Quantum Intrinsically Degenerate and Classical Secular Perturbation Theory

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Abstract

For a class of smooth perturbation of the resonant harmonic oscillator in two degrees of freedom the quantum degenerate perturbation theory is explicitly constructed by quantization of the corresponding classical secular expansion. This result is based upon two identifications, which allow us to apply the quasi-modes construction: the unperturbed subspaces of fixed principal quantum number with the quantization of the Kähler manifolds defined by the classical constant energy surfaces, and the principal symbol of the perturbing operator in this subspace with the classical resonant normal form.

1 Introduction and Statement of the Results

The problem of finding a quantization formula relating the Schrödinger energy levels to the underlying classical motions is still completely open in the general multi-dimensional, non-integrable case. For systems close to the integrable ones the prescription of old quantum theory (see [Bo], §§$41-47$) is the Bohr-Sommerfeld quantization of the classical normal form. In recent times this method has been applied with remarkable success in atomic and molecular physics (see e.g.[Ec],[Gu] and references therein); the problem thus arises of relating this to the Schrödinger eigenvalues, at least at the level of perturbation theory.

For a class of holomorphic perturbations of the *non-resonant* harmonic oscillators the exact relation between classical and quantum perturbation theories has been already established ([GP],[DEGH],[BV]) . It represents (see Formula (1.3) below) each

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term of the quantum series as the Bohr-Sommerfeld quantization of the corresponding term of the classical (Birkhoff) normal form plus corrections in ascending powers of $\hbar$, also expressed under the form of Bohr-Sommerfeld quantization of an explicitly constructed classical observable.

This paper represents a first contribution to the degenerate case, which corresponds to the classically resonant one, and is considerably more difficult as we intend to illustrate in the particular case of a suitable class of perturbations of the resonant harmonic oscillator in two degrees of freedom. Despite its simplicity this example not only describes interesting physical systems such as the Hydrogen Zeeman effect and the Hénon-Heiles model (see e.g.[De]), but already contains all the difficulties of the intrinsic degeneracy.

To state the results we first summarize, within this particular two-dimensional case, what is known in the non-degenerate situation. Let $\mathcal{P}_0$ be the Schrödinger operator, acting in $L^2(\mathbb{R}^2)$, of a two-dimensional, non-resonant harmonic oscillator of frequencies ω_1, ω_2, $\dfrac{\omega_1}{\omega_2} \notin \mathbb{Q}$:

$$\mathcal{P}_0 = -\frac{1}{2}\hbar^2\Delta + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) \tag{1.1}$$

defined on the maximal domain $D(\mathcal{P}_0)$ of the differential expression (1.1); let $x \mapsto V(x)$, $x \in \mathbb{R}^2$ be a smooth potential leaving $D(\mathcal{P}_0^n)$ invariant for all $n \in \mathbb{N}$, i.e. $VD(\mathcal{P}_0^n) \subset D(\mathcal{P}_0^n)$, $n = 1, \ldots$, and let

$$\mathcal{P}_\epsilon = \mathcal{P}_0 + \epsilon V(x) \tag{1.2}$$

be the perturbed Schrödinger operator, defined once more by the maximal action in $L^2(\mathbb{R}^2)$ of the differential expression (1.2). Then given any eigenvalue $E_0(n,\hbar)$ of $\mathcal{P}_0$, $E_0(n,\hbar) = (n_1\omega_1 + n_2\omega_2 + \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2)\hbar$, $n = (n_1, n_2), n_1, n_2 = 0, 1 \ldots$, there is one and only one formal power series $E_0 + \sum_{\ell=1}^{\infty} E_\ell(n,\hbar)\epsilon^\ell$, generated by Rayleigh-Schrödinger perturbation theory. Moreover this series is such that:

(a) Each coefficient $E_\ell(n,\hbar)$ admits an asymptotic expansion of the form

$$E_\ell(n,\hbar) \sim N_k((n+1/2)\hbar) + \sum_{j=2}^{\infty} Q_j^\ell((n+1/2)\hbar)\hbar^j \tag{1.3}$$

(b) The function $A \mapsto N_\ell(A) : \mathbb{R}^2 \to \mathbb{R}$ is the term of order ℓ of the normal form assumed by the classical Hamiltonian h_ϵ corresponding to P_ϵ, and the functions $A \mapsto Q_j^\ell(A) : \mathbb{R}^2 \to \mathbb{R}$, $\ell = 2, \ldots, j = 1, \ldots$ can be explicitly constructed by the algorithm of canonical perturbation theory. The series (1.3) reduces to a finite sum if V is a polynomial.

Here of course

$$h_\epsilon = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2) + \epsilon V(x) : \quad \epsilon \geq 0 \tag{1.4}$$

which we rewrite, under a standard abuse of notation, in the action-angle variables $(I, \chi) = (I_1, I_2, \chi_1, \chi_2)$ of the harmonic oscillators:

$$h_\epsilon = \langle \omega, I \rangle + \epsilon V(I, \chi) \tag{1.5}$$

where, for $k = 1, 2$:

$$\begin{cases} \xi_k = -\sqrt{2\omega_k I_k}\sin(\chi_k) \\ x_k = \sqrt{\dfrac{2I_k}{\omega_k}}\cos(\chi_k) \end{cases} . \tag{1.6}$$

The normal form assumed by the Hamiltonian (1.4) means that that for any $M \in \mathbb{N}$ there is a canonical transformation $C_{M,\epsilon}$ mapping the action-angle variables (I, χ) into new action-angle variables (A, ϕ) in such a way that

$$h_\epsilon\left(C_{M,\epsilon}^{-1}(A, \phi)\right) = \sum_{\ell=0}^{M} N_\ell(A)\epsilon^\ell + O(\epsilon^{N+1}) \tag{1.7}$$

Remark that in (1.3) the term of order zero in $\hbar$ is the Bohr-Sommerfeld quantization of the classical normal form, and thus the quantization prescription of old quantum theory; the "exactness" comes from the fact that all corrections to this formula in ascending powers of $\hbar$ are determined.

We come now the degenerate case $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$. Let therefore exist two relatively prime integers (p, q) such that $p\omega_1 + q\omega_2 = 0$. Although the results can be formulated in exactly the same way as before, the construction of the coefficients $E_\ell(n, \hbar)$ in (1.3) and their meaning is in the two cases quite different. The eigenvalues $E_0(n, \hbar)$ of $\mathcal{P}_0$ are degenerate, because their multiplicity $M(p, q)$ is equal to $\left[\frac{q}{p}\right] n$, the number of positive integers (n_1, n_2) such that $n = pn_2 - qn_1$. Correspondingly the classical unperturbed problem is intrinsically resonant, namely the free Hamiltonian $h_0(I)$ depends only on the "fast" action $J_1 = \frac{I_1}{p} - \frac{I_2}{q}$ and not on the "slow" one $J_2 = \frac{I_1}{p} + \frac{I_2}{q}$. The motion of the "slow" angle $\theta_2 = \frac{1}{2}(p\chi_1 + q\chi_2)$ is entirely determined by the pertubation $V(J, \theta)$ because the free Hamiltonian only generates the linear flow of the "fast" angle $\theta_1 = \frac{1}{2}(p\chi_1 - q\chi_2)$.

Thus in extending (1.3) to the present situation we have in general to construct M sequences $E_\ell^{(k)}(n, \hbar) : k = 1, \dots, M$ and two major additional problems arise in this context:

(1) Since the free part involves only one quantization condition, namely the quantum number n, the second one must entirely come from the perturbation;

(2) The second quantization condition should involve but M integers, and be uniform with respect to the first: namely, for any fixed value of the principal quantum number n, the *same* classical function of the actions, computed at the "fast" action equal to $n\hbar$, has to yield, upon quantization of the "slow" action, the M perturbation coefficients near any unperturbed eigenvalue of multiplicity M.

Collecting in an Appendix, for convenience of exposition, the basic facts about quantization and symbols in the Bargmann representation needed throughout the paper, let us now shortly sketch the strategy to overcome these difficulties. The main result is stated in Theorem 1.1 below.

(1) The construction is recalled (Section 2) of the classical normal form in the resonant case via the Poincaré-von Zeipel algorithm (see e.g.[AKN]). First the dependence on the "fast angle" is eliminated (up to order ϵ^{N+1}) through non resonant canonical perturbation theory, and then the resulting (ordinary differential) Hamilton-Jacobi equation is integrated to eliminate (up to the same order) also the dependence on the slow angle θ_2.

(2) The quantum degenerate perturbation theory is constructed (Section 3) by the natural analog of the above algorithm. The elimination of the fast variable corresponds indeed to the construction of an operator expansion

$$\mathcal{P}_0 + \sum_{\ell=1}^{N} K_\ell \epsilon^\ell + \epsilon^{N+1} R_{N+1} \qquad N = 1, 2, \dots$$

where $K_\ell \, : \, \ell = 1, \dots$ is block diagonal (on the eigenspaces of $\mathcal{P}_0$), and the elimination of the slow one to the solution of the resulting secular equations. The eigenvalues thus obtained are precisely the coefficients $E_\ell^{(k)}(n, \hbar) : k = 1, \dots, M$ of the degenerate Rayleigh-Schrödinger perturbation expansion at order ℓ.

(3) Working in the Bargmann representation the M dimensional eigenspaces $\mathcal{G}_{E_0(n)}$ of $\mathcal{P}_0$ corresponding to the principal quantum number n are identified (Section 4) with the quantization of the Kähler manifolds ([GS1],Ra) defined by taking the quotients of the constant energy surfaces with respect to the (completely periodic) unperturbed classical flow.

(4) We prove (Section 5) that K_ℓ, restricted to $\mathcal{G}_{E_0(n)}$, admits a symbol expansion in ascending powers of $\hbar$, the principal symbol being the classical normal form.

Then the validity of (1.3) becomes a straightforward application of the recent construction of quasi-modes via coherent states[PU].

To state our main result we need the following definition:

Definition 1.1 *Let* $z = (z_1, \dots, z_s) \in \mathbb{C}^s$ *and* $z_k = \dfrac{\omega_k x_k + i\xi_k}{\sqrt{2\omega_k}}$. *Let* $\{\psi_z : z \in \mathbb{C}^s\}$ *be the set of the coherent states in* $L^2(\mathbb{R}^n)$, *namely:*

$$\psi_z(y) = \frac{1}{(\pi\hbar)^{n/4}} e^{-i\frac{\langle x,\xi\rangle}{2\hbar}} \cdot e^{-i\frac{\langle \xi,y\rangle}{\hbar}} \cdot e^{-\frac{(y-x)^2}{\hbar}} \tag{1.8}$$

We say that an $\hbar$ *dependent operator* V *in* $L^2(\mathbb{R}^s)$ *admits a holomorphic semiclassical asymptotic expansion if* $\psi_z \in D(V)$ $\forall z \in \mathbb{C}^s$ *and if, for* $k = 0, 1, \dots$ *there are functions* $V_k(z, \overline{z}') : C^{2s} \to \mathbb{C}$ *holomorphic in a neighbourhood of the manifold* $\Theta \equiv \{(\zeta, \eta) \in \mathbb{C}^{2s} : \overline{\zeta} = \eta\}$ *such that the scalar products* $\langle \psi_z, V\psi_{z'} \rangle$ *admit the following asymptotic expansion*

$$\langle \psi_z, V\psi_{z'} \rangle \sim \langle \psi_z, \psi_{z'} \rangle \sum_{k=0}^{\infty} V_k(z, \overline{z}')\hbar^k \tag{1.9}$$

for $\hbar \to 0$.

Remarks.

(1) In our normalization:

$$\langle \psi_z, \psi_{z'} \rangle = e^{\frac{\langle \overline{z}, z' \rangle}{\hbar}} \cdot e^{-\frac{|z|^2}{2\hbar}} \cdot e^{-\frac{|z'|^2}{2\hbar}} = \rho_z^B(z') \tag{1.10}$$

where $\rho_z^B(z')$ is the reproducing kernel in the Bargmann space (see Lemma A.2(i))

(3) Even though $V_0(z, \overline{z})$ is the Wick symbol of V (see e.g.[BS], §5.2, recalled in Lemma A.4), we do not require the r.h.s. to have the meaning of a symbol expansion for a pseudodifferential operator. The class of operators fulfilling the above definition therefore does *not* necessarily coincide with the class of the pseudodifferential operators of semiclassical type of [Ro]. In particular it contains all maximal differential operators $P(x, -i\hbar\nabla_x)$ in $L^2(\mathbb{R}^s)$, where $P(x, \xi)$ is a polynomial of arbitrary order.

Theorem 1.1 *Let* h_ϵ *be defined as in (1.4),(1.5) and* $\mathcal{P}_0, \mathcal{P}_\epsilon$ *as in (1.1),(1.2), respectively. Assume*

(A) $VD(\mathcal{P}_0^m) \subset D(\mathcal{P}_0^m)$, $m = 1, \dots$;

(B) V *admits a holomorphic semiclassical expansion.*

Then:

(i) *For any $N \in \mathbb{N}$ there exists a smooth canonical transformation $C_{\epsilon,N} : J(J_1,\theta_1) \leftrightarrow (A_1,\phi_1)$ into new action-angle variables (A_1,ϕ_1) mapping $h_\epsilon(I,\chi)$ into the resonant normal form*

$$
\begin{aligned}
&h_\epsilon\left(C_{\epsilon,N}^{-1}(A_1,\phi_1;J_2,\theta_2)\right) = \\
&h_0(A_1,J_2)+\sum_{\ell=1}^{N} k_\ell(A_1;J_2,\theta_2)\epsilon^\ell+\epsilon^{N+1}R_{N+1}(A_1,\phi_1,J_2,\theta_2)
\end{aligned}
\tag{1.11}
$$

(ii) *Let $\Omega_{A_1,\epsilon}$ be the family of open sets parametrized by (A_1,ϵ) defined as follows*

$$
\Omega_{A_1,\epsilon} = \{(J_2 \in \mathbb{R}_+, \theta_2 \in \mathbb{T}^1) | \nabla_{(J_2,\theta_2)} k_1(J_2,\theta_2,\cdot)| > \epsilon \ \forall \theta_2 \in \mathbb{T}^1\}
\tag{1.12}
$$

Then for $\epsilon > 0$ sufficiently small and any $N \in \mathbb{N}$ there is a family of smooth canonical transformations, parametrized by A_1, $D_{\epsilon,N} : (J_2,\theta_2) \leftrightarrow (A_2,\phi_2)$ into new action-angle variables (A_2,ϕ_2), such that, if $\Omega^r_{A_1,\epsilon}$: $r = 1,2\dots$ is any connected component of $\Omega_{A_1,\epsilon}$, then $D_{\epsilon,N}(\Omega^r_{A_1,\epsilon} \times \mathbb{T}^1) = \mathcal{B}_r \times \mathbb{T}^1$. Here the open sets $B_r : r = 1,\dots$ are pairwise disjoint, and

$$
\begin{aligned}
&h_0(A_1)+\sum_{\ell=1}^{N} k_\ell(A_1,D_{\epsilon,N}^{-1}(A_2,\phi_2))+\epsilon^{N+1}R_{N+1}(A_1,D_{\epsilon,N}^{-1}(A_2,\phi_2)) = \\
&h_0(A_1)+\sum_{\ell=1}^{N} \tilde{k}_\ell(A_1,A_2)+\epsilon^{N+1}\tilde{R}_{N+1}(A_1,A_2,\phi_1,\phi_2,\epsilon)
\end{aligned}
\tag{1.13}
$$

(iii) *Let $E_0(n,\hbar)$ be an eigenvalue of $\mathcal{P}_0$ of multiplicity M, and let furthermore $\mathcal{B}(A_1,\epsilon) = \bigcup_r \mathcal{B}_r(A_1,\epsilon)$. Given A_1 and $1 \le s \le M$, let $(s+1/2)\hbar \in \mathcal{B}$. Then there are functions $(A_1,A_2) \mapsto \tilde{k}^j_\ell(A_1,A_2) : j = 1,2,\dots$, defined and holomorphic on $\mathbb{R}_+ \times \mathcal{B}(A_1,\epsilon)$, such that any coefficient $E^s_\ell(n,\hbar)$ of the quantum perturbation theory (4.18) admits the asymptotic expansion*

$$
E^s_\ell(n,\hbar) \sim \tilde{k}_\ell((n+1/2)\hbar,(s+1/2)\hbar)+\sum_{j=2}^{\infty} \tilde{k}^j_\ell((n+1/2)\hbar,(s+1/2)\hbar)\hbar^j
\tag{1.14}
$$

Remarks

(1) The condition (1.12) accounts for the possible existence of separatrices for the Hamiltonian $k_1(J_2,\theta_2,\cdot)$. The integrating canonical transformations are different in the different connected components of Ω_ϵ separated by the separatrices. It is well known that chaotic behaviour develops near the separatrices, and thus Assertion

(iii) tells nothing on levels which, loosely speaking, "live" in the chaotic regions of phase space. In other terms, these levels do not appear in our semiclassical of perturbation theory (even though the recent results of [CP] should make possible to handle them). On the other hand, denoting $\Omega^c(\cdot)$, $\mathcal{B}^c(\cdot)$ the complements of $\Omega_{A_1,\epsilon}$, $\mathcal{B}(A_1,\epsilon)$, respectively, we have $\mathrm{vol}(\mathcal{B}^c(A_1,\epsilon)) \sim \epsilon$.

(ii) The first addend in the r.h.s. of Formula (1.14) is the Bohr-Sommerfeld quantization of the classical normal form, and the second one yields all quantum corrections. Unlike the non-resonant case, however, even if V is a polynomial we cannot conclude that yields an *exact* quantization formula for the Rayleigh-Schrödinger perturbation theory, because the series in the r.h.s. will never reduce to finite sum and, being divergent, has at most the meaning of an asymptotic expansion.

2 The Poincaré-von Zeipel Algorithm

In this Section we construct the classical normal form through the Poincaré-von Zeipel method which. Even though well known (see e.g.[AKN] and references therein), this construction is recalled here in some detail because it enters in an essential way in the quantization procedure. As already mentioned, we consider in this paper only the Hamiltonian (1.5) with $\dfrac{\omega_1}{\omega_2} \in \mathbb{Q}$. There are thus integers p, q (we can assume without loss of generality $p > 0$, $q < 0$) such that

$$p\omega_1 + q\omega_2 = 0 \tag{2.1}$$

It follows from Assumption B of Theorem 1.1 that the corresponding classical potential, also denoted V by the standard abuse of notation, is entire holomorphic with respect to $(x,\xi) \in \mathbb{C}^4$ in a neighbourhood of $\mathbb{R}^4$. In the notation of Definition 1.1 one has indeed $V = V_0(z,\overline{z})$, the Wick symbol of the operator V; since V_0 is holomorphic, it is enough to reexpress the variables $(z,\overline{z})$ in terms of (x,ξ) (see (A.1)). This obviously makes $V(I,\chi)$ entire holomorphic with respect to (I,χ) but for $I = (0,0)$.

Given the perturbed Hamiltonian $h_\epsilon(I,\chi)$ (1.4), introduce the standard decoupling into "fast" and "slow" variables (J,θ) of the resonant harmonic oscillator, defined as follows:

$$\begin{cases} J_1 = \dfrac{I_1}{p} - \dfrac{I_2}{q}, & J_2 = \dfrac{I_1}{p} + \dfrac{I_2}{q} \\ \theta_1 = \dfrac{1}{2}(p\chi_1 - q\chi_2), & \theta_2 = \dfrac{1}{2}(p\chi_1 + q\chi_2) \end{cases} . \tag{2.2}$$

inverted as

$$\left\{\begin{array}{ll} I_1 = \frac{p}{2}(J_1+J_2), & I_2 = \frac{q}{2}(J_2-J_1) \\ \chi_1 = \frac{1}{p}(\theta_1+\theta_2), & \chi_2 = \frac{1}{q}(\theta_2-\theta_1) \end{array}\right. . \tag{2.3}$$

In these variables the unperturbed and perturbed Hamiltonians become (standard abuse of notation)

$$\left\{\begin{array}{l} h_0(J_1,J_2) = \omega_1 p J_1 \\ h_\epsilon(J,\theta) = \omega_1 p J_1 + V(J,\theta) \end{array}\right. . \tag{2.4}$$

The first step is represented by the elimination of the "fast angle" (the normal form thus obtained is sometimes called Birkhoff-Gustavson normal form) and yields the verification of the first Assertion of Theorem 1.1.

Proposition 2.1 *Let $N \geq 1$, $\epsilon > 0$, $J_2 \in \mathbb{C} \setminus \{0\}$, $\theta_2 \in \mathbb{T}^1$. Then there exists a family of holomorphic canonical transformations $C_{N,\epsilon}$, parametrized by (J_2,θ_2), mapping the action-angle variables (J_1,θ_1) into new action-angle variables $(A_1,\phi_1) = C_{N,\epsilon}(J_1,\theta_1)$ such that*

$$\begin{array}{l} h_\epsilon\left(C^{-1}_{N,\epsilon}(A_1,\phi_1;J_2,\theta_2)\right) = \\ h_0(A_1,J_2) + \sum_{\ell=1}^{N} k_\ell(A_1,J_2,\theta_2)\epsilon^\ell + \epsilon^{N+1}R_{N+1}(A_1,J_2,\phi_1,\theta_2,\epsilon) \end{array} \tag{2.5}$$

where $k_\ell(A_1,J_2,\theta_2) : \ell = 1,\ldots,N$ and $R_{N+1}(A_1,J_2,\phi_1,\theta_2,\epsilon)$ are holomorphic functions of their arguments but for $J_2 = 0, A_1 = 0, \epsilon = 0$.

Proof. Denote $\mathcal{L}_w f = \{f,w\}$ the Lie derivative of any observable f with respect to the flow generated by any Hamiltonian w, which for any fixed time defines a canonical transformation. Look for that particular w such that the canonical transformation generated by its flow of initial conditions (J_1,θ_1) at time 1 (we omit the parametric dependence on (J_2,θ_2))

$$(A_1,\phi_1) = \exp[\mathcal{L}_{w_\epsilon}](J_1,\theta_1) \tag{2.6}$$

is such that $\exp[\mathcal{L}_{w_\epsilon}]h_\epsilon = k(A_1,J_2,\theta_2,\epsilon)$ where the new Hamiltonian $k(A_1,J_2,\theta_2,\epsilon)$ has the form of the r.h.s of (2.5). According to the standard procedure of perturbation theory, we look at the formal power series expansion of both w_ϵ and $k(A_1,J_2,\theta_2,\epsilon)$:

$$\left\{\begin{array}{l} w_\epsilon = w_1 + \epsilon w_2 + \epsilon^2 w_3 + .. \\ k(A_1,J_2,\theta_2,\epsilon) = h_0(A_1,J_2) + \epsilon k_1(A_1,J_2,\theta_2) + \epsilon^2 k_2(A_1,J_2,\theta_2) + \ldots \end{array}\right. . \tag{2.7}$$

Therefore a universal expansion in powers of ϵ yields the formal power series equality

$$\exp[\mathcal{L}_{w_\epsilon}]h_\epsilon = h_0 + \epsilon[\{w_1,h_0\} + v] + \frac{\epsilon^2}{2!}[\{w_2,h_0\} + \{w_1,v\} + \{w_1,\{w_1,h_0\}\}] + \ldots$$

$$\frac{\epsilon^\ell}{\ell!}[\{w_\ell, h_0\} + \{w_{\ell-1}, v\} + \{w_{\ell-1}, \{w_1, h_0\}\} + \{w_1, \{w_{\ell-1}, h_0\}\} + p_\ell] + \ldots =$$
$$k_0(A_1, \cdot) + \epsilon k_1(A_1, \cdot) + \epsilon^2 k_2(A_1, \cdot) + \ldots \epsilon^\ell k_\ell(A_1, \cdot) + \ldots$$

where

$$p_\ell(w_1, \ldots, w_{\ell-1}) = \sum_{r=3}^{\ell-1} \sum_{j_1+j_2+..j_r=\ell-1: j_k \geq 1} \{w_{j_1}, \{w_{j_2}, .., \{w_{j_\ell}, h_0\}, .., \}+$$
$$\sum_{r=2}^{\ell-1} \sum_{j_1+j_2+..j_r=\ell-2: j_r \geq 1} \{w_{j_1}, \{w_{j_2}, .., \{w_{j_{\ell-1}}, v\}, .., \}$$

for $\ell \geq 2\,, p_1 = V$. Equating the coefficients of the same powers of ϵ we immediately obtain the homological equations

$$\{h_0, w_\ell\} + p_\ell(w_1, \ldots, w_{\ell-1}) = k_\ell, \quad \ell \geq 1, \quad w_0 = p_0 = 0 \tag{2.8}$$

Given the form (2.4) of h_0, the homological equations are first order ODE's of the form

$$p\omega_1 \partial_{\phi_1} w_\ell(A_1, \phi_1) + p_\ell[(w_1, \ldots, w_{\ell-1}) \circ (A_1, \phi_1)] = k_\ell(A_1, \cdot) \tag{2.9}$$

which admit the immediate solutions

$$k_\ell(A_1, J_2, \theta_2) = \frac{1}{2\pi} \int_0^{2\pi} p_\ell(A_1, \phi_1, \cdot)\, d\phi_1 \tag{2.10}$$

and

$$w_\ell(A_1, \phi_1, J_2, \theta_2) = \frac{1}{p\omega_1} \int_0^{\phi_1} \left[p_\ell(A_1, \phi_1, \cdot) - \frac{1}{2\pi} \int_0^{2\pi} p_\ell(A_1, \phi_1, \cdot)\, d\phi_1 \right] d\phi_1 \tag{2.11}$$

The desired transformation is then generated by

$$w_{\epsilon,N}(A_1, \phi_1, \cdot) = A_1 \phi_1 + \sum_{\ell=1}^{N} w_\ell((A_1, \phi_1), \cdot)$$

This concludes the proof of Proposition 1.1 and hence of Theorem 1.1, Assertion (i).
Remark. As is well known, the above construction holds in the general case of a (smooth) perturbation of an integrable system $h_0(I_1, \ldots, I_d)$ in $d > 2$ degrees of freedom admitting an *intrinsic* resonance relation, i.e. a rational relation among its frequencies valid for *all* values of the actions in $\mathbb{R}^d$. It is well known (see e.g.[Bo]) that in this case it is possible to define an appropriate linear canonical transformation generalizing (2.2) and (2.3) into new action-angle variables $(J, \theta) = (J_1, .., J_{d-1}; J_d; \theta_1, .., \theta_{d-1}; \theta_d)$ such that the unperturbed Hamiltonian can be written under the form $h_0 = h_0(J_1, ..J_{d-1})$ where the components $\omega_r(J) = \partial_{J_r} h_0 : r = 1, .., d-1$ of the *reduced frequency vector*

$\omega_F \equiv (\omega_1, \ldots, \omega_{d-1})$ are rationally independent. Since the unperturbed Hamiltonian does not depend on J_d, the resonance condition thus takes the equivalent form $\omega_d(J) = 0$, i.e. $\omega = (\omega_F; 0)$. As above, the motion of the "slow" canonical variables $(J_d; \theta_d)$ is thus entirely determined by the perturbation. The remaining ones, $(J_1, .., J_{d-1}; \theta_1, .., \theta_{d-1}) \equiv (J_F, \theta_F)$ are obviously called *fast variables.*
The procedure to obtain the normal form is the same as before, up to the following two technical differences (and obvious meaning of the notation (A_F, ϕ_F)):

(a) The homological equations (2.9) are now linear, constant coefficients *PDE* of the form

$$\langle \omega_F, \nabla_{\phi_F} w_\ell(A_F, \phi_F) \rangle + p_\ell[(w_1, \ldots, w_{\ell-1}) \circ (A_F, \phi_F)] = k_\ell(A_F, \cdot) \tag{2.12}$$

(b) The solution of (2.12), obtained through Fourier expansion, is formally written as follows:

$$w_\ell(A_F, \phi_F) = -i \sum_{|\nu| \in \mathbb{Z}^{d-1}} \frac{p_\ell^\nu[(w_1, \ldots, w_{\ell-1}](A_F)}{\langle \omega_F, \nu \rangle} e^{i\langle \nu, \phi_F \rangle} \tag{2.13}$$

If the frequency vector ω_F is, as in the harmonic oscillator example, independent of J, under our holomorphy hypothesis on V the solution is a holomorphic function of $(A_F, \phi_F, \epsilon; \cdot)$ but for $A_F = 0$ in the further assumption that the components of ω_F fulfill a diophantine condition; if ω_F is not constant the holomorphy holds true provided the function $V(J_F, \theta_F; \cdot)$, i.e., the perturbation reexpressed on the canonical variables (J, θ), has a finite number of Fourier components $V_\nu(J; \cdot) : \nu \in \mathbb{Z}^{d-1}$, i.e. $V_\nu(J; \cdot) = 0$, $|\nu| > Q$ if $|\nu| > Q$, for some $Q > 0$.

The elimination of the second angle is now immediate because the system has been reduced to a single degree of freedom. This yields the normal form, and hence the simple verification of the second assertion of Theorem 1.1.

Proof of Theorem 1.1 (Assertion (ii)). Consider once more the family of open sets $\Omega_{A_1,\epsilon}$ parametrized by (A_1, ϵ)

$$\Omega_{A_1,\epsilon} = \{(J_2 \in \mathbb{R}_+, \theta_2 \in \mathbb{T}^1) \mid \nabla_{(J_2,\theta_2)} k_1(J_2, \theta_2, \cdot)| > \epsilon \; \forall \theta_2 \in \mathbb{T}^1\}$$

By (1.12) the Hamiltonian $\sum_{\ell=0}^{N-1} k_\ell(J_2, \theta_2, \cdot)$ has only one degree of freedom and is therefore integrable, in each connected component of Ω_ϵ, by a canonical transformation holomorphic near $\epsilon = 0$, denoted $D_{\epsilon,N}$, generated by the solution of the corresponding Hamilton-Jacobi equation. (We omit the well known details). This means that the image $\Xi^r_{A_1,\epsilon} = D_{\epsilon,N}(\Omega^r_{A_1\epsilon}) : r = 1,\ldots$ of any connected component of $\Omega_{A_1\epsilon}$ under $D_{\epsilon,N}$ has the form $\Xi^r_{A_1,\epsilon} = \mathcal{B}_r \times \mathbb{T}^1$ and that these sets are pairwise disjoint for ϵ sufficiently small. This yields (1.13), the new Hamiltonian being defined on $\mathbb{R}_+ \times \mathcal{B}$, where:

$$\mathcal{B}(A_1,\epsilon) = \cup_r \mathcal{B}_r(A_1,\epsilon) \tag{2.14}$$

and this concludes the proof of the assertion.

3 The Quantum Algorithm

Let us now turn to the quantum algorithm. The spectrum $\sigma(\mathcal{P}_0)$ is discrete:

$$\sigma(\mathcal{P}_0) = \{E_0(n,\hbar) | n = (n_1,n_2) | n_1 = 0,1,\ldots, n_2 = 0,1,\ldots\} \tag{3.1}$$

where by the resonance condition (2.1) we have

$$E_0(n,\hbar) = \left[\omega_1(n_1 + \frac{1}{2}) + \omega_2(n_2 + \frac{1}{2})\right]\hbar = \frac{\omega_1}{|q|}n\hbar + \frac{1}{2}(1 - \frac{p}{q})\omega_1\hbar \tag{3.2}$$

Here as we know $|q|n_1 + pn_2 \equiv n$ is the *principal quantum number*. Hence $E_0(n)$ is degenerate with multiplicity $M(q,p;n)$ given by

$$M = \{\#(n_1,n_2) | -qn_1 + pn_2 = n\} = \left[\frac{|q|}{p}\right] n \tag{3.3}$$

where $[x]$ denotes the integer part of x. We further denote $\mathcal{G}_{E_0(n)}$ the subspace spanned by the (orthonormal) eigenvectors of $\mathcal{P}_0$ corresponding to the eigenvalue $E_0(n,\hbar) = \omega_1\hbar(n_1 - \frac{p}{q}n_2) : qn_1 - pn_2 = n$ of multiplicity $M(q,p)$ and of fixed principal quantum number n.

Denote once more $\mathcal{P}_\epsilon$ the maximal operator generated by $\mathcal{P}_0 + \epsilon V$, $\epsilon \in \mathbb{R}$. The first step is the analog of Proposition 1.1. It consists in this case in the diagonalization outside the constant degeneracy subspaces, and can be stated and proved in full generality:

Proposition 3.1 *Let H_0 be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ with discrete spectrum, $\sigma(H_0) = \{\lambda_0 \le \lambda_1, \ldots\}$, where the eigenvalue λ_n has multiplicity $M(n)$, $n = 0,1,\ldots$ with $M(\lambda_0) = 1$. Denote $\mathcal{H}_n$ the $M-$ dimensional eigenspace of λ_n, and let $|n_j\rangle : j = 1,\ldots,M(n)$ be any orthonormal basis in $\mathcal{H}_n$.*
Let $H_\epsilon : \epsilon \in \mathbb{R}$ be the symmetric operator family in $\mathcal{H}$ defined by the maximal action of $H_0 + \epsilon H_1$, where H_1 is a self-adjoint operator in $\mathcal{H}$ such that

(i) $H_1^k D(H_0) \subset D(H_0)$, $\forall k = 1, \ldots$.

(ii) The linear hull of the eigenvectors of H_0 *is a core of* H_1.

Then for any $N \geq 1$ *there exists a unitary operator* $U_{\epsilon,N}$ *in* $\mathcal{H}$ *such that*

$$U_{\epsilon,N} H_\epsilon U_{\epsilon,N}^{-1} = H_0 + \sum_{\ell=1}^{N} K_\ell \epsilon^\ell + \epsilon^{N+1} \mathcal{R}_{N+1}(\epsilon) \tag{3.4}$$

where K_ℓ *is self-adjoint in* $\mathcal{H}$ *and block-diagonal on* $\bigoplus_{\lambda(n)} \mathcal{H}_n = \mathcal{H}$, *i.e.* $K_\ell \mathcal{H}_n \subset \mathcal{H}_n$, $\langle n_1^i, K_\ell n_2^j \rangle = 0$, $\ell = 1, \ldots, N$, $n_1 \neq n_2$, *and* $\mathcal{R}_{N+1}(\epsilon)$ *is symmetric in* $\mathcal{H}$.

Proof. Following [DEGH], we proceed in analogy to the Lie derivative method employed in the former Section. That is, we look for the self-adjoint generator $W_{\epsilon,N}$ of $U_{\epsilon,N}$:

$$U_{\epsilon,N} = \exp \frac{i}{\hbar} W_{\epsilon,N} \tag{3.5}$$

Once more according to the standard procedure of perturbation theory, as in the proof of Proposition (1.1), we look at the formal power series expansion of both $W_{\epsilon,N}$ and K_ϵ:

$$\begin{cases} W_\epsilon = W_1 + \epsilon W_2 + \epsilon^2 W_3 + \ldots + \epsilon^{N+1} W_{N+1} \\ K_\epsilon = P_0 + \epsilon K_1 + \epsilon^2 k_2 + \ldots + \epsilon^{N+1} P_{N+1} \end{cases} . \tag{3.6}$$

Inserting this in the l.h.s. of (3.11), a universal expansion in powers of ϵ yields the formal power series equality

$$\exp\left[\frac{W_\epsilon}{i\hbar}\right] H_\epsilon \exp\left[-\frac{W_\epsilon}{i\hbar}\right] =$$

$$H_0 + \epsilon\frac{i}{\hbar}([W_1, H_0] + H_1) + \frac{\epsilon^2}{2!}\left(\frac{i}{\hbar}[W_2, H_0] + \frac{i}{\hbar}[W_1, H_1] + \left(\frac{i}{\hbar}\right)^2 [W_1, [H_1, H_0]]\right)$$

$$+ \ldots + \frac{\epsilon^\ell}{\ell!}\left(\frac{i}{\hbar}[W_\ell, H_0] + \frac{i}{\hbar}[W_{\ell-1}, H_1]\right) +$$

$$\frac{\epsilon^\ell}{\ell!}\left(\left(\frac{i}{\hbar}\right)^2 [W_{\ell-1}, [W_1, H_0]] + \left(\frac{i}{\hbar}\right)^2 [W_1, [W_{\ell-1}, H_0]]\right) + P_\ell + \ldots + O(\epsilon^{N+1})$$

$$= H_0 + \epsilon K_1 + \epsilon^2 K_2 + \ldots \epsilon^\ell K_\ell + \ldots$$

where

$$P_\ell = \sum_{r=3}^{\ell-1} \left(\frac{i}{\hbar}\right)^r \sum_{j_1+j_2+\ldots j_r = \ell-1 : j_k \geq 1} [W_{j_1}, [W_{j_2}, .., [W_{j_\ell}, H_0], \ldots,] +$$

$$\sum_{r=2}^{\ell-1} \left(\frac{i}{\hbar}\right)^r \sum_{j_1+j_2+..j_r = \ell-2 : j_r \geq 1} [W_{j_1}, [W_{j_2}, .., [W_{j_{\ell-1}}, H_1], ..,]$$

for $\ell \geq 2$, and $P_1 = H_1$. The unbounded operators on both sides can be considered acting on $D(H_0)$ as we shall immediately verify. Equating the coefficients of the same powers of ϵ, and remarking that P_ℓ depends only on $W_1, \dots, W_{\ell-1}$ we obtain the quantum homological equations analogous to (2.8)

$$\frac{i}{\hbar}[H_0, W_\ell] + P_\ell(W_1, \dots, W_{\ell-1}) = K_\ell, \quad \ell \geq 1, \quad W_0 = 0,\, P_1 = H_1 \tag{3.7}$$

which by Assumption (3) can be recursively solved on the eigenvector basis $|n_j\rangle$ of H_0. It is indeed easily verified that the condition $P_1^k|n_j\rangle = H_1^k|n_j\rangle \subset D(H_0)\,\forall n,\, \forall k$ implies that the operators W_ℓ and P_ℓ make sense on the linear hull of the eigenvectors $|n_j\rangle : j = 1, \dots, M(n); n = 0, 1, \dots$ for all $\ell \geq 1$. Therefore

$$\frac{i}{\hbar}[E_0(n) - E_0(m)]\langle m_i|W_\ell|n_j\rangle + \langle m_i|P_\ell|n_j\rangle = \langle m_i|K_\ell|n_j\rangle \tag{3.8}$$

This implies

$$\begin{cases} \langle m_i|K_\ell|n_j\rangle = \langle m_i|P_\ell|n_j\rangle & \forall (m,n) \text{ such that } E_0(m) = E_0(n) \\ \langle m_i|K_\ell|n_j\rangle = 0 & \text{otherwise} \end{cases} \tag{3.9}$$

while

$$\begin{cases} \langle m_i|W_\ell|n_j\rangle = -i\hbar \dfrac{\langle m_i|P_\ell|n_j\rangle}{E_0(m) - E_0(n)} & \forall (m,n) \text{ such that } E_0(m) \neq E_0(n) \\ \langle m_i|K_\ell|n_j\rangle = Q(m,n) & \text{otherwise} \end{cases} \tag{3.10}$$

where $Q(m,n)$ is an arbitrary $M \times M$ matrix. We take from now on $Q = 0$ in agreement with the convention of setting equal to zero the (θ_2-dependent) integration constant for the classical generating function w_ℓ. It remains to be proved that the operators W_ℓ defined through the matrix elements given recursively by (3.9) and (3.10), which are symmetric by construction, are actually self-adjoint and therefore the intertwining operators $U_{\epsilon,N}$ unitary. To see this, denote $\dot{H}_1$ the restriction of H_1 to the linear hull of the eigenvectors of H_0. By Assumption (ii) $\dot{H}_1$ is essentially self-adjoint or, equivalently, the minimal closed operator in $\mathcal{H}$ generated on the eigenvector basis by the infinite matrix $\langle m_i|H_1|n_j\rangle \;: i = 1, \dots, M(m); j = 1, \dots, M(n); m_i, n_j = 0, 1, \dots$ is self-adjoint. Likewise, denote W_1 the minimal closed operator in $\mathcal{H}$ generated by the action of the infinite matrix $\langle m_i|W_1|n_j\rangle$ of (3.10), and by $\dot{W}_1$ its restriction to the linear hull of $|n_j\rangle$. Let $\{x_{k_j}\} \in \ell^2 : k = 0, \dots;\; j = 1, \dots, M(k)$ be such that

$$\sum_{m,k=0}^{\infty} \sum_{i,j=1}^{M(m),M(k)} |\langle m_i|H_1|k_j\rangle x_{k_j}|^2 < +\infty$$

Since by (3.10) $|\langle m_i|W_1|n_j\rangle| \le |\langle m_i|H_1|n_j\rangle$, where the equality sign might hold if and only if $E_0(m)=E_0(n)$, there is $a<1$ independent of $\{x_{k_j}\}$ such that

$$\sum_{m,k=0}^{\infty}\sum_{i,j=1}^{M(m),M(k)} |\langle m_i|W_1|k_j\rangle x_{k_j}|^2 < a \sum_{m,k=0}^{\infty}\sum_{i,j=1}^{M(m),M(k)} |\langle m_i|H_1|k_j\rangle x_{k_j}|^2$$

This implies $D(\dot{H}_1)\subset D(\dot{W}_1)$, and

$$\|\dot{W}_1\psi\| \le a\,\|\dot{H}_1\psi\| \qquad \forall\,\psi\in D(H_1)$$

Since $a<1$, by the Rellich-Kato theorem $\dot{W}_1$ is essentially self-adjoint and hence W_1 is self-adjoint. Since the argument can be repeated without change for any $\ell>1$, this concludes the proof of Proposition 3.1.

Assumption B of Theorem 1.1 in turn ensures that the operators $K_\ell : \ell=1,\dots$ admit a holomorphic semiclassical expansion in the sense of Definition 1.1 in the case under examination, namely $H_0=\mathcal{P}_0$, $H_1=V$. Since $P_1\equiv V$ admits a holomorphic semiclassical expansion by assumption, by the recurrence defined by the homological equations (3.7), and by (3.9), it is indeed enough to prove that $P_\ell : \ell=1,\dots$ admits a holomorphic semiclassical expansion; in turn, this is obviously implied by the following result

Proposition 3.2 *Let the operators V_1 and V_2 in $L^2(\mathbb{R}^2)$ admit a holomorphic semiclassical expansion in the sense of Definition 1.1. Then the operators V_1V_2 and V_2V_1 enjoy the same property.*

Proof. Using the abbreviation $\sum_{k=0}^{\infty} V_k(z,\overline{z}')\hbar^k=\tilde{V}(z,\overline{z},\hbar)$ in (1.9), by the completeness of the coherent states (1.8) we have

$$\begin{aligned}\langle\psi_z,V_1V_2\psi_{z'}\rangle &= \int_{\mathbb{R}^2}\langle\psi_z,V_1\psi_{z''}\rangle\langle\psi_{z''},V_2\psi_{z'}\rangle\,dz''d\overline{z}''\\ &= \int_{\mathbb{R}^2}\langle\psi_z,\psi_{z''}\rangle\langle\psi_{z''},\psi_{z'}\rangle\tilde{V}_1(z,\overline{z}'',\hbar)\tilde{V}_2(z'',\overline{z}',\hbar)\,dz''d\overline{z}''\end{aligned}$$

By (1.10) we can write

$$\langle\psi_z,V_1V_2\psi_{z'}\rangle=\int_{\mathbb{R}^2}e^{-\frac{Q(z,z',z'')}{\hbar}}\tilde{V}_1(z,\overline{z}'',\hbar)\tilde{V}_2(z'',\overline{z}',\hbar)\,dz''d\overline{z}'' \tag{3.11}$$

where

$$Q(z,z',z'')=-\frac{1}{2}\langle z,\overline{z}\rangle-\frac{1}{2}\langle z'',\overline{z}''\rangle+\langle\overline{z},z''\rangle-\frac{1}{2}\langle z'',\overline{z}''\rangle-\frac{1}{2}\langle z',\overline{z}'\rangle+\langle\overline{z}'',z'\rangle \tag{3.12}$$

We compute the integral (3.11) by the stationary phase expansion because it can obviously regarded as an oscillating integral over $\mathbb{R}^{2s}$. The phase $Q(z,z',z'')$ is stationary for $z''=z'$, $\overline{z}''=\overline{z}$. Hence $Q(z,z',z'')|_{z''=z',\overline{z}''=\overline{z}} = -\frac{1}{2}\langle z,\overline{z}\rangle - \frac{1}{2}\langle z',\overline{z}'\rangle + \langle \overline{z}, z'\rangle$ and, once more by (1.10), this yields

$$\langle \psi_z, V_1V_2\psi_{z'}\rangle \sim \langle \psi_z,\psi_{z'}\rangle \sum_{k=0}^{\infty}(V_1V_2)_k(z,\overline{z}')\hbar^k$$

where the coefficients $(V_1V_2)_k(z,\overline{z}')$ can be explicitly written by a straightforward application of the complete stationary phase expansion (see e.g.[Hö], §7.7) to the integrand of (3.12). This proves the Proposition. The analog of the elimination of the slow angle is the diagonalization of the blocks. This yields the standard result of quantum degenerate Rayleigh-Schrödinger perturbation theory, namely:

Proposition 3.3 *Let $E_0(n,\hbar)$ be an eigenvalue of H_0 of multiplicity M. Then for ϵ sufficiently small there exist at most $M(p,q;n)$ distinct formal power series in ϵ of initial point $E_0(n,\hbar)$ of the type*

$$E_0(n,\hbar) + \sum_{\ell=1}^{\infty} E^k_\ell(n,\hbar)\epsilon^\ell, \quad k=1,\dots,M \tag{3.13}$$

the rank-M operator $K_\ell|_{\mathcal{G}_{E_0(n)}}$, namely of the matrix $\langle n_r|P_\ell|n_s\rangle : r,s=1,\dots,M$ where $n_r : r=1,\dots,M$ is any orthonormal basis in $\mathcal{G}_{E_0(n)}$.

Remarks.

(i) If all the eigenvalues $E^k_\ell(n,\hbar) : k=1,\dots,M$ turn out to be simple, we denote $\varphi^k_\ell : k=1,\dots,M$ the corresponding eigenvectors

(ii) In what follows we will need Proposition 3.1 for the particular case of the operator family $\mathcal{P}_\epsilon$ in $L^2(\mathbb{R}^2)$ or for its unitary image $T_\epsilon = \mathcal{F}_2\mathcal{P}_\epsilon\mathcal{F}_2$ in the Bargmann space defined below (see (4.3, 4.4) and Proposition A.1). By the abuse of notation followed throughout this paper we will indifferently denote by the same symbols K_ℓ and P_ℓ the (unitarily equivalent) operators generated by the perturbation recurrence in both cases, i.e. whether the initial conditions in (3.7) are $H_0 = \mathcal{P}_0, H_1 = P_1 = V$ or $H_0 = T_0, H_1 = P_1 = T_1$.

At the classical limit $n\to\infty, \hbar\to 0, n\hbar = A$ we have however to deal with blocks of dimension increasing linearly with $E_0(n,\hbar)$. Therefore an understanding of $K_\ell|_{\mathcal{G}_{E_0(n)}}$ in term of a classical analogue is needed. Now the quantum homological equations (3.7) are formally the quantization of the corresponding classical ones (2.8). This will imply that the operators K_ℓ and P_ℓ, once written in the Bargmann representation, admit as principal symbols (see e.g.[BS]) the classical functions k_ℓ and p_ℓ, written in the corresponding Bargmann canonical variables. The relation between the diagonalization of $K_\ell|_{\mathcal{G}_{E_0(n)}}$ and the integration of its classical counterpart $k_\ell(J_2, \theta_2, \cdot)$ is the heart of the relation between classical and quantum perturbation theory briefly summarized in the Introduction and will be obtained in the next two Sections.

4 The space $\mathcal{G}_{E_0(n)}$

In this Section the finite-dimensional Hilbert space $\mathcal{G}_{E_0(n)}$ is realized under the Bargmann form of a Hilbert space of holomorphic functions on a phase space isomorphic to a compact Kähler manifold. Moreover it turns out that the underlying reduced phase space (the phase space parametrized by the action-angle variables (A_2, ϕ_2)), the quotient of the energy surface $h_0 = E_0$ by the completely periodic unperturbed flow, can be identified with the sphere S^2 if $p = -q = 1$. This implies the identity between $\mathcal{G}_{E_0(n)}$ and the Hilbert space associated to S^2 by geometric quantization ([So, Ka, Ra] and also [GS2] for a general construction of the Hilbert spaces corresponding to the above quotient manifolds) and their approximate identity (up to terms of order $\hbar$) in the general case.

Referring the reader to the Appendix for more detail on operators in Bargmann space, we just recall here that the Bargmann space $\mathcal{F}_2$ is defined as the set of all entire holomorphic functions $f(z_1, z_2) : \mathbb{C}^2 \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{F}_2} \equiv \int_{\mathbb{C}^2} |f(z_1, z_2)|^2 e^{-\frac{1}{\hbar}\left[|z_1|^2+|z_2|^2\right]}\, dz_1 dz_2 d\overline{z}_1 d\overline{z}_2 < +\infty \tag{4.1}$$

and that the operators $\mathcal{P}_0$ and $\mathcal{P}_\epsilon$ are unitarily equivalent, when V is a polynomial, to the operators T_0, $T_\epsilon = T_0 + \epsilon T_1$ respectively generated by the maximal action in $\mathcal{F}_2$ of the differential expressions

$$T_0 f = \hbar \langle \omega z, \nabla_z \rangle\, f$$

and of

$$T_1 f = V\left(\frac{z_1 + \hbar\partial_{z_1}}{\sqrt{2}\omega_1}, \frac{z_2 + \hbar\partial_{z_2}}{\sqrt{2}\omega_2}\right) \tag{4.2}$$

representation the subspace $\mathcal{G}_{E_0(n)} \subset \mathcal{F}_2$ is obviously spanned by the set of vectors

$$\begin{cases} g_{n_1,n_2}(z_1, z_2) = \dfrac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}} \\ (n_1, n_2) \text{ such that } n_1 - \dfrac{p}{q} n_2 = \dfrac{E_0}{\hbar\omega_1} \end{cases} . \tag{4.3}$$

Let us begin by introducing the change of variables $u = U(z)$ on $\mathbb{C}^2$ defined as follows:

$$\begin{cases} u_1 = z_1^p z_2^{-q} \equiv U_1(z_1, z_2) \\ u_2 = z_1^p z_2^q \equiv U_2(z_1, z_2) \end{cases}. \tag{4.4}$$

which can be inverted as

$$\begin{cases} z_1 = (u_1 u_2)^{\frac{1}{2p}} \equiv U_1^{-1}(u_1, u_2) \\ z_2 = \left(\dfrac{u_1}{u_2}\right)^{-\frac{1}{2q}} \equiv U_2^{-1}(u_1, u_2) \end{cases}. \tag{4.5}$$

The Jacobian is

$$J(u) = \frac{\partial z}{\partial u} = \frac{1}{2pq}|u_1|^{\frac{1}{2p}-\frac{1}{2q}-1}|u_2|^{\frac{1}{2p}+\frac{1}{2q}-1} \tag{4.6}$$

Hence (standard abuse of notation)

$$g_{n_1,n_2}(U^{-1}(u)) \equiv \tilde{g}_{n_1,n_2}(u) = \frac{u_1^{\frac{n_1}{2p}-\frac{n_2}{2q}} u_2^{\frac{n_1}{2p}+\frac{n_2}{2q}}}{\sqrt{n_1! n_2!}} \tag{4.7}$$

Denoting now

$$\begin{cases} \Omega_{E_0} = \left\{ (n_1, n_2) \text{ such that } n_1 - \dfrac{p}{q} n_2 = \dfrac{E_0}{\hbar\omega_1} \right\} \\ \alpha = \displaystyle\min_{n_1, n_2 \in \Omega_{E_0}} \left\{ \frac{n_1}{2p} + \frac{n_2}{2q} \right\}, \qquad M = \#\Omega_{E_0} \end{cases}. \tag{4.8}$$

we have

Lemma 4.1 *Let Ω_{E_0} and α be as above. Then*

(i)

$$\left\{ \frac{n_1}{2p} + \frac{n_2}{2q} - \alpha | (n_1, n_2) \in \Omega_{E_0} \right\} = \{0, 1, \ldots, M\} \tag{4.9}$$

(ii) There is λ independent of $n = (n_1, n_2)$ such that

$$\frac{n_1}{2p} - \frac{n_2}{2q} = \frac{\lambda}{\hbar} \in \mathbb{Z} \tag{4.10}$$

Proof. Let $(n, m) \in \Omega_{E_0} \times \Omega_{E_0}$. Then $\langle \omega, (n - m) \rangle = 0$ entails $n - m = (rp, rq)$ for some $r \in \mathbb{Z}$, whence

$$\frac{1}{2}\left[\left(\frac{n_1}{p} + \frac{n_2}{q} \right) - \left(\frac{m_1}{p} + \frac{m_2}{q} \right) \right] = r$$

and this proves (i). To see (ii) it is enough to remark that $\dfrac{n_1}{2p} - \dfrac{n_2}{2q} = \dfrac{E_0}{2\hbar\omega_1 pq^2}$ by definition whence the assertion with $\lambda = \dfrac{E_0}{2\omega_1 pq^2}$. This proves Lemma (4.1).

Corollary 4.1 *Let*

$$\gamma_{(n)}(u_2) = \frac{g_n(U^{-1}(u))}{u_1^{\frac{\lambda}{\hbar}} u_2^{\alpha}} \tag{4.11}$$

Then $\{\gamma_{(n)}(x) : n \in \Omega_{E_0}\} = \{P_M(x)\}$, *where* $\{P_M(u_2)\}$ *is the set of polynomials of degree not exceeding* M.

Remark Since by its very definition (4.4) the variable u_2 describes a $p-q$ covering of $\mathbb{C}$, $\gamma_{(n)}(u_2)$ a priori cannot be considered as an analytic function on $\mathbb{C}$. Since however it is a polynomial in u_2 it has a natural extension to a holomorphic function on $\mathbb{C}$. We will eventually see that its natural domain of holomorphy is the symplectic manifold defined by the quotient of the constant energy E_0 surface with respect to the unperturbed harmonic flow.

The above results yield the Bargmann-type characterization of $\mathcal{G}_{E_0(n)}$:

Proposition 4.1 *Let* $\mathcal{G}_{E_0(n)}$ *be the eigenspace corresponding to the eigenvalue* $E_0(n)$ *of* T_0, *spanned by the set of vectors (4.5) reexpressed on the* (u_1, u_2) *variables through Lemma 4.1 and Corollary 4.1. Then*

(i) $\mathcal{G}_{E_0(n)}$ *is the set of all entire holomorphic functions* $\gamma(u_2)$ *such that*

$$\int_{\mathbb{C}} |\gamma(u_2)|^2 \nu(u_2, \overline{u}_2)\, du_2 d\overline{u}_2 < +\infty \tag{4.12}$$

Here:

$$\nu(u_2, \overline{u}_2) = |u_2|^{\frac{1}{2p}+\frac{1}{2q}-1+2\alpha} F_{p,q,\lambda}(|u_2|) \tag{4.13}$$

where

$$F_{p,q,\lambda}(|u_2|) = \frac{\pi}{pq}\int_0^\infty x^{\frac{1}{2p}-\frac{1}{2q}} e^{-\frac{1}{\hbar}\left(x^{\frac{1}{p}}|u_2|^{\frac{1}{p}} + x^{-\frac{1}{q}}|u_2|^{\frac{1}{q}} - 2\lambda \ln x\right)}\, dx \tag{4.14}$$

(ii) The (orthogonal) projection operator on the subspace $\mathcal{G}_{E_0(n)}$ *spanned by the vectors (4.3) admits the representation*

$$(P_{E_0} f)(z_1, z_2) = \frac{1}{T}\int_0^T f(z_1 e^{i\omega_1 t}, z_2 e^{i\omega_2 t}) e^{-i\frac{E_0 t}{\hbar}}\, dt \tag{4.15}$$

where $T = 2\pi\dfrac{p}{\omega_1} = -2\pi\dfrac{q}{\omega_2}$ *is the common period of the classical and quantum flows.*

(iii) Let

$$\rho(u_2, \overline{u}_2') = (u_2\overline{u}_2')^{\frac{1}{2p}+\frac{1}{2q}-1} \int_\Gamma \theta^{\frac{\lambda}{\hbar}-1} e^{\frac{1}{\hbar}\left[(\theta u_2\overline{u}_2')^{\frac{1}{2p}} + \left(\frac{\theta}{u_2\overline{u}_2'}\right)^{\frac{1}{2q}}\right]}\, d\theta \tag{4.16}$$

where Γ is the unit circumference in $\mathbb{C}$. Then $\rho(u_2, \overline{u}'_2)$ is holomorphic in $\mathbb{C}^2$ and is a reproducing kernel in $\mathcal{G}_{E_0(n)}$; namely, $\forall \gamma(u_2) \in \mathcal{G}_{E_0(n)}$,

$$\gamma(u_2) = \int_{\mathbb{R}^2} \rho(u_2, \overline{u}'_2)\nu(u'_2, \overline{u}'_2)\gamma(u'_2)\, du'_2\, d\overline{u}'_2 \tag{4.17}$$

(iv) The function $x \mapsto f(x)$ implicitly defined as

$$\frac{1}{2p}(xf(x))^{\frac{1}{2p}} - \frac{1}{2q}\left(\frac{x}{f(x)}\right)^{\frac{1}{2q}} = \lambda \tag{4.18}$$

is analytic in a neighborhood of $\mathbb{R}$. Moreover, setting

$$\varphi_1(y) = [yf(y)]^{\frac{1}{2p}} \qquad \varphi_2(y) = \left[\frac{y}{f(y)}\right]^{\frac{1}{2q}} \tag{4.19}$$

the kernels $\rho(u_2, \overline{u}'_2)$ and $\nu(u_2, \overline{u}_2)$ admit the following asymptotic expansions

$$\rho(u_2, \overline{u}'_2) = R(u_2, \overline{u}'_2) \cdot \exp\frac{1}{\hbar}\left[\varphi_1(u_2\overline{u}'_2) + \varphi_2(u_2\overline{u}'_2)\right] \cdot \left[1 + \sum_{\ell=1}^{\infty} d_\ell(u_2\overline{u}'_2)\hbar^\ell\right] \tag{4.20}$$

where

$$R(u_2, \overline{u}'_2) = \frac{2pq\sqrt{2\pi f(u_2\overline{u}'_2)}(\varphi_1(u_2, \overline{u}'_2))^{\frac{p}{q}-2p\lambda+2p}}{\sqrt{4q^2(1-2p)\varphi_1(u_2\overline{u}'_2) + 4p^2(1+2q)\varphi_2(u_2\overline{u}'_2) + \lambda}} \tag{4.21}$$

and

$$\nu(u_2, \overline{u}_2) = N(u_2, \overline{u}'_2) \cdot \exp -\frac{1}{\hbar}\left[\varphi_1(|u_2|) + \varphi_2(|u_2|)\right] \cdot \left[1 + \sum_{\ell=1}^{\infty} c_\ell(u_2\overline{u}'_2)\hbar^\ell\right] \tag{4.22}$$

where

$$N(u_2, \overline{u}'_2) = \frac{\pi\sqrt{2\pi}|u_2|^{\frac{1}{2p}+\frac{1}{2q}-1+2\alpha}(f(|u_2|))^{\frac{1}{2p}-\frac{1}{2q}-\lambda+1}}{\sqrt{4q^2(1-2p)\varphi_1(|u_2|) + 4p^2(1+2q)\varphi_2(|u_2|) + \lambda}} \tag{4.23}$$

Proof. By (4.4), (4.5), (4.6), (4.7), (4.11) the requirement $g_n(U^{-1}(z)) \in \mathcal{F}_2$ yields, after performing the change of variables $z \leftrightarrow u$:

$$\frac{1}{2pq}\int_{\mathbb{R}^4} |\gamma_{(n)}(u_2)u_1^{\frac{\lambda}{\hbar}}u_2^{\alpha}|^2\, G_{p,q}(u_1, u_2)\, du_1 d\overline{u}_1 du_2 d\overline{u}_2 < +\infty \tag{4.24}$$

where

$$G_{p,q}(u_1, u_2) = e^{-\frac{1}{\hbar}\left[|u_1u_2|^{\frac{1}{p}} + \left|\frac{|u_2|}{|u_1|}\right|^{\frac{1}{q}}\right]}|u_1|^{\frac{1}{2p}-\frac{1}{2q}-1}|u_2|^{\frac{1}{2p}+\frac{1}{2q}-1} \tag{4.25}$$

Setting $x = |u_1|$ and integrating over $du_1 d\overline{u}_1$ in polar coordinates this immediately yields, on account of (4.8), Assertion (i) with $\nu(u_2, \overline{u}_2)$ given by (4.13,4.14). The asymptotic behaviour of $\nu(u_2, \overline{u}_2)$, useful for later reference, is also immediately verified

$$\nu(u_2, \overline{u}_2) \sim |u_2|^{2\alpha + \frac{1}{q} - \frac{2\lambda}{\hbar}}, \qquad |u_2| \to \infty \tag{4.26}$$

In order to prove Assertion (ii) we expand the vector $f \in \mathcal{F}_2$ on the orthonormal basis $\dfrac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}} : n_1, n_2 = 0, 1, \ldots$:

$$f(z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} f_{n_1, n_2} \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}}$$

and interchange summation with integration (the justification is obvious) in (4.15); since $\dfrac{E_0}{\hbar} = \omega_1(n_1 - \dfrac{p}{q} n_2)$ we see that the only nonzero terms are those for which $\omega_1(n_1 - \dfrac{p}{q} n_2) = \dfrac{E_0}{\hbar\omega_1}$, namely, $(P_{E_0} f)(z_1, z_2) \in \mathcal{G}_{E_0(n)}$. The symmetry property $P = P^*$ is obvious.

To see (iii), denote

$$\phi_t(z) = (z_1 e^{i\omega_1 t}, z_2 e^{i\omega_2 t}) \tag{4.27}$$

the classical unperturbed flow of initial data (z_1, z_2) (see Appendix). Therefore, if

$$\rho_z^B(z') = e^{-\frac{|z|^2}{2\hbar} - \frac{|z'|^2}{2\hbar}} \sum_{n_1, n_2 = 0}^{\infty} \frac{1}{n_1! n_2!} \left(\frac{z_1 \overline{z_1}'}{\hbar} \right)^{n_1} \left(\frac{z_2 \overline{z_2}'}{\hbar} \right)^{n_2} \tag{4.28}$$

is the reproducing kernel in $\mathcal{F}_2$ (see (A.14)), it is not difficult to verify, performing the change of variables (4.5) and applying (4.15) that the reproducing kernel $\rho = P_{E_0} \rho_z^B$ in $\mathcal{G}_{E_0(n)}$ has the following expression

$$\rho(u_2, \overline{u}_2') = \frac{\overline{u}_1^{\frac{-\lambda}{\hbar}} \overline{u}_2^{-\alpha}}{T^2} \int_0^T \int_0^T \rho_z^B(\phi_t(z), \overline{\phi_{t'}(z')}) J(u) J(u') e^{-\frac{i}{\hbar} E_0 (t - t')} \, dt dt' \tag{4.29}$$

evaluated at $z = U(u)$ and $z' = U(u')$. A short computation based on this formula and (A.15) proves (4.16) and hence Assertion (iii). Finally (iv) follows by a straightforward application of the stationary phase expansion (see e.g.[Hö]) to (4.14) and (4.16). This concludes the proof of Proposition 4.1.

We turn now to the realization of $\mathcal{G}_{E_0(n)}$ as the quantization of the constant energy surface for the unperturbed flow. Let us begin by remarking that, in the coordinates u_1, u_2:

(a) The unperturbed Hamiltonian reads

$$h_0(u,\overline{u}) = \omega_1|u_1u_2|^{\frac{1}{p}} + \omega_2|\frac{u_2}{u_1}|^{\frac{1}{q}} \tag{4.30}$$

(b) Let $x \to f(x)$ solve the algebraic equation (4.18), where, we recall, $\lambda = \dfrac{E_0}{p\omega_1}$. Then on the constant energy surface $|u_1|^2 = f(|u_2|^2)$.

(c) The classical flow (4.27) is

$$\phi_t(u) = \begin{cases} u_1(t) = e^{i(p\omega_1 - q\omega_2)t}u_1(0) \\ u_2(t) = u_2(0) \end{cases} \tag{4.31}$$

Denote $\tilde{\Sigma}$ the constant energy surface

$$\tilde{\Sigma} = \{(u_1,u_2)|\, h_0(u_1,u_2) = E_0\} \tag{4.32}$$

Identifying on $\tilde{\Sigma}$ the points connected by the classical flow, i.e. taking the quotient with respect to (4.31) we obtain the manifold

$$\Sigma = \tilde{\Sigma}/\phi_t(u) \tag{4.33}$$

which is parametrized only by u_2 because $|u_1|$ is determined by (4.30) and the dependence on $\arg u_1$ is eliminated by the quotient with respect to the classical flow (4.31).

We now discuss the relation between $\mathcal{G}_{E_0(n)}$ and the Hilbert space associated to Σ by geometric quantization (see e.g.[GS1, So, Ka, Ra]).

Proposition 4.2 *Let $\tilde{\Sigma}$ and Σ be as above. Then Σ is the two-dimensional Kähler manifold defined by $\mathbb{C}$ equipped with the symplectic form*

$$\omega_{E_0} = g_{E_0}(u_2,\overline{u}_2)\, du_2 \wedge d\overline{u}_2 \tag{4.34}$$

where

$$g_{E_0}(u_2,\overline{u}_2) = \frac{2}{q^2}(u_2\overline{u}_2)^{\frac{1}{2q}-1} f(u_2\overline{u}_2)^{-\frac{1}{2q}-1}\left[f(u_2\overline{u}_2) - f'(u_2\overline{u}_2)u_2\overline{u}_2\right] \tag{4.35}$$

f being given by (4.18).

Remark If $\omega_1 = \omega_2 = 1$, namely $p = -q = 1$, (4.18) becomes

$$(xf(x))^{\frac{1}{2}} + \left(\frac{x}{f(x)}\right)^{-\frac{1}{2}} = E_0 \tag{4.36}$$

and hence

$$\omega_{E_0} = 4\sqrt{E_0}\frac{du_2 \wedge d\overline{u}_2}{(1+u_2\overline{u}_2)^2} \tag{4.37}$$

which is the Kähler symplectic structure of the two-dimensional sphere S^2 of radius $\sqrt[6]{3E_0}$, identified with $\mathbb{C}$ after stereographic projection.

Proof. We have already pointed out that Σ can be parametrized by u_2, so let us prove that u_2, when restricted to Σ, actually describes $\mathbb{C}$. Clearly $|u_2| = |z_1|^p|z_2|^q$ describes $\mathbb{R}^+$. Now $\arg u_2 = p\chi_1 + q\chi_2$, and the classical flow ϕ_t over T^2 with coordinates (χ_1, χ_2) is linear with slope $-\frac{p}{q}$. It is immediately verified that

$$T^2/\phi_t = \{(\chi_1, \chi_2)\,|\, 0 \le \chi_1 \le \frac{2\pi}{p},\ \chi_2 = 0\}, \quad \text{if } p \ge -q \tag{4.38}$$

$$T^2/\phi_t = \{(\chi_1, \chi_2)\,|\, \chi_1 = 0,\ 0 \le \chi_2 = -\frac{2\pi}{q}\}, \quad \text{if } p \le -q \tag{4.39}$$

In both cases we have indeed $0 \le p\chi_1 + q\chi_2 \le 2\pi$.

The computation of g_{E_0} is straightforward: one has first to reexpress $dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2$ in terms of $(u_1, u_2; \overline{u}_1, \overline{u}_2)$ and their differentials through (4.5) and then notice that

$$u_1\, d\overline{u}_1 + \overline{u}_1\, du_1 = f'(u_2\overline{u}_2)(u_2\, d\overline{u}_2 + \overline{u}_2\, du_2) \tag{4.40}$$

since $u_1\overline{u}_1 = f(u_2\overline{u}_2)$ on Σ. This proves the Proposition.

Remarks.

(i) Denote $\Xi \equiv U^{-1}(\Sigma)$ the inverse image of Σ under U. Then the mapping U is canonical when considered between $(\Xi; dz \wedge d\overline{z})$ and $(\Sigma; \omega_{E_0})$.

(ii) The geometric quantization procedure (see e.g.[So],[Ko],[Ra]) associates to the manifold Σ the Hilbert space $\mathcal{H}_{E_0}$ defined by all entire holomorphic functions $f(u_2)$ on Σ such that

$$\int_{\mathbb{C}} |f(u_2)|^2 \mu(u_2, \overline{u}_2) g_{E_0}(u_2, \overline{u}_2)\, du_2\, d\overline{u}_2 < +\infty \tag{4.41}$$

where now $g_{E_0}(u_2, \overline{u}_2)\, du_2\, d\overline{u}_2$ is the volume form on Σ and the weight $\mu(u_2, \overline{u}_2)$ is related to g_{E_0} by the following formula

$$\hbar\partial_u\partial_{\overline{u}} \ln \mu(u, \overline{u}) = g_{E_0}(u, \overline{u}) \tag{4.42}$$

order terms in $\hbar$ and therefore up to terms of the same order $\mathcal{G}_{E_0(n)}$ and $\mathcal{H}_{E_0}$ can be identified. If moreover $p = -q = 1$ one actually has $\mathcal{G}_{E_0(n)} \equiv \mathcal{H}_{E_0}$ because, by (4.37), $g_{E_0} = 4\sqrt{E_0}(1 + u_2\overline{u}_2)^{-2}$ and, by (4.13),(4.14), one immediately obtains

$$\ln \nu(u_2, \overline{u}_2) = \frac{E_0}{2\hbar} \ln(1 + u_2\overline{u}_2) + \text{ const.}$$

5 Semiclassical analysis on the space $\mathcal{G}_{E_0(n)}$

In this final Section we deal with the relation between the diagonalization of the operator $K_{\epsilon,N} = \sum_{\ell=0}^{N} K_\ell$ in $\mathcal{G}_{E_0(n)}$ and the classical flow generated by $k_{\epsilon,N} = \sum_{\ell=0}^{N} k_\ell \epsilon^\ell$. This represents the most intriguing point because $\mathcal{G}_{E_0(n)}$ is a finite-dimensional Hilbert space and there is no way of realizing $K_{\epsilon,N}$ as a pseudodifferential operator on some (unitarily equivalent to) L^2 space. Nevertheless the Kähler structure of Σ allows for an appropriate semiclassical analysis[BHSS],[BMS],[KL] in terms of Töplitz operators and Szegö projections[BG]. In this case however, owing to the Bargmann structure of the quantization inherited by the projection into $\mathcal{G}_{E_0(n)}$, we can immediately write a symbolic calculus for the projection of the quantum perturbation on $\mathcal{G}_{E_0(n)}$, whose principal symbol coincides with the resonant part of the potential, written in suitable canonical variables. This calculus makes possible to solve the spectral problem in $\mathcal{G}_{E_0(n)}$, and hence to obtain the desired quantization formula, through a direct application of the quasi-modes technique as recently developed in a Bargmann setting [PU], following earlier work on WKB methods in the Bargmann representation[Vo].

More specifically, the structure of $\mathcal{G}_{E_0(n)}$ analyzed in the former Section motivates the introduction of the following setting:

(1) Let $\mathcal{M}$ be a Kähler manifold of complex dimension one with symplectic form

$$\omega = g(v,\overline{v})\, dv \wedge d\overline{v} \tag{5.1}$$

where g is the Kähler metric of $\mathcal{M}$; furthermore, denote by $\alpha = G(v,\overline{v})\, dv - \overline{G(v,\overline{v})}\, d\overline{v}$ a purely imaginary potential $1-$ form of ω, namely

$$\frac{1}{2}\overline{[G(v,\overline{v})\, dv - \overline{G(v,\overline{v})}\, d\overline{v}]} = -\frac{1}{2}[G(v,\overline{v})\, dv - \overline{G(v,\overline{v})}\, d\overline{v}]$$

with $d\alpha = \omega$;

(2) Let $\mathcal{H}^\hbar$, $0 < \hbar < 1$ be the family of (Bargmann type) Hilbert spaces on $\mathcal{M}$ defined by all functions $f(v)$ holomorphic on $\mathcal{M}$ such that

$$\|f\|^2 \equiv \int_{\mathbb{R}^2} |f(v)|^2 \mu_\hbar(v,\overline{v})\, dv d\overline{v} < +\infty \tag{5.2}$$

where $\mu_\hbar(v,\overline{v}) :\ 0 < \hbar < 1$ is a given family of weight functions;

(3) Let $\mathcal{H}^\hbar$ admit a reproducing kernel, i.e. let there exist $\rho_\hbar(v,\overline{v})$ holomorphic on $\mathcal{M} \times \overline{\mathcal{M}}$ such that

$$\rho_\hbar^v(\cdot) \equiv \rho_\hbar(\cdot,\overline{v}) \in \mathcal{H}^\hbar \tag{5.3}$$

enjoying $\forall\, f \in \mathcal{H}^\hbar$ the following property

$$\int_{\mathbb{R}^2} f(v)\rho_\hbar(v',\overline{v})\mu_\hbar(v,\overline{v})\, dvd\overline{v} = f(v') \tag{5.4}$$

Remark. The manifold Σ and the Hilbert space $\mathcal{G}_{E_0(n)}$ enjoy the above properties of $\mathcal{M}$ and $\mathcal{H}^\hbar$ by Propositions 4.1 and 4.2. Remark moreover that we recover from (5.2) with $\nu(u_2,\overline{u}_2)$ in place of $\mu_\hbar(v,\overline{v})$ the fact that $\mathcal{G}_{E_0(n)}$ has dimension $M(n)$. As a matter of fact, the asymptotic behaviour (4.26) of $\nu(u_2,\overline{u}_2)$ and the holomorphy condition imply, given Lemma 4.1, that $f(u_2)$ has to be a polynomial of degree at most $M-1$.

Now, in the spirit of the usual definition of operators of semiclassical type recalled in Appendix, we have

Definition 5.1 *Let $S_\hbar$ be any bounded operator on $\mathcal{H}^\hbar$. Then*

(i) The (Wick) symbol $\sigma_\hbar(v,\overline{v})$ of $S_\hbar$ is the holomorphic function $\sigma_\hbar(v,\overline{v})$ on $\mathcal{M}\times\overline{\mathcal{M}}$ given by

$$\sigma_\hbar(v,\overline{v}) = \frac{\langle \rho_\hbar^v, S_\hbar \rho_\hbar^{\overline{v}}\rangle}{\langle \rho_\hbar^v, \rho_\hbar^{\overline{v}}\rangle} \tag{5.5}$$

(ii) $S_\hbar$ is of semiclassical type if there exist holomorphic functions $\sigma_\ell(v,\overline{v})$: $\ell = 0,1\ldots$ on $\mathcal{M}\times\overline{\mathcal{M}}$ such that its symbol admits an asymptotic expansion of the form

$$\sigma_\hbar(v,\overline{v}) \sim \sigma_0(v,\overline{v}) + \sum_{\ell\geq 1}\sigma_\ell(v,\overline{v})\hbar^\ell \tag{5.6}$$

Since $(v,\overline{v})$ are canonical coordinates on $\mathcal{M}$, the *principal symbol* $\sigma_0(v,\overline{v})$ defines a Hamiltonian flow, which is clearly periodic on any regular energy surface $\sigma_0 = E$. Moreover, it is equally clear that the Hamiltonian $\sigma_0(v,\overline{v})$ is holomorphically integrable away from separatrices: namely, if $(v,\overline{v}) \in \mathcal{V}$, where $\mathcal{V} = \{(v,\overline{v}) \in \mathcal{M} \mid \nabla\sigma_0(v,\overline{v}) \neq 0\}$ there is a holomorphic canonical transformation $\mathcal{C}$ into action-angle variables $(A,\phi) = \mathcal{C}(v,\overline{v}) \in \mathcal{I}\times\mathbb{T}^1$, where $\mathcal{I}$ is a finite disjoint union of open intervals in $\mathbb{R}_+$, such that $\sigma_0(\mathcal{C}^{-1}(A,\phi)) = E(A)$.

As in [PU], we can now associate to the pair $(\gamma(t),a(t))$ formed by any regular trajectory $\gamma(t)$ of period T on the regular energy surface $\sigma_0 = E$ and by any $T-$antiperiodic C^∞ function $a(t)$ the following vector in $\mathcal{H}^\hbar$

$$\psi_{\gamma,a} = \frac{1}{T}\int_0^T a(t)\xi(t)\rho_\hbar^{v(t)}[\mu_\hbar(v(t),\overline{v}(t)]^{\frac{1}{2}}\, dt \tag{5.7}$$

where

$$\xi(t) = \exp\frac{i}{\hbar}\int_0^t [G(v,\overline{v})\dot{v} - \overline{G(v,\overline{v})}\dot{\overline{v}}](\tau)\, d\tau \tag{5.8}$$

and take over to the present case the construction of the quasi-modes:

Proposition 5.1 *Assume:*

(i) The function $-\hbar\partial_v\partial_{\overline{v}}\ln\mu_\hbar(v,\overline{v})$ *admits an asymptotic expansion in powers of* $\hbar$ *with leading term* $g(v,\overline{v})$*; namely,* $\mu(v,\overline{v})$ *admits an asymptotic expansion of the form*

$$\mu_\hbar(v,\overline{v}) \sim e^{-\frac{1}{\hbar}\nu(v,\overline{v})}\left[1+\sum_{\ell\geq 1}\mu_\ell(v,\overline{v})\hbar^\ell\right] \tag{5.9}$$

with $\partial_v\partial_{\overline{v}}\nu(v,\overline{v}) = g(v,\overline{v})$.

(ii) $\nu(v,\overline{v})$ *can be continued as a holomorphic function* $\nu(v,v')$ *defined on a neighbourhood* $\Omega\subset\mathbb{C}^2$ *containing the diagonal* $\{(v,v')\in\mathbb{C}^2 : v'=\overline{v}\}$*, and* $\rho_\hbar(v',\overline{v})$ *admits in* Ω *an asymptotic expansion of the form*

$$\rho_\hbar(v',\overline{v}) \sim e^{\frac{1}{\hbar}\nu(v',\overline{v})}\left[1+\sum_{\ell\geq 1}\rho_\ell(v',\overline{v})\hbar^\ell\right] \tag{5.10}$$

(iii) $\partial_{v'}\nu(v',\overline{v})$ *is injective on* Ω.

Then we can find a sequence of numbers $\{c_k\}$ *and a sequence of smooth* $T-$ *antiperiodic functions* $a_k(t)$ *such that, setting for any* $N\in\mathbb{N}$

$$E_N(\hbar) = E + \sum_{k=1}^N c_k\hbar^k \tag{5.11}$$

and

$$a_N(t) = \sum_{k=0}^N a_k(t)\hbar^k \tag{5.12}$$

(1) There exists $C_N>0$ *such that, for* $\hbar$ *small of the form*

$$\hbar = \frac{\int_\gamma \Im G(v(t),\overline{v}(t))\dot{v}\,dt}{2\pi(p+\frac{1}{2})} = \frac{A}{p+\frac{1}{2}}, \qquad p=0,1,\ldots \tag{5.13}$$

one has

$$\begin{gathered}\|\psi_{\gamma,a_N}\|_{\mathcal{H}^\hbar} = 1+O(\hbar)\\ \|(S_\hbar - E_N)\psi_{\gamma,a_N}\|_{\mathcal{H}^\hbar} \leq C_N\hbar^{N+1}\end{gathered} \tag{5.14}$$

and there is $\lambda\in\sigma(S_\hbar)$ *such that*

$$|E_N-\lambda| \leq C_N\hbar^{N+1} \tag{5.15}$$

(2) If there exists a constant $\beta>0$ *such that the set* $\sigma(S_\hbar)\cap[E-\beta\hbar, E+\beta\hbar]$ *contains only simple eigenvalues at a distance* $\geq cst\times\hbar^{N+1}$ *of each other, and if* ψ_λ *is the eigenvector corresponding to any such eigenvalue* λ*, then there is* $D_N>0$ *such that*

$$\|\psi_{\gamma,a_N}-\psi_\lambda\|_{\mathcal{H}^\hbar} \leq D_N\hbar^{N+1} \tag{5.16}$$

(3) Let λ_p be any simple eigenvalue of $S_\hbar$, of eigenvector ψ_{λ_p}, such that $\lambda_p \to \lambda = E(A)$ as $\hbar \to 0$, $p \to \infty$, $(p+\frac{1}{2})\hbar = A$, and let $Q_\hbar$ be a continuous operator of semiclassical type on $\mathcal{H}^\hbar$, with symbol $q(z,\overline{z},\hbar)$ admitting asymptotic expansion denoted $q(z,\overline{z},\hbar) \sim q_0(v,\overline{v}) + \sum_{j=1}^{\infty} q^j(z,\overline{z})\hbar^j$. Then the scalar products $\langle \psi_{\lambda_m}, Q_\hbar \psi_{\lambda_n}\rangle$ admit the following asymptotic expansion in powers of $\hbar$

$$\langle \psi_{\lambda_m}, Q_\hbar \psi_{\lambda_n}\rangle \sim \sum_{j=0}^{\infty} \beta^j_{n-m}(A)\hbar^j \tag{5.17}$$

where the leading term has the following expression

$$\beta^0_{n-m}(A) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{i(m-n)\phi} q_0(\mathcal{C}^{-1}(A,\phi))\, d\phi \tag{5.18}$$

$\beta_{m-n}(A)$ and hence on A.

Remarks

(i) The assumptions of Proposition (5.1) are fulfilled by $\mathcal{G}_{E_0(n)}$ and its reproducing kernel. As a matter of fact:
Assumption (i) is fulfilled by the weight $\nu_\hbar(u_2,\overline{u}_2)$ defined by (4.13) because of the asymptotic expansions (4.20),(4.22);
Assumption (ii) follows from the analyticity of $f(x)$ defined by (4.18), which is in turn implied by the analytic implicit function theorem because

$$\begin{gathered}\partial_y\left[\frac{1}{2p}(xy)^{\frac{1}{2p}} - \frac{1}{2q}\left(\frac{x}{y}\right)^{\frac{1}{2q}}\right] = \\ \left(\frac{1}{2p}\right)^2 x^{\frac{1}{2p}} y^{\frac{1}{2p}-1} + \left(\frac{1}{2q}\right)^2 x^{\frac{1}{2q}} y^{\frac{1}{2q}-1} > 0\end{gathered} \tag{5.19}$$

Assumption (iii) is once more implied by the asymptotic expansion (4.22-23) at least for $\hbar$ suitably small.

(ii) The integral $A = \int_\gamma \Im G(v(t),\overline{v}(t))\dot{v}\, dt$ is obviously the action of the periodic trajectory $\gamma(t)$ and thus (5.13) has the meaning of a Bohr-Sommerfeld quantization condition. It will be seen shortly that in our case ($\mathcal{H}^\hbar = \mathcal{G}_{E_0(n)}$, $\mathcal{M} = \Sigma$) it involves at most $M(n)$ integers.

Proof. The proof of this Proposition can be taken over without change from [PU] §3, 4, up to three minor variants:

(a) The different expression in the phase appearing in $\mu_\hbar(v,\overline{v})$ and in $\rho_\hbar(v,\overline{v'})$;

(b) The different form of the "transport" equations due to higher order terms in the semiclassical expansion of $\sigma_\hbar(v,\overline{v})$ and of $\rho_\hbar(v,\overline{v})$;

(c) The possible finite dimensionality of $\mathcal{H}^\hbar$ or, equivalently, the possible compactness of the Kähler manifold.

The first variant will be dealt with upon verification that the critical points of these new phases are the same ones of [PU]; the second one by writing the new "transport" equations and showing that they can be still recursively solved, and the third by a straightforward adaptment of the argument of [PU]. To see all this, first remark that in the present case (see formula (29) of [PU]) the object to estimate obviously reads

$$([S_\hbar - E_N]\psi_{\gamma,a_N})(v') \equiv I(v') \tag{5.20}$$

Likewise, the the subsequent formulas (30) and (31) are immediately adapted to the present situation. By the expansions (5.6), (5.10) and (5.11) we get (remark that all truncations of the symbols to any finite order in $\hbar$ yield bounded operators because the action takes place on a compact set):

$$\begin{aligned} I(v') = &\left(\frac{\pi\hbar}{2T^2}\right)^{1/4} \int_0^T e^{-\frac{\hbar}{2}\partial_{\overline{v}(t)}\partial v'} \left[\sigma_\hbar\left((v'+\overline{v}(t))/\sqrt{2}, (v'-\overline{v}(t))/\sqrt{2}i\right) - E_N\right]\cdot \\ &\cdot a_N(t)\xi(t)\rho_\hbar(v',v(t))[\mu_\hbar((v(t),\overline{v}(t))]^{\frac{1}{2}}\, dt = \\ = &\frac{1}{\sqrt{2\pi\hbar T}} \exp -\frac{|\nu(v',\overline{v}')|^2}{2\hbar} \int_0^T [A(v',\overline{v}(t)) - E_N] B(v',\overline{v}(t)) a_N(t)\xi(t)\cdot \\ &\cdot \exp \frac{\nu(v',\overline{v}(t))}{\hbar} \exp -\frac{|\nu(v(t),\overline{v}(t))|^2}{2\hbar}\, dt \end{aligned} \tag{5.21}$$

of the kernel $A(v',\overline{v}(t))$, and all terms in the expansions (5.10) and (5.11) to that of $B(v',\overline{v}(t))$:

$$\begin{aligned} A(v',\overline{v}(t)) &= e^{-\frac{\hbar}{2}\partial_{\overline{v}(t)}\partial v'} \sigma_\hbar\left((v'+\overline{v}(t))/\sqrt{2}, (v'-\overline{v}(t))/\sqrt{2}i\right) \sim \sum_{k=0}^{\infty} A_k(\cdot)\hbar^k \\ A_k(v',\overline{v}(t)) &= \left(\frac{1}{2}\right)^k \sum_{m=0}^{k} \frac{1}{(k-m)!} \partial^{2(k-m)}_{\overline{v}(t)v'} \sigma_{k-m}(v',\overline{v}(t)) \end{aligned} \tag{5.22}$$

$$\begin{aligned} B(v',\overline{v}(t)) &\sim \sum_{k=0}^{\infty} B_k(v',\overline{v}(t))\hbar^k \\ B_k(\cdot) &= \sum_{m=0}^{k} \rho_m(\cdot)\mu_{k-m}(\cdot) \end{aligned} \tag{5.23}$$

We proceed now to the verification that the phase $\Phi(t)$ appearing in the integral (5.23), depending parametrically on v', has the same stationary points as in [PU]. The phase has indeed the following general form

$$\Phi(t) = i\int_0^t \Im g(v,\overline{v})\,dv + \nu(v',\overline{v}(t)) - \frac{1}{2}\nu(v(t),\overline{v}(t)) \tag{5.24}$$

so that we obtain the direct analog of [PU], formula (32):

$$\frac{\partial\Phi(t)}{\partial t} = \frac{1}{2}[g(v,\overline{v})\dot{v}(t) - \overline{g(v,\overline{v})\dot{v}(t)}] - \tag{5.25}$$

$$\frac{1}{2}[\partial_v\nu(v(t),\overline{v}(t))\dot{v} - \partial_{\overline{v}}\nu(v(t),\overline{v}(t))\dot{\overline{v}}] + \partial_{\overline{v}}\nu(v',\overline{v})\dot{\overline{v}}$$

However, by assumption, $\partial_v\nu(v,\overline{v}) = g(v,\overline{v})$ and hence

$$\frac{\partial\Phi(t)}{\partial t} = [\partial_{\overline{v}}\nu(v',\overline{v}(t)) - \partial_{\overline{v}}\nu(v(t),\overline{v}(t))]\dot{\overline{v}} \tag{5.26}$$

Since E is regular and $\dot{\overline{v}} \neq 0$, we have $\dfrac{\partial\Phi(t)}{\partial t} = 0$ iff

$$\partial_{\overline{v}}\nu(v',\overline{v}(t)) - \partial_{\overline{v}}\nu(v(t),\overline{v}(t)) = 0$$

and by the injectivity of $\partial_{\overline{v}}\nu$ this implies that, as in [PU], the phase is actually stationary exactly for the value of t, denoted t_1, such that $v(t_1) = v'$.

Let us come now to the transport equations, which determine the coefficients c_k and $a_k(t)$ in the expansions (5.11) and (5.12). Once more as in [PU], they are obtained by a direct application of the complete complex stationary phase expansion (see e.g.[Hö], Theorem 7.7.12), this time to (5.21). One has indeed

$$\begin{aligned} \Phi(t)|_{t=t_1} &= i\int_0^{t_1} \Im g(v(t),\overline{v}(t))\dot{v}\,dt + \frac{1}{2}\nu(v(t_1)\overline{v}(t_1)) \\ \frac{\partial^2\Phi}{\partial t^2}|_{t=t_1} &= -|\dot{v}(t_1)|^2\frac{\partial\nu(v(t),\overline{v}(t))}{\partial v\partial\overline{v}}|_{t=t_1} \end{aligned} \tag{5.27}$$

and therefore we can assert, as in Lemma 3.3, Formula (33) of [PU] the existence of coefficients $\beta_k(t_1,v') : k = 0,\dots,N$ such that, up to order $\hbar^{N+1}$

$$I(v') = \frac{(2\pi\hbar)^{1/4}\, e^{-\frac{|\nu(v',\overline{v}')|^2}{2\hbar}}}{\sqrt{T}|\dot{v}(t_1)|\partial^2_{v\overline{v}}\nu(v(t),\overline{v}(t))|_{t=t_1}} \cdot \exp i\int_0^{t_1} \Im g(v(t),\overline{v}(t))\dot{v}\,dt \cdot \left[\sum_{k=0}^N \beta_k(t_1,v')\right]$$

One easily checks that even in this situation the coefficients coming from the stationary phase expansion on all terms of (5.21), once rearrranged according to the increasing powers of $\hbar$, are such that

$$\begin{cases} \beta_0(t_1,v') \equiv 0 \\ \beta_k(t_1,v') = \dfrac{\partial a_{k-1}(t)}{\partial t}|_{t=t_1} - c_k a_0(t_1) + g_k(t_1;a_0,\dots,a_{k-1};c_0,\dots,c_{k-1};\dot{\overline{v}},\ddot{\overline{v}}) \end{cases} \tag{5.28}$$

where $g_k(\cdot)$ is a smooth function of its arguments. The form (5.28) implies that the equations

$$\beta_k(t_1, v') = 0, \qquad k = 1, \dots$$

together with the periodicity (antiperiodicity) and the initial conditions ($c_1 = \frac{\pi}{T}$ for the periodic case and $c_1 = 0$ for the antiperiodic one) can be recursively solved as in [PU] to yield the coefficients $a_k(t)$ and c_k.
Finally, recall that the Bohr-Sommerfeld condition (5.13) requires the trajectory γ to have an action which should be a (half)integer multiple of $\hbar$. Clearly, since the volume of $\mathcal{M}$ is bounded, there can be at most M such trajectories. This concludes the proof of Proposition 5.1.

$K_\ell|_{\mathcal{G}_{E_0(n)}}$ is of semiclassical type in the sense of Definition 5.1 and to identify its principal symbol with the classical resonant normal form (expressed in the Bargmann canonical variables). This is the object of the next statement.
By the standard abuse of notation, we denote

$$\begin{array}{ll} k_\ell(z_1, \overline{z}_1, z_2, \overline{z}_2), & p_\ell(z_1, \overline{z}_1, z_2, \overline{z}_2) \\ k_\ell(I_1, I_2; \chi_1, \chi_2), & p_\ell(I_1, I_2; \chi_1, \chi_2)J \\ k_\ell(J_1, J_2, \theta_2), & p_\ell(J_1, J_2; \theta_1\theta_2) \\ k_\ell(u_1, \overline{u}_1, u_2, \overline{u}_2), & p_\ell(A_1; u_1, \overline{u}_1, u_2, \overline{u}_2) \end{array} \tag{5.29}$$

the classical functions k_ℓ and p_ℓ, constructed in Section 2, written in the corresponding Bargmann canonical variables (A.1) $(z, \overline{z})$, in the action-angle variables $(I, \chi), (J, \theta)$, and in the variables u_1, u_2, respectively. We then have

Proposition 5.2 *Let Σ be the Kähler manifold of Proposition 4.2. Then:*

(i) When restricted to Σ the normal form $k_\ell(\cdot)$ depends only on the canonical coordinates $(u_2, \overline{u}_2)$, namely

$$k_\ell(u_1, \overline{u}_1, u_2, \overline{u}_2) = k_\ell(E_0; u_2, \overline{u}_2) \tag{5.30}$$

(ii) The operator $P_{E_0} K_\ell P_{E_0}$ on $\mathcal{G}_{E_0(n)} \cong \mathcal{H}_{E_0}$ admits a holomorphic semiclassical expansion, with principal symbol $k_\ell(E_0; u_2, \overline{u}_2')$.

Proof. The coefficients $k_\ell(\cdot)$ of the normal form (2.5), Proposition 2.1, depend on (A_1, J_2, θ_2) and hence, by the sequence of transformations (A.1), (2.2), (2.3), (4.4), (4.5), on $(A_1; |u_1|, |u_2|, \arg u_2)$. Now recall that in the (u_1, u_2) variables $\tilde{\Sigma}$ is defined by (4.30,4.32) which is equivalent to $\omega_1 p J_1 = E_0$ in the (J, θ) variables. Hence $|u_1|$ is a function of $|u_2|$. Taking the quotient with respect to the flow $\phi_t(u)$ we eliminate the

dependence on $\arg u_1$ as above. Hence $k_\ell(\cdot)$ depends only on u_2 (and parametrically on $E_0 = \omega_1 p J_1 = \omega_1 p A_1$) or, equivalently, on $(u_2, \overline{u}_2)$. The canonicity is preserved because (see Remark (i) after Proposition 4.2) the variables $(u_2, \overline{u}_2)$ are canonical on Σ. This proves Assertion (i).
To prove Assertion (ii) we first remark that, by Proposition 3.2, the operators K_ℓ and P_ℓ, written in the Bargmann representation,admit a holomorphic semiclassical expansion (in the sense of Definition 1.1) in $\mathcal{F}_2$. This is an equivalent definition of holomorphic semiclassical expansion because

$$\langle \psi_z, K_\ell \psi_{z'} \rangle = \langle \rho_z, \mathcal{F}_2 K_\ell \mathcal{F}_2^{-1} \rho_{z'} \rangle, \qquad \langle \psi_z, P_\ell \psi_{z'} \rangle = \langle \rho_z, \mathcal{F}_2 P_\ell \mathcal{F}_2^{-1} \rho_{z'} \rangle$$

Let us call $\rho_{u_2}^{E_0}$ the reproducing vectors of $\mathcal{G}_{E_0(n)}$, namely:

$$\rho_{u_2}^{E_0}(u) \ =: \ \rho(\overline{u}_2, u) \tag{5.31}$$

where $\rho(\overline{u}_2, u)$ is defined by 4.29.

To compute the principal symbol of $P_{E_0} K_l P_{E_0}$ and prove Assertion (ii) we have to show that:

$$\frac{\langle \rho_{u_2}^{E_0}, P_{E_0} K_l P_{E_0} \rho_{u_2}^{E_0} \rangle}{\langle \rho_{u_2}^{E_0}, \rho_{u_2}^{E_0} \rangle} \ = \ k_l(E_0; u_2, \overline{u}_2) + \sum_{j=1} k_l^j(E_0; u_2, \overline{u}_2) \hbar^j \tag{5.32}$$

Since $\rho_{u_2}^{E_0} = \overline{u}_1^{-\frac{\lambda}{\hbar}} \overline{u}_2^{-\alpha} P_{E_0} \rho_z^B$ (under the standing tacit assumption of considering u reexpressed as a function of z through the change of variable $z = U^{-1}(u)$), the result will be a consequence of the following lemma:

Lemma 5.1

$$\langle \rho_z^B, P_{E_0} K_l P_{E_0} \rho_z^B \rangle \sim \begin{cases} O(\hbar^\infty) & \text{if} z \notin \Sigma_{E_0} \\ \langle P_{E_0} \rho_z^B, P_{E_0} \rho_z^B \rangle (k_l(E_0; u_2, \overline{u}_2) + \sum_{j=1}^{\infty} k_l^j(E_0; u_2, \overline{u}_2) \hbar^j) & \text{if} z \in \Sigma_{E_0} \end{cases} \tag{5.33}$$

Proof: Thanks to the representation (4.27) for P_{E_0}, we can write:

$$\langle \rho_z^B, P_{E_0} K_\ell P_{E_0} \rho_z^B \rangle = \frac{1}{T^2} \int_0^T \int_0^T e^{-i \frac{E_0}{\hbar}(t-t')} \langle \rho_{\phi_t(z)}^B, K_\ell \rho_{\phi_{t'}(z)}^B \rangle \rangle dt dt' \tag{5.34}$$

Performing the change of variable $s = t' - t$ we get:

$$\langle \rho_z^B, P_{E_0} K_\ell P_{E_0} \rho_z^B \rangle \ = \frac{1}{T^2} \int_0^T \int_{t-T}^{T-t} e^{i E_0 s/\hbar} \langle \rho_{\phi_t(z)}^B K_l \rho_{\phi_t(\phi_s(z))}^B \rangle \, ds \, dt \tag{5.35}$$

where $\phi_{t+s}(z)) = \phi_t(\phi_s(z))$ by the autonomy of the flow. Remark now that, construction that:

$$k_l^\hbar(z,\overline{z}) \;=:\; \frac{\langle\rho_z^B K_l \rho_z^B\rangle}{\langle\rho_z^B,\rho_z^B\rangle} \;\sim\; k_l(z,\overline{z}) + \sum_{j=1}^{\infty} k_l^j(z,\overline{z})\hbar^j \tag{5.36}$$

This obviously yields, given the expression (4.28) or (A.14) of ρ_z^B:

$$\begin{aligned}&\langle\rho_z^B, P_{E_0}K_\ell P_{E_0}\rho_z^B\rangle = \\ &\frac{1}{T^2}\int_0^T\int_{t-T}^{T-t} e^{iE_0 s/\hbar}k_l(\phi_t(z),\phi_t(\phi_s(\overline{z}))e^{i(E_0 s - i\langle\phi_s(z),z\rangle)/\hbar}\,ds\,dt\end{aligned} \tag{5.37}$$

7.7.12) is sufficient to get the result: the phase is indeed $E_0 s - \langle\phi_s(z),z\rangle$ whose derivative with respect to s obviously vanishes only if $s=0$. The leading term of the stationary phase expansion is therefore

$$\frac{1}{T}\sqrt{\hbar}\frac{1}{\sqrt{\langle\omega z,z\rangle}}\hat{k}_l(z,\overline{z})$$

where

$$\hat{k}_l(z,\overline{z}) = \frac{1}{T}\int_0^T k_l(\phi_t(z),\phi_t(\overline{z}))\,dt \tag{5.38}$$

Moreover, the leading term of $\langle P_{E_0}\rho_z^B, P_{E_0}\rho_z^B\rangle k_l(E_0;u_2,\overline{u}_2)$ can be computed by the same stationary phase expression and is $\frac{1}{T}\sqrt{\hbar}\frac{1}{\sqrt{\langle\omega z,z\rangle}}k_l(E_0;u_2,\overline{u}_2)$.

Finally remark that $\hat{k}_l(z,\overline{z})\mid_{z\in\Sigma_{E_0}} = k_l(E_0;u_2,\overline{u}_2)$. This concludes the proof of the lemma.

Assertion (ii) is an immediate consequence of the lemma. This concludes the proof of Proposition 5.2.

Remarks.

(i) The symbols $k_\ell^j(E_0;u_2,\overline{u}_2)$, $k_\ell^0(\cdot)\equiv k_\ell(\cdot)$ can obviously be reexpressed on the coordinates (J,θ), and have the form (standard abuse of notation) $k_\ell^j(E_0;J_2,\theta_2)$ where $E_0 = 2p\omega_1 J_1 = 2p\omega_1 A_1$, $J_1 = A_1 = (n+\frac{1}{2})\hbar$. Since k_ℓ is obtained by eliminating the dependence on θ_1, Assertion (ii) means that, expressed on symbols, the operation of projecting on $\mathcal{G}_{E_0(n)}$ amounts to averaging over the fast angle. This is of course a reflection of the coincidence between quantum and classical evolution of the harmonic oscillators in Bargmann space.

(ii) The canonical transformation obtained by composing the canonical transformations (A.3) and U with the integrating canonical transformation $D_{\epsilon,N}$ introduced in the proof of Theorem 1.1, Assertion (ii), Section 2, maps holomorphically

$k_\ell(E_0;u_2,\overline{u}_2)$ into the (ℓ-th term of) the normal form $\tilde{k}_\ell(A_1,A_2)$ of Theorem 1.1. Clearly the same transformations maps the higher order symbols $k_\ell^j(E_0;u_2,\overline{u}_2)$ into holomorphic functions $\tilde{k}_\ell^j(A_1,A_2)$.

*Proof of Theorem 1.1 (Conclusion).*The last Proposition shows that the conditions for the validity of Assertion (3) in Proposition 5.1 are fulfilled with $Q_\hbar = P_{E_0}K_\ell P_{E_0} : \ell = 1,\ldots$. Moreover $P_{E_0}K_\ell P_{E_0}$ has only simple eigenvalues because its principal symbol under Wick quantization is the holomorphic function $k_\ell(E_0;u_2,\overline{u}'_2)$. Now (see Remark (i) after Proposition 3.3), we have

$$E_\ell^k(n,\hbar) = \langle \varphi_\ell^k, P_{E_0}K_\ell P_{E_0}\varphi_\ell^k\rangle$$

On the other hand formula (5.18), with $P_{E_0}K_\ell P_{E_0}$ in place of $Q_\hbar$ and hence $k_\ell(E_0,u_2,\overline{u}_2)$ in place of $q_0(z,\overline{z})$, $k_\ell^j(E_0;u_2,\overline{u}_2)$ in place of $q^j(z,\overline{z})$ and φ_ℓ^k in place of ψ_{λ_n} yields, on account of Remark (ii) above

$$E_\ell^{(k)}(n,\hbar) \sim \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\left[k_\ell((n+\frac{1}{2})\hbar,k\hbar,\theta_2)+\sum_{j=1}^{\infty}k_\ell^j((n+\frac{1}{2})\hbar,k\hbar,\theta_2)\hbar^j\right]\,d\theta_2$$

Therefore

$$E_\ell^{(k)}(n,\hbar) \sim \tilde{k}_\ell((n+\frac{1}{2})\hbar,(k+\frac{1}{2})\hbar)+\sum_{j=1}^{\infty}\tilde{k}_\ell^j((n+\frac{1}{2})\hbar,(k+\frac{1}{2})\hbar)\hbar^j$$

and this concludes the proof of Theorem 1.1.

6 Appendix: Symbols and Quantization in the Bargmann Representation

Definition A.1 *The Bargmann canonical variables are the complex coordinates defined by the linear complex transformation*

$$\mathcal{C} = \begin{cases} z_k = \dfrac{\omega_k x_k + i\xi_k}{\sqrt{2\omega_k}} \equiv \zeta_k \\ \overline{z}_k = \dfrac{\omega_k x_k - i\xi_k}{\sqrt{2\omega_k}} \equiv \eta_k \end{cases} . \tag{A.1}$$

where $k = 1, \dots, s$, inverted as

$$\mathcal{C}^{-1} = \begin{cases} q_k = \dfrac{z_k + \overline{z}_k}{\sqrt{2\omega_k}} = \dfrac{\zeta_k + \eta_k}{\sqrt{2\omega_k}} J \\ p_k = -i\dfrac{\sqrt{\omega_k}(z_k - \overline{z}_k)}{\sqrt{2}} = -i\dfrac{\sqrt{\omega_k}(\zeta_k - \eta_k)}{\sqrt{2}} \end{cases} . \tag{A.2}$$

$k = 1, \dots, s$. The real phase space $\mathbb{R}^{2s}$ is mapped under $\mathcal{C}$ in the complex manifold $\Theta \equiv \{(\zeta, \eta) \in \mathbb{C}^{2s} : \overline{\zeta} = \eta\}$.

Concerning this definition, we remark

(1) The coordinates $(z, \overline{z})$ are canonical: $\{z_k, \overline{z}_j\} = i\delta_{k,j}$. The action-angle variables $(I_k, \chi_k) : k = 1, 2$ of Section 2 are just modulus and phase of z_k. In terms of $(z, \overline{z})$ the change of variables (3.3) becomes

$$\begin{cases} J_1 = \dfrac{z_1\overline{z}_1}{p} - \dfrac{z_2\overline{z}_2}{q}, \quad J_2 = \dfrac{z_1\overline{z}_1}{p} + \dfrac{z_2\overline{z}_2}{q} \\ \theta_1 = \arg z_1^p z_2^{-q}, \quad \theta_2 = \arg z_1^p z_2^q \end{cases} \tag{A.3}$$

(2) The Hamiltonian family $h_0 + \epsilon V$ becomes, once reexpressed on $z, \overline{z}$:

$$h_\epsilon(z, \overline{z}) = \omega_1 z_1 \overline{z}_1 + \omega_2 z_2 \overline{z}_2 + \epsilon V(z, \overline{z}) \equiv \langle \omega z, \overline{z} \rangle + \epsilon V(z, \overline{z}) \tag{A.4}$$

where, by the usual abuse of notation, $V(z, \overline{z}) = V(\mathcal{C}^{-1}(z, \overline{z}))$.

(3) Consider q_k and p_k as complex variables. Correspondingly, the canonical variables (ζ, η), as well as the Hamiltonian $h_\epsilon(\zeta, \eta)$, can be considered as defined in $\mathbb{C}^4$ and this will be tacitly assumed in what follows. The complex manifold $\Theta \subset \mathbb{C}^4$ is invariant under the flow of the real Hamiltonian (A.4). Remark also that the appearence of the i in the symplectic form implies that the Hamilton equations in the $(z, \overline{z})$ coordinates have the form

$$\dot{z}_k = -i\frac{\partial h_\epsilon(z, \overline{z})}{\partial \overline{z}_k} \quad \dot{\overline{z}}_k = i\frac{\partial h_\epsilon(z, \overline{z})}{\partial z_k} \quad k = 1, \dots$$

whence, for $\epsilon = 0$, the expression (4.27) of the harmonic flow in the $(z, \overline{z})$ coordinates

$$\phi_t(z) = \begin{cases} z \mapsto ze^{-i\omega t} \\ \overline{z} \mapsto \overline{z}e^{i\omega t} \end{cases}$$

Let us turn now to the quantum case. To this end, let us first recall the basic results on the Bargmann representation of the canonical commutation rules, which represents the quantization of the classical canonical variables (ζ, η).

Lemma A.1 (Bargmann) *Let (ζ, η) be the canonical variables defined in (A.1). Then the CCR representation problem of finding a Hilbert space $\mathcal{H}$ and operators $Y_k, Z_k \in \mathcal{L}(\mathcal{H}) : k = 1, \dots, s$ such that*

(i) $[Y_j, Z_k] = i\hbar\{\eta_j, \zeta_k\}$

(ii) $Y_k^\star = Z_k$

admits the following solution:

(a) $\mathcal{H} = \mathcal{F}$, where $\mathcal{F}$ is the Fock-Bargmann space of all entire holomorphic functions $u : \mathbb{C}^s \to \mathbb{C}$ such that

$$\|u\|_{\mathcal{F}} \equiv \int_{\mathbb{R}^{2s}} \mid u(z) \mid^2 e^{-\frac{|z|^2}{\hbar}} \prod_{k=1}^{s} dx_k dy_k < +\infty \tag{A.5}$$

(b) $Z_k, Y_k : k = 1, \dots, s$ are the maximal multiplication operator by z_k and the maximal differentiation operator generated by $\hbar\partial_{z_k}$ in $\mathcal{F}$, respectively.

(c) The Bargmann representation of the CCR is unitarily equivalent to the Schrödinger one, namely $\forall \psi \in L^2(\mathbb{R}^s)$ the Bargmann transform

$$(\mathcal{B}\psi)(z) \equiv f(z) = (\sqrt{\pi}\hbar\omega)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{(z^2+\omega^2 q^2)}{2\hbar} + 2\frac{\sqrt{2}\langle \omega z, q\rangle}{\hbar}} \psi(q)\, dq \tag{A.6}$$

is a unitary map between $L^2(\mathbb{R}^s)$ and the Fock-Bargmann space $\mathcal{F}_s$ and the following unitary equivalences hold:

$$\begin{cases} \mathcal{B}a_k^\star\mathcal{B}^{-1} = Z_k \qquad \mathcal{B}a_k\mathcal{B}^{-1} = Y_k \\ \mathcal{B}Q_k\mathcal{B}^{-1} = \dfrac{1}{\sqrt{2\omega_k}}(Z_k + Y_k) \qquad \mathcal{B}P_k\mathcal{B}^{-1} = -i\dfrac{\sqrt{2\omega_k}}{2}(Z_k - Y_k) \end{cases} . \tag{A.7}$$

Here $a_k^\star, a_k$ are the standard creation and annihilation operators of the k-th harmonic oscillator, and P_k, Q_k the standard momentum and position operators in $L^2(\mathbb{R}^d)$.

(d) *The Schrödinger operator corresponding to the harmonic oscillator $h_0 = \langle \omega\zeta, \eta\rangle$ is the operator T_0 defined as the maximal operator in $\mathcal{F}$ generated by the maximal action of the differential expression $\hbar\langle \omega z, \nabla_z\rangle$, i.e*

$$(T_0 f)(z) = (\mathcal{B}\mathcal{P}_0\mathcal{B}^{-1} f)(z) = (\langle \omega Z, Y\rangle f)(z) \tag{A.8}$$

The (orthonormal) eigenvectors of T_0 are

$$\psi_{n_1,n_2} = \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1! n_2!}} \tag{A.9}$$

Given the quantization of the canonical variables (ζ, η), the classical, mixed, monomials of the type $\zeta^k \eta^m$, and thus all polynomial observables, are quantized according to the Weyl prescription, which can be generated by the Berezin-Shubin symmetric operator producy[BS],§5.2. It follows that (see ([DEGH]), if the function $F(q) : \mathbb{R}^s \to \mathbb{R}$ is a polynomial (by the usual abuse of notation F denotes also the maximal multiplication operator by $F(q)$ in $L^2(\mathbb{R}^s)$), the unitary image $\mathcal{B}F\mathcal{B}^{-1}$ is the maximal differential operator $\hat{F}$ of action

$$(\hat{F}f)(z_1, \ldots, z_s) = F\left(\frac{Z_1 + Y_1}{\sqrt{2\omega_1}}, \ldots, \frac{Z_s + Y_s}{\sqrt{2\omega_s}}\right) f(z_1, \ldots, z_s) \tag{A.10}$$

If in particular we consider the perturbation operator V, (V denotes the maximal multiplication operator by $V(q)$ in L^2), we get the definition of the perturbation operator in $\mathcal{F}_2$

$$T_1 = \mathcal{B}V\mathcal{B}^{-1} \tag{A.11}$$

and this realizes the Schrödinger operator family $T_\epsilon \equiv \mathcal{B}\mathcal{P}_\epsilon\mathcal{B}^{-1}$ in $\mathcal{F}_2$ of (4.3),(4.4):

$$(T_\epsilon f)(z) = ((T_0 + \epsilon T_1)f)(z) = (\langle \omega\hat{\zeta}, \eta\rangle f)(z) + (\hat{V}f)(z) \tag{A.12}$$

The well known relation among coherent states in the Bargmann representation, operators and symbols (see e.g.[BS], §5.2) is summarized in the following

Lemma A.2 *Let*

$$\Phi_{\overline{z}}(z') = \frac{1}{(\pi\hbar)^{s/2}} e^{\frac{\langle z', \overline{z}\rangle}{\hbar}}, \quad z' \in \mathbb{C}^s \tag{A.13}$$

be the family of the Poisson (or principal) vectors in $\mathcal{F}_s$. Then:

(i) *Set*

$$\rho_z^B(z') = \cdot e^{-\frac{|z|^2}{2\hbar}} \cdot e^{-\frac{|z'|^2}{2\hbar}} \Phi_{\overline{z}}(z') \tag{A.14}$$

Then $\rho_z^B(z')$ is a self-reproducing kernel in $\mathcal{F}_s$; namely, $\forall f \in \mathcal{F}_s$ one has

$$f(z) = \int_{\mathbb{R}^{2s}} \rho_z^B(z') \prod_{k=1}^{s} dx'_k dy'_k \tag{A.15}$$

(ii) Let $\psi_z(y)$ *be the coherent states in* $L^2(\mathbb{R}^S)$ *of (1.8). Then*

$$\mathcal{B}(\psi_z) = (\pi\hbar)^{s/2}\rho_z \tag{A.16}$$

(iii) Given a bounded operator A *on* $\mathcal{F}_s$*, its Wick symbol* $\mathcal{A}_W(z,\overline{z},\hbar)$ *is given by*

$$\mathcal{A}_W(z,\overline{z},\hbar) = \langle A\rho_z, \rho_z\rangle \tag{A.17}$$

(iv) The Weyl symbol $\mathcal{A}$ *of* A *is related to the Wick one in the following way*

$$\mathcal{A}(z,\overline{z},\hbar) = \exp\frac{\hbar}{2}\Delta\mathcal{A}_W(z,\overline{z},\hbar) = \tag{A.18}$$

$$\int_{\mathbb{R}^{2s}} \mathcal{A}_W(z,\overline{z},\hbar)\, e^{-\frac{2}{\hbar}[(\overline{z}-\overline{z'})(z-z')]}\, dz'd\overline{z'}$$

(v) If an operator S *acting in* $L^2(\mathbb{R}^s)$ *admits a (Weyl) symbol* $s(q,p;\hbar)$*, its unitary image* $A = \mathcal{B}S\mathcal{B}^{-1}$ *in the Bargmann space* $\mathcal{F}_s$ *admits the Weyl symbol* $\mathcal{A}(z,\overline{z};\hbar) = a(\mathcal{C}^{-1}(z,\overline{z});\hbar)$*. The action of the operator* A *in* $\mathcal{F}_s$ *in terms of its Weyl symbol is defined as follows*

$$(Af(z)) = \sqrt{\pi}\hbar^{-\frac{s}{2}}\int_{\mathbb{C}^s} \mathcal{A}(z,\overline{v};\hbar)\exp\left[\frac{i}{\hbar}(\langle\overline{v},z\rangle - |v|^2)\right]f(v)\,dv\,d\overline{v} \tag{A.19}$$

(vi) The operator A *is said of semiclassical type if its (Wick and hence Weyl) symbol admits an asymptotic expansion of the type*

$$\mathcal{A}_W(z,\overline{z},\hbar) = a_0(z,\overline{z}) + \sum_{\ell\geq 1} a_\ell(z,\overline{z})\hbar^\ell \tag{A.20}$$

where the principal symbol $a_0(z,\overline{z})$ *as well as the higher order symbols* $a_\ell(z,\overline{z})$: $\ell = 1,\ldots$ *are holomorphic functions of* $(z,\overline{z})$.

(vii) If A *and* B *are operators of semiclassical type, with principal symbols* $a_0(z,\overline{z})$, $b_0(z,\overline{z})$ *respectively, then so are the operators* AB, BA, $i\hbar^{-1}[A,B]$*, and their principal symbols are* $a_0(z,\overline{z})\cdot b_0(z,\overline{z})$, $b_0(z,\overline{z})\cdot a_0(z,\overline{z}) = a_0(z,\overline{z})\cdot b_0(z,\overline{z})$, $\{a_0(z,\overline{z})\, b_0(z,\overline{z})\}$*, respectively.*

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