# UPPER BOUND ON THE TRUNCATED CONNECTIVITY IN ONE-DIMENIONAL $\beta/|x-y|^2$ PERCOLATION MODELS AT $\beta > 1$

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**Abstract**

We consider one-dimensional Fortuin-Kasteleyn percolation models generated by the bond occupation probabilities

$$p_{(xy)} = \begin{cases} p & \text{if } |x-y| = 1 \\ 1 - e^{-\beta/|x-y|^2} & \text{otherwise} \end{cases}$$

and a real parameter $\kappa$. We prove that for any $\beta > 1$ and $\kappa \geq 1$ the percolation density $M$ is strictly positive provided $p$ is sufficiently close to 1. We also prove, under the same assumptions, that the following upper bound for the truncated connectivity

$$\tau'(x,y) \leq C|x-y|^{-\overline{\theta}}$$

holds with $\overline{\theta} = \min(2(\beta\eta - 1), 2)$ where $\eta = \eta(p) \nearrow 1$ as $p \nearrow 1$.

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# 1 Introduction

We consider bond percolation models on $\mathbf{Z}$ with the occupation probability $p_b$ of a bond $b = (xy)$, $x, y \in \mathbf{Z}$, given by

$$p_b = p_{(xy)} = \begin{cases} p & \text{if } |x-y| = 1 \\ 1 - e^{-\beta/|x-y|^2} & \text{otherwise} \end{cases} . \tag{1}$$

The bond $b$ is vacant with probability $q_b = 1 - p_b$. Two parameters, $\beta$ and $p$, control the behavior of this system and will be restricted to the percolating regime, i.e., $\beta > 1$ and $1 - p \ll 1$.

A configuration $n$ of bond occupation is a function

$$n : \{b = (xy)\}_{x,y\in\mathbf{Z}} \longrightarrow n_b \in \{0, 1\} \tag{2}$$

with $n_b = 1$ ($n_b = 0$) meaning the bond $b$ is occupied (vacant).

The Fortuin-Kasteleyn random-cluster models [FK] are percolation models defined by a measure on the configuration space of occupied bonds generated by $\{p_b\}$ and a real parameter $\kappa$. Let us first introduce the independent percolation model. It is described by a measure $\mu$ on the configuration space, given by

$$\mu(n) = \prod_{b:n_b=1} p_b \prod_{b:n_b=0} q_b. \tag{3}$$

For a finite interval $I_L = [-L/2, L/2] \cap \mathbf{Z}$, the "wired" measure $\mu_L$ is defined by assigning in (3) the occupation probability $p_b = 1$ for all bond $b = (xy)$ with $x, y \in \mathbf{Z}/I_L$.

Given a configuration $n$ we may regard the occupied bonds as connecting and decompose $\mathbf{Z}$ on *clusters* $X_1, \ldots, X_m$, according to the following relation: two sites are connected if there is a path of occupied bonds between them. A cluster $X$ is a maximal set of mutually connected sites.

The random cluster models are thus defined by the measure

$$\mu_\kappa(n) = \frac{1}{Z_\kappa}\ \mu(n)\ \kappa^m \qquad , \kappa \geq 0 \tag{4}$$

where

$$Z_\kappa = \sum_n \mu(n)\ \kappa^m \tag{5}$$

is the generating function and $m = m(n)$ is the number of distinct clusters in $n$.

"Wired" measure $\mu_{L,\kappa}$ is defined by replacing $\mu(n)$ by $\mu_L(n)$ in (4) and (5). We denote by $\mu^w_\kappa$ the infinite limit measure which always exists [F, ACCN].

**Remark:** The Fortuin-Kasteleyn random-cluster model at $\kappa = 1$ is the independent percolation model. For this model the infinite measure $\mu$ is unique, independently of the boundary conditions [F]. For $\kappa = 2, 3, \ldots$, the measure (4) is equivalent to the Gibbs measure of the $\kappa$-state Potts model (see [FK] for precise meaning).

Percolation occurs when at least one cluster is infinite. By ergodicity, percolation occurs if and only if the percolation density $M$, defined by

$$M = \mu_\kappa^w (0 \leftrightarrow \infty), \tag{6}$$

is strictly positive. By $0 \leftrightarrow \infty$ we mean the event that the origin is connected to the infinite cluster. We shall use the same symbol for the event and for the indicator of that event. We set

$$\beta^* = \sup\{\beta : M = 0, \ \forall p < 1\}.$$

Our first result is Theorem 3.1 of ref. [IN] extended to any real value $\kappa \geq 1$.

**Theorem 1** *For any $\beta > 1$, $\kappa \geq 1$, there exist $0 < p_0 < 1$ such that, if $1 > p > p_0$, then*

$$\mu_{L,\kappa}(0 \not\leftrightarrow \infty) \leq \epsilon \tag{7}$$

*holds uniformly in $L$ with $\epsilon = \epsilon(p) \to 0$ as $p \to 1$.*

**Remark:** Theorem 1 implies $M > 0$ for $p$ sufficiently close to 1 and consequently $\beta^* \leq 1$. When combined with the opposite bound $\beta^* \geq 1$ [ACCN, AN], gives a direct proof of $\beta^* = 1$ for all $\kappa \geq 1$. Theorem 1 was proved by Imbrie and Newman [IN] for $\kappa = 2, 3, \ldots$, (and in [Sch] for $\kappa = 1$). In the ref. [IN] the result was extended to all $\kappa \geq 1$ by using Fortuin's comparison principle [F].

The truncated connectivity function $\tau'(x, y)$, is defined to be the probability of the event that $x$ and $y$ belong to the same finite cluster, denoted by $x \leftrightarrow y \not\leftrightarrow \infty$, i.e.,

$$\tau'(x, y) = \mu_\kappa^w (x \leftrightarrow y \not\leftrightarrow \infty) \tag{8}$$

Our next result gives an upper bound on $\tau'$.

**Theorem 2** *Under the assumptions of Theorem 1, we have*

$$\mu_{L,\kappa}(x \leftrightarrow y \not\leftrightarrow \infty) \leq C \ |x - y|^{-\overline{\theta}} \tag{9}$$

*where $\overline{\theta} = \min(2(\beta\eta - 1), 2)$, $\eta = \eta(p) \nearrow 1$ as $p \nearrow 1$ and $C < \infty$ uniformly in $L$.*

**Remarks:** 1. Theorem 2 and Theorem 1.4 of ref. [IN] would imply

$$\tau'(x, y) \asymp \ |x - y|^{-\theta}$$

with $\theta = \min\big(2(\beta M^2 - 1), 2\big)$for all $\kappa \geq 1$ if one shows that $\eta \geq M^2$. Our estimates do not allow us such conclusion.

2. In [ACCN] it was shown that the truncated correlation function $G^T$ for the $\kappa$-state Potts model can be written as

$$G^T(x,y) = \tau'(x,y) + \widehat{\tau}(x,y) \tag{10}$$

where $\widehat{\tau}$ is another truncated connectivity given by

$$\widehat{\tau}(x,y) = \mu_\kappa^w(x \leftrightarrow \infty, y \leftrightarrow \infty) - \mu_\kappa^w(x \leftrightarrow \infty)\ \mu_\kappa^w(y \leftrightarrow \infty).$$

An upper and lower bounds on $G^T$ can be found in ref. [IN]. Here we were not able to find a direct treatment for the function $\widehat{\tau}$. However the function (10) could be seen as a simple correlation by the standard method of doubling the system and use the technique developed in [M] to rederive an upper bound as in (9).

To prove Theorems 1 and 2 we modify the procedure in the Imbrie-Newman proof. In their proof of (7) the entropy-energy estimate in the Frohlich-Spencer's generalized Peierls argument [FS] (extended to the Potts models) was sharpened to work just above the critical "inverse temperature" $\beta_c$ (i.e. $\beta > 1$). Correlations are then controlled by a Peierls expansion followed by a Mayer expansion. Here, for any $\beta > 1$, we perform a multiscale expansion directly in the random cluster models whose ideas we begin to describe.

We note that the measure $\mu_\kappa$ induces a probability measure on the random partition of $\mathbf{Z}$. Let $\mathcal{O}_{\mathbf{X}}$ be the set of all occupation configurations $n$ which generate the partition of $\mathbf{Z}$ into connected clusters $\mathbf{X} = (X_1, \ldots, X_m)$. Then

$$\nu_\kappa(\mathbf{X}) = \sum_{n \in \mathcal{O}_{\mathbf{X}}} \mu_\kappa(n) \tag{11}$$

can be written as

$$\nu_\kappa(\mathbf{X}) = \frac{1}{Z_\kappa} \prod_{i=1}^{m} \lambda_{X_i}\ e^{-\beta \sum_{i<j} U(X_i, X_j)} \tag{12}$$

where

$$U(X,Y) = \sum_{\substack{x \in X \\ y \in Y}} v(x-y) \tag{13}$$

is the two-body energy interaction of $X$ and $Y$ with

$$v(z) = \begin{cases} \infty & \text{if } z = 0 \\ -\frac{1}{\beta}\ln(1-p) & \text{if } |z| = 1 \\ 1/|z|^2 & \text{otherwise} \end{cases}$$

and $\lambda_X$ is the "a priori" weight of $X$ (or activity of $X$) given by $\lambda_X = \kappa$ if $X = \{i\}$ and

$$\lambda_X = \kappa\ e^{-\beta U(X,X)} \sum_G \prod_{b \in G} \frac{p_b}{1-p_b} \tag{14}$$

otherwise, with the summation running over the connected graphs $G$ on $X$.

We want to evaluate expectations of the form

$$\nu_\kappa(f) = \sum_{\mathbf{X}} f(\mathbf{X})\ \nu_\kappa(\mathbf{X}) \tag{15}$$

where $f$ is an indicator of events as in (7) and (8). Note that (12) is a measure of a polymer gas and, due to the range of the interaction (13), the usual Mayer expansion of (15) is not expected to converge (see e.g. ref. [B], Theorem 3.5).

We thus proceed as in the ref. [M]. After a resummation, the expectation $\nu_\kappa(f)$ can be written as a convex combination

$$\nu_\kappa(f) = \sum_{\gamma} d_\gamma\ \nu_{\kappa,\gamma}(f) \tag{16}$$

of expectations with respect to a measure $\nu_{\kappa,\gamma}(\cdot) = Z_{\kappa,\gamma}(\cdot)/Z_{\kappa,\gamma}(1)$ of a "diluted" gas of clusters of variable sizes, where $Z_{\kappa,\gamma}(\cdot)$ is a generating function suitably defined in Section 2.

The advantage of (16) is that $\nu_{\kappa,\gamma}(f)$ can be evaluated by using a Peierls-like cancellation developed by Fröhlich- Spencer [FS1] (see also [M, MK]) which avoids to perform its Mayer expansion. Our cancellation will be presented in Section 3.

Using (16) the truncated connectivity function $\tau'$, can be bounded by

$$\sup_{\gamma}\ \nu_{\kappa,\gamma}(x \leftrightarrow y \not\leftrightarrow \infty) \leq C \sup_{X \ni x,y} \zeta_X$$

where $\zeta_X$ is the activity of a finite "admissible" cluster $X$. The estimate on $\zeta_X$ thus gives an upper bound to the function $\tau'$.

This paper is organized as follows. In Section 2 the generating function (5) is written as a convex combination of $k$-regular generating function for any scale $k$ (Theorem 9). In Section 3 we prove Theorems 1 and 2. In Section 4 we prove Lemma 11 and other energy estimates.

# 2 The Generating Function

Our treatment of the "wired" generating function $Z_{\kappa,L}$ on the interval $I_L = [-L/2, L/2] \cap \mathbf{Z}$, involves a sequence of scales $\ell_1, \ldots, \ell_M$ defined as follows: we pick $\alpha > 1$ and an initial scale $\ell_1 = 2^{n_1}$, where $n_1 \in \mathbf{N}$; the successive scales are given by $\ell_k = 2^{n_k}$, where $n_k = [\alpha\, n_{k-1}]$ ($[t]$ is the largest integer $\leq t$); and $M$ is such that $\ell_{M-2} < L \leq \ell_{M-1}$.

By $I_\ell(x)$ we denote a interval in $\mathbf{Z}$, centered at $x$ with size $\ell$, i.e.,

$$I_\ell(x) = \left\{ y \in \mathbf{Z} : |y - x| \leq \frac{\ell}{2} \right\}; \tag{17}$$

we set $I_k(x) = I_{\ell_k}(x)$ for all $x \in I_{L,k} \equiv I_L \cap \ell_k \mathbf{Z}$ and $I_k^{k'}(y) = I_k(y) \cap \ell_{k'} \mathbf{Z}$ for $k' < k$. Note that nearest neighbor intervals share a common site, i.e., $I_k(x) \cap I_k(x + \ell_k) = x + \frac{\ell_k}{2}$, for all $x \in I_{L,k}, k = 1, \ldots M$ (recall $\ell_k$ is even).

By a $n$-block $\Delta$, we mean a set of sites on $I_L$ whose diameter $d(\Delta)$ satisfies $\ell_{n-1} < d(\Delta) \leq \ell_n$ with $n = \inf \{k \in \mathbf{N} : \Delta \subseteq I_k(x) for some x \in I_{L,k}\}$. We may think of a $n$-block consisting of $n-1$-blocks, each of which consisting of $n-2$-blocks, and so on.

Our cluster expansion will successively connect sites according to their hierarchy: at the $n$-th step of the iteration procedure described below, only sites between distinct $n$-blocks (but not between $n+1$-blocks or larger ones) may be connected.

## 2.1 The First Step

Following [M], we start by writing $Z_{L,\kappa}$ as a convex combination of conveniently defined regular generating functions.

We now introduce some notation. We shall use capital letters $X, Y, W, \ldots$, for clusters (i.e., connected set of sites); calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$, for collection of bonds; lower case letters $o, v, e, \ldots$, for events. By $|X|, |\mathcal{A}|, \ldots$, we denote the cardinality of the sets $X, \mathcal{A}, \ldots$.

For each bond $b \in \mathcal{B}$ let $o_b(v_b)$ be the elementary event "$b$ is occupied (vacant)" and let $\Omega_{\mathcal{B}}$ be the space of events consisting of products of elementary events of the form

$$e^{\mathcal{C}} = v^{\mathcal{C}} o^{\mathcal{B}/\mathcal{C}} \qquad , \mathcal{C} \subseteq \mathcal{B}$$

where $o^{\mathcal{D}} = \prod_{b \in \mathcal{D}} o_b$ (same for $v^{\mathcal{D}}$). We shall use the same symbol to their indicator function, e.g., $o_b = \delta_{n_b,1}$, $v_b = \delta_{n_b,0}$, $e^{\mathcal{C}} = \prod_{b \in \mathcal{C}} \delta_{n_b,0} \prod_{b \in \mathcal{B}/\mathcal{C}} \delta_{n_b,1}$. We shall also denote by $1_X$ the indicator function of the event "$X$ is a connected cluster" and by $1_{\mathbf{X}} = 1_{X_1} \cdots 1_{X_m}$ the indicator function of a partition $\mathbf{X} = (X_1, \ldots, X_m)$ into connected clusters. Notice that if $X \cap Y \neq \emptyset$, then

$$1_X 1_Y = 1_{X \cup Y} \tag{18}$$

indicates the event "$X \cup Y$ is a connected cluster".

With this notation, the generating function $Z_{L,\kappa}$ is writing as

$$Z_{L,\kappa} = \sum_n \prod_{b \in \mathcal{B}_0} (p_b \, o_b + q_b \, v_b) \, \kappa^{\gamma(n)} \tag{19}$$

where $\gamma(n)$ is a function which counts the number of distinct clusters in $n$ and $\mathcal{B}_0 = \{b = (xy) : x, y \in \mathbf{Z}, \ x \neq y\}$. We shall use the following basic lemma to block events.

**Lemma 3** *[M] Let $\mathcal{I}$ be a index set of $N$ elements. Let $a_i, b_i$, be distinct events and $\zeta_i > 0$, for each $i \in \mathcal{I}$. Then*

$$\prod_{i\in\mathcal{I}} (a_i + \zeta_i b_i) = \sum_{\substack{\mathcal{X}\subseteq\mathcal{I}: \\ \mathcal{X}\neq\emptyset}} c_{\mathcal{X}} \left(a^{\mathcal{I}} + \zeta_{\mathcal{X}}\ e^{\mathcal{X}}\right) \tag{20}$$

*where $e^{\mathcal{X}} = b^{\mathcal{X}} a^{\mathcal{I}/\mathcal{X}}$,*

$$\zeta_{\mathcal{X}} = \prod_{i\in\mathcal{X}} K_N\ \zeta_i$$

*for a constant $K_N \leq \frac{N}{\ln 2}$, and $0 < c_{\mathcal{X}}$, $\sum_{\mathcal{X}} c_{\mathcal{X}} = 1$.*

Later we shall apply Lemma 3 with $a_i, b_i$, replaced by $1_{\Delta_i}, 1_{\mathbf{X}_i}$, respectively, with $\mathbf{X}_i = (X_{i,1}, \ldots, X_{i,m_i})$ being a nontrivial partition of $\Delta_i$.

For each site $x \in I_{L,1}$, we define a triple $(I, \mathcal{B}, N)$, where $I = I_1(x)$ is the interval of size $\ell_1$ centered at $x$, $\mathcal{B} = \mathcal{B}_I$ is the set of bonds in $I$ and $N = \{n : b \in \mathcal{B} \longrightarrow n_b \in \{0,1\}\}$ is the configuration set restrict to $I$.

Let $\mathcal{P}_I$ be the collection of partitions of $I$. By the following similarity relation

$$n \sim n' \Longleftrightarrow 1_{\mathbf{X}}(n) = 1_{\mathbf{X}}(n'). \tag{21}$$

$N$ is decomposed into equivalence classes according to $\mathbf{X} \in \mathcal{P}_I$ and denoted by $[n]$.

Using lemma 3, we write

$$\prod_{b=\langle xy\rangle\in\mathcal{K}} (p\ o_b + (1-p)\ v_b) = p^{|\mathcal{K}|} \sum_{\substack{\mathcal{X}\subseteq\mathcal{K}: \\ \mathcal{X}\neq\emptyset}} c_{\mathcal{X}} \left(o^{\mathcal{K}} + \widehat{\zeta}_{\mathcal{X}}\ e^{\mathcal{X}}\right) \tag{22}$$

where $\mathcal{K} \subset \mathcal{B}$ is the set of "nearest neighbor bonds" in $I$, and

$$\widehat{\zeta}_{\mathcal{X}} \leq \left[\frac{(1-p)|\mathcal{K}|}{p\ \ln 2}\right]^{|\mathcal{X}|}. \tag{23}$$

Notice that $e^{\mathcal{X}}$ already define a nontrivial partition $\mathbf{X} \in \mathcal{P}_I$ and we now use the remaining bonds $\mathcal{W} = \mathcal{B}/\mathcal{K}$, to further connect $\mathbf{X}$. Denoting by $\mathbf{Y} \succ \mathbf{X}$ if $\mathbf{X}$ is a refinement of $\mathbf{Y}$, we have

$$\sum_n \prod_{b\in\mathcal{W}} (p_b\, o_b + q_b\, v_b) \left(o^{\mathcal{K}} + \widehat{\zeta}_{\mathcal{X}}\, e^{\mathcal{X}}\right) = \sum_{[n]} \left(\left(1 + \widehat{\zeta}_{\mathcal{X}}\, \lambda_I\right) 1_I + \widehat{\zeta}_{\mathcal{X}} \sum_{\mathbf{Y}} \lambda_{\mathbf{Y}} 1_{\mathbf{Y}}\right) \tag{24}$$

with the summations on the right hand side over the equivalence classes defined by (21) and over the nontrivial partitions $\mathbf{Y} \in \mathcal{P}_I$ such that $\mathbf{Y} \succeq \mathbf{X}$, respectively. Moreover,

$$\lambda_I + \sum_{\mathbf{Y}} \lambda_{\mathbf{Y}} = 1. \tag{25}$$

One can easily write expressions for the activities $\lambda_I$ and $\lambda_{\mathbf{Y}}$ similar to (14) which will not be necessary here.

Now, let $\{\xi_{\mathbf{Y}}\}_{\mathbf{Y}\in\mathcal{P}_I}$ be a sequence of positive numbers such that

$$\sum_{\mathbf{Y}\succeq\mathbf{X}} \frac{\xi_{\mathbf{Y}}\,\lambda_{\mathbf{Y}}}{1+\widehat{\zeta}_{\mathcal{X}}\,\lambda_I} = 1$$

and notice that $\xi_{\mathbf{Y}}$ can be chosen $\geq 1$. Then expression (24) can be written as a convex combination of terms of the form

$$\sum_{[n]} W\ (1_I + \zeta_{\mathbf{Y}}\ 1_{\mathbf{Y}}) \tag{26}$$

where $W = (1+\widehat{\zeta}_{\mathcal{X}}\,\lambda_I) > 1$ is a constant which will not be specified during our procedure, and

$$\zeta_{\mathbf{Y}}\ = \frac{\widehat{\zeta}_{\mathcal{X}}}{\xi_{\mathbf{Y}}} \leq \widehat{\zeta}_{\mathcal{X}} \leq \left[\frac{(1-p)\ell_1}{p\ \ln 2}\right]^{m-1} \tag{27}$$

provided $p$ is sufficiently close to 1. Here $m = m(\mathbf{Y}) \geq 2$ is the number of connected clusters in $\mathbf{Y}$.

Since the product of convex combinations is a convex combination, we have proven the following theorem:

**Theorem 4** *Let $\ell_1 \geq 2$ be fixed. Then, if $(1-p)\ell_1 < p\ln 2$, the "wired" generating function $Z_{L,\kappa}$, can be written as a convex combination of generating functions of the form*

$$W_1 \sum_{[n]_1} \prod_{b\in\mathcal{B}_1} (p_b\, o_b + q_b\, v_b) \prod_{x\in I_{L,1}} \left(1_{I_1(x)} + \zeta_{\mathbf{Y}(x)} 1_{\mathbf{Y}(x)}\right) \kappa^{\gamma(n)} \tag{28}$$

*where we sum over all configurations $n$ which restricted to the interval $I_1(x)$, $x \in I_{L,1}$, counts only once each class of equivalence given by the relation (21); $\mathcal{B}_1 = \mathcal{B}_0 / \bigcup_{x\in I_{L,1}} \mathcal{B}_{I_1(x)}$; $W_1$ is a positive constant and $\mathbf{Y}(x)$ is a nontrivial partition of $I_1(x)$ with activity $\zeta_{\mathbf{Y}(x)}$ satisfying bond (27).*

(28) and (27) give the initial conditions for the inductive procedure described in the following.

## 2.2 The Inductive Step

**Definition 5** *Let $k \in \mathbf{N}$, $y \in I_{L,k}$ and $\delta > 0$. A $(k,y,\delta)$-admissible partition* $\mathbf{X}$ *consists of:*

**(i)** *A $k$-block $\Delta$, i.e.,*

$$\Delta \subset \overline{I_k}(y) \equiv I(y, 3\ell_k)$$

*whose diameter is such that $d(\Delta) > \ell_{k-1}$.*

**(ii)** *A partition of* $\Delta$,

$$\mathbf{X} = (X_1, \ldots, X_m),$$

*in connected clusters.*

**(iii)** *A positive activity function* $\zeta = \zeta_{\mathbf{X}}$, *satisfying the following bound*

$$\zeta \leq \kappa^{-m(\mathbf{X})} \ \ell_k^{-\delta}. \tag{29}$$

By admissible partition we always mean a $(k, y, \delta)$-admissible partition for some $(k, y, \delta)$.

**Definition 6** *Let* $k \in \mathbf{N}$, $y \in I_{L,k}$ *and* $\delta > 0$. *A collection* $\mathcal{S}_{(k,y,\delta)}$ *of admissible partitions will be called a* $(k, y, \delta)$*-sparse partition ensemble if:*

**(i)** *For* $k = 1$, $\mathcal{S}_{(1,y,\delta)} = \emptyset$.

**(ii)** *For* $k = 2, 3, \ldots$, *we have*

$$\mathcal{S}_{(k,y,\delta)} = \left( \bigcup_{y' \in I_k^{k-1}(y)} \mathcal{S}_{(k-1,y',\delta)} \right) \bigcup \{\mathbf{X}\} \tag{30}$$

*where* $\mathcal{S}_{(k-1,y',\delta)}$ *is a* $(k-1, y', \delta)$*-sparse partition ensemble and* $\mathbf{X} = (\Delta, \mathbf{X}, \zeta)$ *is a* $(k-1, y'', \delta)$*-admissible partition for some* $y'' \in I_k^{k-1}(y)$. $\mathbf{X}$ *may have* $\Delta = \emptyset$ *with* $\zeta = 0$.

We let

$$F(\mathcal{S}_{(k,y,\delta)}; n) = \prod_{\mathbf{X} \in \mathcal{S}_{(k,y,\delta)}} (1_\Delta(n) + \zeta \, 1_{\mathbf{X}}(n)) \tag{31}$$

**Definition 7** *Given* $k \in N$, $\delta > 0$, *a* $(k, \delta)$*-regular partition assignment is a collection* $\left\{\mathcal{S}_{(k,y,\delta')}, \mathbf{X}_y\right\}_{y \in I_{L,k}}$, *where* $\mathcal{S}_{(k,y,\delta')}$ *is a* $(k, y, \delta')$*-sparse partition ensemble with* $\delta' = \delta - \alpha + 1$, *and* $\mathbf{X}_y = (\Delta_y, \mathbf{X}_y, \zeta_y)$ *is a* $(k, y, \delta)$*-admissible partition. Except by the left and right endpoints,* $\Delta_y$*'s are pairwise disjoint and*

$$\bigcup_{y \in I_{L,k}} \left( \bigcup_{\mathbf{X} \in \mathcal{S}_{(k,y,\delta')}} \Delta \bigcup \Delta_y \right) = I_L \tag{32}$$

**Definition 8** *A* $(k, \delta)$*-regular generating function is a generating function of the form*

$$Z_{(k,\delta)} = W_k \sum_{[n]_k} \prod_{b \in \mathcal{B}_k} (p_b o_b + q_b v_b) \prod_{y \in I_{L,k}} K_{(k,y,\delta)} \ \kappa^{\gamma(n)} \tag{33}$$

*where $W_k$ is a positive constant (an upper bound on the percolation density $M$);*

$$K_{(k,y,\delta)} = F(\mathcal{S}_{(k,y,\delta')};n)\ \left(1_{\Delta_y}(n) + \zeta_y\ 1_{\mathbf{X}_y}(n)\right) \tag{34}$$

*with $\left\{\mathcal{S}_{(k,y,\delta')}, \mathbf{X}_y\right\}_{y\in I_{L,k}}$ being a $(k,\delta)$-regular partition assignment; $\mathcal{B}_k$ is a collection of bonds $b = (xy)$ with endpoints in distinct blocks $\Delta, \Delta'$, from the partition assignment satisfying the distance condition*

$$|x-y| \geq \frac{1}{3}\min\left(d(\Delta), d(\Delta')\right)^{\alpha}; \tag{35}$$

*the summation runs over all configurations $n$ which restrict to any block $\Delta$ of the partition assignment counts once each equivalence class given by the relation (21).*

**Theorem 9** *Let $1 < \alpha < 2$ and let $\delta$ be such that*

$$\frac{\alpha(\alpha-1)}{2-\alpha} < \delta < \beta(1-a) - \alpha \tag{36}$$

*with $a = a(\alpha, \ell_1) \to 0$ as $\ell_1 \to \infty$. Suppose $1 > p \geq (1+\ln 2\,\kappa^{-2}\ell_1^{-\delta-1})^{-1}$. Then, if the initial scale $\ell_1$ is sufficiently large, the generating function $Z_{L,k}$, can be always written as a convex combination of $(k,\delta)$-regular generating functions for any $k = 1, 2, \ldots, M$ [1].*

**Proof.** The proof is by induction. The initial step follows immediately from Theorem 4. The proof of the inductive step uses two basic lemmas. The first is just Lemma 3 with $\{a_i, b_i, \zeta_i\}_{i\in\mathcal{I}}$ replaced by $\{1_{\Delta_y}, 1_{\mathbf{X}_y}, \zeta_y\}_{y\in I^k_{k+1}(u)}$, for each $u \in I_{L,k+1}$.

We let $k \in \{1,\ldots,M-1\}$ and let $\left\{\mathcal{S}_{(k,y,\delta')}, \mathbf{X}_y\right\}_{y\in I_{L,k}}$ be a $(k,\delta)$-regular partition assignment. Let $Z_{(k,\delta)}$ be given by (33). As in [M, MK], we can use Lemma 3 to write $Z_{(k,\delta)}$ as a convex combination of generating function of the form

$$W_k \sum_{[n]_k} \prod_{b\in\mathcal{B}_k} (p_b o_b + q_b v_b) \prod_{u\in I_{L,k+1}} K^*_{(k+1,u,\delta)}\ \kappa^{\gamma(n)} \tag{37}$$

where for each $u \in I_{L,k+1}$,

$$K^*_{(k+1,u,\delta)} = F(\mathcal{S}^*_{(k+1,u,\delta')};n)\ \left(1_{\Delta^*_u}(n) + \zeta^*_u\ 1_{\mathbf{X}^*_u}(n)\right)$$

where

$$\mathcal{S}^*_{(k+1,u,\delta')} = \bigcup_{y\in I^k_{k+1}(u)} \mathcal{S}_{(k,y,\delta')} \tag{38}$$

[1]The proof of Theorem 2.7 can be slightly modified to replace $\delta' = \delta - \alpha + 1$ in Definition 7 by $\delta' = \frac{2}{\alpha}\delta - \alpha + 1$.

is a $(k+1,u,\delta')$-sparse partition ensemble and $\mathbf{X}_u^* = (\Delta_u^*, \mathbf{X}_u^*, \zeta_u^*)$ is a partition consisting of a $k+1$-block

$$\Delta_u^* = \bigcup_{y \in I_\tau(u)} \Delta_y \tag{39}$$

where $I_\tau(u) = \sum_{t=u,u\pm\ell_{k+1}} \tau_t \, I_{k+1}^k(t)$ for some $\tau = \{\tau_t = 0,1\}$; a partition $\mathbf{X}_u^*$ of $\Delta_u^*$ determined by the following equation

$$1_{\mathbf{X}_u^*} = \prod_{y \in C} 1_{\mathbf{X}_y} \prod_{y' \in I_\tau(u)/C} 1_{\Delta_{y'}} \tag{40}$$

for some $C \subseteq I_\tau(u),\ C \neq \emptyset$; and an activity $\zeta_u^*$ satisfying

$$\zeta_u^* \leq \frac{4}{(\ln 2)^2} \prod_{y \in C} \frac{\left|I_{k+1}^k(u)\right|}{\ln 2} \zeta_y. \tag{41}$$

All the $\Delta_u^*,\ u \in I_{L,k+1}$, are disjoint except by the end points, and we arranged these blocks so that the following condition is always valid:

**Condition 10** *If for some $u_0 \in I_{L,k+1}$ a partition $\mathbf{X}_{u_0}^* = (X_1, \ldots, X_m)$ given by equation (40) has $C = \{y\}$ for some $y \in I_\tau(u_0)$, then*

$$I(y, \frac{1}{3}\ell_{k+1}) \cap \Delta_u^* = \emptyset \tag{42}$$

*for all $u \in I_{L,k+1}$, $u \neq u_0$.*

Moreover, we have

$$\bigcup_{u \in I_{l,k+1}} \left( \bigcup_{\mathbf{X} \in \mathcal{S}^*_{(k+1,u,\delta')}} \Delta \bigcup \Delta_u^* \right) = I_L \tag{43}$$

**Remark:** We recall that nearest neighbor blocks have a common end point which implies that the union of successive $\Delta_y$'s (in the sense of (18)) is a single cluster. Because of $\mathcal{S}^*_{(k+1,u,\delta)}$ is sparse, $\Delta_u^*$ (again, in the sense of (18)) is a dense connected cluster spread over the interval $\overline{I_{k+1}}(u)$. Now, if $\mathbf{X}_{u_0}^*$ satisfies Condition 10 either there is a unique cluster, say $X_1$, extending outside $\overline{I_k}(y)$, or there are two clusters, say $X_1$ and $X_2$, extending over $\Delta_{u_0}^*$, one at each side of $y$, with diameters, $d(X_1)$ and $d(X_2)$, of at least $\frac{1}{3}\ell_{k+1} - \ell_k$. We call this two situations "type 1" and "type 2", respectively.

To propagate our bounds on the activities to the next scale we will need, in some cases, to extract a small factor from the probability of keeping the partition

$\mathbf{X}_u^*$ essentially unchanged after the bonds $b \in \mathcal{B}_k$ in $\Delta_u^*$ is connected.This will be done as in Lemma 3.3 of ref. [M].

For each $u \in I_{L,k+1}$, we define a triple $(\Delta^*, \mathcal{B}^*, N^*)$ where $\Delta^* = \Delta_u^*$, $\mathcal{B}^* = \mathcal{B}_u^* = \{b = (xy) \in \mathcal{B}_k : x, y \in \Delta_u^*\}$ and $N^* = \{n : b \in \mathcal{B}^* \longrightarrow \{0,1\}\}$ and decompose $N^*$ in equivalence classes, denoted by $[n]^*$, according to the similarity relation (21) with $\mathbf{X} \in \mathcal{P}_{\Delta^*}$.

If $\mathbf{X}^* = (\ \Delta^*, \mathbf{X}^*, \zeta^*)$, we write

$$\sum_{n \in N^*} \prod_{b \in \mathcal{B}^*} (p_b o_b + q_b v_b) \left(1_{\Delta^*} + \zeta^* 1_{\mathbf{X}^*}\right) = \sum_{[n]^*} \left( (1 + \zeta^* \lambda_{\Delta^*}) 1_{\Delta^*} + \zeta^* \sum_{\mathbf{Y} \succeq \mathbf{X}^*} \lambda_{\mathbf{Y}} 1_{\mathbf{Y}} \right). \tag{44}$$

**Lemma 11** *Let $1 < \alpha < 2$ and let $\left\{\mathcal{S}^*_{(k+1,u,\delta')}, \mathbf{X}^*\right\}$ be given by (38)-(41). Suppose that for $u_0 \in I_{L,k+1}$, $\mathbf{X}^*_{u_0}$ satisfies Condition 10 and $\mathbf{Y} \succeq \mathbf{X}^*_{u_0}$ is of type 2. If $\lambda_{\mathbf{Y}}$ is given by equation (44), then the following upper bound*

$$\lambda_{\mathbf{Y}} \leq \left(\frac{\ell_k}{\ell_{k+1}}\right)^{\beta(1-a)} \tag{45}$$

*holds with $a = a(\ell_1, \alpha) \to 0$ as $\ell_1 \to \infty$.*

Lemma 11 will be proven in the Section 4.

We shall now finish the proof of theorem 9. Let us fix $u \in I_{L,k+1}$ and let $\mathbf{X}_u^*$ be as in (40). We consider two cases depending on the number $|C|$ of $\mathbf{X}_y$'s necessary to build $\mathbf{X}_u^*$:

1. $|C| \geq 2$.

   In this case we define $\mathcal{S}_{(k+1,u,\delta')} = \mathcal{S}^*_{(k+1,u,\delta')}$ and notice that (29) and (41) gives us

   $$\zeta_u^* \leq \frac{4}{(\ln 2)^3} \kappa^{-m(\mathbf{X}_u^*)} \, \ell_k^{-2\delta - 2(\alpha-1)} \leq \kappa^{-m(\mathbf{X}_u^*)} \ell_{k+1}^{-\delta} \tag{46}$$

   if $\delta \geq \frac{\alpha(\alpha-1)}{2-\alpha}$, $\kappa \geq 1$ and $\ell_1$ is sufficiently large. Note $\sum_{y \in C} m(\mathbf{X}_y) \geq m(\mathbf{X}_u^*)$.

   Following the first step, given $\{\lambda_{\mathbf{Y}}\}$ as in (44), we pick a sequence $\{\xi_{\mathbf{Y}}\}$, such that $\xi_{\mathbf{Y}} \geq 1$ and

   $$\sum_{\mathbf{Y} \succ \mathbf{X}_u^*} \frac{\xi_{\mathbf{Y}} \lambda_{\mathbf{Y}}}{1 + \zeta_u^* \lambda_{\Delta_u^*}} = 1. \tag{47}$$

   Then (44) can be written as a convex combination of terms of the form

   $$\sum_{[n]_*} W^* \left(1_{\Delta_u^*} + \zeta_u 1_{\mathbf{Y}_u}\right) \tag{48}$$

where $[n]^*$ is given by (21), $W^* = 1 + \zeta_u^* \lambda_{\Delta_u^*}$, $\mathbf{Y}_u$ is a nontrivial partition of $\Delta_u^*$ and

$$\zeta_u \le \frac{\zeta_u^*}{\xi_{\mathbf{Y}_u}}. \tag{49}$$

We define $(\Delta_u, \mathbf{X}_u, \zeta_u) = (\Delta_u^*, \mathbf{Y}_u, \zeta_u)$ and notice that it is a $(k+1,u,\delta)$-admissible partition (note $m(\mathbf{X}_u^*) \ge m(\mathbf{Y}_u)$).

2. $|C| = 1$.

   Here we consider two sub cases depending on $\mathbf{X}_u^*$ is of type 1 or 2:

   **(i)** *Type 1.*

   We pick $\{\xi_{\mathbf{Y}}\}$ as before and write (44) as a convex combination of the form (48). For $\mathbf{Y}_u = (\Delta_u^*, \mathbf{Y}_u, \zeta_u)$ where $\mathbf{Y}_u = (Y_1, Y_{2,} \ldots, Y_m)$ is such that $Y_1$ extends over $\Delta_u^*$, let $\mathbf{X}^1 = (\Delta^1, \mathbf{X}^1, \zeta^1)$ and $\mathbf{X}^2 = (\Delta^2, \mathbf{X}^2, \zeta^2)$ be two partitions given by:

$$\Delta^1 = Y_1 \qquad ; \zeta^1 \equiv 0 \tag{50}$$

   and

$$\Delta^2 = \bigcup_{j \ge 2} Y_j \bigcup \widetilde{Y}_1 \qquad ; \mathbf{X}^2 = (\widetilde{Y}_1, Y_2, \ldots, Y_m) \qquad ; \zeta^2 = \zeta_u \tag{51}$$

   for some $\widetilde{Y}_1 \subset \Delta^1 \cap \overline{I_k}(y)$.
   We have

$$1_{\Delta_u^*} + \zeta_u 1_{\mathbf{Y}_u} = \left(1_{\Delta^1} + \zeta^1 1_{\mathbf{X}^1}\right)\left(1_{\Delta^2} + \zeta^2 1_{\mathbf{X}^2}\right).$$

   We define $\mathbf{X}_u = \mathbf{X}^1$and

$$\mathcal{S}_{(k+1,u,\delta')} = \mathcal{S}^*_{(k+1,u,\delta')} \bigcup \{\mathbf{X}^2\}$$

   notice that the former is a $(k+1,u,\delta)$-admissible partition and, in view of the bound

$$\zeta^2 \le \frac{4}{(\ln 2)^3} \kappa^{-m(\mathbf{X}^2)} \ell_k^{-\delta+\alpha-1} \tag{52}$$

   due to (41) and (49), the latter is a $(k+1,u,\delta')$-sparse partition ensemble.

   **(ii)** *Type 2.*

   We introduce $(\widetilde{\Delta}, \widetilde{\mathcal{B}}, \widetilde{N})$ where $\widetilde{\mathcal{B}} = \left\{b = (zz') \in \mathcal{B}_u^* : z, z' \in \overline{I_k}(y)\right\}$, $\widetilde{\Delta} = \Delta^*$ and $\widetilde{N} = \left\{n : b \in \widetilde{B} \longrightarrow \{0,1\}\right\}$ and by (47)-(49) we write the left hand side of (44) as a convex combination of terms of the form

$$\sum_{\widetilde{[n]}} \widetilde{W} \prod_{b \in \mathcal{B}^*/\widetilde{\mathcal{B}}} (p_b o_b + q_b v_b) \left(1_{\widetilde{\Delta}} + \widetilde{\zeta} 1_{\widetilde{\mathbf{Y}}}\right) \tag{53}$$

where $\widetilde{[n]}$ runs over all configurations in $N^*$ which counts only once each equivalence classes given by (21) according to the partitions on $\overline{I_k}(y)$; $\widetilde{W}$ is a positive constant; $\widetilde{\mathbf{Y}}$ is a nontrivial partition of $\widetilde{\Delta}$ such that $\widetilde{\mathbf{Y}} \succeq \mathbf{X}^*$; and $\widetilde{\zeta}$ is the activity of $\widetilde{\mathbf{Y}}$ satisfying (52).

If $\widetilde{\mathbf{Y}}$ is of type 1, we repeat item (i) with (44) replaced by (53).

If $\widetilde{\mathbf{Y}}$ is of type 2, we write

$$\prod_{b \in \mathcal{B}^*/\widetilde{\mathcal{B}}} (p_b o_b + q_b v_b) \left(1_{\widetilde{\Delta}} + \widetilde{\zeta} 1_{\widetilde{\mathbf{Y}}}\right) = \frac{1}{2}\left(1_{\widetilde{\Delta}} + \widetilde{\zeta}^1 1_{\widetilde{\mathbf{Y}}^1}\right) + \frac{1}{2}\left(1_{\widetilde{\Delta}} + \widetilde{\zeta}^2 1_{\widetilde{\mathbf{Y}}^2}\right) \tag{54}$$

where $\widetilde{\mathbf{Y}}^1$ is of type 1 and $\widetilde{\mathbf{Y}}^2$ is of type 2,

$$\widetilde{\zeta}^{1(2)} = 2\,\widetilde{\zeta}\,\lambda_{\widetilde{\mathbf{Y}}^{1(2)}} \tag{55}$$

with $\lambda_{\widetilde{\mathbf{Y}}^1} + \lambda_{\widetilde{\mathbf{Y}}^2} = 1$.

(54) is a convex combination of two terms involving admissible partition. For the first term, we let $\widetilde{\mathbf{Y}}^{1,1}$ and $\widetilde{\mathbf{Y}}^{1,2}$ be defined as (50) and (51). Again, we define $\mathbf{X}_u = \widetilde{\mathbf{Y}}^{1,1}$ and

$$\mathcal{S}_{(k+1,u,\delta')} = \mathcal{S}^*_{(k+1,u,\delta')} \bigcup \left\{\widetilde{\mathbf{Y}}^{1,2}\right\}, \tag{56}$$

which in view of (55) and (52), is a $(k+1,u,\delta)$-admissible partition and $(k+1,u,\delta')$-sparse partition ensemble, respectively.

For the second term, we notice that $\widetilde{\mathbf{Y}}^2$ satisfies the assumptions of Lemma 11. We write $\mathbf{X}_u = (\Delta_u, \mathbf{X}_u, \zeta_u) = (\widetilde{\Delta}, \widetilde{\mathbf{Y}}^2, \widetilde{\zeta}^2)$ and use (45) in (55) to get

$$\zeta_u \leq \left(\frac{2}{\ln 2}\right)^3 \kappa^{-m(\mathbf{X}_u)} \ell_k^{-\delta+\alpha-1} \left(\frac{\ell_k}{\ell_{k+1}}\right)^{\beta(1-a)} \leq \kappa^{-m(\mathbf{X}_u)} \ell_{k+1}^{-\delta}$$

if $\delta < \beta(1-a) - \alpha$. Thus $\mathbf{X}_u$ is a $(k+1,u,\delta)$-admissible partition.

We conclude the proof of Theorem 9 by setting $\mathcal{B}_{k+1} = \mathcal{B}_k / \cup_u \mathcal{B}^*_u$ and defining the equivalence classes, denoted by $[n]_{k+1}$, in accordance to the partitions of the block $\Delta_u$ for all $u \in I_{L,k+1}$. Notice that (50), (51) and Remark after (43) guarantee any bond $b \in \mathcal{B}_{k+1}$ satisfies the distance condition (35).

# 3 The Truncated Connectivity Function

We want to extend the treatment given in the Section 2 to the truncated connectivity $\tau'(x,y)$. We start by writing the expectation of the event $f$ in the random cluster model (with "wired" boundary conditions on $I_L$) as a ratio of two generating functions:

$$\mu_{L,\kappa}(f) = \frac{Z_{L,\kappa}(f)}{Z_{L,\kappa}} \tag{57}$$

where,

$$Z_{L,\kappa}(f) = \sum_n \sum_{b\in\mathcal{B}_0} f \ \ (p_b o_b + q_b v_b)\, \kappa^{\gamma(n)} \tag{58}$$

differs from (19) only by the indicator function of the event $f$. We then notice that the treatment in the Section 2 can be extended to $Z_{L,\kappa}(f)$ by simply replacing $Z_{(k,\delta)}$ in Definition 7 by $Z_{(k,\delta)}(f)$ defined by (33) with $f$ included.

For $\beta > 1$, $\kappa \geq 1$, we pick $1 < \alpha < 2$, $\ell_1$ sufficiently large, $p$ and $\delta$ such that

$$1 > p \geq \frac{1}{1 + \ln 2 \ \kappa^{-2} \ell_1^{-\delta-1}} \tag{59}$$

and

$$\frac{\alpha(\alpha-1)}{2-\alpha} < \delta < \beta(1-a) - \alpha \tag{60}$$

hold with $a = a(\alpha, \ell_1)$ as in Lemma 11. By Theorem 9 the generating function (58) can be written as a convex combination

$$Z_{L,\kappa}(f) = \sum_\gamma c_\gamma \ Z^\gamma_{(M,\delta)}(f) \tag{61}$$

where

$$Z^\gamma_{(M,\delta)} = W^\gamma_M \ \widehat{Z}^\gamma_{(M,\delta)}(f),$$
$$\widehat{Z}^\gamma_{(M,\delta)}(f) = \sum_{[n]_M} \prod_{b\in\mathcal{B}_\infty} f \ \ (p_b o_b + q_b v_b)\, K^\infty_{(M,0,\delta)}\, \kappa^{\gamma(n)}, \tag{62}$$

with

$$K^\infty_{(M,0,\delta)} = 1_{\Delta_\infty} \ F(\mathcal{S}_{(M,0,\delta)}; n),$$

is a $(M,0,\delta)$-regular generating function. Here we used the "wired" boundary condition,

$$p_b = 1 \qquad , \forall b \in \mathcal{B}_L = \{b = (xy) \in \mathcal{B}_0 : x, y \in \mathbf{Z}/I_L\},$$

to write explicitly the infinity cluster $\Delta_\infty$ into the assignment $\{\mathcal{S}_{(M,0,\delta)}, \Delta_\infty\}$ and define $\mathcal{B}_\infty = \mathcal{B}_M/\mathcal{B}_L$, i.e., $\mathcal{B}_\infty$ is the collection of bonds connecting distinct blocks in the assignment. Notice that $\mathbf{Z}/I_L \subset \Delta_\infty$ and we have any $b \in \mathcal{B}_\infty$ satisfying the distance condition (35).

Hence, the expectation (57) can be written as

$$
\begin{aligned}
\mu_{L,\kappa}(f) &= \frac{\sum_\gamma c_\gamma \, Z^\gamma_{(M,\delta)}(f)}{\sum_\gamma c_\gamma \, Z^\gamma_{(M,\delta)}} \\
&= \sum_\gamma d_\gamma \, \frac{\widehat{Z}^\gamma_{(M,\delta)}(f)}{\widehat{Z}^\gamma_{(M,\delta)}} \\
&\equiv \sum_\gamma d_\gamma \, \nu_{\kappa,\gamma}(f)
\end{aligned}
\tag{63}
$$

where $d_\gamma \geq 0$ is such that $\sum_\gamma d_\gamma = 1$.

Since $\zeta_{\mathbf{X}} \geq 0$ for any $\mathbf{X} \in \mathcal{S}^\gamma_{(M,0,\delta)}$ it follows that $\widehat{Z}^\gamma_{(M,\delta)}(f) \geq 1 > 0$ and the resummation in (63) is well defined.

Our proof of Theorems 1 and 2 requires to perform a cancellation between the numerator and the denominator of $\nu_{\kappa,\gamma}(f)$. To do that we decouple clusters whose participation on the expectation of the event $f$ is only marginal. Our tools are the three lemmas stated in the sequel.

We fix $\gamma$ and write $Z(\mathcal{A},\mathcal{B};f)$ for the generating function $\widehat{Z}^\gamma_{(M,\delta)}(f)$ which is described by the pair $(\mathcal{A},\mathcal{B})$ of partition assignment $\mathcal{A} = \{\mathcal{S}_{(M,0,\delta)}, \Delta_\infty\}$ and connecting set $\mathcal{B} = \mathcal{B}_\infty$, respectively.

We let $f_1 = 0 \not\leftrightarrow \infty$ and $f_2 = x \leftrightarrow y \not\leftrightarrow \infty$ and notice that their indicator function are locally decreasing functions [F]. A function $f$ is a locally decreasing function on the space of events $\Omega$, if for any two partitions $\mathbf{X}, \mathbf{X}'$ of $\mathbf{Z}$ such that $\mathbf{X} \succeq \mathbf{X}'$,

$$1_{\mathbf{X}}(n) \; f(n) \leq 1_{\mathbf{X}'}(n) \; f(n).$$

**Lemma 12** *Let $\kappa \geq 1$ and let $f$ be a locally increasing function. If $\mathcal{B}' \subseteq \mathcal{B}$, then*

$$Z(\mathcal{A},\mathcal{B};f) \leq Z(\mathcal{A},\mathcal{B}';f). \tag{64}$$

**Proof.** For $b \in \mathcal{B}/\mathcal{B}'$, we have

$$
\begin{aligned}
Z(\mathcal{A},\mathcal{B};f) &= q_b Z(\mathcal{A},\mathcal{B}-\{b\};f) + p_b Z(\mathcal{A},\mathcal{B}-\{b\};f\,o_b) \\
&= Z(\mathcal{A},\mathcal{B}-\{b\};f) - p_b\left(Z(\mathcal{A},\mathcal{B}-\{b\};f\,) - Z(\mathcal{A},\mathcal{B}-\{b\};f\,o_b)\right)
\end{aligned}
\tag{65}
$$

which is bounded by

$$\leq Z(\mathcal{A},\mathcal{B}-\{b\};f) \tag{66}$$

since , from the assumptions, the term in (65) between parenthesis is positive. Lemma 12 follows by iteration of (66).

A partition assignment $\mathcal{A}$ may be reduced by setting $\zeta_{\mathbf{X}} = 0$ for some $\mathbf{X} \in \mathcal{A}$. When $\mathcal{A}'$ is reduced from $\mathcal{A}$, we write $\mathcal{A}' \leq \mathcal{A}$.

**Lemma 13** *Let $\mathcal{A}' \leq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$. Then*

$$Z(\mathcal{A}, \mathcal{B}; 1) \geq q^{\mathcal{B}/\mathcal{B}'} \, Z(\mathcal{A}', \mathcal{B}'; 1)$$

*where $q^{\mathcal{D}} = \prod_{b \in \mathcal{D}} q_b$.*

**Proof.** From (65), we have

$$Z(\mathcal{A}, \mathcal{B}; 1) \geq q_b \, Z(\mathcal{A}, \mathcal{B} - \{b\}; 1)$$

which can be iterated to get

$$\geq q^{\mathcal{B}/\mathcal{B}'} \, Z(\mathcal{A}, \mathcal{B}'; 1).$$

This implies Lemma 13 since by reducing the assignment it cancels out terms in the expansion of $Z(\mathcal{A}, \mathcal{B}'; 1)$.

**Lemma 14** *Given $\mathcal{A}$ and $y \in I_L$, let $\mathcal{D} = \mathcal{D}_y$ be a connecting set of bonds $b = (zz')$ in $\mathcal{A}$ such that $z \leq y < z'$ and satisfying the distance condition (35). If $y$ is an end point for some $\Delta \in \mathcal{A}$, then*

$$\frac{1}{q^{\mathcal{D}}} \leq C_1$$

*where $C_1 = C_1(\alpha, \ell_1) \searrow 1$ as $\ell_1 \to \infty$.*

Lemma 14 will be proved in Section 4.

## 3.1 Proof of Theorem 1.1

We now prove Theorem 1. Given a pair $(\mathcal{A}, \mathcal{B})$ of partition assignment $\mathcal{A} = \{\mathcal{S}, \Delta_\infty\}$ and connecting set $\mathcal{B} = \mathcal{B}_\infty$. If $I_{\mathbf{X}}$ denotes the interval between the end points of $\Delta$, let

$$\mathcal{S}_0 = \{\mathbf{X} = (\mathbf{X}, \Delta, \zeta_{\mathbf{X}}) \in \mathcal{S} : I_{\mathbf{X}} \ni 0\} \, . \tag{67}$$

and

$$\overline{\mathcal{S}}_0 = \{\mathbf{X} \in \mathcal{S} : I_{\mathbf{X}} \cap I_{\mathbf{X}'} \neq \emptyset forall \mathbf{X}' \in \mathcal{S}_0\} \, . \tag{68}$$

Let $\mathcal{A}_0$, $\overline{\mathcal{A}}_0$, $\mathcal{A}_0'$ and $\mathcal{A}_0''$ be reduced from $\mathcal{A}$ by setting $\zeta_{\mathbf{X}} = 0$ for all $\mathbf{X} \in \mathcal{L}$, with $\mathcal{L} = \mathcal{S}_0$, $\overline{\mathcal{S}}_0$, $\mathcal{A}/\overline{\mathcal{S}}_0$ and $(\mathcal{A}/\overline{\mathcal{S}}_0) \cup \mathcal{S}_0$, respectively. Notice that $\mathcal{A}_0$ makes the origin to be connected to the infinite cluster $\Delta_\infty$ throughout a chain of connected end points (recall any $\mathbf{X} \in \mathcal{S}$ is derived from type 1 partition which makes their end points to be connected with each other). Thus $Z(\mathcal{A}_0, \mathcal{B}; f_1) = 0$.

Let $\mathcal{B}_0$, $\mathcal{B}_0'$ be given by

$$\{b = (zz') \in \mathcal{B} : z \in \Delta, \, z' \in \Delta' forall \Delta, \Delta' \in \mathcal{M}\} \tag{69}$$

with $\mathcal{M} = \mathcal{A}/\overline{\mathcal{S}}_0$, $\overline{\mathcal{S}}_0$, respectively, and set $\mathcal{B}' = \mathcal{B}/\mathcal{B}_0 \cup \mathcal{B}'_0$.

¿From (62) we have the following factorizations:

$$\kappa\ Z(\mathcal{A}, \mathcal{B}_0 \cup \mathcal{B}'_0; f_1) = Z(\mathcal{A}'_0, \mathcal{B}'_0; f_1)\ Z(\overline{\mathcal{A}}_0, \mathcal{B}_0; 1) \tag{70}$$

and

$$\kappa\ Z(\mathcal{A}_0, \mathcal{B}_0 \cup \mathcal{B}'_0; 1) = Z(\mathcal{A}''_0, \mathcal{B}'_0; 1)\ Z(\overline{\mathcal{A}}_0, \mathcal{B}_0; 1). \tag{71}$$

Notice that the product of generating functions counts twice the infinite cluster which is compensated by the factor $\kappa$.

¿From Lemmas 12, 13 and 14, expressions (70), (71) and (62) we have

$$\begin{aligned}
\frac{Z(\mathcal{A}, \mathcal{B}; f_1)}{Z(\mathcal{A}, \mathcal{B}; 1)} &\leq \frac{1}{q^{\mathcal{B}'}} \frac{Z(\mathcal{A}, \mathcal{B}_0 \cup \mathcal{B}'_0; f_1)}{Z(\mathcal{A}_0, \mathcal{B}_0 \cup \mathcal{B}'_0; 1)} \\
&\leq 2C_1 \frac{Z(\mathcal{A}'_0, \mathcal{B}'_0; f_1)}{Z(\mathcal{A}''_0, \mathcal{B}'_0; 1)} \\
&\leq 2C_1 \left\{ \prod_{\mathbf{X} \in \mathcal{S}_0} \left(1 + \zeta_{\mathbf{X}} \kappa^{m(\mathbf{X})}\right) - 1 \right\} \\
&\leq 2C_1 \left\{ \prod_{n=1}^{M} \left(1 + \ell_n^{-\delta'}\right) - 1 \right\} \\
&\leq \epsilon
\end{aligned}$$

where $\epsilon = \epsilon(\alpha, \ell_1, \kappa) \searrow 0$ as $\ell_1 \to \infty$ uniformly in $M$.

This concludes the proof of Theorem 1 since to satisfy (59) we must let $p \nearrow 1$ as $\ell_1 \to \infty$.

## 3.2 Proof of Theorem 1.2

The proof of Theorem 2 follows the same steps. Let

$$\mathcal{S}_{xy} = \{\mathbf{X} = (\mathbf{X}, \Delta, \zeta_{\mathbf{X}}) \in \mathcal{S} : I_{\mathbf{X}} \ni x, y\} \tag{72}$$

and

$$\overline{\mathcal{S}}_{xy} = \{\mathbf{X} \in \mathcal{S} : I_{\mathbf{X}} \cap I_{\mathbf{X}'} \neq \emptyset forall \mathbf{X}' \in \mathcal{S}_{xy}\}.$$

Let $\mathcal{S}_x$ and $\overline{\mathcal{S}}_x$, $\mathcal{S}_y$ and $\overline{\mathcal{S}}_y$ be given by (67) and (68) with the origin replaced by $x$, $y$, respectively. We set $\overline{\mathcal{S}}'_{xy} = \mathcal{S}_x \cup \mathcal{S}_y$, $\widehat{\mathcal{S}}_x = \overline{\mathcal{S}}'_{xy}/\mathcal{S}_y$ and $\widehat{\mathcal{S}}_y = \overline{\mathcal{S}}'_{xy}/\mathcal{S}_x$.

Let $\mathcal{A}_{xy}$, $\overline{\mathcal{A}}_{xy}$, $\mathcal{A}'_{xy}$ and $\mathcal{A}''_{xy}$ be reduced from $\mathcal{A}$ by setting $\zeta_{\mathbf{X}} = 0$ for all $\mathbf{X} \in \mathcal{N}$, with $\mathcal{N} = \mathcal{S}_{xy}$, $\overline{\mathcal{S}}_{xy}$, $\mathcal{S}/\overline{\mathcal{S}}_{xy}$ and $(\mathcal{S}/\overline{\mathcal{S}}_{xy}) \cup \mathcal{S}_{xy}$, respectively. Notice that $Z(\mathcal{A}_{xy}, \mathcal{B}; f_2) = 0$ as we have $x \leftrightarrow y \leftrightarrow \infty$ for all coefficients.

We now must be carefull in the factorization. We define $\mathcal{B}_{xy}$, $\mathcal{B}'_{xy} \subseteq \mathcal{B}$ by

$$\{b = (zz') : z \in \Delta,\ z' \in \Delta'\ foreither\ \Delta, \Delta' \in \mathcal{Q}\} \tag{73}$$

with $\mathcal{Q} = \overline{\mathcal{S}}_{xy}$, $\mathcal{A}/\overline{\mathcal{S}}_{xy}$.

In analogy to (70) and (71), we have

$$\kappa\ Z(\mathcal{A}, \mathcal{B}_{xy} \cup \mathcal{B}'_{xy}; f_2) = Z(\mathcal{A}'_{xy}, \mathcal{B}'_{xy}; f_2)\ Z(\overline{\mathcal{A}}_{xy}, \mathcal{B}_{xy}; 1) \tag{74}$$

and

$$\kappa\ Z(\mathcal{A}_{xy}, \mathcal{B}_{xy} \cup \mathcal{B}'_{xy}; 1) = Z(\mathcal{A}''_{xy}, \mathcal{B}'_{xy}; 1)\ Z(\overline{\mathcal{A}}_{xy}, \mathcal{B}_{xy}; 1). \tag{75}$$

¿From Lemmas 12, 13 and 14 and (74),(75) and (62)

$$\begin{aligned}
\frac{Z(\mathcal{A}, \mathcal{B}; f_2)}{Z(\mathcal{A}, \mathcal{B}; 1)} &\leq \frac{1}{q^{\mathcal{B}/(\mathcal{B}_{xy} \cup \mathcal{B}'_{xy})}} \frac{Z(\mathcal{A}, \mathcal{B}_{xy} \cup \mathcal{B}'_{xy}; f_2)}{Z(\mathcal{A}_{xy}, \mathcal{B}_{xy} \cup \mathcal{B}'_{xy}; 1)} \\
&\leq 2C_1 \frac{Z(\mathcal{A}'_{xy}, \mathcal{B}'_{xy}; f_2)}{Z(\mathcal{A}''_{xy}, \mathcal{B}'_{xy}; 1)} \\
&\leq 2C_1 \left\{ \prod_{\mathbf{X} \in \mathcal{S}_{xy}} \left(1 + \zeta_{\mathbf{X}}\ \kappa^{m(\mathbf{X})}\right) - 1 \right\} (1 + Z_1) + Z_2 \\
&\leq 2C_1 \left\{ \prod_{n=n_{xy}}^{M} \left(1 + \ell_n^{-\delta'}\right) - 1 \right\} (1 + Z_1) + Z_2 \\
&\leq C_2 \ell_{n_{xy}}^{-\delta'} (1 + Z_1) + Z_2
\end{aligned} \tag{76}$$

where

$$n_{xy} = \inf \{n \in \mathbf{N} : |x - y| \leq \ell_n\} \tag{77}$$

and $Z_1, Z_2$ are generating functions depending on the partitions in $\widehat{\mathcal{S}}_x$, $\widehat{\mathcal{S}}_y$ and the probabilities $p_b$ for $b$ connecting them. $Z_i = Z_i(x, y)$, $i = 1, 2$, decays to zero as $|x - y| \to \infty$ and we now will estimate its rate.

Let $\Delta \in \widehat{\mathcal{S}}_x$ and $\Delta' \in \widehat{\mathcal{S}}_y$. As a consequence of the distance condition (35), we show in the Section 4 that

$$\sum_{\substack{z \in \Delta \\ z' \in \Delta'}} p_{(zz')} \leq C_3 \frac{|I||I'|}{\mathrm{dist}(\Delta, \Delta')^2} \tag{78}$$

for some constant $C_3 < \infty$, where $I, I'$ are segments determined by their end points. Since

$$\prod_{\Delta \in \widehat{\mathcal{S}}_z} \left(1 + \zeta_{\mathbf{X}}\ \kappa^{m(\mathbf{X})}\right) \leq C_4 < \infty, \qquad z = x, y$$

it follows that $Z_1$ and $Z_2$ can be bounded by

$$\begin{aligned}
&\leq C_5 \sup_{\mathbf{X} \in \widehat{\mathcal{S}}_x, \mathbf{X}' \in \widehat{\mathcal{S}}_y} \left\{ \zeta_{\mathbf{X}}\ \kappa^{m(\mathbf{X})} \frac{|I||I'|}{\mathrm{dist}(\Delta, \Delta')^2} \zeta_{\mathbf{X}'}\ \kappa^{m(\mathbf{X})} \right\} \\
&\leq C_6 \sup_{n, n' < n_{xy}} \left\{ \ell_n^{-\delta'+1} \frac{1}{d_{n,n'}^2} \ell_{n'}^{-\delta'+1} \right\}
\end{aligned}$$

where $d_{n,n'} = d_{n,n'}(x,y) = \max(1/3(\ell_{n+1} + \ell_{n'+1}), |x-y| - \ell_n - \ell_{n'})$, which can be bounded by

$$\leq C_7 \begin{cases} \ell_{n_{xy}-1}^{-2\delta'} & \text{if } \delta' \leq 2 \\ |x-y|^{-2} & \text{otherwise} \end{cases} \tag{79}$$

By setting $\delta' = 2(\beta\eta - 1)$, equation (60) can be replaced by

$$\eta \in \left( \frac{1}{\beta} \left[ 1 + \left( \frac{\alpha(\alpha-1)}{2(2-\alpha)} \right) \right] , \frac{1}{\alpha}(1-a) - \frac{1}{2\beta}(\alpha - 1) \right).$$

Notice that it requires $\beta > \frac{\alpha}{(1-a)(2-\alpha)}$ which can be always satisfied for $\beta > 1$ by picking $\alpha$ sufficiently close to 1 and then $\ell_1$ sufficiently large.

This concludes the proof of Theorem 2.

# 4 Energy Estimates

## 4.1 Proof of Lemma 2.9

Given $u_0 \in I_{L,k+1}$, let $\mathbf{X}^*_{u_0} \equiv \mathbf{X}^* = (\Delta^*, \mathbf{X}^*, \zeta^*)$ be given by (40) and satisfying Condition 10. Let $\mathcal{B}^* = \mathcal{B}^*_{u_0}$,

$$\mathcal{S}^* = \mathcal{S}^*(u_0) = \bigcup_{\substack{t=u_0, u_0 \pm \ell_{k+1}: \\ \tau_t = 1}} \mathcal{S}^*_{(k+1,t,\delta)} \tag{80}$$

and $\Sigma = \bigcup_{\mathbf{X} \in \mathcal{S}^*} \Delta$.

Clearly,

$$I^* \equiv \Delta^* \cup \Sigma \subseteq \overline{I_{k+1}}(u_0)$$

is a segment of $\mathbf{Z}$ delimited by the end points of $\Delta^*$ (recall (32)).

For $y \in I_\tau(u_0)$ as in the Condition 10, we set

$$\Sigma^*_{<(>)} = \left\{ x \in I^*/\overline{I_k}(y) : x < (>) \, y \right\} \tag{81}$$

and

$$\Sigma_{<(>)} = \left\{ x \in \Sigma/\overline{I_k}(y) \cap \Sigma : x < (>) \, y \right\}.$$

Notice that, if $\mathbf{Y} \succeq \mathbf{X}^*$ is of type 2, two components of $\mathbf{Y}$, say $Y_<$ and $Y_>$, are such that

$$\Sigma^*_</\Sigma_< \subseteq Y_< \qquad \textit{and} \qquad \Sigma^*_>/\Sigma_> \subseteq Y_>. \tag{82}$$

¿From equation (44) and (82), we have

$$\begin{aligned} \lambda_{\mathbf{Y}} &\leq \prod_{b=(xy)} q_b^{\chi_{Y_<}(x) \cdot \chi_{Y_>}(y)} \\ &\leq \prod_{b=(xy)} q_b^{\chi_{\Sigma^*_<}(x) \cdot \chi_{\Sigma^*_>}(y)} \Bigg/ \prod_{b=(xy)} q_b^{\left[ \chi_{\Sigma^*_<}(x) \cdot \chi_{\Sigma_>}(y) + \chi_{\Sigma_<}(x) \cdot \chi_{\Sigma^*_>}(y) \right]} \end{aligned} \tag{83}$$

where $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.

Let $I, I'$ be two segment of $\mathbf{Z}$. If $d = \mathrm{dist}(I, I')$ is sufficiently large, there exist constants $C_1, C_2$, such that

$$C_1 \; J(I, I') \leq \sum_{\substack{x \in I \\ y \in I'}} \frac{1}{|x-y|^2} \leq C_2 \; J(I, I') \tag{84}$$

where

$$J(I, I') = \int_{I_1 \times I_2} dx \, dy \frac{1}{|x-y|^2} = \ln \frac{(|I| + d)(|I'| + d)}{d\,(|I| + |I'| + d)} \tag{85}$$

We let $I = \Sigma^*_<$, $I' = \Sigma^*_>$ and notice that (42) implies that $\min(|I|, |I'|) \geq \ell_{k+1}/3$ and by (81) $\mathrm{dist}(I, I') = 3\ell_k$.

In view of this, the numerator of (83) can be estimated by

$$\leq \left( \frac{\ell_k}{\ell_{k+1}} \right)^{\beta(1-b)} \tag{86}$$

with $b = b(\alpha, \ell_1) \searrow 0$ as $\ell_1 \to \infty$.

Now, we decompose $\mathcal{S}^*$ according to the scale of its components, i. e.,

$$\mathcal{S}^* = \bigcup_{n=1}^{k-1} \mathcal{S}^*_n$$

so,

$$\Sigma_{<(>)} = \bigcup_{n=1}^{k-1} \Sigma^n_{<(>)} \tag{87}$$

where $\Sigma^n_{<(>)} = \left( \bigcup_{\mathbf{X} \in \mathcal{S}^*_n} \Delta \right) \bigcap \Sigma_{<(>)}$.

In view of (84), (87) and (30), we have (see [M] for more details)

$$\begin{aligned}
\sum_{x,y} \chi_{\Sigma^*_<}(x) \frac{1}{|x-y|^2} \chi_{\Sigma^*_>}(y) &= \sum_{n=1}^{k-1} \sum_{x,y} \chi_{\Sigma^*_<}(x) \frac{1}{|x-y|^2} \chi_{\Sigma^n_>}(y) \\
&\leq C \left( \sum_{n=1}^{k-1} \frac{\ell_n}{\ell_{n+1}} \right) \ln \frac{\ell_{k+1}}{\ell_k} \\
&\leq b' \ln \frac{\ell_{k+1}}{\ell_k}
\end{aligned}$$

where $b' = b'(\alpha, \ell_1) \searrow 0$ as $\ell_1 \to \infty$.

Since $\sum_{x,y} \chi_{\Sigma_<}(x) \frac{1}{|x-y|^2} \chi_{\Sigma^*_>}(y)$ satisfies the same estimate, the denominator of (83) can be bounded by

$$\geq \left( \frac{\ell_k}{\ell_{k+1}} \right)^{2\beta b'}$$

This concludes the proof of Lemma 11.

### 4.2 Proof of (3.22)

Let $\Delta \in \widehat{\mathcal{S}}_x$ and $\Delta' \in \widehat{\mathcal{S}}_y$. Let $I, I'$ be the segment defined by their end points. from (35) we have $\mathrm{dist}(\Delta,\Delta') \geq 1/3\,[\min(|I|,|I'|)]^{\alpha}$. If $\ell_1$ is sufficiently large we then can Taylor expand (85) in power of $\frac{|I|}{d}$ and $\frac{|I'|}{d}$ to get

$$J(\Delta,\Delta') \leq C'' \frac{|I||I'|}{\mathrm{dist}(\Delta,\Delta')^2} \tag{88}$$

which proves (78).

### 4.3 Proof of Lemma 3.3

Given $\mathcal{A}$ and $\mathcal{D}$ as in the Lemma 14, let us denote by $\chi_{\mathcal{A}}$ the indicating function of the distance condition, i. e., $\chi_{\mathcal{A}}(b) = 1$ if $b = (xy)$ satisfies (35) for any $\Delta$,$\Delta' \in \mathcal{A}$ and 0 otherwise.

¿From (84), (88) and (35),

$$K(\Delta,\Delta') = \sum_{b=(zz')\in\mathcal{D}} \chi_\Delta(z) \frac{1}{|x-y|^2} \chi_{\Delta'}(z')\ \chi_{\mathcal{A}}(b)$$

can be bounded by

$$\leq C \min\left\{ \frac{|I||I'|}{\mathrm{dist}(\Delta,\Delta')^2}, [\min(|I|,|I'|)]^{1-\alpha} \right\} \tag{89}$$

where $I(I')$ is the segment in $\mathbf{Z}$ between the end points of $\Delta(\Delta')$.

(89) implies

$$q^{\mathcal{D}} = \prod_{\Delta,\Delta'\in\mathcal{A}} e^{-\beta K(\Delta,\Delta')} \geq e^{-\beta C' \ell_1^{1-\alpha}}$$

which proves Lemma 14.

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