INFINITE KNEADING MATRICES AND WEIGHTED ZETA FUNCTIONS OF INTERVAL MAPS

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Abstract. We consider a piecewise continuous, piecewise monotone interval map and a weight of bounded variation, constant on homtervals and continuous at periodic points of the map. With these data we associate a sequence of weighted Milnor-Thurston kneading matrices, converging to a countable matrix with coefficients analytic functions. We show that the determinants of these matrices converge to the inverse of the correspondingly weighted zeta function for the map. As a corollary, we obtain convergence of the discrete spectrum of the Perron-Frobenius operators of piecewise linear approximations of Markovian, piecewise expanding and piecewise C^{1+BV} interval maps.

Introduction

Let f be a transformation of a compact interval, say $I = [0,1]$. Assume that f is piecewise continuous, and piecewise strictly monotone, with a finite number N of pieces defined by turning points $0 = a_0 < a_1 < a_2 < \ldots < a_{N-1} < a_N = 1$. Let $g : I \to \mathbb{C}$ be a weight (a natural choice is $g = 1/|f'|$, if f is piecewise differentiable and without critical points). If each iterate f^n has only finitely fixed points, we can formally define the weighted Ruelle zeta function

$$\zeta_g(t) = \exp \sum_{n\geq 1} \frac{t^n}{n} \sum_{x\,:\,f^n x = x} \prod_{k=0}^{n-1} g(f^k x)\,. \tag{0.1}$$

In Section 1 we shall introduce a *reduced zeta function* which can also be defined when some iterate(s) of f have infinitely many fixed points. We shall also denote it by $\zeta_g(t)$, no confusion should arise since all our results are about the reduced function.

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If the weight g is of bounded variation, the analytic properties of $\zeta_g(t)$ in some domain of the complex plane can often be studied with the help of the transfer operator $\mathcal{L}_g$ defined by

$$\mathcal{L}_g\varphi(x) = \sum_{y\,:\,fy=x} g(y)\cdot\varphi(y) \tag{0.2}$$

acting on the Banach space BV of functions $\varphi : I \to \mathbb{C}$ of bounded variation (see Baladi-Keller [1990] for the case where the partition is generating, and Ruelle [1993a, 1993b] for more general results). In particular, one shows under suitable assumptions that the essential spectral radius of $\mathcal{L}_g$ is not larger than ρ, and that $\zeta_g(t)$ admits a meromorphic extension to the open disc of radius $1/\rho$, where

$$\rho = \rho(g) = \lim_{n\to\infty} (\sup_x |\prod_{k=0}^{n-1} g(f^k x)|)^{1/n}\,. \tag{0.3}$$

Furthermore, the poles of $\zeta_g(t)$ in this disc are exactly the inverses of eigenvalues of $\mathcal{L}_g$.

Zeta functions may also be studied via the approach initiated by Milnor and Thurston [1988] who express a slightly different reduced (unweighted) zeta function as the determinant of a finite "kneading matrix," whose coefficients are power series. In a previous paper (Baladi-Ruelle [1993]), we defined weighted kneading matrices and determinants, and extended Milnor and Thurston's main result to the case of reduced weighted zeta functions ζ_g *under the strong assumption that the map g be constant on each interval* $[a_{i-1}, a_i]$. With this assumption, the weighted kneading matrix is again a finite matrix.

In the present paper we extend this result to the case where the weight g is of bounded variation. (We need technical assumptions on g: it should be constant in homtervals, and continuous at the periodic points of f.) In this case, one has to consider infinite (countable) kneading matrices.

More precisely, we construct a sequence of finite weighted kneading matrices $K^{(n)}(t)$, of strictly increasing sizes, corresponding to better and better locally constant approximations $g^{(n)}$ of g. Set $\rho^{(n)} = \rho(g^{(n)}) \le \rho$, and write $\bar\rho = \lim_{n\to\infty}\rho^{(n)}$. By the previous result in Baladi-Ruelle [1993] the kneading determinant $\Delta^{(n)}(t) = \det(K^{(n)}(t))$, viewed as a power series, coincides with $1/\zeta_{g^{(n)}}(t)$ up to a trivial factor. (By slightly changing the kneading matrices, we are now able to replace this trivial factor by the constant 1.) Also, the series $\Delta^{(n)}(t)$ converges in the disc of radius $1/\rho^{(n)}$. Because of the assumptions on g, the power series $\zeta_{g^{(n)}}(t)$ converges to $\zeta_g(t)$ as n goes to infinity, and the analytic function defined by $\zeta_{g^{(n)}}(t)$ converges to $\zeta_g(t)$ on some nonempty open disc. From this, it follows that $\lim_{n\to\infty}\Delta^{(n)}(t)$ exists and defines an analytic function $\Delta(t)$, in this disc. However, the properties of the limits of the meromorphic functions $\zeta_{g^{(n)}}(t)$ and of the analytic functions $\Delta^{(n)}(t)$ on the (possibly larger) disc of radius $1/\bar\rho$, are not clear a priori: we must show that the analytic functions $\zeta_{g_n}^{-1}(t) = \Delta^{(n)}(t)$ are uniformly bounded iby applying the classical Hadamard inequality to matrices $M^{(n)}(t) = 1 - L^{(n)}(t)$, where 1 is the identity matrix and $L^{(n)}(t)$ is obtained from $K^{(n)}(t)$ by elementary row operations. This yields our main result: the two functions $\Delta(t)$ and $1/\zeta_g(t)$ admit analytic extensions to the disk D of radius $1/\bar{\rho}$, where they coincide, and $\zeta_{g_n}(t)$ converges to $\zeta_g(t)$ as a meromorphic function in D. A precise statement is given in Section 1. Section 2 contains the proofs.

In the cases where the relationship between the spectrum of the transfer operator and the properties of the zeta function have been established (for example, when the partition into intervals of monotonicity is generating, see Baladi–Keller [1990]), our main result implies that in the domain $\rho < |z|$ the spectrum of the transfer operator $\mathcal{L}_{g^{(n)}} : BV \to BV$ converges to that of $\mathcal{L}_g : BV \to BV$. *If we assume further that the map f has a finite Markov partition* (i.e., all the turning points are (pre)periodic), each coefficient of the finite matrix $K^{(n)}(t)$ is by definition a polynomial in t multiplied by a geometric series in (a power of) t, and the matrix is therefore a "finite" object. Also, the Markov property implies that the transfer operator preserves a finite-dimensional vector space on which it can be described by a finite matrix. In the Markov case, our results hence yield two procedures to approach the discrete spectrum of the transfer operator acting on BV.

Under more restrictive assumptions (the map f is piecewise $\mathcal{C}^2$, the weight is $g = 1/|f'|$, and $\rho < 1$), Mori [1990, 1992] has introduced "Fredholm" matrices $\Phi_n(t)$, and has related the limit of the determinants of these matrices to the correspondtransfer operators (acting on BV) and weighted zeta functions. Mori uses different techniques and does not relate his results with the Milnor-Thurston theory. However, he is able to define Banach spaces on which his limiting matrix $\Phi(t)$ defines a bounded operator, and to show that if the limit of the determinants has a zero at t, the countable matrix $\Phi(t)$ has a fixed point. In our case, the limit $K^{(\infty)}(t)$ of the matrices $K^{(n)}(t)$ is also well defined, and we conjecture that $K^{(\infty)}(t)$ has a fixed point in a suitable Banach space whenever $\Delta(t)$ vanishes.

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1. Definitions and statement of results

The basic ingredients: the map f and the weight g.

We start by fixing notation, which we have tried to keep essentially consistent with Baladi-Ruelle [1993]. Let $1 \le N < \infty$, let $a_0 < a_1 < \ldots < a_N$, and let $f_i : [a_{i-1}, a_i] \to [a_0, a_N]$ be strictly monotone and continuous maps for $i = 1, \ldots, N$. (The intervals are not necessarily maximal intervals of monotonicity or continuity.) We write $f =$

$(f_1, \dots, f_N)$. By abuse of language we also denote by f the multivalued transformation of $[a_0, a_N]$ whose graph is the union of the graphs of the f_i. We let $\epsilon_i = \pm 1$ depending on whether f_i is increasing or decreasing, and we define a function ϵ on $[a_0, a_N]$ such that it has the constant value ϵ_i on (a_{i-1}, a_i) and the value 0 on $\{a_0, a_1, \dots, a_N\}$. Let $g = (g_1, \dots, g_N)$ where each $g_i : [a_{i-1}, a_i] \to \mathbb{C}$ is a function of bounded variation. We also view g as a multivalued function on $[a_0, a_N]$.

Denote by $\mathcal{Z} = \mathcal{Z}_1$ the "partition" of I into N closed intervals given by the a_i. For $n \geq 2$, let $\mathcal{Z}_n$ be the n^{th} refinement of $\mathcal{Z}$ under f, i.e., the partition into intervals $\eta_0 \cap f^{-1}\eta_1 \cap \dots \cap f^{-n+1}\eta_{n-1}$ with $\eta_j \in \mathcal{Z}$. Recall that the partition $\mathcal{Z}$ is called *generating* if the maximal length of the intervals in $\mathcal{Z}_n$ tends to zero as n tends to infinity (this is an intrinsic property of the map f which does not vary by considering a finer partition $\mathcal{Z}'$). We define a *$\mathcal{Z}$-homterval* to be a maximal nontrivial interval J such that for each $n \geq 1$ there is $\eta \in \mathcal{Z}_n$ with $J \subset \eta$. (The partition is generating if and only if there are no $\mathcal{Z}$-homtervals.)

We make the additional assumption that g is constant on each $\mathcal{Z}$-homterval of f.

The locally constant approximations $g^{(n)}$.

For $n \geq 1$, and $\eta \in \mathcal{Z}_n$, let $\{x_{\eta,m}, m \geq 1\} \subset \eta$ be such that $\lim_{m\to\infty} |g(x_{\eta,m})| = \operatorname{ess\,inf}_\eta |g|$, and such that $\lim_{m\to\infty} g(x_{\eta,m})$ existsmultivalued function whose restriction to each closed interval $\eta \in \mathcal{Z}_n$ is equal to the constant $\lim_{m\to\infty} g(x_{\eta,m})$. The constants $\rho = \rho(g)$ and $\rho^{(n)} = \rho(g^{(n)}) \leq \rho$ are defined as in Equation (0.3), and we write $\bar\rho = \lim_{n\to\infty} \rho^{(n)}$. If g is (piecewise) continuous, $\bar\rho = \rho$.

The kneading matrices $K^{(n)}(t)$, $L^{(n)}(t)$, the kneading determinants $\Delta^{(n)}(t)$.

We introduce a sequence of finite kneading matrices $K^{(n)}(t) = K^{(n)}(g)(t)$ of sizes $(N_n + 1) \times (N_n + 1)$, where $N_n + 1$ is the number of endpoints of the partition $\mathcal{Z}_n$. Let us start with $K^{(1)}(t)$, which is essentially the matrix considered in Baladi-Ruelle [1993] (we add two rows and two columns in order to suppress the "trivial factor" mentioned in the introduction). We need some preliminary definitions: the *first address* of $x \in [a_0, a_N]$ is the vector

$$\vec\alpha^{(1)}(x) = (\operatorname{sgn}(x - a_0), \dots, \operatorname{sgn}(x - a_N)) \in \{-1, 1, 0\}^{N+1};$$

the *first weighted invariant coordinate* of $x \in [a_0, a_N]$ is the $(N+1)$-tuple of power series

$$\vec\theta^{(1)}(x) = \vec\theta^{(1)}_f(g)(x)(t) = \sum_{m=0}^{\infty} t^m \cdot \Big(\prod_{k=0}^{m-1} (\epsilon g^{(1)})(f^k x)\Big) \cdot \vec\alpha^{(1)}(f^m x) \in \mathbb{C}[[t]]^{N+1}.$$

Note that $\vec\theta^{(1)}(x)$ is single-valued as a function of x (because if $f^k x \in \{a_0, \dots, a_N\}$ for some $k \leq m$, then $\epsilon(f^m x) = 0$). Writing

$$\phi(a\pm) = \lim \phi(x) \text{when } x \downarrow a\,, \text{ respectively } x \uparrow a\,,$$

and setting

$$\vec\theta^{(1)}(a_0-) = (-1, \dots, -1)\,, \qquad \vec\theta^{(1)}(a_N+) = (1, \dots, 1)\,,$$

we define the rows of the

$$\vec{K}_i^{(1)}(t) = \frac{1}{2}\Big[\vec{\theta}^{(1)}(a_i+) - \vec{\theta}^{(1)}(a_i-)\Big]$$
$$= (K_{i,0}^{(1)}, \dots, K_{i,N}^{(1)}), \quad \text{for } i = 0, \dots, N\,.$$

The determinant of $K^{(1)}(t)$

$$\Delta^{(1)}(t) = \Delta_f^{(1)}(g)(t) = \det\big[K_{ij}^{(1)}(t)\big] \in \mathbb{C}[[t]]$$

is called the *first kneading determinant.* Since $K_{ij}^{(1)}(t) = \delta_{ij} + \text{higher order in } t$, we have $\Delta^{(1)}(t) = 1 + \text{higher order in } t$. By the definition of $\rho^{(1)}$, the power series $\Delta^{(1)}(t)$ converges in the disc of radius $1/\rho^{(1)} \geq 1/\rho$.

We now give the general step for the construction of the n^{th} *kneading matrix.* We first order the $N+1$ endpoints of the partition $\mathcal{Z}_1$ according to $a_0 < \ldots < a_N$. For $n \geq 2$, we order the $N_n + 1$ endpoints $a_0^{(n)} < \ldots < a_{N_n}^{(n)}$ of the partition $\mathcal{Z}_n$ inductively: we decompose this set of endpoints into $A \cup B$, where A is the set of endpoints of the partition $\mathcal{Z}_{n-1}$, we keep the order on A given by the preceding inductive step, order the points in B according to their position on the real line, and impose $\max A \prec \min B$. Let π denote the permutation on $\{0, \dots, N_n\}$ induced by this reordering of the endpoints:

$$a_{\pi(0)}^{(n)} \prec a_{\pi(1)}^{(n)} \prec \cdots \prec a_{\pi(N_n)}^{(n)}\,.$$

The permutation π allows us to handle the bookkeeping of rows and columns inductively.

We define the n^{th} *sign function* $\epsilon^{(n)} : I \to \{-1, 1, 0\}$ which vanishes at the points $a_i^{(n)}$ and otherwise coincides with ϵ, and the n^{th} *address* of $x \in I$ by

$$\vec{\alpha}^{(n)}(x)\mathrm{sgn}\,(x - a_{\pi(N_n)}^{(n)})) \in \{-1, 1, 0\}^{N_n+1}\,.$$

Similarly as above, using $\epsilon^{(n)}$, $\vec{\alpha}^{(n)}$, and $g^{(n)}$, we define the n^{th} *weighted invariant coordinate*

$$\vec{\theta}^{(n)}(x) \in \mathbb{C}[[t]]^{N_n+1}\,,$$

and the rows of the n^{th} $(N_n + 1) \times (N_n + 1)$ *kneading matrix,* for $i = 0, \dots, N_n$:

$$\vec{K}_i^{(n)}(t) = \frac{1}{2}\Big[\vec{\theta}^{(n)}(a_{\pi(i)}+) - \vec{\theta}^{(n)}(a_{\pi(i)}-)\Big] \in \mathbb{C}[[t]]^{N_n+1}\,,$$

and finally the n^{th} *kneading determinant*

$$\Delta^{(n)}(t) = \det\big[K_{ij}^{(n)}(t)\big] \in \mathbb{C}[[t]]\,.$$

The power series $\Delta^{(n)}(t)$ converges in the disc of radius $1/\rho^{(n)}$. The permutation π on the rows and columns has no effect on the determinant of the matrix, and it allows us to define a countable matrix $K^{(\infty)}(t)$: for each finite index pair (i,j), we define the *infinite kneading matrix*

$$K_{ij}^{(\infty)}(t) = \lim_{n\to\infty} K_{ij}^{(n)}(t)$$

(if the partition is generating, this limit exists because g is of bounded variation; when it is not generating, use also the fact that g is constant on $\mathcal{Z}$-homtervals).

In the proof of our main theorem (see (2.5,2.6,2.7)) we shall perform elementary row operations on the sequence $K^{(n)}(t)$ to obtain another sequence of kneading matrices $L^{(n)}(t)$ (with $\det L^{(n)}(t) = \det K^{(n)}(t)$), converging to a countable matrix $L^{(\infty)}(t)$. Note that, although the countable matrix $M^{(\infty)}(t_0) = 1 - L^{(\infty)}(t_0)$, for t_0 in the disc of radius $1/\bar\rho$, defines a Fredholm operator in the sense of Grothendieck [1956] on the Banach space $l^1(\mathbb{N})$, and hence the determinant of $L^{(\infty)}(t_0)$ is a well-dobject (see the Appendix), this determinant does *not* coincide in general with $\Delta(t_0)$ (we give a counter-example after the proof of the main theorem in Section 2).

The reduced zeta function $\zeta_g(t)$.

For the convenience of the reader, we recall the definition of the weighted *reduced zeta function* associated with f, introduced in Baladi-Ruelle [1993] for a locally constant weight g. Denote by $\mathrm{Fix}\, f^m$ the set of fixed points of f^m which have an orbit disjoint from the endpoints $\{a_0, \ldots, a_N\}$ of the partition $\mathcal{Z}$. Since a periodic orbit is disjoint from $\{a_0, \ldots, a_N\}$ if and only if it is disjoint from the endpoints of all partitions $\mathcal{Z}_n$ for $1 \le n$, the set $\mathrm{Fix}\, f^m$ is invariant under the successive refinements of the partition.

We first assume that for each m the set $\mathrm{Fix}\, f^m$ is finite (we shall see below how to remove this assumption). For $x \in \mathrm{Fix}\, f^m$, we introduce a Lefschetz index:

$$L(x, f^m) = \begin{cases} 0 & \text{if the graph of } f^m \text{ doesn't cross the diagonal at } x, \\ \lim_{y\to x} \frac{\mathrm{sgn}\,(f^m y - y)}{\mathrm{sgn}\,(x-y)} & \text{otherwise,} \end{cases}$$

and we set

$$\nu(x, f^m) = -L(x, f^m) \cdot \prod_{k=0}^{m-1} \epsilon(f^k x) \in \{-1, 1, 0\}\,.$$

(If the graph of f^m crosses the diagonal at $x \in \mathrm{Fix}\, f^m$, the point x is either attracting or repelling, and $\nu(x, f^m) = -1$ if and only if f^m is increasing and attracting at x.)

We extend now the set $\mathrm{Fix}\, f^m$ to a set $\mathrm{Fix}\star f^m$ containing *all* the periodic orbits, obtained by adding symbols $x*$ where $x \in [a_0, a_N]$, and $*$ is $+$ or $-$:

$$\mathrm{Fix}\star f^m = \mathrm{Fix}\, f^m \bigcup \{x* \,:\, f^m(x*) = x\,,$$

$$\exists\, k, i \text{ with } f^k(x*) = a_i \text{ and } \prod_{s=0}^{m-1} \epsilon(f^s(x$$

Again, $\mathrm{Fix}{\star}f^m$ is invariant under refinements of $\mathcal{Z}$. For $x* \in \mathrm{Fix}{\star}f^m \setminus \mathrm{Fix}\, f^m$, let

$$L(x*, f^m) = \begin{cases} 0 & \text{if } x* \text{ is (one-sided) repelling} \\ 1 & \text{if } x* \text{ is (one-sided) attracting} \end{cases}$$

and $\nu(x*, f^m) = -L(x*, f^m)$.

The *reduced zeta function* (Baladi-Ruelle [1993]) is:

$$\zeta_g(t) = \exp \sum_{m=1}^{\infty} \frac{t^m}{m} \sum_{x \in \mathrm{Fix}\star f^m} \nu(x, f^m) \cdot \prod_{k=0}^{m-1} g(f^k x) \, .$$

If all periodic points are repelling, and if $f^m a_i * \neq a_i$, for all $m \geq 1$ and $1 \leq i \leq N-1$, we recover the usual (weighted) zeta function (0.1).

Removing the assumption that $\# \mathrm{Fix}\, f^m < \infty$, we introduce another definition of the reduced zeta function (again following Baladi-Ruelle [1993]). For $m \geq 1$ we define $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1})$ to be:

-1 if the left end of the graph of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is $<$ the diagonal and the right end $>$ the diagonal,
$+1$ if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is increasing and the left end of the graph is $\geq$ the diagonal and the right end is $\leq$ the diagonal,
$+1$ if $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is decreasing and the left end of the graph is $>$ the diagonal and the right end is $<$ the diagonal,
0 in all other cases (in particular when the domain of $f_{\ell_m} \circ \cdots \circ f_{\ell_1}$ is empty or reduced to a point).

When $\mathrm{Fix}\, f^m$ is finite we have thus

$$\sum_{x \in \mathrm{Fix}\star f^m} \nu(x, f^m) = \sum_{\ell_1, \dots, \ell_m} \sum_{x \in \mathrm{Fix}\star f_{\ell_m} \circ \cdots \circ f_{\ell_1}} \nu(x, f_{\ell_m} \circ \cdots \circ f_{\ell_1})$$

$$= - \sum_{\ell_1, \dots, \ell_m} (\epsilon_{\ell_1} \cdots \epsilon_{\ell_m}) \cdot L(f_{\ell_m} \circ \cdots \circ f_{\ell_1}) \, ,$$

and it is natural to define:

$$\tilde{\zeta}_g(t) = \exp - \sum_{m=1}^{\infty} \frac{t^m}{m} \Bigg(\sum_{\ell_1, \dots, \ell_m} (\epsilon_{\ell_1} \cdots \epsilon_{\ell_m}) L(f_{\ell_m} \circ \cdots \circ f_{\ell_1}) \cdot g_{\ell_1, \dots, \ell_m} \Bigg) \, ,$$

where for any $\ell_1, \dots, \ell_m$ such that $L(f_{\ell_m} \circ \cdots \circ f_{\ell_1}) \neq 0$, we set $g_{\ell_1, \dots, \ell_m} = g_\eta$, where $\eta \in \mathcal{Z}_m$ is the interval corresponding to $\ell_1, \dots, \ell_m$, and g_η is chosen arbitrarily if $\mathrm{Fix}\, f^m \cap \eta$ is empty, and equal to the constant value of g on $\mathrm{Fix}\, f^m \cap \eta$ otherwise (if $\mathcal{Z}$ is generating this set contains at most one point, otherwise use the fact that g is constant on $\mathcal{Z}$-homtervals). If $\mathrm{Fix}\, f^m$ is finite for all $m \geq 1$, we have $\tilde{\zeta}_g(t) = \zeta_g(t)$. We set $\zeta_g(t) := \tilde{\zeta}_g(t)$ if $\mathrm{Fix}\, f^m$ is infinite for some m.

The results.

We can now state our main result.

Theorem. *Let f and g be as defined in the first paragraph of this section. With the above notations we have:*

(1) *For each $n \geq 1$, the following equality holds between analytic functions in the open disc D_n of radius $1/\rho^{(n)}$:*

$$\Delta^{(n)}(t) = \frac{1}{\zeta_{g^{(n)}}(t)} \,. \tag{1.1}$$

If the partition $\mathcal{Z}$ is generating, the zeroes of $\Delta^{(n)}(t)$ in D_n are exactly the inverses of the eigenvalues of the operator $\mathcal{L}_{g^{(n)}} : BV \to BV$

$$(\mathcal{L}_{g^{(n)}}\varphi)(x) = \sum_{fy=x} \varphi(y) \cdot g^{(n)}(y) \,,$$

outside of the closed disc of radius $\rho^{(n)}$. The order of a the corresponding eigenvalue.

Assume additionally that g is continuous at each point of $Fix\, f^m$, and one-sided continuous at each point of $Fix\star f^m \setminus Fix\, f^m$ for all $m \geq 1$. Then:

(2) *The sequence $\Delta^{(n)}(t)$ of analytic functions in D_n converges to a function $\Delta(t)$, analytic in the open disc D of radius $1/\bar{\rho}$. We have the following identity between analytic functions in D:*

$$\Delta(t) = \frac{1}{\zeta_g(t)} \,.$$

In particular $\zeta_g(t)$ is meromorphic in D, and the set of poles of $\zeta_{g^{(n)}}(t)$ in D converges to the set of poles of $\zeta_g(t)$ in D.

If the partition $\mathcal{Z}$ is generating, the zeroes of $\Delta(t)$ in the open disc of radius $1/\rho$ are exactly the inverses of the eigenvalues of the operator $\mathcal{L}_g : BV \to BV$ outside of the closed disc of radius ρ. The order of a zero coincides with the multiplicity of the corresponding eigenvalue.

Remarks.

(1) It may happen that both members of (1.1) admit analytic extensions to larger domains, possibly the whole complex plane, in particular when the map f has a finite Markov partition.

(2) If $\mathcal{Z}$ is generating, the fact that the zeta function $\zeta_g(t)$ admits a meromorphic extension to the disc D of radius $1/\rho$ was obtained by Baladi-Keller [1990] (see Baladi-Ruelle [1994] for the removal of the piecewise continuity assumption on g) by relating the meromorphic properties of $\zeta_g(t)$ to the spectral properties of $\mathcal{L}_g : BV \to BV$. In particular, they prove that the essential spectral radius

of $\mathcal{L}_g$ (or $\mathcal{L}_{g^{(n)}}$) acting on BV is bounded above by ρ (respectively $\rho^{(n)}$). The claims of the above theorem relating the zeroes of the determinantspectrum of $\mathcal{L}_g$ when $\mathcal{Z}$ is generating follow essentially from these previous results.

(3) When the partition $\mathcal{Z}$ is not necessarily generating, Ruelle [1993a, 1993b] introduces a different reduced zeta function, $\hat{\zeta}_g(t)$, using the notion of a representative set of periodic points. He relates the poles of $\hat{\zeta}_g(t)$ to the discrete spectrum of $\mathcal{L}_g$. The two reduced zeta functions $\zeta_g(t)$ and $\hat{\zeta}_g(t)$ in general do not coincide. When the weight g is locally constant, Ruelle [1993c] obtains in some cases a relationship between the spectrum of $\mathcal{L}_g$ and the poles of $\zeta_g(t)$. It would be interesting to see whether this holds also for weights g of bounded variation, with the aims of suppressing the assumption that $\mathcal{Z}$ is generating in all statements of our theorem, and comparing the zeta functions $\zeta_g(t)$ and $\hat{\zeta}_g(t)$.

(4) The assumption that g is (one-sided) continuous at periodic points of f can be relaxed to the requirement that the intersection of the set of discontinuities of g with the periodic points of f is finite. The equation $\Delta(t) \cdot \zeta_g(t) = 1$ must then be corrected by a finite product.

We have the following positive answer to a conjecture of S. Isola:

Corollary. *Let f and g be as in the first paragraph of this section. Assume further that each g_i is continuous, and that the partition $\mathcal{Z}$ is generating and Markovian, i.e., $f(a_i\pm) \in \{a_0, \ldots, a_N\}$ for all i.*

For $n \geq 1$, let $f^{(n)} : I \to I$ be a (possibly $\mathcal{Z}_n$-multivalued) transformation which is monotone on each interval of $\mathcal{Z}$, and which coincides with f at the endpoints of the intervals of $\mathcal{Z}_n$. Write ζ_g, $\zeta_{g^{(n)}}$, and $\zeta_{g^{(n)},functionsoffandg,offandg^{(n)}}$, and of $f^{(n)}$ and $g^{(n)}$. Write $\mathcal{L}_g$, $\mathcal{L}_{g^{(n)}}$, and $\mathcal{L}_{g^{(n)},f^{(n)}}$ for the corresponding transfer operators acting on BV.

Then:

(1) *As power series, $\zeta_{g^{(n)}} = \zeta_{g^{(n)},f^{(n)}}$. Thus, in the open disc D of radius $1/\rho$, the poles of $\zeta_{g^{(n)},f^{(n)}}$ and $\zeta_{g^{(n)}}$ converge to the poles of ζ_g.*

(2) *Let L_n be the restriction of $\mathcal{L}_{g^{(n)},f^{(n)}}$ to the finite-dimensional vector space E_n of functions $\varphi : I \to \mathbb{C}$ which are constant on the intervals of $\mathcal{Z}_n$. Outside of the closed disc of radius ρ the following spectra coincide (including the multiplicities) and converge to the spectrum of $\mathcal{L}_g$ as $n \to \infty$:*
   a) that of L_n;
   b) that of $\mathcal{L}_{g^{(n)},f^{(n)}}$;
   c) that of $\mathcal{L}_{g^{(n)}}$.

The interest of the above corollary is that we approach the discrete spectrum of $\mathcal{L}_g$ by the spectra of the finite matrices L_n, or by the zeroes of the determinants associated with f (or $f^{(n)}$) and $g^{(n)}$ (which, under the assumptions of the corollary, are "finite" objects, as pointed out in the introduction). Of course, this is only really useful when ρ is strictly smaller than the spectral radius of $\mathcal{L}_g$. One case where this holds is when f is piecewise C^{1+BV}, the weight $g = 1/|f'|$, and $|f'| \geq \lambda > 1$. One can then choose $f^{(n)}$ to be the piecewise affine approximations of f which coincide with f at the endpoints of $\mathcal{Z}^n$ (and are affine on these intervals). In that case, since g is piecewise continuous, and $f^{(n)\prime}$ is piecewise constant, one can set $g^{(n)} = 1/|f^{(n)\prime}|$, by the mean value theorem.

Note that our result doesthe eigenspaces corresponding to the discrete spectrum of $\mathcal{L}_g$.

2. Proofs

The main ingredient of our proof of the theorem is the following lemma:

Key Lemma. *Let $M^{(n)}$, $n \geq 1$ be a sequence of $m_n \times m_n$ matrices ($1 \leq m_n < m_{n+1} < \infty$) with complex coefficients. Write 1_{m_n} for the $m_n \times m_n$ identity matrix. Assume that there exist a constant $V > 0$, and for each $n \geq 1$ an m_n-dimensional vector $v^{(n)} = (v_1^{(n)}, \dots, v_{m_n}^{(n)}) \in \mathbb{R}_+^{m_n}$ with*

$$\begin{cases} \sum_{i=1}^{m_n} v_i^{(n)} \leq V\,, \\ |M_{ij}^{(n)}| \leq v_i^{(n)}\,, \qquad \forall\, 1 \leq i, j \leq m_n\,. \end{cases} \tag{2.1}$$

Then for each $K > 0$, there exists a constant $C(K, V)$ such that the determinants $d^{(n)}(\lambda) := \det(1_{m_n} - \lambda M^{(n)})$ satisfy $|d^{(n)}(\lambda)| \leq C(K, V)$, for all λ in the disc of radius K and $n \geq 1$. In particular, $|d^n(1)|$ is bounded by an expression depending only on V.

Proof of the lemma.

We follow the classical Fredholm argument (see e.g. Riesz-Sz.-Nagy [1955, page 172]). The determinant can be developed as follows:

To prove (2.3), we apply the classical Hadamard inequality (see e.g. Riesz-Sz.-Nagy [1955, page 176]), which says that the determinant $\det C$ of a $k \times k$ matrix with coefficients $C_{i,j} \in \mathbb{C}$ satisfies the bound

$$|\det C| \leq \|C_1\|_2 \cdots \|C_k\|_2\,, \tag{2.4}$$

where $\|C_i\|_2 = (\sum_{j=1}^k |C_{ij}|^2)^2$ is the euclidean length of the i^{th} row of the matrix. Indeed, let $1 \leq \ell_1, \dots, \ell_k \leq m_n$ and set $C_{ij} = M_{\ell_i \ell_j}^{(n)}$. The assumption on the matrices implies that $\|C_i\|_2 = (\sum_{j=1}^k |M_{\ell_i \ell_j}^{(n)}|^2)^{1/2} \leq k^{1/2} v_{\ell_i}^{(n)}$. By the Hadamard inequality (2.4), the determinant of the $k \times k$ minor C_{ij} satisfies:

$$|\det C| \leq k^{k/2} \prod_{i=1}^k v_{\ell_i}^{(n)}\,.$$

Summing over the ℓ_i, it suffices to use the upper bound V in (2.1) to obtain (2.3). $\square$

We can now prove our main result:

Proof of the theorem.

(1) Equality (1.1) is a direct application of Theorem 1.1 in Baladi-Ruelle [1993]. We leave to the reader the verification that the two additional rows and columns in the kneading matrices defined here lead to the suppression of the factor $1-\frac{1}{2}(\epsilon_1 z_1+\epsilon_N z_N)$ in the previous paper (it suffices to check that this factor disappears in Lemma 2.1 of Baladi-Ruelle [1993]). Each $\Delta^{(n)}(t)$ is analytic in the disc of radius $1/\rho^{(n)}$ because each coefficient of the matrix $K^{(n)}(t)$ converges in this disc.

As mentioned in Section 1, the relationship between the zeroes of $\Delta^{(n)}$ (or poles of $\zeta_g^{(n)}$) and the discrete spectrum of $\mathcal{L}_{g^{(n)}}$ follows from Hofbauer-Keller [1984] and Baladi-Keller [1990]. The special treatment reserved for the periodic orbits through the turning points does not affect the discrete spectrum of the transfer operator.

(2) Since g is continuous at periodic points and constant on $\mathcal{Z}$-homtervals, for each fixed $m \geq 1$, the expression $g^{(n)}_{\ell_1,\dots,\ell_m}$ in the definition of $\tilde\zeta_{g^{(n)}}(t)=\zeta_{g^{(n)}}(t)$ converges to $g_{\ell_1,\dots,\ell_m}$, the corresponding expression in $\tilde\zeta_g(t)=\zeta_g(t)$, as $n\to\infty$. Also, if we define

$$\zeta_{m,g}=\sum_{\ell_1,\dots,\ell_m}|L(f_{\ell_m}\circ\cdots\circ f_{\ell_1})|\cdot|g_{\ell_1,\dots,\ell_m}|\,,$$

(and analogously $\zeta_{m,g^{(n)}}$), denoting by R_n the radius of convergence of $\zeta_{g^{(n)}}$, and R the radius of convergence of $\zeta_g(t)$, we have $R_n \geq \lim_{m\to\infty}\zeta_{m,g^{(n)}}^{-1/m} \geq \lim_{m\to\infty}\zeta_{m,g}^{-1/m}$. Hence, since

$$R_\infty := \lim_{m\to\infty}\zeta_{m,g}^{-1/m}\in($$

(recall that g is bounded and the number of laps of f^n is bounded by $(N+1)^n$), the analytic functions $\zeta_{g^{(n)}}(t)$ converge to $\zeta_g(t)$ in the disk of radius R_∞. Therefore, if we can show that the analytic functions $\Delta^{(n)}(t)$ are uniformly bounded in the (possibly larger) disc $\tilde D$ of radius $1/\tilde\rho$ for each $\tilde\rho>\bar\rho$, then we have proved (2).

We shall perform elementary row operations on the matrices $K^{(n)}(t)$ for $1\leq n$, obtaining new matrices $L^{(n)}(t)$ with $\det K^{(n)}(t)=\det L^{(n)}(t)$. For $1\leq n$, write $m_n=N_n+1$ and define

$$M^{(n)}(t)=1_{m_n}-L^{(n)}(t)\,, \tag{2.5}$$

where 1 is the $m_n\times m_n$ identity matrix. We shall prove that for any fixed $t_0\in\tilde D$ the conditions of the key lemma are satisfied for the sequence of matrices

$M^{(n)}(t_0)$ and a constant V independent of $t_0 \in \tilde{D}$. This yields the needed uniform bound.

Let $t_0 \in \tilde{D}$ and $2 \leq n < \infty$ be fixed. We leave unchanged the rows which correspond to the initial partition of the interval:

$$L_i^{(n)}(t_0) = K_i^{(n)}(t_0)\,, \text{ for } i = 0, \ldots, N_1\,. \tag{2.6}$$

We now compute the corresponding bounds $v_i^{(n)}$. For any $0 \leq i, j \leq N_n$, we have $M_{ij}^{(n)}(t_0) = \sum_{k=1}^{\infty} a_{ij}(k) t_0^k$ with the $a_{ij}(k) \in \mathbb{C}$ satisfying:

$$|a_{ij}(k)| \leq \sup_x \prod_{s=0}^{k-1} |g^{(n)}(f^s x)| \leq \mathcal{K} \cdot \bar{\rho}^k\,,$$

where $\mathcal{K} > 0$ is a suitable constant. Setting $v_i^{(n)} = \mathcal{K}/((\tilde{\rho}/\bar{\rho}) - 1)$, for $i = 0, \ldots, N_1$, we have

$$\sum_{i=0}^{N_1} v_i^{(n)} \leq (N_1 + 1) \cdot \frac{\mathcal{K}}{(\tilde{\rho}/\bar{\rho}) - 1}\,.$$

We now use the recursive construction of the matrix to perform the elementary operations on the remaining rows of $K^{(n)}(t_0)$We start with the bottom rows and fix $m_{n-1} \leq i < m_n$. By definition

$$\vec{K}_i^{(n)}(t_0) = \frac{1}{2}\left[\vec{\theta}^{(n)}(u_i+) - \vec{\theta}^{(n)}(u_i-)\right],$$

where u_i is an endpoint of the partition $\mathcal{Z}_n$ which is not an endpoint of the partition $\mathcal{Z}_{n-1}$. We have $\epsilon(u_i+) = \epsilon(u_i-)$, we denote the common value ϵ_i. By definition, if $\epsilon_i = +1$ (respectively -1) $f(u_i-)$ and $f(u_i+)$ are limits from the left and from the right (respectively right and left) to an endpoint $u_{\ell(i)}$ of the partition $\mathcal{Z}_{n-1}$ (which is not an endpoint of the partition $\mathcal{Z}_{n-2}$). The index $\ell(i)$ thus corresponds to a *higher* row $m_{n-2} \leq \ell(i) < m_{n-1}$ of the matrix $K^{(n)}(t_0)$. The crucial remark is:

$$\begin{aligned}
\vec{K}_i^{(n)}(t_0) = (\delta_{ij}, j = 0, \ldots, N_n)& \\
&+ \epsilon_i t_0 \cdot g^{(n)}(u_i+) \cdot (\vec{\theta}^{(n)}(u_{\ell(i)}[\epsilon_i]) - \vec{\theta}^{(n)}(u_{\ell(i)}[-\epsilon_i]))/2 \\
&+ \epsilon_i t_0 \cdot g^{(n)}(u_i+) \cdot \vec{\theta}^{(n)}(u_{\ell(i)}[-\epsilon_i])/2 \\
&- \epsilon_i t_0 \cdot g^{(n)}(u_i-) \cdot \vec{\theta}^{(n)}(u_{\ell(i)}[-\epsilon_i])/2 \\
=&(\delta_{ij}, j = 0, \ldots, N_n) \\
&+ t_0 \cdot g^{(n)}(u_i+) \cdot K_{\ell(i)}^{(n)}(t_0) \\
&+ \epsilon_i t_0 \cdot \frac{g^{(n)}(u_i+) - g^{(n)}(u_i-)}{2} \cdot \vec{\theta}^{(n)}(u_{\ell(i)}[-\epsilon_i])\,.
\end{aligned}$$

For $m_{n-1} \le i < m_n$, we define:

$$\vec{L}_i^{(n)}(t_0) = (\delta_{ij}, j = 0, \dots, N_n) \\ + \epsilon_i t_0 \cdot \frac{(g^{(n)}(u_i+) - g^{(n)}(u_i-))}{2} \cdot \vec{\theta}^{(n)}(u_{\ell(i)}[-\epsilon_i]) \,. \tag{2.7}$$

(If the weight g is in fact locally constant on some refinement $\mathcal{Z}_k$, the rows defined by (2.7) are of the form $(\delta_{ij}, j = 0, \dots, N_n)$ $n \ge k$ and $i \ge m_k$.) We proceed similarly for rows $m_{n-p} \le i < m_{n-p+1}$ with $2 \le p \le n-1$. By the same analysis as for the $N_1 + 1$ first rows, we have for $N_1 < i < m_n$

$$|\theta^{(n)}(u_i *)_j(t_0)| \le \frac{\mathcal{K}}{(\tilde{\rho}/\bar{\rho}) - 1} \,, \text{ for } 0 \le j \le N_n \,.$$

Thus for $N_1 + 1 < i < m_n$ we can set

$$v_i^{(n)} = |g^{(n)}(u_i+) - g^{(n)}(u_i-)| \cdot \frac{\mathcal{K} \cdot \tilde{\rho}/2}{(\tilde{\rho}/\bar{\rho}) - 1} \,.$$

Since the points u_i are all distinct, we have

$$\sum_{i=N_1+1}^{N_n} v_i^{(n)} \le \operatorname{var} g^{(n)} \cdot \frac{\mathcal{K} \cdot \tilde{\rho}/2}{(\tilde{\rho}/\bar{\rho}) - 1} \le \operatorname{var} g \cdot \frac{\mathcal{K} \cdot \tilde{\rho}/2}{(\tilde{\rho}/\bar{\rho}) - 1} \,.$$

The key lemma yields the needed uniform bound on $\Delta^{(n)}(t)$ for $t \in \tilde{D}$.

The claim on the discrete spectrum of $\mathcal{L}_g$ follows from Baladi-Keller [1990] (and Baladi-Ruelle [1994] to remove the assumption that the g_i are continuous). □

A counter-example to $\lim_{n\to\infty} \det(1 - M^{(n)}(t)) = \det(1 - M^{(\infty)}(t))$.

If g is continuous, the vectors $v^{(n)}$ constructed in the proof of the theorem satisfy $v_i^{(n)} \to 0$ as $n \to \infty$, for each $i > N_1$. Hence each matrix row $M_i^{(n)}(t_0)$, with $i > N_1$, converges to the countable zero vector as $n \to \infty$. This allows us to construct a counter-example which shows that, in general, $\det(1 - M^{(\infty)}(t)) \ne \Delta(t)$ (where $M^{(\infty)}(t)$ is viewed as a Fredholm operator acting on $l^1(\mathbb{N})$ as described in the appendix). Indeed, let $f : [0,1] \to [0,1]$ be the tent-map which has slope 2 on $[0, 1/2]$ and -2 on $[1/2, 1]$ (with turning points $a_0 = 0$, $a_1 = 1/2$, $a_2 = 1$), and let g be a strictly positive continuous function of bounded variation with $i \ge 3$ in the matrix $M^{(\infty)}(t)$ vanish, the determinant of $1 - M^{(\infty)}(t)$ is equal to the determinant of the 3×3 matrix obtained by considering the first three rows and columns of $1 - M^{(\infty)}(t)$. ¿From the definition of $M^{(n)}(t)$, we see that this determinant is a rational fraction in t, which depends only on (the orbit of a_1 under f and) the three parameters $g(0)$, $g(1/2)$ and $g(1)$. However, the zeta function $\zeta_g(t)$ of f (which is equal to the ordinary Artin-Mazur zeta function multiplied by a factor

$(1-g(0)t)$ to take into account the fixed point at 0), is not invariant when we consider different weights having the same values at 0,1/2 and 1 (take for example a strictly positive continuous weight h of bounded variation with $h(0)=g(0)$, $h(1/2)=g(1/2)$ and $h(1)=g(1)$, but with $h(x)\neq g(x)$, where $1/2<x<1$ is the other fixed point of f — then the coefficients of order one in the Taylor series of $\zeta_g(t)$ and $\zeta_h(t)$ are different).

Note that when g has finitely or countably many discontinuities, the matrix $M^{(\infty)}(t)$ contains information not only on the values of the weights at the turning points, but also on these discontinuities.

Proof of the corollary.

(1) Since f is Markovian, the kneading determinant $\Delta^{(n)}(t)$ only depends on f evaluated at the endpoints of $\mathcal{Z}_n$. Since f and $f^{(n)}$ coincide on these points by construction, the n^{th} kneading determinant of $f^{(n)}$ coincides with $\Delta^{(n)}(t)$. The claimed equality thus follows from (1.1), and the assertion about convergence of the poles follows from (2) in the theorem.

(2) We first show that the spectra a) and b) are the same. We follow the argument of Pollicott [1986,since the spectrum of $\mathcal{L}_{g^{(n)},f^{(n)}}$ is a subset of the union of the spectrum of $\mathcal{L}_{g^{(n)},f^{(n)}}$ restricted to E_n and the spectrum of the quotient operator $\mathcal{L}_{g^{(n)},f^{(n)}}/E_n : BV/E_n \to BV/E_n$, it suffices to check that the spectral radius of $\mathcal{L}_{g^{(n)},f^{(n)}}/E_n$ is not larger than $\rho^{(n)}$. We have not been able to find a proof of this (certainly well-known) fact in the literature, and we include it here. The induced norm on BV/E_n is $\|\varphi\|_{E_n}=\sum_{\eta\in\mathcal{Z}_n}\operatorname{var}_\eta(\varphi)$. The key observation is that if $\eta\in\mathcal{Z}_n$ and $m>n$, then for any $\varphi\in BV$,

$$\begin{aligned}\operatorname{var}_\eta\Big(\sum_{\xi\in\mathcal{Z}_m}\mathcal{L}^m_{g^{(n)},f^{(n)}}(\varphi\cdot\chi_\xi)\Big) = \\ \sum_{\xi\,:\ \text{interior}\ ((f^{(n)})^m\xi\cap\eta)\neq\emptyset} \operatorname{var}_\xi(\varphi\cdot g^{(n)}_m) \\ + \sum_{x\in\mathcal{D}(\eta,m)} |(\varphi\cdot g^{(n)}_m(x+)-(\varphi\cdot g^{(n)}_m(x-)|\,,\end{aligned}$$

where the set $\mathcal{D}(\eta,m)$ is the set of endpoints of the connected components of ${f^{(n)}}^{-m}(\eta)$ which are not endpoints of η, and where we have written

$$g^{(n)}_m(x)=g^{(n)}(x)\cdot g^{(n)}(f^{(n)}(x))\cdot\cdots\cdot g^{(n)}({f^{(n)}}^{m-1}(x))\,.$$

Thus for $m\geq n$:

$$\begin{aligned}\|\mathcal{L}^m_{g^{(n)},f^{(n)}}\varphi\|_{E_n} &= \sum_{\eta\in\mathcal{Z}_n}\operatorname{var}_\eta(\mathcal{L}^m_{g^{(n)},f^{(n)}}\varphi) \\ &= \sum_{\eta\in\mathcal{Z}_n}\operatorname{var}_\eta\Big(\mathcal{L}^m_{g^{(n)},f^{(n)}}(\sum_{\xi\in\mathcal{Z}_m}\varphi\cdot\chi_\xi)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\eta \in \mathcal{Z}_n} \operatorname{var}_\eta \Big(\sum_{\xi \in \mathcal{Z}_m} \mathcal{L}^m_{g^{(n)}, f^{(n)}} (\varphi \cdot \chi_\xi) \Big) \\
&= \sum_{\xi \in \mathcal{Z}_m} \operatorname{var}_\xi (\varphi \cdot g_m^{(n)}) \\
&\qquad + \sum_{\eta \in \mathcal{Z}_n} \sum_{x \in \mathcal{D}(\eta, m)} |(\varphi \cdot g_m^{(n)} \\
&\le \sup g_m^{(n)} \cdot \sum_{\eta \in \mathcal{Z}_n} \operatorname{var}_\eta (\varphi) = \sup g_m^{(n)} \cdot \|\varphi\|_{E_n} \, .
\end{aligned}
$$

Hence, $\lim_{m\to\infty} (\|\mathcal{L}^m_{g^{(n)}, f^{(n)}}\|_{E_n})^{1/m} \le \rho^{(n)}$, as desired.

The spectra b) and c) are the same because of part (1) of this corollary combined with the second assertion of claim (1) in the theorem.

Since the partition is generating, outside of the closed disc of radius $\rho^{(n)}$, the spectrum of $\mathcal{L}_{g^{(n)}, f^{(n)}}$ acting on BV consists of the inverses of the poles of $\zeta_{g^{(n)}, f^{(n)}}$. Similarly, outside of the closed disk of radius ρ, the spectrum of $\mathcal{L}_g$ acting on BV consists of the inverses of the poles of ζ_g. It hence suffices to apply part (1) of this corollary again to get the convergence. $\square$

Appendix: Viewing the countable matrix $M^{(\infty)}(t)$ as a Fredholm operator

In this appendix, we will indicate how the theory of Grothendieck yields:

Proposition. *Let f and g be as in Section 1. Assume that g is continuous at each point of $Fix\, f^m$, and one-sided continuous at each point of $Fix{\star} f^m \setminus Fix\, f^m$, for all $m \ge 1$. Let $M^{(\infty)}(t) = \lim_{n\to\infty} M^{(n)}(t)$ be the countable matrix defined by (2.5), (2.6) and (2.7). Then for each $t_0 \in D$, the open disc of radius $1/\bar\rho$, the matrix $M^{(\infty)}(t_0)$, when viewed as a linear operator $u = u(t_0)$:*

$$
u : l^1(\mathbb{N}) \to l^1(\mathbb{N}) \, , \qquad u : (x_j)_j \mapsto (\sum_j M^{(\infty)}_{ij}(t_0) x_j)_i
$$

is a Fredholm operator (in the sense of Grothendieck [1956]) with trace norm $\|u\|_1 = \|M^{(\infty)}(t_0)\| = \sum_{i=0}^\infty v_i$, wherv$_i := \sup_j |M^{(\infty)}_{ij}(t_0)|$.

In particular, for each $t_0 \in D$, the operator u has a Fredholm determinant $d(t_0)(\lambda) =$ $\det(1-\lambda u)$ which is an entire function of the complex variable λ, and an entire function of the operator $u \in l^1(\mathbb{N})' \hat\otimes l^1(\mathbb{N})$. It follows that $d(t_0)(\lambda)$ is analytic in t_0 for $t_0 \in D$.

Proof of the proposition. Let $v_i^{(n)}$, for $n \ge 1$ be the sequence of vectors introduced in the proof of (2) in the main theorem (for some fixed $\tilde\rho > \bar\rho$). Note that $v_i^{(n)}$ does not depend on n, for $i \le N_1$, and that $v_i^{(n)}$ converges to $|g(u_i+) - g(u_i-)| \cdot \mathcal{K} \cdot \tilde\rho / (2 \cdot ((\tilde\rho/\bar\rho) - 1))$, as $n \to \infty$, for $i > N_1$. Writing $v_i^{(\infty)} := \lim_{n\to\infty} v_i^{(n)}$, we have $\sum_{i=0}^\infty v_i^{(\infty)} < \infty$. Also, the numbers v_i introduced in the statement of this proposition satisfy $v_i \le v_i^{(\infty)}$. (Indeed, for any fixed i,j we have, just as in the proof of the theorem in Section 2, $M_{ij}^{(\infty)}(t_0) = \sum_k a_{ij}(k)t_0^k$, with $|a_{ij}(k)| \le \mathcal{K}\cdot\bar{\rho}^k$ for $i \le N_1$, and $|a_{ij}(k)| \le |g(u_i+)-g(u_i-)|\cdot\mathcal{K}\cdot\tilde{\rho}\cdot\bar{\rho}^k/2$ for $i > N_1$.)

We now wish to show that a result in Grothendieck [1956, III.1, Corollary 1] can be applied. For the convenience of the reader we give the statement below.

Let $\mathcal{I}$ be a locally compact space endowed with a measure μ and E a Banach space; the Fredholm applications $u : E \to \mathcal{L}^1(\mu)$ are those defined by an integrable map $h : X \to E'$ by $(u(x))(i) =< x, h(i) >$. The trace norm of u is the norm of h in $\mathcal{L}^1_{E'}(|\mu|(i)$.

We apply the result above to the case $\mathcal{I} = \mathbb{N}$, μ the discrete measure, $E = \mathcal{L}^1(\mu) = l^1(\mathbb{N})$, and $h(i) = M_{i\cdot}^{(\infty)}(t_0)$. The row $M_{i\cdot}^{(\infty)}(t_0)$ acts on $l^1(\mathbb{N})$ by $< x, M_{i\cdot}^{(\infty)}(t_0) >= \sum_j M_{ij}(t_0)x_j$. We have

$$\begin{aligned} \|h(i)\| = \|M_{i\cdot}^{(\infty)}(t_0)\| &= \sup_{x\in l^1(\mathbb{N}), \|x\|=1} \sum_j |M_{ij}^{(\infty)}(t_0)\cdot x_j| \\ &= \sup_j |M_{ij}^{(\infty)}(t_0)| = \sum_{i=0}^{\infty} v_i < \infty\,, \end{aligned} \tag{A.1}$$

which shows that $h(i)$ is of integrable norm. The chain of equalities (A.1) also gives the formula for the trace norm. $\square$

We note that for each $t_0 \in D$, the transposition of the matrix $M^{(\infty)}(t_0)$, when viewed as a linear operator $u' = u'(t_0)$:

$$u' : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})\,, \qquad u' : (y_i)_i \mapsto (\sum_i M_{ij}^{(\infty)}(t_0)y_i)_j$$

is a Fredholm operator with trace norm $\|u'\|_1 = \|M^{(\infty)}(t_0)\| = \sum_{i=1}^\infty v_i$. (Apply Proposition 1 in Grothendieck [1956, III.1] to the measure space $\mathbb{N}$, the Banach space $E = F = l^\infty(\mathbb{N})$, and the functions $k \in \mathcal{L}^\infty_{E'_s}(\mathbb{N}) = l^\infty_{E'_s}(\mathbb{N})$ and $h \in \mathcal{L}^1_E(\mu) = l^1_E(\mathbb{N})$:

$$\begin{aligned} k(i) : l^\infty(\mathbb{N}) \to \mathbb{C}, \quad i \in \mathbb{N}, &\qquad k(i) : (y_j)_j \mapsto y_i\,, \\ h : \mathbb{N} \to l^\infty(\mathbb{N})\,, &\qquad h : i \mapsto M_{i\cdot}^{(\infty)}(t_0)\,, \end{aligned}$$

using again (A.1) to see that the sequence $\|h(i)\|$ is summable. The operator $u' : E \to E$ is the one noted $\int k(i)\otimes h(i)\, d\mu(i) = \sum_i k(i)\otimes h(i)$ by Grothendieck, to whom we refer for details.) The expressions by Grothendieck [1956, III.1 (9)] (see also Grothendieck [1956, III.2 (4)]) for the coefficients α_n in the Taylor series of $d(t_0)(\lambda)$ and $d'(t_0)(\lambda)$ (see also Grothendieck [1956, II.2. Proposition 1]) coincide. Hence the Fredholm determinants $\det(1 - \lambda \cdot u)$ and $\det(1 - \lambda \cdot u')$ coincide as analytic functions in $t_0 \in D$ and $\lambda \in \mathbb{C}$.

Note that the fact that a matrix such that $\sum_i \sup_j |M_{ij}^{(\infty)}| < \infty$, acting to the right on $l^1(\mathbb{N})$ or to the left on $l^\infty(\mathbb{N})$, defines a *bounded* operator is an easy exercise (see, e.g. Kato [1984, III.2.1 Ex. 2.3-2.4, III.3.1 Ex 3.1]).

References

1. V. Baladi and G. Keller, *Zeta functions and transfer operators for piecewise monotone transformations*, Comm. Math. Phys. **127** (1990), 459–477.
2. V. Baladi and D. Ruelle, *An extension of the theorem of Milnor and Thurston on the zeta functions of interval maps*, IHES preprint, to appear Ergodic Theory Dynamical Systems (1993).
3. V. Baladi and D. Ruelle, *Some properties of zeta functions associated with maps in one dimension*, in preparation (1994).
4. A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France **84** (1956), 319–384.
5. F. Hofbauer and G. Keller, *Zeta-functions and transfer-operators for piecewise linear transformations*, J. reine angew. Math. **352** (1984), 100–113.
6. T. Kato, *Perturbation theory for linear operators*, Second printing of the second edition, Springer-Verlag, Berlin Heidelberg New York, 1984.
7. J. Milnor and W. Thurston, Dynamical Systems (Maryland 1986-87) (J.C. Alexander, ed.), Lecture Notes in Math. Vol. 1342, Springer-Verlag, Berlin Heidelberg New York, 1988.
8. M. Mori, *Fredholm determinant for piecewise linear transformations*, Osaka J. Math **27** (1990), 81–116.
9. M. Mori, *Fredholm determinant for piecewise monotonic transformations*, Osaka J. Math. (1992), 497–529.
10. M. Pollicott, *Meromorphic extensions of generalised zeta functions*, Invent. Math. **85** (1986), 147–164.
11. F. Riesz and B. Sz.-Nagy, *Leçons d'Analyse Fonctionnelle*, 3ème édition, Académie des Sciences de Hongrie, 1955.
12. D. Ruelle, *Dynamical zeta functions for maps of the interval*, IHES preprint, to appear Bull. Amer. Math. Soc. (N.S.) (1993a).
13. D. Ruelle, *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval*, Aisenstadt Lecture 1993, Univ. de Montréal, to be published (1993b).
14. D. Ruelle, *Functional equation for dynamical zeta functions of Milnor-Thurston type*, IHES preprint (1993c).

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