Exact large deviation bounds up to T_c for the Ising model in two dimensions

Dmitry Ioffe
Departement of Mathematics, Northwestern University
2033 Sheridan Rd., Evanston, IL 60208
e-mail: ioffe@math.nwu.edu

Abstract

We prove an upper large deviation bound for the block spin magnetization in the 2D Ising model in the phase coexistence region. The precise rate given by the Wulff construction is shown to hold true for all $\beta > \beta_c$. Combined with the lower bounds derived in [I] those results yield an exact second order large deviation theory up to the critical temperature.

Key words and phrases: Large deviations, Ising model, Wulff construction, FK percolation.

1 Introduction.

This work is a continuation of [I], where we derived precise lower large deviation bounds for the block spin magnetization in the 2D Ising model in the phase coexistence region up to the critical temperature. We, thereby, refer to the latter paper as well as for the basic manuscripts [DKS] and [P] for a more comprehensive discussion of the underlying problem. To set up notations let $\mathbf{Z}^2$ be a 2D integer lattice, d a lattice distance; $d((x_1,x_2),(y_1,y_2)) = max\{|x_1 - y_1|, |y_1 - y_2|\}$ and $\mathbf{B}_N(x)$ a lattice box ; $\mathbf{B}_N(x) = \{y \in \mathbf{Z^2} : d(y,x) \le N\}$. Also, let $\Lambda(L) \subset \mathbf{Z^2}$ be a square box of volume L^2 and let $P^\beta_{\Lambda,+}$ to denote the Gibbs measure on Λ with plus boundary conditions at inverse temperature β, that is the Hamiltonian $H_{\Lambda,+}$ of a spin configuration σ on $\{-1,1\}^\Lambda$ is given by

$$-H_{\Lambda,+}(\sigma) \;=\; \frac{1}{2}\sum_{\substack{<x,y>\\ x,y\in\Lambda}} \sigma_x\sigma_y \;+\; \sum_{\substack{<x,y>\\ x\in\Lambda, y\in\Lambda^c}} \sigma_x,$$

where the summation is over all (ordered) pairs of nearest neighbours x and y. The probability distribution of $\sigma \in \{-1,1\}^\Lambda$ is defined, then, via

$$P^\beta_{\Lambda,+}(\sigma) \;=\; \frac{1}{Z^\beta_{\Lambda,+}} e^{-\beta H_{\Lambda,+}(\sigma)} \;,$$

where $Z^\beta_{\Lambda,+}$ is the corresponding partition function.

In a similar fashion one can define $P^\beta_{\Lambda,-}$. When $L \to \infty$ the measures $P^\beta_{\Lambda,+}$ and $P^\beta_{\Lambda,-}$ weakly converge to limits P^β_+ and P^β_- respectively, both measures being ergodic on $\{1,-1\}^{\mathbf{Z}^2}$ [Pr]. Then the critical value β_c, such that

$$P^\beta_+ = P^\beta_- \; for \; \beta \le \beta_c \;\; and \;\; P^\beta_+ \ne P^\beta_- \; for \; \beta > \beta_c$$

is well defined. In our particular case $\beta_c \in (0,\infty)$ and two alternative definitinions of β_c are the Cramer-Wannier relation

$$tanh(\beta_c) \;=\; e^{-2\beta_c} \tag{1.1}$$

and the order parameter relation

$$\beta \;>\; \beta_c \;\; iff \;\; E^\beta_+\sigma_0 \;=\; m^* \;>\; 0, \tag{1.2}$$

where m^* is a spontaneous magnetization. Define now the block-spin magnetization X_Λ on Λ as

$$X_\Lambda \;=\; \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_x \;,$$

where $\sigma_x = \pm 1$ is the value of the spin at the site $x \in \Lambda$.

Since P_+^β is ergodic, X_Λ converges to m^* P_+^β-a.s.. The purpose of this article is to derive a corresponding large deviation theorem. More precisely we are going to find the main term in the logarithmic asymptotics of

$$P_{\Lambda,+}^\beta\{X_\Lambda \;\in\; [a,b]\}. \tag{1.3}$$

Such large deviations results have a very different flavour depending on whether $[a,b]\cap[-m^*,m^*]$ is empty or not. In the former case (1.3) decays exponentially with the volume $|\Lambda|$ and the precise nontrivial answer is provided within the framework of a more general theory developed in [FO] and [O]. On the other hand, points in $[-m^*,m^*]$ correspond to Gibbs states which have zero specific relative entropy with respect to P_+^β. Consequently the exponential decay of (1.3) in this region is at most of the surface order. It turns out that the main mechanism responsible for large deviations in the phase coexsistence region $[-m^*,m^*]$ is that of the phase separation. Schonmann [S1] was the first to derive nontrivial surface order bounds for large values of β, which were subsequently extended to all $\beta > \beta_c$ in [CCS]. Those bounds were, however, imprecise. Precise bounds, their relation to the phase separation and many more remarkable results were recently obtained in the extensive works of [DKS] and [P]. Both works, however, relied on the method of cluster expansions. Consequently their results are valid only for large values of β, although being almost exhaustive when they apply. A weak version of their theorem states that for $m \in [-m^*,m^*]$ which is sufficiently close to m^* and for $\beta > \beta_c$ large enough,

$$\lim_{L\to\infty}\frac{1}{L}\log P_{\Lambda,+}^\beta\{X_\Lambda \;\le\; m\} \;=\; -2(\lambda_F\alpha(m))^{1/2}\;, \tag{1.4}$$

where $\alpha(m) \;=\; (m^*-m)/(2m^*)$ and λ_F is the volume of the (unnormalized) Wulff shape, generated by the surface tension function F of the model.

In order to explain (1.4) recall that the surface tension (see [A],[P] for the definition and properties) is a function $F : \mathbf{S}^1 \to \mathbf{R}_+$ and it's value $F(n)$ is designed to measure the free energy of the $\pm$-interface in the direction othogonal to the unit vector n. It is also known [P] that the homogeneous extension of F,

$$F(x) \;=\; |x|F(\frac{x}{|x|})$$

is convex. Being such it is the support function of some (uniquely defined) convex body K_F, which is called the Wulff shape. K_F can be recovered from F via

$$K_F \;=\; \{x\in\mathbf{R}^2 \;:< x,n > \le\; F(n)\;\forall n\},$$

where $< \bullet,\bullet >$ above denotes the scalar product in $\mathbf{R}^2$. By the definition, $\lambda_F = Vol(K_F)$. Yet, there is another description of K_F - a variational one. To this end let γ_F be the boundary of K_F; $\gamma_F = \partial K_F$, and let $\mathcal{A}$ be the class of all closed rectifiable curves in $\mathbf{R}^2$ without selfintersections . For $\gamma\in\mathcal{A}$ define the Wulff functional as

$$\mathcal{W}_F(\gamma) \;=\; \int_\gamma F(n_s)ds\;,$$

where n_s is the unit normal to γ at s. Then,

$$2\lambda_F \;=\; \min_{\substack{\gamma\in\mathcal{A}\\ vol(\gamma)=\lambda_F}} \mathcal{W}_F(\gamma) \;=\; \mathcal{W}_F(\gamma_F).$$

Roughly speaking the least energy consuming way to emerse an island of " $-$ " phase of the volume λ_F into the sea of " $+$ " phase is to pack it into the Wulff shape. Since the Wulff functional scales with the length of the boundary,

$$\mathcal{W}_F(a\gamma) \;=\; a\mathcal{W}_F(\gamma),$$

its minimum on simple Jordan curves of volume v is attained on the corresponding dilatation of the Wulff shape, $(v/\lambda_F)^{1/2}K_F$ and is equal to $2(v\lambda_F)^{1/2}$. In particular the left hand side of (1.4) is nothing but the minimal value of the Wulff functional on shapes of the volume $\alpha(m)$.

The meaninig of (1.4) becomes more transparent now: the main contribution to the event $\{X_\Lambda \leq m\}$ comes from the formation of the Wulff drop of the " $-$ " phase of the relative volume $\alpha(m)$. In its turn the choice of $\alpha(m)$ is based on the following identity:

$$(1-\alpha(m))m^* \;-\; \alpha(m)m^* \;=\; m.$$

In the present paper we establish (1.4) for all values of $\beta > \beta_c$. Half of the work was done in the earlier paper [I], where the corresponding lower bound was verified. Ou main result here is the following

Theorem 1.1 *Let $m \in [-m^*, m^*]$ and $\beta > \beta_c$. Then,*

$$\limsup_{L\to\infty} \frac{1}{L} \log P^{\beta}_{\Lambda,+}\{X_\Lambda \;\leq\; m\} \;\leq\; -2(\lambda_F\alpha(m))^{1/2}. \tag{1.5}$$

As in [I] our global strategy of the proof as well as some notations we use is inherited from [P] . The novel feature is , however, that we've been able to avoid cluster expansions while deriving all the necessary estimates in the phase of small contours (see section 3). Our technical tools to do so are those of percolation theory, the principal inspiration being drawn from the works of [ACC] and [Pi]. The main technical idea is to translate events stated in terms of small contours into the events stated in terms of small FK (Fortuin-Kasteleyn) clusters. In section 2 we briefly review some relevant facts about dependent percolation (FK measures) and prove our main large deviation bound for the FK percolation. Section 3 is devoted to the basic reduction to the FK language. The proof of Theorem 1.1 is concluded in section 4.

2 FK percolation and an estimate on the density of small clusters.

Let $A \in \mathbf{Z}^2$ be a finite set of sites. The bond(edge) set of A , $\mathcal{B}(A)$, is defined to be the set of all lattice bonds $< xy >$ with at least one end belonging to A. Given $p \in (0,1)$ set $\mathcal{P}^p_{A,w}$ and $\mathcal{P}^p_{A,f}$ to denote the FK (random cluster) measures on bond configurations $n \in \mathcal{D}_A \stackrel{def}{=} \{0,1\}^{\mathcal{B}(A)}$ with wired and free boundary conditions respectively. We refer to [ACCN] for definitions and a comprehensive account of properties of the FK measures as well as to an excellent review in a recent paper [Pi]. Since $\mathcal{D}_A$ has the natural partial order one may consider FKG (Fortuin-Kasteleyn-Ginibre) properties of $\mathcal{P}^p_{A,w}$ and $\mathcal{P}^p_{A,f}$. Also note that it is possible to view those measures as being defined on $\mathcal{D}_{\mathbf{Z}^2}$ by requiring all the bonds $b \notin \mathcal{B}(A)$ to be open (i.e $n(b) = 1$) or closed $(n(b) = 0)$ respectively. Below we list all the properties of random cluster measures which we are going to use later on.

P1.(FKG relations)[ACCN]. Let $\succ$ denote the FKG order relation. Both $\mathcal{P}^p_{A,w}$ and $\mathcal{P}^p_{A,f}$ are FKG measures. Moreover, if $A \subset B$, then

$$\mathcal{P}^p_{A,w} \succ \mathcal{P}^p_{B,w} \;\; and \;\; \mathcal{P}^p_{A,f} \prec \mathcal{P}^p_{B,f}.$$

Also,

$$\lim_{L\to\infty} \mathcal{P}^p_{\Lambda(L),w} \stackrel{def}{=} \mathcal{P}^p_w$$

exists and is an FKG translation invariant measure on $\mathcal{D}_{\mathbf{Z}^2}$.

P2.(Relations to the Ising model)[ACCN],[ES]. Let $\Omega_A = \{-1,1\}^A$ be the set of spin configurations on A. Choose $p = e^{-2\beta}$ and define $\mathbf{Q}_{A,+}$ to be a joint site-bond measure on $\Omega_A \times \mathcal{D}_A$, given by

$$\mathbf{Q}_{A,+}\{\sigma,n\} = \frac{1}{K_{A,+}} \prod_{\substack{b=<xy>\\ n(b)=0}} (1-p) \prod_{\substack{b=<xy>\\ n(b)=1}} p\delta_{\sigma_x=\sigma_y},$$

where $K_{A,+}$ is the normalizing constant and all the spins on ∂A are set to be $+1$. Then the bond marginal of $\mathbf{Q}_{A,+}$ is precisely $\mathcal{P}^p_{A,w}$, whereas the site marginal is $P^\beta_{A,+}$. Moreover for each site $x \in A$,

$$E^\beta_{\Lambda,+}\sigma_x = \mathcal{P}^p_{A,w}\{x \leftrightarrow \partial A\},$$

where as usual $\{x \leftrightarrow \partial A\}$ denotes the event that x belongs to the open cluster of the boundary ∂A (x is connected to ∂A by open bonds). In particular, for any $\beta > \beta_c$ we have,

$$m^* = E^\beta_+\sigma_x = \mathcal{P}^p_w\{|C(x)| = \infty\},$$

where $C(x)$ is the cluster of x and $|C(x)|$ is its cardinality.

P3.(Duality relations)[CCS] Each bond b of the direct lattice is intersected by precisely one bond b^* of the dual lattice $\mathbf{Z}^2 + (1/2,1/2)$. Consequently, to a given bond configuration $n \in \mathcal{D}_A$ one can correspond the dual bond configuration $n^* \in \mathcal{D}_{A^*}$ (where $A^* \subset \mathbf{Z}^2 + (1/2,1/2)$) via

$$n^*(b^*) = 1 - n(b).$$

The above correspondence is obviously one to one. The measure $\mathcal{P}^p_{A,w}$,therefore, induces some measure on $\mathcal{D}_{A^*}$ which happens to be nothing but $\mathcal{P}^{p^*}_{A^*,f}$, where $p^* = 1 - p/(2-p)$. Note that if $p^* = 1 - e^{-2\beta^*}$, then $e^{-2\beta^*} = tanh(\beta)$. Consequently, dual FK measures correspond to dual Ising models. Likewise the dual measure induced by $\mathcal{P}^p_{A,f}$ on $\mathcal{D}_{A^*}$ is $\mathcal{P}^{p^*}_{A^*,w}$, where p^* is as above.

P4.(Strong FKG and decoupling)[K],[Pi] Let $B \subset \mathcal{B}(A)$ and η_1, η_2 are two bond configurations on B such that $\eta_1 \geq \eta_2$. Then,

$$\mathcal{P}^p_{A,w}\{\bullet \mid n|_B = \eta_1\} \succ \mathcal{P}^p_{A,w}\{\bullet \mid n|_B = \eta_2\}.$$

Assume now that $B \subset \mathcal{B}(A)$ doesn't contain sites belonging to the boundary ∂A and after the removal of all sites attached to B, A is splitted into two disjoint components - the inner componet A_1 and the outer component A_2; A_2 is assumed to be connected to ∂A by the bonds from $\mathcal{B}(A) \setminus B$. Then,

$$\mathcal{P}^p_{A,w}\{n|_{\mathcal{B}(A_1)} = \bullet \mid n|_B \equiv 0\} = \mathcal{P}^p_{A_1,f}\{ \bullet \}$$

and

$$\mathcal{P}^p_{A,w}\{n|_{\mathcal{B}(A_1)} = \bullet;\ n|_{\mathcal{B}(A_2)} = \bullet \mid n|_B \equiv 1\} = \mathcal{P}^p_{A_1,w}\{\bullet\}\mathcal{P}^p_{A_2,w}\{\bullet\}.$$

We are now in a position to prove our first large deviation result about the infinite volume FK measure $\mathcal{P}^p_w$. Let $p > p_c \stackrel{def}{=} 1 - e^{-2\beta_c}$. Pick a number N, $0 < N < \infty$, and set

$$\eta_x = \mathbf{1}_{\{|c(x)| \leq N\}},$$

where $\mathbf{1}_{\{\bullet\}}$ is the indicator function. Set also $d(N) = E\eta_x = \mathcal{P}_w^p\{|C(x)| \leq N\}$. Then, with $\Lambda(L)$ as above,

$$\frac{1}{|\Lambda(L)|}\sum_{x\in\Lambda}\eta_x$$

converges to $d(N)$ as L tends to infinity. The next lemma is an FK analog of lemma 4.2 in [ACC].

Lemma 2.1 *There exists a function $c = c(\epsilon) > 0$, such that,*

$$\mathcal{P}_w^p\{|\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\eta_x \ - \ d(N)| \ \geq \ \epsilon\ \} \ \leq \ exp\{\ -\frac{L^2}{N^2}c(\epsilon)\}.$$

Proof: The main observation is that the field $\{\eta_x\}$ has nice mixing properties. Going along the lines of [S2] one can in principle use this to derive the desired estimate. We'll, however, follow an elegant approach of [Pi], based on renormalization and comparison with an independent site percolation.

Note first of all that η_x is a decreasing function of the bond configuration n, consequently the field $\{\eta_x\}$ is positively associated, i.e. it posseses the FKG property. We'll show next that

$$0 \ \leq \ Cov(\eta_x,\eta_y) \ \leq \ exp\{\ -c'd(x,y)\}\ , \tag{2.1}$$

where c' is some constant. Let us consider $d(x,y)$ so large that the boxes $\mathbf{B}_{2N}(x)$ and $\mathbf{B}_{2N}(y)$ are far apart. Since,

$$Cov(\eta_x,\eta_y) \ = \ \mathcal{P}_w^p\{\eta_x = 1;\ \eta_y = 1\} \ - \ \mathcal{P}_w^p\{\eta_x = 1\}\mathcal{P}_w^p\{\eta_y = 1\}$$

the nonnegativity part of (2.1) is implied by the FKG property. To derive the upper bound in (2.1) we'll split $\mathcal{P}_w^p\{\eta_x = 1;\ \eta_y = 1\}$ according to whether $\mathbf{B}_{2N}(x)$ and $\mathbf{B}_{2N}(y)$ are separated by open bonds of a configuration n or not. To be more precise let us say that a closed contour (Jordan curve) γ separates $\mathbf{B}_{2N}(x)$ from $\mathbf{B}_{2N}(y)$ if $\mathbf{B}_{2N}(y) \subseteq int(\gamma)$ and $\mathbf{B}_{2N}(x) \subseteq ext(\gamma)$. Set

$$\mathcal{A} \ = \ \{n \in \mathcal{B}(\mathbf{Z^2}) : \exists\ \gamma - closed\ contour\ of\ open\ bonds\ which\ separates\ \mathbf{B}_{2N}(x)\ from\ \mathbf{B}_{2N}(y)\ \}$$

and

$$\mathcal{A}_\gamma \ = \ \{n \in \mathcal{A} : \gamma\ is\ the\ smallest\ contour\ of\ n\ which\ separates\ \ \mathbf{B}_{2N}(x)\ from\ \mathbf{B}_{2N}(y)\ \}$$

Obviously, $\mathcal{A}$ is the disjoint union of $\mathcal{A}_\gamma$. On the other hand $\mathcal{A}_\gamma \ = \ \mathbf{1}_{\{n|_\gamma\equiv 1\}} \cap \mathcal{A}'_\gamma$, where the event $\mathcal{A}'_\gamma$ depends only on bond configurations inside $int(\gamma)$. Thus, by the strong FKG and decoupling properties P4,

$$\mathcal{P}_w^p\{\eta_x = 1;\ \eta_y = 1;\ \mathcal{A}_\gamma\} \ = \ \mathcal{P}_w^p\{\eta_x = 1 \mid \mathbf{1}_{\{n|_\gamma\equiv 1\}}\}\mathcal{P}_w^p\{\mathcal{A}_\gamma \mid \eta_y = 1\ \}\mathcal{P}_w^p\{\eta_y = 1\ \} \ \leq$$

$$\mathcal{P}_w^p\{\eta_x = 1\ \}\mathcal{P}_w^p\{\eta_y = 1\ \}\mathcal{P}_w^p\{\mathcal{A}_\gamma \mid \eta_y = 1\ \}$$

Consequently,

$$\mathcal{P}_w^p\{\eta_x = 1;\ \eta_y = 1;\ \mathcal{A}\} \ \leq \ \mathcal{P}_w^p\{\eta_x = 1\ \}\mathcal{P}_w^p\{\eta_y = 1\ \}\mathcal{P}_w^p\{\mathcal{A} \mid \eta_y = 1\ \} \ \leq$$

$$\leq \ \mathcal{P}_w^p\{\eta_x = 1\ \}\mathcal{P}_w^p\{\eta_y = 1\ \}\mathcal{P}_w^p\{\mathcal{A}\ \}.$$

The argument above may look formal since the infinite volume measure $\mathcal{P}_w^p$ is involved, but we can reiterate it for $\mathcal{P}_{\Lambda,w}^p$ with $\Lambda \nearrow \infty$ to obtain the desired estimate.

We claim, furthermore, that

$$\mathcal{P}_w^p\{\mathcal{A}^c\} \;\leq\; exp\{\,-c'd(x,y)\} \tag{2.2}$$

(2.1), then, follows immediately.

The best way to see (2.2) is to go to the dual model, the pass being secured by the property P3. Indeed,

$$\mathcal{P}_w^p\{\mathcal{A}^c\} \;=\; \mathcal{P}_f^{p^*}\{\mathbf{B}_{2N}^*(x) \;\leftrightarrow\; \mathbf{B}_{2N}^*(y)\}\;,$$

where $\mathbf{B}_{2N}^*(x)(\;\mathbf{B}_{2N}^*(y))$ is the corresponding dual box. But,

$$\mathcal{P}_f^{p^*}\{\mathbf{B}_{2N}^*(x) \;\leftrightarrow\; \mathbf{B}_{2N}^*(x)\} \;\leq\; 64N^2 \max_{u\in\mathbf{B}_{2N}^*(x),\; v\in\mathbf{B}_{2N}^*(y)} \mathcal{P}_f^{p^*}\{\;u\leftrightarrow v\;\}.$$

However,

$$\mathcal{P}_f^{p^*}\{\;u\leftrightarrow\;u\}\;=<\sigma_u,\sigma_v>_{\beta^*},$$

where the latter quantity is the correlation function of the infinite volume Ising Gibbs measure at the inverse dual temperature $\beta^* < \beta_c$, which is known to decay exponentially with $d(x,y)$ [P]. (2.3), thereby, follows.

We'll use a renormalization procedure to finish the proof of lemma 2.1. Note that because of (2.1),

$$Var(\;\frac{1}{|\mathbf{B}_M(x)|}\sum_{y\in\mathbf{B}_M(x)}\eta_y\;)$$

tends to zero as M tends to ∞. Therefore, by the Chebyshev's inequality,

$$\mathcal{P}_w^p\{\;|\frac{1}{|\mathbf{B}_M(x)|}\sum_{y\in\mathbf{B}_M(x)}\eta_y \;-\; d(N)|\;\geq\;\epsilon/2\}$$

becomes arbitrary small provided only that M is large enough. Set $M \;=\; NK^2$ and choose K in such a fashion that,

a)

$$\mathcal{P}_w^p\{\;|\frac{1}{|\mathbf{B}_M|}\sum_{y\in\mathbf{B}_M(x)}\eta_y \;-\; d(N)|\;\geq\;\epsilon/2\}\;\leq\;\epsilon/16$$

b) $K\;\geq\;1/\epsilon$ *and* $N(K^2\;+\;2K)$ *divides* L

c) $exp\{\;-c'KN\}\;\leq\;\epsilon/16,$

where the constant c' in *c)* is to be specified below. Let us define a sublattice $\mathbf{Z}_K^2 \subset \mathbf{Z}^2$ with the step $2N(K+K^2)$, i.e. $\mathbf{Z}_K^2 \;=\; 2N(K^2+K)\mathbf{Z}^2$. Consider a partition of the large box $\Lambda(L)$ into smaller boxes $\{\mathbf{B}_M(u)\}_{u\in\mathbf{Z}_K^2\cap\Lambda}$ and the corresponding corridors of the length $2KN$. The total number of small boxes in $\Lambda(L)$ equals, then, to

$$T \stackrel{def}{=} \frac{|\Lambda|}{4[N(K^2+K)]^2}.$$

Also condition *b)* implies that the relative volume of the corridors is bounded above by $\epsilon/4$. Consequently,

$$\{\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\eta_x \geq d(N)+\epsilon\}\;\subseteq\;\{\frac{1}{T}\sum_{u\in\mathbf{Z}_K^2\cap\Lambda}\frac{1}{|\mathbf{B}_M|}\sum_{x\in\mathbf{B}_M(u)}\eta_x \geq d(N)+3\epsilon/4\}.$$

and

$$\{\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\eta_x \le d(N)-\epsilon\} \;\subseteq\; \{\frac{1}{T}\sum_{u\in \mathbf{Z}^2_K\cap\Lambda}\frac{1}{|\mathbf{B}_M|}\sum_{x\in \mathbf{B}_M(u)}\eta_x \le d(N)-3\epsilon/4\}.$$

Now, at each site $u\in\mathbf{Z}^2_K$ define random variables Y^i_u; $i=1,2$; as

$$Y^1_u \;=\; \mathbf{1}_{\{\frac{1}{|\mathbf{B}_M|}\sum_{y\in\mathbf{B}_M(u)}\eta_y \,\ge\, d(N)\,+\,\epsilon/2\}}.$$

$$Y^2_u \;=\; \mathbf{1}_{\{\frac{1}{|\mathbf{B}_M|}\sum_{y\in\mathbf{B}_M(u)}\eta_y \,\le\, d(N)\,-\,\epsilon/2\}}.$$

By the property $a)$, $\mathcal{P}^p_w\{Y^i_u \;=\; 1\} \;\le\; \epsilon/16$; $i=1,2$. We claim, moreover, that

$$\mathcal{P}^p_w\{\;Y^i_u \;=\; 1\;|\Sigma_u\} \;\le\; \epsilon/8;\; i=1,2\;, \tag{2.3}$$

where Σ_u is the $\sigma-$algebra generated by the variables $\{Y_v\}_{v\neq u}$. As in [Pi] it is easy to see that under (2.3) the fields $\{Y^i_u\}$ are FKG dominated by the process of independent site percolation on $\mathbf{Z}^2_K$, $\{S_u\}$, with

$$P\{\;S_u \;=\; 1\} \;=\; \epsilon/8 \;=\; 1 \;-\; P\{\;S_u \;=\; 0\}.$$

Note that,

$$\{\frac{1}{T}\sum_{u\in \mathbf{Z}^2_K\cap\Lambda}\frac{1}{|\mathbf{B}_M|}\sum_{x\in \mathbf{B}_M(u)}\eta_x \ge d(N)+3\epsilon/4\} \;\subseteq\; \{\frac{1}{T}\sum_{u\in\Lambda\cap\mathbf{Z}^2_K} Y^1_u \;\ge\; \epsilon/4\}$$

and

$$\{\frac{1}{T}\sum_{u\in \mathbf{Z}^2_K\cap\Lambda}\frac{1}{|\mathbf{B}_M|}\sum_{x\in \mathbf{B}_M(u)}\eta_x \le d(N)-3\epsilon/4\} \;\subseteq\; \{\frac{1}{T}\sum_{u\in\Lambda\cap\mathbf{Z}^2_K} Y^2_u \;\ge\; \epsilon/4\}.$$

However, for $i=1,2$,

$$\mathcal{P}^p_w\{\frac{1}{T}\sum_{u\in\Lambda\cap\mathbf{Z}^2_K} Y^i_u \;\ge\; \epsilon/4\} \;\le\; P\{\frac{1}{T}\sum_{u\in\Lambda\cap\mathbf{Z}^2_K} S_u \;\ge\; \epsilon/4\} \;\le$$

$$\le\; exp\{-T\phi(\epsilon/4)\} \;=\; exp\{\;-\frac{L^2}{4[N(K^2+K)]^2}\phi(\epsilon/4)\}\;,$$

where

$$\phi(a) \;= alog(\frac{8a}{\epsilon}) \;+\; (1-a)log(\frac{8(1-a)}{8-\epsilon})$$

is the rate function which governs large deviations for the Bernoulli i.i.d.-s with $p\;=\;\epsilon/8$. In particular, $\phi(\epsilon/4)\;\ge\;k\epsilon^2$, where k is a positive constant. Since the choice of K depended in essential way only on ϵ this implies the claim of the lemma. Thus it remains to prove (2.3).

Set $\mathbf{B}_1 \;=\; \mathbf{B}_{M+N}(u)$ and $\mathbf{B}_2 \;=\; \mathbf{B}_{M+(2K-1)N}(u)$. Note that the random variables $Y^i_u; i=1,2$; depend only on bond configurations from $\mathcal{D}_{\mathbf{B}_1}$, whereas the rest of $\{Y^i_v\}_{i=1,2;v\neq u}$ depends only on bond configurations from $\mathcal{D}_{(\mathbf{Z}^2\setminus\mathbf{B}_2)}$. By the strong FKG and decoupling properties P4,

$$\mathcal{P}^p_w\{\;Y^1_u=1\mid\Sigma_u\} \;\le\; \mathcal{P}^p_w\{\;Y^1_u=1\mid n|_{\mathcal{B}(\mathbf{Z}^2\setminus\mathbf{B}_2)}\equiv 0\} \;=\; \mathcal{P}^p_{\mathbf{B}_2,f}\{\;Y^1_u=1\}$$

Set $\mathcal{A} \;=\; \{n\in\mathcal{D}_{\mathbf{B}_2}:\;\exists\; closed\ contour\ of\ open\ bond\ lying\ in\ \mathbf{B}_2\setminus\mathbf{B}_1\}$. $\mathcal{A}$ is an increasing event. As it was done above one can use the FKG property and decoupling to show that

$$\mathcal{P}^p_{\mathbf{B}_2,f}\{Y^1_u \;=\; 1\mid\mathcal{A}\} \;\le\; \mathcal{P}^p_{\mathbf{B}_2,w}\{Y^1_u \;=\; 1\} \;\le \mathcal{P}^p_w\{Y^1_u \;=\; 1\} \;\le\; \epsilon/16.$$

Similarly,

$$\epsilon/16 \ \geq \mathcal{P}^p_w\{Y^2_u \ = \ 1\} \ \geq \mathcal{P}^p_w\{Y^2_u \ = \ 1 \mid \mathcal{A}\}\mathcal{P}^p_w\{\mathcal{A}\} \ \geq \mathcal{P}^p_w\{ \ Y^2_u = 1 \mid \Sigma_u\}\mathcal{P}^p_w\{\mathcal{A}\}.$$

On the other hand

$$\mathcal{P}^p_{\mathbf{B}_2,f}\{\mathcal{A}^c\} \ = \ \mathcal{P}^{p^*}_{\mathbf{B}^*_2,w}\{ \ \partial\mathbf{B}^*_2 \ \leftrightarrow \ \partial\mathbf{B}^*_1\} \ \leq$$

$$\leq \ |\partial\mathbf{B}^*_1| \max_{z\in\partial\mathbf{B}^*_1} \mathcal{P}^{p^*}_{\mathbf{B}^*_2,w}\{z \ \leftrightarrow \ \partial\mathbf{B}^*_2\} \ \leq \ 9N(K^2+1) \max_{z\in\partial\mathbf{B}^*_1} E^{\beta^*}_{\mathbf{B}^*_2,+}\sigma_z.$$

Since $\beta^* < \beta_c$ there exists a constant $c_1 = c_1(\beta) > 0$ ([CCS],pp. 442,443), such that

$$\max_{z\in\partial\mathbf{B}^*_1} E^{\beta^*}_{\mathbf{B}^*_2,+}\sigma_z \ \leq \ exp\{ \ -c_1 d(\partial\mathbf{B}^*_1, \partial\mathbf{B}^*_2)\} \ \leq \ exp\{ \ -c_1(2K-3)N\}.$$

Consequently, for KN sufficeinly large,

$$\mathcal{P}^p_{\mathbf{B}_2,f}\{\mathcal{A}^c\} \ \leq \ exp\{ \ -c_1 KN\}.$$

As far as Y^2_u is concerned we can proceed as above using estimates on the infinite volume correlation function, which were already employed in the verification of (2.2). We, thereby, obtain:

$$\mathcal{P}^p_w\{\mathcal{A}^c\} \ \leq \ exp\{ \ -c_2 KN\},$$

where c_2 is another positive constant.

To complete the proof of the lemma define the constant c' in c) as $c' = min(c_1, c_2)$.

Below we'll use a slight generalization of lemma 2.1. Namely, given $\alpha > 0$ let A be a set of sites, $A \subset \Lambda(L)$, such that

$$|A| \ \geq \ \alpha L^2, \tag{2.4}$$

and let $\{\eta_x\}$ be as before.

Lemma 2.2 *There exists a function $c = c(\epsilon,\alpha) > 0$, such that,*

$$\mathcal{P}^p_w\{|\frac{1}{|A|}\sum_{x\in A}\eta_x \ - \ d(N)| \ \geq \epsilon \ \} \ \leq \ exp\{ \ -\frac{L^2}{N^2}c(\epsilon,\alpha)\}.$$

for all sets $A \subset \Lambda(L)$ which satisfy (2.4).

We omit the proof of this lemma since it essentially repeats the proof of lemma 2.1.

3 Basic reduction to the FK case.

We are using two parameters ν and b, $0 < \nu < b < 1/4$; ν to set up the notion of droplets and b to define the phase of small contours. As in [I] our contours are simple Jordan curves composed of dual bonds and rounded corners via the splitting rule of [DKS, chpt.3] or [P, sec.10]. A contour γ is said to be small if

$$Vol(\gamma) \ \leq \ L^{2b}.$$

Given a set $A \subseteq \Lambda(L)$ define $P^{\beta,b}_{A,+}$ to be the Ising Gibbs measure with +-boundary conditions on A at inverse temperature β, conditioned on the event that all $\pm$-contours of $\sigma \in \{-1,1\}^A$ are small.

Lemma 3.1 *Let $\alpha, \epsilon > 0$. Then there exists a function $c = c(\alpha,\epsilon) > 0$, such that for any $A \subset \Lambda(L)$ with $|A| \geq \alpha L^2$,*

$$P^{\beta,b}_{A,+}\{ X_A \leq m^* - \epsilon\} \leq exp\{ -cL^{2-4b}\} \tag{3.1}$$

provided only that L is large enough (i.e. $L \geq l(\beta,\alpha,\epsilon)$). As usual, $X_A = 1/|A| \sum_{x\in A} \sigma_x$.

Remark: Since $b \in (0, 1/4)$ (3.1) yields a supersurface order of decay in the phase of small contours.

Proof: Set $\mathcal{A} = \{\sigma : \textit{all contours of } \sigma \textit{ are small}\}$. We'll see in the next section that

$$P^{\beta}_{A,+}\{\mathcal{A}\} \to 1 \textit{ as } L \to \infty. \tag{3.2}$$

Consequently,

$$P^{\beta,b}_{A,+}\{ X_A \leq m^* - \epsilon\} = P^{\beta}_{A,+}\{ X_A \leq m^* - \epsilon; \mathcal{A}\}(1 + o(1)).$$

Let us consider joint site-bond configurations

$$(\sigma, n) \in \Omega_A \times \mathcal{D}_A,$$

as defined in the previous section. Also let $\mathbf{Q}_{A,+}$ be the corresponding joint site-bond measure. Since,

$$(\mathcal{A} \times \mathcal{D}_A) \cap supp(\mathbf{Q}_{A,+}) \subseteq \{(\sigma,n) : \textit{all } " - " \textit{ FK clusters are small}\} \stackrel{def}{=} \mathcal{N}_-,$$

we have,

$$P^{\beta}_{A,+}\{ X_A \leq m^* - \epsilon;\ \mathcal{A}\} \leq \mathbf{Q}_{A,+}\{X_A \leq m^* - \epsilon;\ \mathcal{N}_-\}. \tag{3.3}$$

Set ϕ_A to denote the density of small FK clusters,

$$\phi_A = \frac{1}{|A|}\sum_{x\in A} \mathbf{1}_{\{|C(x)| \leq L^{2b}\}}.$$

Note that

$$\{X_A \leq m^* - \epsilon;\ \mathcal{N}_-\} \subseteq \{\phi_A \geq \delta\}$$

where $\delta = (1 - m^* + \epsilon)/2$. Indeed, the only place where " $-$ " spins can appear if large " $-$ " clusters are prohibited are small FK clusters. Then in the absence of large " $-$ " FK clusters the density of large " $+$ " FK clusters is given by $1 - \phi_A$. Consequently,

$$\mathbf{Q}_{A,+}\{X_A \leq m^* - \epsilon;\ \mathcal{N}_-\} \leq$$

$$\leq \mathbf{Q}_{A,+}\{\frac{1}{|A|}\sum_{x\in A} \sigma_x \mathbf{1}_{\{|C(x)| \leq L^{2b}\}} \leq -\epsilon/2 \mid \phi_A \geq \delta\} + \mathbf{Q}_{A,+}\{ 1 - \phi_A \leq m^* - \epsilon/2\} \leq$$

$$\leq \mathbf{Q}_{A,+}\{\frac{1}{|A|}\sum_{x\in A} \sigma_x \mathbf{1}_{\{|C(x)| \leq L^{2b}\}} \leq -\epsilon/2\ ;\ \phi_A \geq \delta\} + \mathcal{P}^{p}_{A,w}\{\phi_A \geq 1 - m^* + \epsilon/2\} = (I) + (II).$$

In order to estimate (II) note that the event $\{\phi_A \geq 1 - m^* + \epsilon/2\}$ is decreasing. Therefore,

$$(II) \leq \mathcal{P}^{p}_{w}\{\phi_A \geq 1 - m^* + \epsilon/2\}.$$

Furthermore, observe that

$$\lim_{L\to\infty} \mathcal{P}^p_w\{|C(x)| \le\ L^{2b}\} \ =\ 1-m^*.$$

Therefore for large enough values of L, the $\mathcal{P}^p_w$ expectation of ϕ_A is less than $1-m^*+\epsilon/4$. Lemma 2.2 ,then, implies that for large enough values of L,

$$(II)\ \le\ exp\{\ -cL^{2-4b}\}. \tag{3.4}$$

In order to treat (I) set

$$\phi^N_A \ =\ \frac{1}{|A|}\sum_{x\in A}\mathbf{1}_{\{|C(x)|\le N\}}.$$

Pick $N=N(\epsilon)$ so large that $d(N)\ge 1-m^*-\epsilon/16$. Then, by the virtue of lemma 2.2,

$$\mathcal{P}^p_w\{\phi^N_A \le\ 1-m^*-\epsilon/8\}\ \le\ e^{-c\frac{L^2}{N^2}}$$

for L large enough. One more application of lemma 2.2 then, reveals that for L sufficiently large,

$$\mathcal{P}^p_w\{\ \phi_A-\phi^N_A\ \ge\ \epsilon/4\}\ \le\ e^{-cL^{2-4b}},$$

where $c=c(\epsilon)$ is a positive constant. Consequently, (I) can be estimated above as

$$(I)\ \le\ \mathbf{Q}_{A,+}\{\frac{1}{|A|}\sum_{x\in A}\sigma_x\mathbf{1}_{\{|C(x)|\le N\}}\ \le\ -\epsilon/2\ ;\ \phi^N_A\ge\delta/2\}\ +\ e^{-cL^{2-4b}}.$$

Finally the $\mathbf{Q}_{A,+}$ term above can be estimated by means of a generalization of the usual Cramer's approach to large deviations of normalized sums of i.i.d. random variables. Such an estimate may be found in full details in [Pi], here we only sketch it for the sake of completeness. For each bond configuration with $\phi^N_A\ge\delta/2$ we have at least $\alpha\delta L^2/2N$ small clusters, which are painted independently into " $+$ " or " $-$ " with probability 1/2 each. Thus the problem of the estimation of (I) reduces to the following situation:

Given independent random variables $Z_1,Z_2,...,Z_M$ with $M\ge\frac{\alpha\delta L^2}{2N}$ and

$$P\{\ Z_k=\pm a_k\}\ =\ \frac{1}{2};\ \ 0\le a_k\le N,$$

find an upper bound for

$$P\{\ \frac{1}{M}\sum_{k=1}^{M}Z_k\ \ge\ \epsilon/4\}.$$

Note that the log-moment generating function of Z_k;

$$h_k(t)\ =\ log(Ee^{tZ_k})\ =\ h_B(a_kt)\ =\ log(\frac{e^{a_kt}+e^{-a_kt}}{2}),$$

where h_B is the log-moment generating function of the ±1 Bernoulli random variable. Note also that for $t>0$ $h_B(at)$ is increasing in $a\ge0$. By the exponential Chebyshev's inequality,

$$P\{\ \frac{1}{M}\sum_{k=1}^{M}Z_k\ \ge\ \epsilon/4\}\ \le\ exp\{\ -M\sup_{t>0}\{t\epsilon/4\ -\ \frac{1}{M}\sum_{k=1}^{M}h_k(t)\}\}.$$

But,

$$\sup_{t>0}\{t\epsilon/4\ -\ \frac{1}{M}\sum_{k=1}^{M}h_k(t)\}\ \ge\sup_{t>0}\{t\epsilon/4\ -\ h_B(Nt)\}\ =\ I_B(\frac{\epsilon}{4N}),$$

where I_B is the large devaition rate function for the ± 1 Bernoulli i.i.d.-s. Obviously, there exist a constant $c > 0$, such that

$$I_B(\frac{\epsilon}{4N}) \ \geq \ c\frac{\epsilon^2}{N^2}.$$

Consequently,

$$P\{ \ \frac{1}{M}\sum_{k=1}^{M} Z_k \ \geq \ \epsilon/4\} \ \leq \ exp\{ \ -c\frac{M\epsilon^2}{N^2}\}.$$

Combined with (3.4) the above estimate implies the claim of the lemma.

4 Proof of the main result.

We define a droplet to be an ordered sequence of dual sites $S \ = \ (u_1, ..., u_{N+1})$, with $u_1 \equiv u_{N+1}$ such that:

a) $L^\nu \ \leq \ d(u_k, u_{k+1}) \ \leq \ 2L^\nu$; $k = 1, ..., N$.
b) The polygonal curve $P(S)$ *through* $u_1, ..., u_{N+1}$ *is a simple Jordan curve.*

Given a contour γ, we say that γ is compatible with a droplet $S = \{u_1, ..., u_{N+1}\}$, $\gamma \sim S$, if each u_k is at distance no more than $1/3$ (to include rounded corners) from γ and $\rho(\gamma, P(S)) \ \leq \ L^\nu$, where ρ is the Hausdorff distance. We'll also say that S is a large droplet if $Vol(P(S)) \ \geq \ L^{2b}/2$. It is not hard to see that for any large contour γ there exists a large droplet S, which is compatible with γ. A collection of droplets $\mathbf{S} \ = \ \{S_1, ..., S_n\}$ is called an arrangement of droplets if there exists a configuration σ such that for $\{\gamma_1, ..., \gamma_n\}$- the only large contours of σ we have

$$S_k \ \sim \ \gamma_k; \ k = 1, ..., n \ .$$

If this is the case, we'll say that $\mathbf{S}$ is compatible with σ; $\sigma \sim \mathbf{S}$. Likewise a collection of large droplets $\mathbf{S} \ = \ \{S_1, ..., S_n\}$ is called an arrangement of exterior droplets, if there exists a configuration σ such that for $\{\gamma_1, ..., \gamma_n\}$ - the only large exterior contours of σ we have $S_k \sim \gamma_k$; $k = 1, ..., n$. We denote this as $\mathbf{S} \overset{ext}{\sim} \sigma$.

For an arrangement $\mathbf{S}$ the value of the Wulff functional $\mathcal{W}_F(\mathbf{S})$ is defined via

$$\mathcal{W}_F(\mathbf{S}) \ = \ \sum_{k=1}^{n} \mathcal{W}_F(P(S_k)).$$

With each set of arrangements E one can associate an event

$$\mathcal{E} \ = \ \{\sigma : \ \exists \ \mathbf{S} \in E \ such \ that \ \mathbf{S} \sim \sigma\}.$$

Similarly, with each set of exterior arrangements E one can associate an event $\mathcal{E}_{ext}$. Thus starting with the probability measure $P^\beta_{A,+}$ on a space of configurations $\sigma \in \Omega_A$ we have a well defined (subprobability) set function on a space of arrangements (exterior arrangements) of droplets. Now a key lemma of [P, Lemma 10.1] states that given an arrangement $\mathbf{S}$,

$$P^\beta_{A,+}\{\sigma : \ \sigma \sim \mathbf{S}\} \ \leq \ exp\{ \ -\mathcal{W}_F(\mathbf{S}) + cN(\mathbf{S})\}, \tag{4.1}$$

where $c = c(\beta)$ is a constant and $N(\mathbf{S})$ is the number of vertices in all droplets of the arrangement $\mathbf{S}$. Though the original proof of [P] makes use of cluster expansions one can easily verify that that the claim in the form of (4.1) remains true for all values of $\beta > \beta_c$. Note that

$$N(\mathbf{S}) \ \leq \ \frac{1}{L^\nu}\sum_{k=1}^{n} |\partial P(S_k)| \ \leq \ \frac{\mathcal{W}_F(\mathbf{S})}{L^\nu \inf_{n \in \mathbf{S}^1} F(n)} \ = \ o(\mathcal{W}_F(\mathbf{S}). \tag{4.2}$$

Following [P] one can use (4.1) to derive the following estimate:

Let $A \subseteq \Lambda(L)$ and $\delta > \nu$, then

$$P^{\beta}_{A,+}\{\mathbf{S}:\ \mathcal{W}_F(\mathbf{S}) \ \geq\ kL^{\delta}\} \ \leq\ exp\{\ -kL^{\delta}(1-o(1))\}. \tag{4.3}$$

Indeed, let us rewrite (4.2) as

$$P^{\beta}_{A,+}\{\sigma:\ \sigma \sim \mathbf{S}\} \ \leq\ exp\{\ -\mathcal{W}_F(\mathbf{S})(1-o(1)) \ -\ a(N(\mathbf{S})\},$$

where

$$o(1) \ =\ \frac{cN(\mathbf{S}}{\mathcal{W}_F(\mathbf{S})} \ =\ \frac{a(N(\mathbf{S})}{\mathcal{W}_F(\mathbf{S})}$$

and $a = a(n)$ is some function yet to be determined. Now,

$$P^{\beta}_{A,+}\{\mathbf{S}:\ \mathcal{W}_F(\mathbf{S}) \ \geq\ kL^{\delta}\} \ =\ \sum_n P^{\beta}_{A,+}\{\mathbf{S}:\ \mathcal{W}_F(\mathbf{S}) \ \geq\ kL^{\delta};\ N(\mathbf{S}) = n\} \ =$$

$$= \sum_n \sum_{\substack{\mathbf{S}:N(\mathbf{S})=n \\ \mathcal{W}_F(\mathbf{S})\geq kL^{\delta}}} P^{\beta}_{A,+}\{\sigma:\ \sigma \sim \mathbf{S}\} \ \leq$$

$$exp\{\ -kL^{\delta}(1-o(1))\}\sum_n e^{-a(n)}\#\{\mathbf{S}:\ N(\mathbf{S}) = n\}.$$

But a very rough estimate on the number of droplet arrangements with n vertices, $\#\{\mathbf{S}:\ N(\mathbf{S}) = n\}$ yields:

$$\#\{\mathbf{S}:\ N(\mathbf{S}) = n\} \ \leq\ (L^2)^n n^n.$$

Thus, if we pick $a(n) = 2nlogL \ +\ nlogn \ +\ n$, then

$$\sum_n e^{-a(n)}\#\{\mathbf{S}:\ N(\mathbf{S}) = n\} \ \sum_n e^{-n} \ <\ \infty.$$

It remains to verify that $a(N(\mathbf{S}))/\mathcal{W}_F(\mathbf{S}) = o(1)$ on $\mathbf{S}$ such that $\mathcal{W}_F(\mathbf{S}) \geq kL^{\delta}$. But this is clear, since $\mathcal{W}_F(\mathbf{S})$ is always bounded above by cL^2, and (4.3) follows.

In particular, picking $\delta = b$ and k the optimal constant in the lattice isoperimetric inequality over $infF$, we see that, because $\nu < b$,

$$P^{\beta}_{A,+}\{\ \exists\ large\ \pm -contours\} \ \leq\ exp\{\ -cL^{b}(1-o(1))\},$$

which implies (3.2) of the previous section.

We now turn to the proof of Theorem 1.1. The test event for the upper bound is $\mathcal{E}^m_{ext}$, which is generated by the set of external arrangements

$$E^m_{ext} \ =\ \{\mathbf{S}:\ \mathcal{W}_F(\mathbf{S}) \ \geq\ 2(\alpha(m)\lambda_F)^{1/2}L\}.$$

For any $\delta > 0$,

$$P^{\beta}_{\Lambda,+}\{\ X_{\Lambda} \ \leq\ m\} \ \leq\ P^{\beta}_{\Lambda,+}\{\ \mathcal{E}^{m+\delta}_{ext}\} \ +\ P^{\beta}_{\Lambda,+}\{\ X_{\Lambda} \ \leq\ m\ /\ (\mathcal{E}^{m+\delta}_{ext})^c\}. \tag{4.4}$$

The first term in the right hand side of (4.4) can be estimated above by means of (4.3) with $\delta = 1$ and $k = 2(\alpha(m+\delta)\lambda_F)^{1/2}$ as

$$\limsup_{L\to\infty} \frac{1}{L} log P^{\beta}_{\Lambda,+}\{\mathcal{E}^{m+\delta}_{ext}\} \ \leq\ -2(\alpha(m+\delta)\lambda_F)^{1/2}.$$

Thus if we show that the second term in (4.3) can be neglected, i.e. it has a supersurface order of decay, the claim of the theorem follows by letting δ tend to zero.

Lemma 4.1 *For any $\delta > 0$,*

$$P^{\beta}_{\Lambda,+}\{X_\Lambda \ \le \ m \ / \ (\mathcal{E}^{m+\delta}_{ext})^c\} \ \le \ exp\{ \ -cL^{2-6b}\},$$

where $c \ = \ c(m,\delta,\beta) \ > \ 0.$

Remark: Since $2-4b > 1$, the above lemma provides the required supersurface estimate.

Let $\gamma_1, ..., \gamma_n$ be a collection of external contours in Λ. Set

$$\Gamma(\gamma_1, ..., \gamma_n) \ = \ \{\sigma : \ \gamma_1, ..., \gamma_n \ are\ the\ only\ external \ \pm -contours\ of\ \sigma\}.$$

Note that if we remove sites adjacent to $\gamma_1, ..., \gamma_n$ then the remaining sites of Λ will be splitted into two disjoint components; the external component Λ_{ext} and the internal component Λ_{int}. Moreover, if $\{\gamma_1, ..., \gamma_n\}$ is compatible with some arrangement of external droplets $\mathbf{S} \in (E^{m+\delta})^c$, then

$$\frac{1}{2}Vol(\Lambda \setminus (\Lambda_{int} \cup \Lambda_{ext})) \ \le \ \sum_{k=1}^{n} |\gamma_k| \ \le \ \frac{2L^{1+2\nu}(\alpha(m+\delta)\lambda_F)^{1/2}}{\inf_{n\in \mathbf{S}^1} F(n)} \ \le \ cL^{1+2\nu} \tag{4.5}$$

and

$$Vol(\Lambda_{int}) \ \le \ L^2\alpha(m+\delta) \ + \ o(L^2), \tag{4.6}$$

where $c = c(m)$ is a constant. Consequently, for L large enough,

$$P^{\beta}_{\Lambda,+}\{ \ X_\Lambda \ \le \ m \ / \ \Gamma(\gamma_1, ..., \gamma_n)\} \ \le$$

$$\le \ P^{\beta}_{\Lambda,+}\{ \ \frac{|\Lambda_{int}|}{|\Lambda|}X_{\Lambda_{int}} \ + \ \frac{|\Lambda_{ext}|}{|\Lambda|}X_{\Lambda_{ext}} \ \le \ m \ + \ \delta/3 \ / \ \Gamma(\gamma_1, ..., \gamma_n)\} \ \le$$

$$P^{\beta,b}_{\Lambda_{ext},+}\{ \ X_{\Lambda_{ext}} \ \le \ m^* \ - \delta/3\} \ + \ \mathbf{1}_{\{|\Lambda_{int}| \ge \delta L^2/3\}} P^{\beta}_{\Lambda_{int},-}\{ \ X_{\Lambda_{int}} \ \le \ -m^* - \delta/3\},$$

where the first inequality follows by (4.5). By lemma 3.1 and in a view of (4.6) the first term in the rightmost expression is $o(e^{-cL^{2-4b}})$. Thus it remains to estimate the second term. We claim that for all collections $\gamma_1, ..., \gamma_n$ as above,

$$\mathbf{1}_{\{|\Lambda_{int}| \ge \delta L^2/3\}} P^{\beta}_{\Lambda_{int},+}\{ \ X_{\Lambda_{int}} \ \le \ -m^* \ - \ \delta/3\} \ \le \ exp\{ \ -cL^2\}, \tag{4.7}$$

where $c = c(\delta, \beta) > 0$ is a constant which doesn't depend on a particular collection. The claim of the lemma will, then, follow easily, since

$$P^{\beta}_{\Lambda,+}\{ \ X_\Lambda \ \le \ m \ / \ (\mathcal{E}^{m+\delta}_{ext})^c\} \ \le \ \max_{\{\gamma_1,...,\gamma_n\}\sim \mathbf{S} \in (E^{m+\delta}_{ext})^c} P_{\Lambda,+}\beta\{ \ X_\Lambda \ \le \ m \ / \ \Gamma(\gamma_1, ..., \gamma_n)\}.$$

Proof of (4.7): We'll establish a slightly more general result:

Let $A \subseteq \Lambda(L)$ with $|A| \ \ge \ \delta L^2$ and $|\partial A| \ \le \ L^\rho$; $\rho \ < \ 2$ and let $\epsilon \ > \ 0$. Then there exists $c = c(\epsilon, \delta, \rho, \beta) > 0$ which doesn't depend on A, such that

$$P^{\beta}_{A,+}\{X_A \ > \ m^* \ + \ \epsilon\} \ \le \ exp\{ \ -cL^2\}. \tag{4.8}$$

Fix a number N and tile as much as possible of A by disjoint boxes $\mathbf{B}_N(x_k) \subset A$. Set $\mathbf{B} = \cup \mathbf{B}_N(x_k)$. Clearly,

$$\frac{1}{|A|}|A \setminus \mathbf{B}| \ \to \ 0 \ as \ L \to \infty.$$

Thus, for L large enough,

$$P^{\beta}_{A,+}\{\ X_A\ \geq\ m^* + \epsilon\}\ \leq\ P^{\beta}_{\mathbf{B},+}\{\ X_{\mathbf{B}}\ \geq\ m^* + \epsilon/2\}.$$

Setting $M = |\mathbf{B}|/4N^2$, i.e. M is the number of elementary boxes in $\mathbf{B}$, we see that

$$\{\ X_{\mathbf{B}}\ \geq\ m^* + \epsilon\}\ \subseteq\ \{\ X_{\mathbf{B}_N}(x_k)\ \geq\ m^* + \epsilon/4\ \textit{for at least } M\epsilon/4 \textit{ elementary boxes}\}.$$

To complete the proof of (4.5) just choose N so large, that

$$P^{\beta}_{\mathbf{B}_N(0),+}\{\ X_{\mathbf{B}_N(0)}\ \geq\ m^*\ +\ \epsilon/4\}\ \leq\ \epsilon/8,$$

and use the renormalization technique already employed in section 2.

Acknowledgement: My thanks go to Ago Pisztora for providing me with a preliminary version of [Pi] as well as for several useful discussions we had. I'm also grateful to Roberto Schonmann for a very careful reading of the preliminary draft and a number of valuable remarks he made, to Tom Mountford who pointed out a mistake in the original proof of lemma 2.1 and to Bao Nguyen for the interest he expressed in this work.

References

[A] Abraham,D.B. (1987), *Surface Structures and Phase Transitions - Exact Results*, Phase Transitions and Critical Phenomena Vol 10, (C.Domb and J.L.Leibowitz, eds.), Academic Press, London, 1-74.

[ACCN] Aizenman,M.,Chayes,J.T.,Chayes,L.,Newman,C.M. (1988), *Discontinuity of the magnetization in one-dimensional* $1/|x-y|^2$ *Ising and Potts models,* JSP, 50,1, 1-40.

[ACC] Alexander,K.,Chayes,J.T.,Chayes,L. (1990), *The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation,* Commun.Math.Phys. 131, 1-50.

[CCS] Chayes,J.T.,Chayes,L.,Schonman,R.M. (1987), *Exponential decay of connectivities in the two-dimens,onal Ising model.*, J.Stat.Phys. 49, 433-445.

[DKS] Dobrushin,R.L.,Kotecky,R.,Shlosman,S. (1992), *Wulff construction: a global shape from local interaction,* AMS translations series.

[ES] Edwards,R.G.,Sokal,A.D. (1988), *Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm,* Phys.Rev.D 38 .

[FO] Föllmer,H. and Orey,S. (1988),*Large deviations for the empirical field of a Gibbs measure,* Ann.Prob. 16, 3, 961-977.

[I] Ioffe,D. (1994), *Large deviations for the 2D Ising model: a lower bound without cluster expansions,* J.Stat.Phys. 74, 411-432.

[K] Kesten,H. (1991), *Asymptotics in high dimensions for the Fortuin-Kasteleyn random cluster model,* in: Progress in Probability vol 19, Birkhäuser.

[O] Olla, S. (1988), *Large deviations for Gibbs random fields,* Prob.Th.Rel.Fields 77, 343-357.

[P] Pfister,C.E. (1991), *Large deviations and phase separation in the two-dimensional Ising model,* Helv.Phys.Acta 64, 953-1054.

[Pi] Pisztora,A. (1994), *Surface order large deviations of the Ising model below T_c,* preprint.

[Pr] Preston,C. (1976), *Random Fields,* LNM 534, Springer

[S1] Schonmann, R.H.(1987), *Second order large deviation estimates for ferromagnetic system in the phase coexistence region,* Commun.Mat.Phys. 112, 409-422.

[S2] Schonmann, R.H. (1989), *Exponential convergence under mixing,* Prob.Th.Rel.Fields. 81, 235-238.