasing for β equal to zero, and hence, by continuity, for β near zero. Consider first $d = 1$, so there is no need for renormalization. Then

$$D'(0) = \frac{d}{d\beta}\bigg|_{\beta=0} \frac{E[B(1)^2 e^{\beta\gamma}]}{E e^{\beta\gamma}} = E[B(1)^2 \gamma] - [E B(1)^2][E\gamma]. \tag{4.2}$$

Writing now $p_t(x) = (2\pi t)^{-d/2} \exp[-x^2/2t]$ for the Brownian motion transition function, this gives

$$\begin{aligned} D'(0) &= \int_{0 \le s < t \le 1} ds\, dt \int dx\, dy\, y^2\, p_s(x) p_{t-s}(0) p_{1-t}(y - x) \\ &\qquad -1 \cdot \int_{0 \le s < t \le 1} ds\, dt \int dx\, dy\, p_s(x) p_{t-s}(0) p_{1-t}(y - x) \\ &= \int_{0 \le s < t \le 1} ds\, dt (s + (1 - t)) p_{t-s}(0) - \int_{0 \le s < t \le 1} ds\, dt\, p_{t-s}(0) \\ &= -\int_{0 \le s < t \le 1} ds\, dt\, (t - s)\, p_{t-s}(0). \end{aligned}$$

The last expression is clearly negative, so the diffusion constant is strictly decreasing at $\beta = 0$. For $d = 2$, formally $\underline{\gamma} = \gamma - E\gamma$ and the constant $E\gamma$ drops out in the right side of (4.2), so that the same argument applies also for $d = 2$. This becomes a correct argument by a suitable introduction of cutoffs.

For $\beta \in [0, \beta_0)$, we now prove that $D(\beta)$ is strictly decreasing, as follows. The following proposition shows that for $\beta \in (0, \beta_0)$, the inequality $D'(\beta) \le 0$ is a consequence of a correlation inequality of Fröhlich and Park [14]. Combined with the analyticity of $D(\beta)$ for $\beta \in (-\infty, \beta_0)$, and the fact that D is strictly decreasing in a neighbourhood of $\beta = 0$, this implies that D is strictly decreasing for $\beta \in (0, \beta_0)$. We believe, but have not proved, that D is strictly monotone for all $\beta < \beta_0$.

Proposition 4.1 *Let $d \geq 1$, $\epsilon > 0$ and $f(x) = (2\pi\epsilon)^{-d/2}e^{-x^2/2\epsilon}$. For any $0 < \beta < \infty$, $t_m \in [0,1]$, $n = 0, 1, \ldots$, and $k_m \in \mathbf{R}^d$,*

$$\langle \prod_{m=1}^{n} \left[e^{ik_m \cdot B(t_m)}\right] \rangle_{f,\beta} \equiv \frac{E \prod_{m=1}^{n} \left[e^{ik_m \cdot B(t_m)}\right] e^{\beta \gamma_f(\mathcal{T})}}{E e^{\beta \gamma_f(\mathcal{T})}} \tag{4.3}$$

is monotone nondecreasing in β. In addition, for $d = 1$ or 2 and $0 < \beta < \beta_0$, $D(\beta)$ is nonincreasing in β.

Proof. In dimensions $d = 1$ or 2 the expectation (4.3) has a limit as $f \to \delta$ (that is, as $\epsilon \to 0$), so the limiting expectations are also nondecreasing. For the diffusion constant, we use

$$D(\beta) = 2d \lim_{k \to 0} k^{-2}(1 - \langle e^{k \cdot B(1)} \rangle_{\delta,\beta}) \tag{4.4}$$

to conclude that D is nonincreasing. This is derived using Euclidean symmetry and $\langle B(1) \rangle = 0$.

Thus it suffices to obtain (4.3), for arbitrary dimensions. We will show that this is a consequence of the general Ginibre correlation inequalities proved by Fröhlich and Park in [14] using duplicate variables. In particular, we will apply their Theorem 3.1(5). For this, we need to rewrite (4.3) in the formalism of [14]. Accordingly, we let $\mathcal{H}$ denote the Hilbert space obtained by completing $\mathbf{R}^d \otimes \mathcal{C}[0,1]$ with the inner product

$$C(g,h) = \frac{1}{d} \int_0^1 \int_0^1 ds\, dt \sum_{i=1}^{d} g^{(i)}(s) \min\{s,t\} h^{(i)}(t), \tag{4.5}$$

where the superscripts (i) denote components. Let ϕ denote the Gaussian process with mean 0 and covariance C indexed by $\mathcal{H}$, and let $d\mu_C$ denote the associated Gaussian measure on the space $\mathcal{S}'$ of tempered distributions. By definition of f, $\hat{f}(q) = e^{-\epsilon q^2/2}$. We define

$$\begin{aligned} X &= \{x \equiv (q,s,t) : q \in \mathbf{R}^d, 0 \leq s < t \leq 1\}, \\ d\rho(x) &= \frac{1}{2\pi} dq\, ds\, dt\, \hat{f}(q), \end{aligned}$$

and a mapping $l : X \to \mathcal{H}$ by

$$l_x = q(\delta(\cdot - t) - \delta(\cdot - s)), \quad x = (q,s,t) \in X.$$

Then $C(l_x, l_x) = d^{-1}q^2|t-s|$. (Fröhlich and Park's hypothesis that the integral $\int_X d\rho(x) \exp[\frac{1}{2}C(l_x,l_x)]$ be finite is thus not satisfied for small ϵ, but, in fact, this hypothesis is not necessary for part (5) of their Theorem 3.1.)

By construction, the Gaussian random variable $\phi(l_x)$ has the same distribution as $q \cdot (B(t) - B(s))$; in fact,

$$\int e^{iq \cdot (B(t)-B(s))} dW = e^{-q^2|t-s|/2d} = e^{-\frac{1}{2}C(l_x,l_x)} = \int_{\mathcal{S}'} e^{i\phi(l_x)} d\mu_C. \tag{4.6}$$

Using this fact, as well as the $B \to -B$ symmetry, rewriting γ_f in terms of the Fourier transform gives

$$\begin{aligned}
\gamma_f(\mathcal{T}) &= \frac{1}{2\pi} \int dq\, \hat{f}(q) \int\int_{0\le s<t\le 1} ds\, dt\, e^{iq\cdot(B(t)-B(s))} \\
&= \frac{1}{2\pi} \int dq\, \hat{f}(q) \int\int_{0\le s<t\le 1} ds\, dt\, \cos\left(q\cdot(B(t)-B(s))\right) \\
&= \int_X d\rho\, \cos(\phi(l_x)).
\end{aligned}$$

In (4.3), by the $B \to -B$ symmetry we can replace $\prod_{m=1}^n \left[e^{ik_m\cdot B(t_m)}\right]$ by $\cos(\sum_{j=1}^n k_j \cdot B(t_j))$. Define $m \in \mathcal{H}$ by $m = \sum k_j \delta(\cdot - t_j)$. Then the Gaussian random variable $\phi(m)$ has the same distribution as $\sum k_j \cdot B(t_j)$, and therefore

$$\langle \prod_{j=1}^n e^{k_j\cdot B(t_j)} \rangle_{f,\beta} = \Xi(f,\beta)^{-1} \int d\mu_C(\phi)\, \cos(\phi(m)) e^{\beta \int_X d\rho\, \cos(\phi(l_x))}, \tag{4.7}$$

where

$$\Xi(f,\beta) = \int d\mu_C(\phi)\, e^{\beta \int_X d\rho\, \cos(\phi(l_x))}. \tag{4.8}$$

Let

$$\langle\langle\, F \,\rangle\rangle_{f,\beta} = \Xi(f,\beta)^{-1} \int d\mu_C(\phi)\;\, F\, e^{\beta \int_X d\rho\, \cos(\phi(l_x))}. \tag{4.9}$$

The derivative of $\langle \prod_{j=1}^n e^{k_j\cdot B(t_j)} \rangle_{f,\beta}$ with respect to β can then be written as

$$\int_X d\rho(x) \left(\langle\langle \cos(\phi(l_x)) \cos(\phi(m)) \rangle\rangle_{f,\beta} - \langle\langle \cos(\phi(l_x)) \rangle\rangle_{f,\beta} \langle\langle \cos(\phi(m)) \rangle\rangle_{f,\beta}\right). \tag{4.10}$$

This is nonnegative, by Theorem 3.1(5) of [14]. □

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