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# Comparison of Dynamical Entropies for the Noncommutative Shifts

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**Abstract**

It is shown that two definitions of dynamical entropy for noncommuative shifts give different results.

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The usefulness of the concept of dynamical entropy in classical ergodic theory was encouraging to look for a generalization of the definition to automorphisms on nonabelian algebras. There are several possibilities. In this note we want to concentrate on the one given in [CS] and its generalization in [CNT,ST] and compare it with that of [AF1]. Both can be calculated for the noncommutative shifts and give in some cases completely different answers, varying between zero and infinity. This supports the interpretation that the entropy proposed in [AF1] is related to how quickly by repeated measurements the information on the system grows, whereas the entropy proposed in [CS,CNT,ST] controls how quickly operators become independent from one another.

## The Definitions

We first give the two definitions of dynamical entropies we want to compare. We start with a von Neumann algebra $\mathcal{M}$, $\Theta$ an automorphism acting on it and $\omega$ a $\Theta$–invariant state over $\mathcal{M}$.

The definition offered in [AF1] which is related to some earlier ideas of [L] is the following:

**Definition 1:** Let $\mathcal{A}_0$ be a subalgebra in $\mathcal{M}$, $\Theta\ \mathcal{A}_0 \subset \mathcal{A}_0$. A finite operational partition of unity of size $k$ is a set $X = \{x_1, \ldots, x_k\}$ consisting of elements of $\mathcal{A}_0$ such that

$$\sum_{i=1}^{k} x_i^*\ x_i = 1. \tag{1}$$

A composition of two partitions $X$ and $Y$ is defined by

$$X \circ Y = \{x_i y_j | i = 1, \ldots, k, j = 1, \ldots, \ell\}. \tag{2}$$

To a given partition of size $k$ we assign a state on the $k$–dimensional matrix algebra $M_k$

$$[\rho_X]_{ij} = \mathrm{Tr}\ \rho_X\ e_{ji} = \omega(x_j^*\ x_i) \qquad e_{ij} \text{ matrix unit in } M_k. \tag{3}$$

Consider the partition $\Theta^{m-1}X \circ \ldots \circ \Theta\ X \circ X$ which is of size $km$. Then the dynamical entropy $h^I$ is given as

$$h^I \equiv h^I(\omega, \Theta, \mathcal{A}_0) = \sup_X \limsup_{n\to\infty} \frac{1}{n}\ S(\rho_{\Theta^{m-1}X\circ\ldots\circ X}). \tag{4}$$

**Remark:** It has not been shown that $h^I$ might be independent on the chosen subalgebra $\mathcal{A}_0$. The restriction to $\mathcal{A}_0$ is so far necessary to have control on the convergence in explicit examples. For all known examples there exists a natural choice of $\mathcal{A}_0$.

Another definition for a dynamical entropy was proposed in [CS] for type $\mathrm{II}_1$ algebras and generalized to arbitrary algebras in [CNT]. In [ST] an alternative definition was proposed and it was shown that for hyperfinite algebras the so obtained dynamical entropies coincide with the one proposed in [CNT]. We follow the definition in [ST].

**Definition 2:** Let $(\mathcal{M}, \Theta, \omega)$ be given. We call $(\mathcal{M}\otimes\mathcal{B}, \Theta\otimes\sigma, \lambda)$ stationary couplings of $\mathcal{M}$ with abelian algebras $\mathcal{B}$ with automorphism $\sigma$ where $\lambda$ is a state on $\mathcal{M}\otimes\mathcal{B}$ with $\lambda\circ\Theta\otimes\sigma = \lambda$ and $\lambda|_{\mathcal{M}} = \omega,\ \lambda|_{\mathcal{B}} = \mu$. Let $P$ be a finite subalgebra of $\mathcal{B}$ and

$$h_\mu(\sigma, P) = \lim \frac{1}{n}\, H_\mu\left(\bigvee_{k=0}^{n-1} \sigma^k P\right)$$

the Kolmogorov–Sinai entropy of $\sigma$ with respect to the partition $P$.

With $S(\lambda|\lambda_i)_{\mathcal{M}}$ the relative entropy of $\lambda, \lambda_i$ considered as states over $\mathcal{M}$ [OP] we define

$$H(P|\mathcal{M}) = \sup_{\sum\lambda_i=\lambda} \sum S(\lambda|\lambda_i)_P - S(\lambda|\lambda_i)_{\mathcal{M}} = S_\mu(P) - S(\omega\otimes\mu|\lambda)_{P\otimes\mathcal{M}}$$

(see [N]). Then

$$h^{II}(\Theta) = \sup_{\mathcal{B},P,\lambda} [h_\mu(\sigma, P) - H(P|\mathcal{M})]$$

where the supremum has to be taken over all possible stationary couplings.

The entropies satisfy

$$\begin{array}{rcll} h^I(\omega, \Theta^n, \mathcal{A}_0) & \geq & n\, h^I(\omega, \Theta, \mathcal{A}_0) & \\ h^{II}(\Theta^n) & = & n\, h^{II}(\Theta) & \mathcal{M} \text{ hyperfinite} \\ h^{II}(\Theta) & \leq & h^{II}(\Theta^n) \leq n\, h^{II}(\Theta) & \mathcal{M} \text{ arbitrary.} \end{array}$$

Both entropies have been evaluated [AF1,CNT] for the shift on a lattice algebra $\mathcal{M} = \left\{\bigotimes_{n=-\infty}^{+\infty} M_d^{(n)}\right\}''$ with $\omega$ extremal translationally invariant state

$$h^I = s(\omega) + \ln d, \qquad h^{II} = s(\omega)$$

where $s(\omega)$ is the entropy density. A similar relation holds for CAR quasifree automorphisms [NT1,NT2,SV,AF2].

Now we want to compare them for the noncommutative Powers–Price shifts to demonstrate their different sensitivity on commutativity.

**Example 1:** We consider the algebra $\mathcal{A}_0$ introduced in [P,P]: It is generated by operators $e_i$, $i \in Z$, satisfying

(i) $e_i^* = e_i$, $e_i^2 = 1$,

(ii) $e_i\, e_j = e_j\, e_i(-1)^{g(|i-j|)}$, $g: N \to \{0, 1\}$.

On $\mathcal{A}_0$ the shift acts as $\Theta\ e_i = e_{i+1}$. The elements of $\mathcal{A}_0$ are linearly spanned by words $w_I = e_{i_1}\ldots e_{i_n}$ with $I = \{i_1 < i_2 < \ldots < i_n\}$. Again words either commute or anticommute. On $\mathcal{A}_0$ we define the state

$$\begin{array}{rcll} \omega(1) & = & 1 & \\ \omega(w_I) & = & 0 & I \neq \emptyset. \end{array}$$

This state is tracial and invariant under the shift.

**Lemma:** If $g \equiv 0$, then $\{\mathcal{A}_0'', \Theta, \omega\}$ corresponds to the baker transformation. If $g \not\equiv 0$, then $\mathcal{A}_0''$ is the hyperfinite $\mathrm{II}_1$ factor.

**Theorem 1:**

$$h^I(\omega, \Theta, \mathcal{A}_0) = \ln 2 \qquad \text{independently of } g.$$

**Proof:** To get the lower bound we choose the partition

$$X = \left\{ \frac{1+e_1}{2}, \frac{1-e_1}{2} \right\} = \{p_{11}, p_{12}\}. \tag{5}$$

Define $\mathcal{A}_{[n,\ell]}$ to be the algebra generated by $\{e_i, n \leq i \leq \ell\}$. Then there exists a conditional expectation $E_{[n,\ell]}$ of $\mathcal{A}_0$ (and therefore $\mathcal{M}$) onto $\mathcal{A}_{[n,\ell]}$, preserving the state $\omega$

$$E_{[n,\ell]}(w_I) \begin{array}{ll} = w_I & \qquad I \subset [n,\ell] \\ = 0 & \qquad I \cap [n,\ell]^c \neq \emptyset. \end{array}$$

Especially $E_{2,\infty} p_{1i} = 1/2$. Therefore

$$\begin{aligned} \omega(p_{ki_k} \dots p_{1i_1} p_{1j_1} \dots p_{kj_k}) &= \omega(E_{[2,\infty)}\, p_{ki_k} \dots p_{1i_1} p_{1j_1} \dots p_{kj_k}) \\ &= \frac{1}{2}\, \delta_{i_1 j_1}\, \omega(p_{ki_k} \dots p_{2i_2} p_{2j_2} \dots p_{kj_k}) \\ &= 2^{-k}\, \delta_{i_1 j_1} \dots \delta_{i_k j_k} \\ &= [\rho_{\Theta^{k-1} X \circ \dots \circ X}]_{j_1 \dots j_k, i_1 \dots i_k} \end{aligned} \tag{6}$$

so that $S(\rho_{\Theta^{k-1} X \circ \dots \circ X}) = k \ln 2$ or $h^I \geq \ln 2$.

To estimate the upper bound we take a partition of unity $X = \{x_1, x_2, \dots, x_n\}$ with $x_j \in \mathcal{A}_{[1,\ell]}$. Hence $X, \Theta X, \Theta^{m-1} X \in \mathcal{A}_{[1,\ell+m-1]}$. Generally any algebra $\mathcal{A}_{[1,r]}$ is isomorphic to a certain matrix algebra $\sum \bigoplus_\nu M_{n_\nu}$ where $M_{n_\nu}$ denotes a full $n_\nu \times n_\nu$ matrix algebra. Moreover

$$\sum_\nu n_\nu^2 = \dim \mathcal{A}_{[1,r]} = 2^r.$$

Using the above representation the faithful tracial state $\omega$ restricted to $\mathcal{A}_{[1,r]}$ can be written as

$$\omega|_{\mathcal{A}_{[1,r]}}(\cdot) = \sum_\nu \delta_\nu \frac{1}{n_\nu}\ \mathrm{tr}_{M_{n_\nu}}(\cdot)$$

where $\mathrm{tr}_{M_{n_\nu}}$ denotes the usual trace on $M_{n_\nu}$ and

$$\sum_\nu \delta_\nu = 1, \qquad \delta_\nu > 0.$$

Hence the relevant density matrix can be decomposed as follows

$$\rho_{\Theta^{m-1}(X) \circ \dots \circ X} = \sum_\nu \delta_\nu\ \rho^\nu_{(\Theta^{m-1} X)^\nu \circ \dots \circ X^\nu}$$

with

$$[\rho^{\nu}_{Y^\nu}]_{ij} = \frac{1}{n_\nu}\ \mathrm{tr}_{M_{n_\nu}}(y_j^{\nu *} y_i^\nu).$$

Here for any $x \in \mathcal{A}_{[1,r]}$, $x = \bigoplus x^\nu$, $x^\nu \in M_{n_\nu}$, $Y^\nu = \{y_1^\nu, \ldots, y_k^\nu\}$ is the partition of unity in $M_{n_\nu}$. In order to estimate $S(\rho_{\Theta^{m-1}(X)\circ\ldots\circ X})$ we use first the inequality [W,OP]

$$S\left(\sum_\nu \lambda_i\ \rho_i\right) \leq \sum_\nu \lambda_i\ S(\rho_i) - \sum_i \lambda_i \ln \lambda_i$$

for $\lambda_i \geq 0$, $\sum \lambda_i = 1$ and $\rho_i$–density matrices, then the estimation in the proof of Theorem 4.1 [AF1], which in our case leads to

$$S(\rho^\nu_{(\Theta^{m-1}X)^\nu\circ\ldots\circ X^\nu}) \leq S\left(\omega|_{M_{n_\nu}}\right) + \ln n_\nu = 2\ln n_\nu,$$

and finally the convexity of the function $\ln x$. The result is

$$\begin{aligned} S(\rho_{\Theta^{m-1}(X)\circ\ldots\circ X}) &\leq \sum_\nu \delta_\nu\ 2\ln n_\nu - \sum_\nu \delta_\nu \ln \delta_\nu \\ &= \sum_\nu \delta_\nu \ln \frac{n_\nu^2}{\delta_\nu} \leq \ln \sum_\nu n_\nu^2 = [\ell + m - 1]\ln 2. \end{aligned}$$

The upper bound $h^I \leq \ln 2$ follows and together with the lower bound we get equality.

**Theorem 2:**

a) For $g \equiv 0$, $h^{II}(\Theta) = \ln 2$.

b) For $g \not\equiv 0$, $h^{II}(\Theta) \leq \ln 2$ (Conjecture $\leq \frac{1}{2}\ln 2$).

c) There are $g$ (e.g. $g = (1,0,0,\ldots)$, $g = (1,1,\ldots,1,\ldots)$) with

$$h^{II}(\Theta) = \frac{1}{2}\ln 2.$$

d) For $g$ sufficiently irregular (for all $w_I \exists$ a set $J(I)$ of infinite cardinality such that for all $n, m \in J(I)$ ($[\Theta^m w_I, \Theta^n w_I]_+ = 0$)

$$h^{II}(\Theta) = 0.$$

**Proof:**

a) $g \equiv 0$ is the classical baker transformation.

b) The dynamical entropy is bounded from above by the entropy density [CS], such that

$$h^{II}(\Theta) \leq \lim \frac{1}{n}\ S(\omega|_{\mathcal{A}_{[1,n]}}) \leq \ln 2.$$

Here we have used that the linear dimension of $\mathcal{A}_{[1,n]} = 2^n$. In fact this result can be improved by imbedding $\mathcal{A}_{[1,n]}$ into a matrix algebra which dimension will increase like $2^{k(n)}$. We conjecture that $k(n) \leq n/2 + k_0$, where $k_0$ is determined by the size of the center of $\mathcal{A}_{[1,n]}$ which is supposed to stay uniformly bounded (compare [NT3]).

c) $g = (1, 0, 0, \ldots)$. Here $s(\omega) = \frac{1}{2}\ln 2$ and $h^{II}(\Theta) \geq \frac{1}{2}h^{II}_{\mathcal{A}^0}(\Theta^2)$ with $\mathcal{A}^0$ generated by $e_{2n}$ and thus abelian.

$g = (1, 1, \ldots)$. Again $s(\omega) = \frac{1}{2}\ln 2$ and $h^{II}(\Theta) \geq \frac{1}{2}h^{II}_{\mathcal{A}^0}(\Theta^2)$ with $\mathcal{A}^0$ now built by $e_{2n}\, e_{2n+1}$, again abelian.

d) Assume $\mathcal{B}$ is an abelian model. Consider words in $\mathcal{M} \otimes \mathcal{B}$ of the form $w_I \otimes p$, where $p$ varies over the projections in $\mathcal{B}$. According to our assumption on $g$ to every $I$ there exists a set $J(I)$ such that

$$[\Theta^n w_I \otimes \sigma^2 p, \Theta^k w_I \otimes \sigma^k p]_+ = 0 \quad \forall\, k, n \in J,\ k \neq n.$$

Then (compare [NT4]) for $\lambda = \lambda \circ \Theta \otimes \sigma$

$$\begin{aligned} |\lambda(w_I)|^2 &= \left|\lambda\left(\frac{1}{N}\sum_{i=1,k_i\in J}^{N} \Theta^{k_i} w_I \otimes \sigma^{k_i} p\right)\right|^2 \\ &\leq \frac{1}{2N^2}\sum_{i=1 k_i,k_j\in J}^{N} \lambda([\Theta^{k_i} w_I \otimes \sigma^{k_i} p, \Theta^{k_j} w_I \otimes \sigma^{k_j} p]_+) \\ &= \frac{1}{2N}\,\lambda(w_I^*\, w_I \otimes \cdot p^2) \end{aligned}$$

and therefore since by assumption $N$ can be arbitrarily large, $\lambda(w_I \otimes p) = 0 = \lambda(w_I)$ for any word in $\mathcal{M}$ and any projection $p \in \mathcal{B}$. Therefore the coupling $\lambda = \omega \otimes \mu$ is trivial, $H(P|\mathcal{M}) = H(P)$ and $H(P) \geq h_\mu(\sigma, P)$.

**Example 2:** In [S] the following example was considered: Let $\mathcal{A}$ be the $\mathrm{II}_1$ factor $L(F_\infty)$ obtained from the left regular representation of the free group $F_\infty$ in infinite number of generators, in connection with the free shift as automorphism.

We can think of this algebra constructed in a similar way as in the previous case, only now $e_i^2 = 1$ but $w_I = e_i e_j e_i e_j \ldots$, i.e. words with finitely many alternating letters, are linearly independent operators. The state $\omega$ again reads $\omega(1) = 1$, $\omega(w_I) = 0$ for $I \neq \emptyset$. The shift $\Theta$ is defined by $\Theta\, e_i = e_{i+1}$. In [S] it was shown that in spite that the algebra increases enormously under the shift the noncommutativity is so strong that conditional expectations are essentially always trivial so that a decomposition can only be felt in a finite area which implies that $h^{II}(\Theta) = 0$, as well as $h^{II}(\Theta^k)$. But

**Theorem 3:** Let $\mathcal{A}_0$ be spanned by finite words. Then

$$h^I(\omega, \Theta^2, \mathcal{A}_0) = \infty.$$

**Proof:** Consider $\mathcal{B}_0 = \{e_1, e_2\}''$. This algebra is infinite and therefore $S(\omega|_{B_0})$ is infinite. To every $n$ we can therefore find a partition $X \subset \mathcal{B}_0$ such that $S(\rho_X) > n$. Let the partition be written as $x_i = \sum_j \alpha_{ij} w_{I_j}$. Then $\Theta^2 x_i = \sum_j \alpha_{ij} w_{I_j+2}$, so that $(I_j+2)\cap I_j = \emptyset$. Therefore

$$\omega(x_{i_1}^* \Theta^2 x_{i_2}^* \Theta^2 x_{j_2} x_{j_1}) = \omega(x_{i_1}^* \; x_{j_1})\omega(x_{i_2}^* \; x_{j_2})$$

so that

$$S(\rho_{\Theta^2 X \circ X}) = 2S(\rho_X)$$
$$S(\rho_{\Theta^{2m} X \circ \ldots \circ X}) = (m+1)S(\rho_X)$$

and $h^I(\omega, \Theta^2, \mathcal{A}_0) \geq n$ for arbitrary $n$.

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