Invariant Manifolds for Parabolic Partial Differential Equations on Unbounded Domains

C. E. Wayne

Department of Mathematics

Pennsylvania State University

University Park, Pa. 16802, USA

Abstract. In this paper finite dimensional invariant manifolds for nonlinear parabolic partial differential equations of the form

$$\frac{\partial u}{\partial \tau} = \Delta u + F(u) \ \ ; \ u = u(\xi, \tau) \ , \quad \xi \in \mathbb{R}^d \ , \tau \geq 1 \ \ ,$$

are constructed. Such results are somewhat surprising because of the continuous spectrum of the linearized equation. These manifolds control the long time behavior of solutions of these equations and can be used to construct systematic expansions of the long-time asymptotics in inverse powers of τ. They also give a new perspective on the change in the long-time asymptotics of the equation with nonlinear term $F(u) = |u|^{p-1}u$, when p passes through the critical value $p_c = 1 + 2/d$.

1. Introduction and Statement of Results

Invariant manifold theorems have found numerous applications in partial differential equations ([C], [CM], [EW], [H], [K], [M], provide some examples, and the bibliography of [VI] lists many more). These applications, however, have been limited to situations in which either the time or space variables have varied over a bounded domain, or in situations like traveling waves which could be reduced to one of these situations. This restriction results in a linearized problem with pure point spectrum which permits one to identify modes associated with a center, or center stable manifold. In equations of the form

$$\frac{\partial u}{\partial \tau} = \Delta u + F(u) \ \ ; \ u = u(\xi, \tau) \ , \quad \xi \in \mathbb{R}^d \ , \tau \geq 1 \tag{1.1}$$

with $F(u)$ nonlinear, the linearized problem is the heat equation which clearly has continuous spectrum. Nonetheless, (1.1) does possess finite dimensional invariant manifolds which control the long time asymptotics of solutions near the origin.

In order to discuss these results more precisely, consider the weighted Sobolev spaces $H_r^{q,2} = \{u | (1-\Delta)^{q/2}(1+|\xi|^2)^{r/2} u \in L^2(\mathbb{R}^d)\}$. Also assume that $F(a)$ is a C^k function, with $k > 1$, and that there exists $p > 1$, such that $g(\lambda, z) = \lambda^{-p} F(\lambda z)$ is a C^k function of λ and z on some neighborhood of the origin. Note that this implies $|F(a)| \leq K|a|^p$ for a small and some $K > 0$.

For reasons that will become apparent from the proof, it is more natural to state the main result in the extended phase space $\mathcal{P}^{r,s} = H_r^{s,2} \times \{\tau \in \mathbb{R} | \tau \geq 1\}$.

Theorem 1.1. *Fix $\epsilon > 0$, $s > (d/2) + 1$ an even integer, and $n \in \mathbb{N}$. If r is sufficiently large, then there exists a $1 + \sum_{j=0}^{n} \binom{j+d-1}{d-1}$-dimensional manifold in $\mathcal{P}^{r,s}$, which is left invariant by the (semi)-flow generated by (1.1). Given any initial condition in a sufficiently small neighborhood of the origin, if the solution of (1.1) with that initial condition remains in that neighborhood of the origin for all time, then it approaches the invariant manifold at a rate $\mathcal{O}(\tau^{-(n+d-\epsilon)/2})$.*

For the special nonlinear term $F(u) = |u|^{(p-1)} u$, it is not necessary to work in the extended phase space $\mathcal{P}^{r,s}$. In particular, if we specialize even further to consider $d = 1$, we have the following

Theorem 1.2. *Fix $\epsilon > 0$, $s = 2$, and $n \in \mathbb{N}$. If r is sufficiently large, then there exists an $n+2$-dimensional manifold in $H_r^{2,2}$, which is foliated with $n+1$-dimensional leaves, and which is left invariant by the (semi)-flow generated by (1.1). Given any initial condition in a sufficiently small neighborhood of the origin, if the solution of (1.1) with that initial condition remains in that neighborhood of the origin for all time, then it approaches the invariant manifold at a rate $\mathcal{O}(t^{-(n+1-\epsilon)/2})$.*

In the study of ordinary differential equations, invariant manifold theorems have many applications. (See, *e.g.* [R].) In this paper, I exhibit two uses for them in the present context. First they can be used to give a systematic, rigorous expansion of the long time asymptotics of solutions of (1.1), which I illustrate in a concrete example. The second use I will illustrate is to give an explanation based on bifurcation theory of the changes in the long time asymptotics of the equation with nonlinear term $F(u) = |u|^{(p-1)}u$, when p passes through its critical value of $p_c = 1 + 2/d$.

My approach to this problem was inspired by [BK], in which Bricmont and Kupiainian study the stability of certain non-Gaussian solutions of (1.1) with $F(u) = |u|^{(p-1)}u$, and $1 < p < p_c$. Their analysis was based on a change of variables to what might be called "similarity variables". This same change of variables was also used by Escobedo and Kavaian [EK] in their study of this problem, and it is used here in a slightly different context to construct the invariant manifolds. Related changes of variables have found use in other questions connected with the asymptotics of parabolic partial differential equations. For instance, [GK] apply a similar change of variables to study the blow-up of solutions. Indeed, [B], [FK], and [BB] all appeal to invariant manifold theorems as motivation for their results on blow-up in nonlinear heat equations, but technical difficulties prevented the actual construction of such manifolds in those problems.

The remainder of the paper is organized as follows. In the next section I show how invariant manifolds can be constructed in a much smaller space than $H_r^{q,2}$. In section 3, I then extend these results to $H_r^{q,2}$. In section 4, these results are used to give a systematic expansion of the long time asymptotics of (1.1) in inverse powers of t and to analyze the behavior of the equation with $F(u) = |u|^{(p-1)}u$ near $p = p_c$. Finally, section 5 is devoted to conclusions and possible extensions.

2. The Existence of Invariant Manifolds

Begin by making the change of variables

$$u(\xi,\tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log \tau) \quad . \tag{2.1}$$

In terms of the new variables, $x = \xi/\sqrt{\tau}$ and $t = \log \tau$, (1.1) becomes

$$\frac{\partial v}{\partial t} = \Delta_x v + (1/2) x \cdot \nabla_x v + \frac{1}{p-1} v + \tau^{\frac{p}{p-1}} F(\tau^{\frac{-1}{p-1}} v) \quad . \tag{2.2}$$

Now make a further change of variables, and set $w(x,t) = \exp(x^2/8) v(x,t)$. When expressed in terms of these variables, the equation becomes

$$\frac{\partial w}{\partial t} = H_0 w + e^{x^2/8} \tau^{\frac{p}{p-1}} F(\tau^{\frac{-1}{p-1}} e^{-x^2/8} w) \quad , \tag{2.3}$$

where the linear operator $H_0 = \Delta_x - x^2/16 + (\frac{1}{p-1} - \frac{d}{4})$. Noting that H_0 is (up to the additive constant) just the Hamiltonian of the quantum mechanical harmonic oscillator, one sees immediately that its spectrum, considered as an operator on $L^2(\mathbb{R}^d)$ is $\sigma(H_0) = \{\frac{1}{p-1} - \frac{d}{2} - \frac{m}{2} | m = 0, 1, 2, \ldots\}$. It is not convenient to study (2.3) on $L^2(\mathbb{R}^d)$, because the nonlinear term in the equation has too little regularity when considered as a function on this space to apply the invariant manifold theorem. However, if one chooses a constant $c(p)$ sufficiently large, the operator $(c(p)\mathbf{1} - H_0)$ is positive definite. Thus one can take its square root. This allows us to define a family of Sobolev-like Hilbert spaces via

$$W^s = \{w | (c(p)\mathbf{1} - H_0)^{s/2} w \in L^2(\mathbb{R}^d)\} \quad .$$

Remark. While it is not necessary to introduce the "w" variables, the fact that H_0 is self-adjoint on W^s will simplify many of the computations in later sections.

These norms control the "ordinary" Sobolev norms. For instance, just from the definition of the norm, one has the lemma:

Lemma 2.1. *For any non-negative integer s, there exists a constant $c(s)$ such that $||w||_{H_0^{s,2}} \leq c(s) ||w||_{W^s}$.*

As an immediate corollary of this lemma and the Sobolev lemma one has:

Lemma 2.2. *If $s > k + d/2$, there exists a constant $c(s,k)$ such that $||w||_{C^k(\mathbb{R}^d)} \leq c(s,k)||w||_{W^s}$.*

Note that Lemma 2.2 implies that the nonlinear term in (2.3) is well defined for any $w \in W^s$ provided $s > d/2$. In fact, if the function $F(a)$ is smooth, one has the following lemma.

Lemma 2.3. *If $F \in C^k$, with $k > 1$ and $s > d/2$, then the nonlinear term in (2.3) is a C^k map from $H^s \to H^s$, for every $t \geq 0$.*

Proof. This is immediate from Lemma 2.2.

One can now apply the invariant manifold theorem to (2.3). First convert it to an autonomous equation by a standard trick. Set $\eta = \tau^{\frac{-1}{(p-1)}} = \exp(-t/(p-1))$. Then if a " $\dot{}$ " denotes differentiation with respect to t, (2.3) is equivalent to the system of equations

$$
\begin{aligned}
\dot{w} &= H_0 w + \eta^{-p} e^{x^2/8} F(\eta e^{-x^2/8} w) \\
\dot{\eta} &= -(\frac{1}{p-1})\eta \quad .
\end{aligned}
\tag{2.4}
$$

Remark. Under the hypotheses of Lemma 2.3, and the assumption that $\lambda^{-p}F(\lambda a)$ is a C^k function of λ and a, $\eta^{-p}e^{x^2/8}F(\eta e^{-x^2/8}w)$ is a C^k function of η and w.

One can now construct an invariant manifold for (2.4) corresponding to the $n+1$ largest eigenvalues of H_0, $\lambda_m = \frac{1}{p-1} - \frac{d}{2} - \frac{m}{2}$, $m = 0, 1, \ldots, n$. Let $\mathcal{P}^c$ be the projection (in W^s) onto the subspace spanned by the eigenvectors corresponding to these eigenvalues. (Note that the dimension of this space is $\sum_{j=0}^{n} \binom{j+d-1}{d-1}$). Let $\mathcal{P}^s = (\mathcal{P}^c)^{\perp}$. I will refer to this manifold as the "pseudo-center manifold", in analogy with the more common pseudo-stable manifolds [LW]. Assume that n is large enough that $\lambda_n < -(\frac{1}{p-1})$. In this case the η direction will be included in the "pseudo-center" direction. The case in which $\lambda_n > -(\frac{1}{p-1})$ is an easy modification of this argument, because the dynamics in the η direction are trivial in either case. Let $\mathcal{E}^c = \mathbb{R} \oplus range(\mathcal{P}^c)$ and $\mathcal{E}^s = range(\mathcal{P}^s)$, where the summand $\mathbb{R}$ in $\mathcal{E}^c$ corresponds to the η direction. Let $\mathcal{B}^{c,(s)}(\rho)$ be the ball of radius ρ in $\mathcal{E}^{c,(s)}$, $\mathcal{B}(\rho)$ the ball of radius ρ in $\mathbb{R} \oplus W^s$, and let $A^c = -(\frac{1}{p-1}) \oplus \mathcal{P}^c H_0 \mathcal{P}^c$ and $A^s = \mathcal{P}^s H_0 \mathcal{P}^s$. By the spectral theorem, there exist constants $\lambda_n < \lambda_s < \lambda_c < \min(\lambda_{n-1}, -(\frac{1}{p-1}))$ such that

$$\sup_{t \geq 0} ||e^{A^s t}|| e^{-\lambda_s t} < \infty \quad , \quad \sup_{t \in \mathbb{R}} ||e^{A^c t}|| e^{\lambda_c |t|} < \infty \quad . \tag{2.5}$$

One can now apply Theorem 4.1 of [G] to obtain the following theorem.

Theorem 2.4. *If the nonlinear term F in (1.1) is C^k, with $k > 1$ and $s > d/2$, and if $\lambda^{-p} F(\lambda z)$ is a C^k function of λ and z for some $p > 1$, then there exists $\alpha > 0$ and $\rho > 0$, and a $C^{1+\alpha}$ function $h : \mathcal{B}^c(\rho) \to \mathcal{E}^s$, such that that the graph of h is invariant under the (semi)-flow of (2.3). Furthermore, $h(0) = 0$ and $Dh(0) = 0$.*

Proof. Let x^c and x^s be coordinates on $\mathcal{E}^{c,s}$ respectively. Then (2.3) can be rewritten as

$$\begin{aligned} \dot{x}^c &= A^c x^c + f^c(x^c, x^s) \\ \dot{x}^s &= A^s x^s + f^s(x^c, x^s) \end{aligned} \tag{2.6}$$

By the remark following Lemma 2.3, the nonlinear terms f^c and f^s are C^k functions for some $k > 1$, so the hypotheses of Theorem 4.1 of [G] are satisfied, and the theorem follows.

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3. Attractivity of the Pseudo-Center Manifold

In the present section I prove that the pseudo-center manifold attracts all solutions that remain in some neighborhood of the origin. What is somewhat surprising is that this is true for solutions in a much larger space than the space in which this manifold was constructed. For the present construction we revert to the "v" variables, and consider the equation in the form (2.2)

$$\frac{\partial v}{\partial t} = \Delta_x v + \frac{1}{2} x \cdot \nabla_x v + \frac{1}{p-1} v + \tau^{\frac{p}{p-1}} F(\tau^{-\frac{1}{p-1}} v) \quad .$$

We will study the evolution of this equation on the Sobolev spaces $H_r^{q,2}$.

If $\{\varphi_n\}_{n\geq 0}$ are the eigenfunctions of the linear operator H_0 in the previous section, then $\{\psi_n\}_{n\geq 0}$, with $\psi_j(x) = \exp(-x^2/8)\varphi_j(x)$ are eigenfunctions of $\mathcal{L}_0 = \Delta_x + (1/2)x\cdot\nabla_x + 1/(p-1)$, with the same eigenvalues.

Define a projection operator

$$(P_n v)(x) = \sum_{j=0}^{n} \psi_j(x)\langle\langle\psi_j, v\rangle\rangle_s \quad ,$$

where the inner product

$$\langle\langle v, \tilde{v}\rangle\rangle_s \equiv \int (e^{\frac{x^2}{8}}\tilde{v}(x))(e^{\frac{x^2}{8}}(c(p) - \mathcal{L}_0)^s\overline{v(x)})dx \quad ,$$

is what is formally obtained from the inner product $\langle w, \tilde{w}\rangle_s$ in the Hilbert space W^s considered in the previous section upon substituting $w(x) = \exp(x^2/8)v(x)$, and $\tilde{w}(x) = \exp(x^2/8)\tilde{v}(x)$. While the computation leading to the form of $\langle\langle\cdot,\cdot\rangle\rangle_s$ is formal, the following lemma shows that P_n is well defined.

Lemma 3.1. *If $r > n + (d+1)/2$, then there exists $C > 0$ such that for any $v \in H_r^{q,2}$,*

$$\|P_n v\|_{H_r^{q,2}} \leq C\|v\|_{H_r^{q,2}} \quad .$$

Proof. The $H_r^{q,2}$ norm of ψ_j is finite so the lemma follows from the estimate $|\langle\langle\psi_j, v\rangle\rangle_s| \leq C\|v\|_{H_r^{q,2}}$ for $j = 0,\ldots,n$. This in turn follows if we note that

$$\begin{aligned}|\langle\langle\psi_j, v\rangle\rangle_s| =& |\int e^{\frac{x^2}{8}}v(x)e^{\frac{x^2}{8}}(c(p) - \mathcal{L}_0)^s\psi_j(x)dx| \\ \leq& (c(p) + |\lambda_j|)^s \int e^{\frac{x^2}{4}}|v(x)||\psi_j(x)|dx \quad .\end{aligned}$$

We now recall that $\psi_j(x) = e^{\frac{-x^2}{8}}\varphi_j(x) = e^{\frac{-x^2}{8}}(e^{\frac{-x^2}{8}}\tilde{H}(x))$, where $\tilde{H}(x)$ is (up to normalization) a product of Hermite polynomials the sum of whose orders is j. (In particular, if $d = 1$, $\tilde{H}(x) = H_j(x)$). This implies that $|\tilde{H}(x)| \leq C(1+|x|)^j$. Thus,

$$\begin{aligned}|\langle\langle\psi_j, v\rangle\rangle_s| \leq& (c(p) + |\lambda_j|)^s C\int (1+|x|)^j|v(x)|dx \\ \leq& (c(p) + |\lambda_j|)^s C\int (1+|x|)^{-(d+1)}dx \int (1+|x|)^{2j+(d+1)}|v(x)|^2dx \quad .\end{aligned}$$

Thus, if $2r > 2j+d+1$, this expression is bounded by $C\|v\|_{H_r^{q,2}}$ and the lemma follows.

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Remark. The projection operator P_n is the analogue of $\mathcal{P}^c$, defined in section 2. However, at different points in the argument it will be convenient to allow n to vary and for this reason this new notation is useful.

Remark. Note that an additional property of the inner product $\langle\langle\cdot,\cdot\rangle\rangle_s$ is that $\langle\langle\psi_j,\psi_{j'}\rangle\rangle_s = \delta_{j,j'}$. This follows from the orthonormality of the eigenfunctions φ_j.

The key estimate for the present section is the following estimate on the linear evolution operator acting on $H_r^{q,2}$. Recall that s is the index on the Hilbert space which we used in the construction of the invariant manifold in the previous section. Let $Q_n = 1 - P_n$.

Proposition 3.2. *For any $n > 0$ such that $\lambda_{n+1} < 0$, and for $s \geq 1 + (d/2)$ and r sufficiently large, there exists $C > 0$ such that*

$$\|e^{t\mathcal{L}_0} Q_n v\|_{H_r^{s,2}} \leq C e^{-t|\lambda_{n+1}|} \|v\|_{H_r^{s,2}} \quad .$$

Proof. *See Appendix A.*

Note that this is exactly the sort of decay estimate we could expect for vectors in the "stable-subspace" $\mathcal{E}^s \subset W^s$ – what is surprising it that it also holds in the much larger space $H_r^{s,2}$.

Given such an estimate on the linearized evolution, results about the attractivity of the invariant manifold follow very much as in the finite dimensional case. We include proofs of the following Proposition and Theorem for completeness, but they are essentially taken verbatim from [C, pp. 20-25]. If we let η be as in the previous section, $v^c = P_n v$ and $v^s = Q_n v$, then we can rewrite (2.2) as

$$\begin{aligned} \dot{\eta} &= -(\frac{1}{p-1})\eta \quad , \\ \dot{v^c} &= \mathcal{L}_0^c v^c + F^c(\eta, v^c, v^s) \quad , \\ \dot{v^s} &= \mathcal{L}_0^s v^s + F^s(\eta, v^c, v^s) \quad . \end{aligned} \tag{3.1}$$

Here $\mathcal{L}_0^c = P_n \mathcal{L}_0 P_n$, $\mathcal{L}_0^s = Q_n \mathcal{L}_0 Q_n$, while $F^c(\eta, v^c, v^s) = P_n(\eta^p F(\eta^{-p} v))$ and $F^s(\eta, v^c, v^s) = Q_n(\eta^p F(\eta^{-p} v))$. The results of the previous section imply that there exists an invariant manifold tangent at the origin to $\mathbb{R} \oplus Range(P_n)$. This manifold is given (locally) as the graph of a function $v^s = h(\eta, v^c)$. The following results show first that solutions near this manifold are attracted to it, and second that one can approximate such solutions, up to an exponentially small error, by the finite dimensional system of ordinary differential equations corresponding to motion on the manifold.

Proposition 3.3. *Let $(v^c(t), v^s(t))$ be a solution of (3.1) which remains for all time in a sufficiently small neighborhood of the origin. (Note that the evolution of η is trivial.) Then there exist positive constants C_1 and μ such that*

$$\|v^s(t) - h(\eta(t), v^c(t))\|_{\mathcal{E}^s} \leq C_1 e^{-\mu t} \|v^s(0) - h(\eta(0), v^c(0))\|_{\mathcal{E}^s} \quad .$$

Proof. Following Carr, we let $z(t) = v^s(t) - h(\eta(t), v^c(t))$. Then

$$\dot{z}(t) = \mathcal{L}_0^s z + N(\eta, v^c, z) \quad , \tag{3.2}$$

where $N(\eta, v^c, z) = F^s(\eta, v^c, h(v^c)+z) - F^s(\eta, v^c, h(v^c)) + D_{v^c} h(\eta, v^c) \cdot \{F^c(\eta, v^c, h(v^c)) - F^c(\eta, v^c, h(v^c) + z)\}$. Note that $D_{v^c} h(\eta, v^c)$ is a linear operator from $Range(P_n)$ to $Range(Q_n)$. If we rewrite (3.2) in integral form we obtain

$$z(t) = e^{t\mathcal{L}_0^s} z(0) + \int_0^t e^{(t-s)\mathcal{L}_0^s} N(\eta(s), v^c(s), z(s)) ds \quad . \tag{3.3}$$

¿From the definitions of F^c and F^s, there exists a constant $\delta(\rho)$ which goes to zero as $\rho \to 0$, and such that if $\|v^c\|_{\mathcal{E}^c} \leq \rho$, $\|N(\eta, v^c, z)\|_{\mathcal{E}^c} \leq \delta(\rho) \|z\|_{\mathcal{E}^s}$. Here, $\|\cdot\|_{\mathcal{E}^c}$ is the ordinary finite dimensional Euclidean norm on $Range(P_n)$, while $\|\cdot\|_{\mathcal{E}^s}$ is the $H_r^{s,2}$-norm, restricted to $\mathcal{E}^s$.

Applying this estimate of N, and the estimate of Proposition 3.2 to (3.3) one sees that

$$\|z(t)\|_{\mathcal{E}^s} \leq C e^{-t|\lambda_{n+1}|} \|z(0)\|_{\mathcal{E}^s} + C\delta(\rho) \int_0^t e^{-(t-s)|\lambda_{n+1}|} \|z(s)\|_{\mathcal{E}^s} ds \quad . \tag{3.4}$$

The proposition then follows immediately from Gronwall's Lemma.

$\blacksquare$

Remark. Note that the constant $\delta(\rho)$ can be chosen to be $CL(\rho)$ where $L(\rho)$ is the Lipschitz constant of the nonlinearity $F(v)$ on a ball of radius ρ in $H_r^{s,2}$. This coupled with (3.4) implies that the constant μ in Proposition 3.3 can be chosen to be any number less than $||\lambda_{n+1}| - C\delta(\rho)|$. Thus, one has:

Corollary 3.4. *There exists $C > 0$ and $\rho_0 > 0$, such that if the solution of (2.2) remains in a ball of radius $\rho < \rho_0$ centered at the origin for all $t \geq 0$, then $v(t)$ approaches the invariant manifold with a rate $\mathcal{O}(e^{-\mu t})$, where $\mu = |\lambda_{n+1}| - CL(\rho)$.*

We now turn to the second important result, namely that one can approximate any trajectory of (2.2) near the origin by a solution of a finite dimensional system of ordinary differential equations. Let

$$\begin{aligned} \dot{\eta} &= -\left(\frac{1}{p-1}\right)\eta \quad , \\ \dot{\psi} &= \mathcal{L}_0^c \psi + F^c(\eta, \psi, h(\psi)) \quad . \end{aligned} \tag{3.5}$$

Theorem 3.5. *Let $(v^c(t), v^s(t))$ be a solution of (3.1) which remains in a sufficiently small neighborhood of the origin for all $t \geq 0$. Then there exists a solution of $\psi(t)$ of (3.5), and $\mu > 0$, such that as $t \to \infty$,*

$$\begin{aligned} v^c(t) &= \psi(t) + \mathcal{O}(e^{-\mu t}) \quad , \\ v^s(t) &= h(\eta(t), \psi(t)) + \mathcal{O}(e^{-\mu t}) \quad . \end{aligned}$$

Remark. As above, one can choose $\mu = |\lambda_{n+1}| - CL(\rho)$.

Proof. We follow Carr's proof [C, pp. 21-25], merely indicating the differences. We will use his notation without comment. Define $z(t) = v^s(t) - h(v^c(t))$ and $\varphi(t) = v^c(t) - \psi(t)$. Then z and φ satisfy

$$\begin{aligned} \dot{z} &= \mathcal{L}_0^s z + N(\eta, v^c, z) \quad , \\ \dot{\varphi} &= \mathcal{L}_0^c \varphi + R(\eta, \varphi, z) \quad , \end{aligned} \tag{3.6}$$

where N is as in (3.2) and $R(\eta, \varphi, z) = F^c(\eta, \varphi + \psi, z + h(\eta, \varphi + \psi)) - F^c(\eta, \psi, h(\eta, \psi))$. Consider the Banach space of functions

$$X_a = \{\varphi \in C([0, \infty), \mathbb{R}^{n+1}) \mid |||\varphi|||_a = \sup_{t \geq 0} \|\varphi(t) e^{at}\|_{\mathcal{E}^s} < \infty\} \quad .$$

Note that we can rewrite the second equation in (3.6) in integral form as

$$\varphi(t) = e^{(t-t_0)\mathcal{L}_0^c} \varphi(t_0) - \int_t^{t_0} e^{(t-s)\mathcal{L}_0^c} R(\varphi(s), z(s)) ds \quad .$$

If $\varphi \in X_a$, and $a > |\lambda_n|$, then $e^{(t-t_0)\mathcal{L}_0^c} \varphi(t_0) \to 0$ as $t_0 \to \infty$. This motivates the definition of the transformation

$$(T\varphi)(x) = - \int_t^{\infty} e^{(t-s)\mathcal{L}_0^c} R(\varphi(s), z(s)) ds \quad .$$

By the previous Proposition we know that for any $0 < \beta_1 \leq (|\lambda_{n+1}| - CL(\rho))$, $\|z(s)\|_{\mathcal{E}^s} \leq C\|z(0)\| e^{-\beta_1 s}$. Using this, Carr shows that for any $|\lambda_n| < a < \beta_1$, T has a unique fixed point in X_a for any $(\psi(0), z(0))$ sufficiently small. Furthermore, (possibly by shrinking the size of the neighborhood) the fixed point depends continuously on $(\psi(0), z(0))$. (In Carr's case, $|\lambda_n| = 0$ – however, the case $|\lambda_n| \neq 0$ produces no essential change in the argument.)

Now define $S(\psi(0), z(0)) = (v^c(0), z(0))$, where $v^c(0) = \varphi(0) + \psi(0)$. We want to prove that S is one-to-one and hence invertible on a neighborhood of the identity. This will then establish the theorem, with the constant $\mu = a$. As Carr notes, proving that S is one-to-one is equivalent to showing that if $\psi(0) + \varphi(0) = \tilde{\psi}(0) + \tilde{\varphi}(0)$, then $\varphi(0) = \tilde{\varphi}(0)$ and $\psi(0) = \tilde{\psi}(0)$. By the uniqueness of solutions of (3.1) we see that $\psi(t) + \varphi(t) = \tilde{\psi}(t) + \tilde{\varphi}(t)$ for all $t \geq 0$, where $\psi(t)$ and $\tilde{\psi}(t)$ are the solutions of (3.5) which initial conditions $\psi(0)$ and $\tilde{\psi}(0)$ respectively. This equation is equivalent to the equation

$$\varphi(t) - \tilde{\varphi}(t) = \psi(t) - \tilde{\psi}(t) \quad . \tag{3.7}$$

Choose β_2 such that $|\lambda_n| < \beta_2 < a \leq \beta_1$, where a is the constant which determines the exponential decay rate of the space X_a. Then from (3.5) we see that

$$\lim_{t \to \infty} e^{\beta_2 t} \|\psi(t) - \tilde{\psi}(t)\|_{\mathcal{E}^c} = \infty \quad ,$$

unless $\psi(0) = \tilde{\psi}(0)$. On the other hand, from the definition of X_a, $\lim_{t\to\infty} e^{\beta_2 t}\|\varphi(t) - \tilde{\varphi}(t)\|_{\mathcal{E}^c} = 0$. Thus, (3.7) implies $\psi(0) = \tilde{\psi}(0)$, and hence $\varphi(0) = \tilde{\varphi}(0)$.

This implies that S is invertible, or that given any $v^c(0)$ and $z(0)$ sufficiently small, we can find $z(0)$ and $\psi(0)$ such that if $v^c(t)$ and $\psi(t)$ are the solutions of (3.1) and (3.5) with these initial conditions, and $\varphi(t) = v^c(t) - \psi(t)$, then $\|\varphi(t)\|_{\mathcal{E}^c} \leq Ce^{-at}$, and $\|z(t)\|_{\mathcal{E}^s} \leq Ce^{-at}$.

■

Proof. (of Theorem 1.1 and Theorem 1.2)

Combining Theorem 2.4 and Proposition 3.3, and then rewriting the conclusions in terms of the original variables $u(\xi,\tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log\tau)$ immediately gives Theorem 1.1.

Theorem 1.2 follows in a similar fashion. Because of the homogeneous form of the nonlinearity, the equation in the "v"-variables becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \frac{1}{2} x \frac{\partial v}{\partial x} + \frac{1}{p-1} v + |v|^{p-1} v \quad . \tag{3.8}$$

In particular, there is no explicit dependence on τ, hence, no need to introduce the auxiliary variable, η, and hence no need to consider the extended phase space $\mathbb{R} \oplus H_r^{s,2}$. Thus, we obtain an $n+1$-dimensional invariant manifold of (3.8) in $H_r^{s,2}$. Now consider the image of this manifold under the change of variables $u(\xi,\tau) = \tau^{-1/(p-1)} v(\xi/\sqrt{\tau}, \log\tau)$. This will give an $n+2$-dimensional invariant manifold in $H_r^{s,2}$ for (1.1). This manifold will be foliated with $n+1$-dimensional leaves given by the image of the invariant manifold for (3.8) for fixed τ. This completes the proof of Theorem 1.2

■

4. Applications

In the present section two applications of the preceding theory are given. The first is the computation of higher order asymptotics for equations like (1.1). I work through an example in some detail to illustrate the differences between these sorts of corrections, and the lowest order asymptotics discussed in [BKL]. The second example shows that these results, when combined with bifurcation theory give a new perspective on the change in the nature of the long time behavior of solutions of

$$\frac{\partial u}{\partial \tau} = \Delta_\xi u - |u|^{p-1}u \quad , \quad u = u(\xi,\tau) \ , \ \xi \in \mathbb{R}^d \ , \ \tau \geq 1 \quad , \tag{4.1}$$

when p passes through its critical value of $p_c = 1 + 2/d$.

4.1. Higher order asymptotics

As an example, let us consider the initial value problem

$$\frac{\partial u}{\partial \tau} = \Delta_\xi u - u^4 \quad , \ u(\xi,1) = u_0(\xi) \ , \ \xi \in \mathbb{R}^2 \ , \ \tau \geq 1. \tag{4.2}$$

By considering the second order asymptotics, one can prove

Proposition 4.1. *Fix $\epsilon > 0$. For $u_0(\xi)$ in a sufficiently small neighborhood of the origin, there exist constants $\mathbf{z}(0) = (z_{00}, z_{10}, z_{01})$, and $K > 0$, and functions $\zeta_{00}, \zeta_{10}, \zeta_{01}$, such that*

$$\begin{aligned}\|u(\xi_1\sqrt{\tau},\xi_2\sqrt{\tau},\tau) - \{&\frac{1}{\tau}\frac{1}{2\sqrt{\pi}}(\frac{3}{5})^2(z_{00}+\zeta_{00}(\mathbf{z}(0)))e^{-\frac{(\xi_1^2+\xi_2^2)}{4}} \\ &+\frac{1}{\tau^{3/2}}\frac{1}{2\sqrt{2\pi}}(\frac{6}{13})^2(z_{10}+\zeta_{10}(\mathbf{z}(0)))H_1(\xi_1)e^{-\frac{(\xi_1^2+\xi_2^2)}{4}} \\ &\quad+\frac{1}{\tau^{3/2}}\frac{1}{2\sqrt{2\pi}}(\frac{6}{13})^2(z_{01}+\zeta_{01}(\mathbf{z}(0)))H_1(\xi_2)e^{-\frac{(\xi_1^2+\xi_2^2)}{4}}\}\|_{H_r^{s,2}} \leq K\tau^{2-\epsilon} \quad .\end{aligned}$$

Remark. The function H_1 is the first Hermite polynomial. The constants z_{00}, z_{10}, z_{01} can be computed in terms of the initial condition u_0, while the functions $\zeta_{00}, \zeta_{10}, \zeta_{01}$ are determined by the flow on the pseudo-center manifold – a system of three ordinary differential equations in this case.

Proof. Transforming to the "w"-variables used in section 2, the equation becomes

$$\frac{\partial w}{\partial t} = H_0 w + e^{-3|x|^2/8} w^4 \quad , \quad w(x,0) = u_0(x) \ , \ t \geq 0 \quad , \tag{4.3}$$

where $H_0 = \Delta_x - \frac{x^2}{16} - \frac{1}{6}$. The spectrum of H_0 is $\sigma(H_0) = \{-\frac{2}{3} - \frac{m}{2} \mid m = 0, 1, 2, \ldots\}$ and we will construct the pseudo-center manifold corresponding to the two lowest eigenvalues. Proposition 3.3 then implies that all solutions in a sufficiently small neighborhood of the origin, approach this manifold at a rate $\mathcal{O}(1/t^{\frac{5}{3}-\epsilon})$, where one can make ϵ as small as one likes by taking the neighborhood on which one works sufficiently small.

In order to construct the invariant manifold one needs to know the eigenfunctions of H_0, which in this case are $\varphi_{jk}(x,y) = \gamma_{jk} H_j(x) H_k(y) \exp(-(x^2+y^2))/8$, where the constants γ_{jk} are chosen so that $\langle \varphi_{jk}, \varphi_{\tilde{j}\tilde{k}} \rangle_s = \delta_{j\tilde{j}} \delta_{k\tilde{k}}$. In particular, if we choose the constant $c(p)$ in the definition of of the inner product in W^s to be 1, and take $s = 4$, we find that $\gamma_{00} = 9/(50\sqrt{\pi})$, and $\gamma_{10} = \gamma_{01} = 18/(169\sqrt{2\pi})$. The eigenvalue corresponding to φ_{jk} is $\lambda_{jk} = -\frac{2}{3} - \frac{(j+k)}{2}$. Thus, the "central subspace" $\mathcal{E}^c = span\{\varphi_{00}, \varphi_{10}, \varphi_{01}\}$.

By Theorem 3.5, any solution of (4.3) in a sufficiently small neighborhood of the origin will approach a solution of

$$\dot{\psi} = H_0^c \psi + F^c(\psi, h(\psi)) \tag{4.4}$$

Remark. Equation (4.4) is just (3.5), rewritten in the "w"-variables. Also, since the nonlinear term in (4.1) is a homogeneous function, the nonlinear term in (4.4) is independent of η.

Let z_{jk} be coordinates in W^s in the φ_{jk} direction, and let $h_{jk}(z_{00}, z_{01}, z_{10}) \equiv h_{jk}(\mathbf{z})$, $j + k \geq 2$ be the component of h in the φ_{jk} direction. (Recall that h is the function whose graph gives the invariant manifold.) One can compute h by standard techniques (see, *e.g.* [C]) and one finds:

$$h_{jk}(\mathbf{z}) = \sum_{\substack{m_1+m_2+m_3=4 \\ m_j \geq 0}} \frac{-c_{jk}(\mathbf{m})}{(\lambda_{jk} + \frac{8}{3} + \frac{m_2+m_3}{2})} \mathbf{z}^{\mathbf{m}} + \mathcal{O}(|\mathbf{z}|^5) \quad .$$

I have used the standard multi-index notation here, so that $\mathbf{z}^{\mathbf{m}} \equiv z_{00}^{m_1} z_{01}^{m_2} z_{10}^{m_3}$. Also, $c_{jk}(\mathbf{m})$ is the coefficient of $\mathbf{z}^{\mathbf{m}}$ in the eigenfunction expansion of the nonlinear term in

(4.3). More precisely, it is the coefficient of $\mathbf{z}^{\mathbf{m}}$ in the expression $\langle \varphi_{jk}, \exp(-3(x^2+y^2)/8)(z_{00}\varphi_{00}+z_{01}\varphi_{01}+z_{10}\varphi_{10})^4\rangle_s$. Again, these are straight forward to compute and one finds, for example, $c_{00}(4,0,0) = (\frac{1}{\pi})^{3/2}(\frac{1}{4})^{5/2}(\frac{3}{5})^6$, $c_{00}(3,1,0) = 0$, $c_{00}(3,0,1) = 0$, ...

Using this information about h, one can write (4.4) more explicitly as:

(4.5)
$$
\begin{aligned}
\dot{z}_{00} &= -\frac{2}{3}z_{00} + \sum_{\substack{m_1+m_2+m_3=4\\ m_j\geq 0}} c_{00}(\mathbf{m})\mathbf{z}^{\mathbf{m}} + \mathcal{O}(|z|^5) \quad , \\
\dot{z}_{10} &= -\frac{7}{6}z_{10} + \sum_{\substack{m_1+m_2+m_3=4\\ m_j\geq 0}} c_{10}(\mathbf{m})\mathbf{z}^{\mathbf{m}} + \mathcal{O}(|z|^5) \quad , \\
\dot{z}_{01} &= -\frac{7}{6}z_{01} + \sum_{\substack{m_1+m_2+m_3=4\\ m_j\geq 0}} c_{01}(\mathbf{m})\mathbf{z}^{\mathbf{m}} + \mathcal{O}(|z|^5) \quad ,
\end{aligned}
$$

One can easily derive the asymptotics of this system of equations. If one starts from the point $\mathbf{z}(0) = (z_{00}(0), z_{10}(0), z_{01}(0))$, then one finds

$$
\begin{aligned}
z_{00}(t) =& e^{-\frac{2}{3}t}(z_{00}(0) + \zeta_{00}(\mathbf{z}(0)) + \mathcal{O}(e^{-\mu t}) \quad , \\
z_{10}(t) =& e^{-\frac{7}{6}t}(z_{10}(0) + \zeta_{10}(\mathbf{z}(0)) + \mathcal{O}(e^{-\mu t}) \quad , \\
z_{01}(t) =& e^{-\frac{7}{6}t}(z_{10}(0) + \zeta_{00}(\mathbf{z}(0)) + \mathcal{O}(e^{-\mu t}) \quad ,
\end{aligned}
$$

with $\mu \geq 2$. The functions ζ_{jk} can be approximated by their Taylor series and one finds

$$
\zeta_{jk}(\mathbf{z}(0)) = \sum_{\substack{m_1+m_2+m_3=4\\ m_j\geq 0}} \frac{6c_{jk}(\mathbf{m})}{(4m_1+7m_2+7m_3-4)}\mathbf{z}(0)^{\mathbf{m}} + \mathcal{O}(|\mathbf{z}|^5) \quad .
$$

If one now returns to the variables of (4.4), this implies that

$$
\begin{aligned}
\psi(x,y,t) =& e^{-\frac{2}{3}t}(z_{00}(0) + \zeta_{00}(\mathbf{z}(0)))\varphi_{00}(x,t) + e^{-\frac{7}{6}t}(z_{10}(0) + \zeta_{10}(\mathbf{z}(0)))\varphi_{10}(x,t) \\
&+ e^{-\frac{7}{6}t}(z_{01}(0) + \zeta_{01}(\mathbf{z}(0)))\varphi_{01}(x,t) + \mathcal{O}(e^{-\mu_1 t}) \quad ,
\end{aligned}
$$

for some $\mu_1 \geq 2$. In light of Proposition 3.3 and the fact that $h_{jk}(\mathbf{z}) = \mathcal{O}(|\mathbf{z}|^4)$, we see that if $w(t) = (w^r(t), w^s(t))$ is a solution of (4.3) which remains in a neighborhood of the origin for all time, then there exists $\mathbf{z}(0) = (z_{00}(0), z_{10}(0), z_{01}(0))$ such that

$$
w^c(t) = \psi(t) \quad , \quad w^s(t) = \mathcal{O}(e^{-\frac{8}{3}t}) + \mathcal{O}(e^{-\mu_2 t}) \quad ,
$$

where μ_2 can be made as close as we like to $\frac{5}{3}$ $(= \lambda_{02} = \lambda_{11} = \lambda_{20})$. Thus, if we revert to our original variables, we see that for long times, the solution of (4.1) is

$$
\begin{aligned}
u(\xi\sqrt{\tau},\tau) = & \frac{1}{2\sqrt{\pi}}(\frac{3}{5})^2\frac{1}{\tau}(z_{00}+\zeta_{00}(\mathbf{z}(0)))e^{-\frac{(\xi_1^2+\xi_2^2)}{4}} \\
& + \frac{1}{2\sqrt{2\pi}}(\frac{6}{13})^2\frac{1}{\tau^{3/2}}(z_{10}+\zeta_{10}(\mathbf{z}(0)))H_1(\xi_1)e^{-\frac{(\xi_1^2+\xi_2^2)}{4}} \\
& + \frac{1}{2\sqrt{2\pi}}(\frac{6}{13})^2\frac{1}{\tau^{3/2}}(z_{01}+\zeta_{01}(\mathbf{z}(0)))H_1(\xi_2)e^{-\frac{(\xi_1^2+\xi_2^2)}{4}} + \mathcal{O}(\frac{1}{\tau^{2-\epsilon}}) \quad ,
\end{aligned}
$$

where ϵ can be made arbitrarily small by choosing a sufficiently small neighborhood of the origin. This estimate completes the proof of Proposition 4.1.

■

Remark. Note that while the lowest order asymptotics for this equation (see, *e.g* [BKL]) are "universal" – that is regardless of initial conditions they approach the function $u^*(\xi\sqrt{\tau},\tau) = \frac{K}{\tau}e^{-(\xi_1^2+\xi_2^2)/4}$, for some choice of K, the higher order asymptotics are more complicated – in particular, not universal – but depend in a complicated way on the initial conditions. In related equations, which model the propagation of waves on a fluid surface, such dependence may have experimentally observable consequences.

Remark. While this example shows that the second order asymptotics are more complicated than the first, yet more involved phenomena may occur. For examples of such problems, the reader may wish to consider the second order asymptotics of $\partial_\tau u = \partial_{\xi\xi}u - u^4$, or $\partial_\tau u = \partial_{\xi\xi}u - u^3$, for $\xi \in \mathbb{R}$.

4.2. Behavior of solutions for p near $p_c = 1 + 2/d$.

In this subsection we consider the behavior of solutions on (4.1), for $p \approx 1 + 2/d$. This question has already received much attention ([GKS], [KP], [BK], [EK]). I wish to illustrate a new outlook on the change in the behavior of solutions when p passes through p_c using ideas from bifurcation theory, coupled with the invariant manifold theorems of this paper. For simplicity restrict consideration to one spatial dimension. If one rewrites (4.1) in the "w"-variables, one finds

$$
\frac{\partial w}{\partial \tau} = H_0(p_c)w + \frac{(p_c - p)}{(p_c-1)(p-1)}w - e^{-\frac{(p-1)x^2}{8}}|w|^{p-1}w \quad .
$$

Now use a standard trick from bifurcation theory (see, *e.g.* [R]) and introduce $\rho = p_c - p$ as a new variable. Then, using the fact that $p_c = 3$ for $d = 1$, one has:

$$
\begin{aligned}
&\dot{\rho} = 0 \quad , \\
(4.6) \qquad &\frac{\partial w}{\partial \tau} = \left(\frac{d^2}{dx^2} - \frac{x^2}{16} + \frac{1}{4} \right) w + \frac{\rho w}{2(2-\rho)} - e^{-\frac{(2-\rho)x^2}{8}} |w|^{2-\rho} w \quad .
\end{aligned}
$$

Noting that the spectrum of $\left(\frac{d^2}{dx^2} - \frac{x^2}{16} + \frac{1}{4} \right)$ is $\{0, -1/2, -1, -3/2, \ldots\}$ one sees that (4.6) has a two dimensional center manifold, tangent at the origin to the zero eigenspace of $H_0(p_c)$, and the ρ direction.

As in the previous example, if one takes $s = 2$ and $c(p) = 1$, one can readily compute an approximation to the equations in the center manifold and one finds the approximate equations in the center manifold are:

$$
\begin{aligned}
&\dot{\rho} = 0 \quad , \\
(4.7) \qquad &\dot{z} = \frac{\rho}{2(2-\rho)} z - \frac{1}{2} \sqrt{\frac{1}{\sqrt{3\pi}}} z^3 + \mathcal{O}_4(\rho, z) \quad ,
\end{aligned}
$$

where z is the coordinate in the direction of $\varphi_0(x)$, the eigenfunction of H_0 with zero eigenvalue, and $\mathcal{O}_4(\rho, z)$ means that the terms that have been omitted are of the form $\rho^\alpha z^\beta$ with $\alpha + \beta \geq 4$. Note that for $\rho < 0$, (*i.e.* $p > p_c$), the origin is a stable fixed point of (4.7). Thus, $z(t) \approx z(0) \exp(\rho t / 2(2-\rho))$, or reverting to the original variables,

$$
u(\xi \sqrt{\tau}, \tau) \approx \frac{K}{\tau^{\frac{1}{p-1}}} \tau^{\frac{\rho}{2(2-\rho)}} e^{-\xi^2/4} = \frac{K}{\sqrt{\tau}} e^{-\xi^2/4} \quad ,
$$

as expected.

On the other hand, as ρ passes through zero, the origin becomes unstable and a new stable fixed point of (4.7) appears at $z_* \approx \sqrt{\sqrt{3\pi}\rho/(2-\rho)}$. (This is the exact location of the fixed point if one ignores the terms $\mathcal{O}_4(\rho, z)$.) Thus in this case,

$$
z(t) \approx z_* + K(z(0)) e^{-\mu t} \quad ,
$$

where for ρ small, $\mu = -\rho/2 + \mathcal{O}(\rho^2)$. Thus, if $w^*(x)$ is the function corresponding to z_* (the properties of w^* have been studied in [BPT] [GKS] [KP]) one finds that

$$
\| w(x, t) - w^*(x) \|_{W^s} \leq K e^{-\mu t} \quad ,
$$

or if we again revert to our original variables,

$$\|\tau^{\frac{1}{p-1}} u(\cdot\sqrt{\tau},\tau) - e^{-\frac{\xi^2}{8}} w^*(\xi)\|_{H_r^{s,2}} \leq K/\tau^{\mu} \quad ,$$

which is again consistent with the results of [EK] and [BK], but in contrast to those results, gives an explicit estimate of μ.

5. Conclusions and Possible Extensions

One obvious extension of this work is to expand the class of nonlinear terms allowed in (1.1). For instance, the studies of the lowest order asymptotics of such equations (*e.g* [BKL]) admit nonlinearities of the form $F(u, \nabla u)$ which depend on the gradient of u. In the present case we cannot include such terms because of the requirement of Gallay's theorem [G] that the nonlinear term be a C^k function with $k > 1$. Such a condition seems to be necessary if one works with only very weak assumptions on the linear evolution, as Gallay does. In the present situation, however, one has quite detailed knowledge of the linear evolution, and I hope that by taking advantage of that knowledge, one can extend these "pseudo-center" manifolds to equations with derivatives in the nonlinear terms, much as Mielke [M1] was able to do for the ordinary center-manifolds.

Another intriguing question is whether or not results analogous to those described here are also applicable to dispersive equations. Strauss [S] has investigated the long-time behavior of nonlinear perturbations of the Schrödinger equation and linearized KdV equation and found that under appropriate assumptions on the nonlinear term, solutions of such equations approach a solution of the linear equation, just as do solutions of (1.1), when $p > p_c$. It is natural to wonder whether the geometric structures underlying the long time behavior in the dissipative case are also present in the dispersive equations.

APPENDIX

In this appendix I prove Proposition 3.2, which formed the basis of the study of the attractivity of the pseudo-center manifold in section 3. We recall that the weighted Sobolev spaces were defined as

$$H_r^{q,2} = \{v \mid (1-\Delta)^{q/2}(1+|x|^2)^{r/2}v(x) \in L^2(\mathbb{R}^d)\} \quad .$$

For simplicity we assume q is an even integer. Let $\chi_l(x)$ be a smooth function satisfying

$$\chi_\ell(x) = \begin{cases} 0, & \text{if } |x| \leq R \ ; \\ 1, & \text{if } |x| \geq 8R/7, \end{cases}$$

for some constant $R \geq 0$,

The first result shows that the linear evolution $\exp(t\mathcal{L}_0)$ tends to shrink things located far from the origin.

Proposition 5.1. *There exists $C > 0$ such that for any $v \in H_r^{q,2}$,*

$$\|\chi_\ell \exp(t\mathcal{L}_0)v\|^2_{H_r^{q,2}} \leq Ce^{\frac{2t}{p-1}}e^{-\frac{dt}{2}}\{e^{\frac{(q-r)t}{2}} + e^{\frac{tq}{2}}e^{-\frac{3}{16}R^2}\}^2\|v\|^2_{H_r^{q,2}} \quad .$$

Proof. Using Mehler's formula for $\exp(t\mathcal{L}_0)$ as in [BK], one finds:

$$\begin{aligned}\|\chi_\ell \exp(t\mathcal{L}_0)v\|^2_{H_r^{q,2}} = (4\pi a(t))^{-d}e^{\frac{2t}{p-1}} &\int\int dz_1dz_2 e^{-\frac{z_1^2}{4a(t)}}e^{-\frac{z_2^2}{4a(t)}} \times \\ &\int [(1-\Delta)^{q/2}(1+|x|^2)^{r/2}\chi_\ell(x)v(e^{\frac{t}{2}}[x+z_1])\times \\ &\quad [(1-\Delta)^{q/2}(1+|x|^2)^{r/2}\chi_\ell(x)v(e^{\frac{t}{2}}[x+z_2])dx \quad ,\end{aligned}$$

where $a(t) = (1-\exp(-t))$. Applying the Cauchy-Schwartz inequality to the integral over x allows one to estimate it by

$$\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot+z_1])\|_{H_r^{q,2}}\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot+z_2])\|_{H_r^{q,2}} \quad ,$$

so that

$$\|\chi_\ell \exp(t\mathcal{L}_0)v\|^2_{H_r^{q,2}} \leq (4\pi a(t))^{-d}e^{\frac{2t}{p-1}}\Big(\int dz e^{-\frac{z^2}{4a(t)}}\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot+z])\|_{H_r^{q,2}}\Big)^2 \quad . \tag{5.1}$$

To complete the proof, break up the integral over z with the aid of the following lemma.

Lemma 5.2. *There exists a constant $C > 0$ such that for all $v \in H_r^{q,2}$:*

(i) if $|z| \leq 7R/8$, $\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot + z_1])\|^2_{H_r^{q,2}} \leq Ce^{-\frac{dt}{2}}e^{(q-r)t}\|v\|^2_{H_r^{q,2}}$,

(ii) if $|z| > 7R/8$, $\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot + z_1])\|^2_{H_r^{q,2}} \leq Ce^{qt}e^{-\frac{dt}{2}}(1+|z|^2)^r\|v\|^2_{H_r^{q,2}}$.

Applying this lemma to (5.1) we find

$$\|\chi_\ell \exp(t\mathcal{L}_0)v\|^2_{H_r^{q,2}} = (4\pi a(t))^{-d}e^{\frac{2t}{p-1}}Ce^{-\frac{dt}{2}}\|v\|^2_{H_r^{q,2}} \times \\ \{\int_{|z|\leq 7R/8} e^{\frac{(q-r)t}{2}}e^{-\frac{z^2}{4a(t)}}dz + \int_{|z|>7R/8} e^{\frac{qt}{2}}(1+|z|^2)^{r/2}e^{-\frac{z^2}{4a(t)}}dz\}^2 \ .$$

Elementary estimates of these integrals then complete the proof of the proposition.

■

Proof. (of the lemma) First we consider the case $|z| \leq 7R/8$.

$$\begin{aligned}\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot + z])\|^2_{H_r^{q,2}} =& \int[(1-\Delta)^{q/2}(1+|x|^2)^{r/2}\chi_\ell(x)v(e^{\frac{t}{2}}[x+z])]^2dx \ , \\ \leq& C\sum_{\alpha,\beta}\int(1+|x|^2)^r|\partial^\alpha\chi_\ell(x)|^2e^{|\beta|t}|\partial^\beta v(e^{\frac{t}{2}}[x+z])|^2dx \ ,\end{aligned}$$

where the sum runs over all ways of distributing the derivatives over χ and v. This in turn is bounded by:

$$C(q) \sup_{\substack{\alpha,\beta \\ |\alpha|+|\beta|\leq q}} e^{|\beta|t}\int(1+|x|^2)^r(1+e^t|x+z|^2)^{-r}|\partial^\alpha\chi_\ell|^2(1+e^t|x+z|^2)^r|\partial^\beta v(e^{\frac{t}{2}}(x+z))|^2dx \, .$$

Since $\chi_\ell(x) = 0$ if $|x| \leq R$, we see that $|x+z| \geq |x|/8$. Thus, $(1+|x|^2)^r(1+e^t|x+z|^2)^{-r} \leq Ce^{-tr}$ and we have

$$\begin{aligned}\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot + z])\|^2_{H_r^{q,2}} \leq& C(q,\chi)e^{t(q-r)}\sup_{|\beta|\leq q}\int(1+e^t|x+z|^2)^r|\partial^\beta v(e^{\frac{t}{2}}|x+z|)|^2dx \\ \leq& C(q,\chi)e^{t(q-r)}e^{-\frac{dt}{2}}\|v\|^2_{H_r^{q,2}} \ .\end{aligned}$$

Next we consider the case in which $|z| > 7R/8$. In this case we have

$$\begin{aligned}\|\chi_\ell(\cdot)v(e^{\frac{t}{2}}[\cdot+z])\|^2_{H^{q,2}_r} =& C(q,\chi)e^{tq}\sup_{|\beta|\le q}\int(1+|x|^2)^r|\partial^\beta v(e^{\frac{t}{2}}[x+z])|^2 dx \\ =& C(q,\chi)e^{tq}\sup_{|\beta|\le q}\int(1+|e^{-\frac{t}{2}}y-z|^2)^r|\partial^\beta v(y)|e^{-\frac{td}{2}}dy \\ \le& C(q,\chi)e^{tq}e^{-\frac{td}{2}}(1+|z|^2)^r\|v\|^2_{H^{q,2}_r} \quad .\end{aligned}$$

■

With the aid of this proposition we now proceed roughly as follows. In order to estimate $\exp(t\mathcal{L}_0)Q_n v$, we break it into two pieces, one with the support near the origin, and one with support far from the origin. The part with support near the origin will be in W^s, the Hilbert space studied in section 2, and hence its evolution can be studied using the spectral theory of $\exp(tH_0)$. The evolution of the part far from the origin will be controlled using Proposition 5.1. A similar method was used in [BK], which inspired the present approach.

I begin with a few remarks and some notation. Given a positive integer N, define Q_N as in section 3. Note that if $v \in H^{q,2}_r$ is in the range of Q_N, then $Q_N v = v$. Also, given any function f, define $f_>(x) = \chi_\ell(x)f(x)$, and $f_<(x) = (1-\chi_\ell(x))f(x)$. Finally, we have

Lemma 5.3. *Suppose that $s \ge q$, and that $w \in W^s$. Then there exists $C(q,r,s) \ge 0$, such that*

$$\|e^{-\frac{x^2}{8}}w(x)\|_{H^{q,2}_r} \le C(q,r,s)\|w\|_{W^s} \quad .$$

Proof. This follows from the definition of the norms and the fact that for every $r > 0$, there exists $K(r) > 0$, such that $\sup_x(1+|x|^2)^r e^{-\frac{x^2}{8}} \le K(r)$.

■

To estimate $\exp(t\mathcal{L}_0)Q_n v$, we begin by writing it as

$$\exp(t\mathcal{L}_0)Q_n v = \exp(t\mathcal{L}_0)Q_n v_< + \exp(t\mathcal{L}_0)Q_n v_> \quad . \tag{5.2}$$

The first of these two terms is simple to estimate.

Lemma 5.4. *Assume that $q = s$, and $r \geq q$. Then there exists $C > 0$, such that*

$$\| \exp(t\mathcal{L}_0) Q_n v_< \|_{H_r^{q,2}} \leq C e^{\frac{R^2}{6}} e^{-|\lambda_{N+1}|t} \|v\|_{H_r^{q,2}} \quad .$$

Proof.

$$\|e^{(t\mathcal{L}_0)} Q_n v_<\|_{H_r^{q,2}} = \|e^{-\frac{x^2}{8}} e^{tH_0} e^{\frac{x^2}{8}} Q_n v_<\|_{H_r^{q,2}} \leq C \|e^{tH_0}(e^{\frac{x^2}{8}} Q_n v_<)\|_{W^s} \quad ,$$

by the preceding lemmas. But $(e^{\frac{x^2}{8}} Q_n v_<) = \tilde{Q}_N (e^{\frac{x^2}{8}} v_<)$, where $\tilde{Q}_N$ is the projection onto the orthogonal complement of $span\{\varphi_0, \ldots, \varphi_N\}$ in W^s. Thus,

$$\|e^{tH_0} \tilde{Q}_N (e^{\frac{x^2}{8}} v_<)\|_{W^s} \leq C e^{-|\lambda_{N+1}|t} \|(e^{\frac{x^2}{8}} v_<)\|_{W^s} \quad .$$

Since $v_<(x) = 0$ if $|x| \geq 8R/7$, $\|(e^{\frac{x^2}{8}} v_<)\|_{W^s} \leq C(1 + R^r) e^{\frac{8}{49} R^2} \|v\|_{H_r^{q,2}}$ and the lemma follows.

■

The second term in (5.2) is more difficult to control. We begin by noting that if one sets $R = 0$ in Proposition 5.1 one has

Corollary 5.5.

$$\|e^{(t\mathcal{L}_0)} v\|_{H_r^{q,2}}^2 \leq C e^{\frac{2t}{p-1}} e^{-\frac{dt}{2}} e^{tq} \|v\|_{H_r^{q,2}}^2 \quad .$$

Fix $0 < \alpha < 1$. Then,

$$\begin{aligned} e^{t\mathcal{L}_0} Q_N v_> =& e^{\alpha t \mathcal{L}_0} e^{(1-\alpha)t\mathcal{L}_0} Q_N v_> = e^{\alpha t\mathcal{L}_0} Q_N e^{(1-\alpha)t\mathcal{L}_0} v_> \\ =& e^{\alpha t\mathcal{L}_0} Q_N \chi_\ell e^{(1-\alpha)t\mathcal{L}_0} v_> + e^{\alpha t\mathcal{L}_0} Q_N (1 - \chi_\ell) e^{(1-\alpha)t\mathcal{L}_0} v_> \quad . \end{aligned} \tag{5.3}$$

By Lemma 5.4 and Corollary 5.5, one has

$$\begin{aligned} \|e^{\alpha t\mathcal{L}_0} Q_N (1-\chi_\ell) e^{(1-\alpha)t\mathcal{L}_0} v_>\|_{H_r^{s,2}} \leq& C e^{\frac{R^2}{6}} e^{-\alpha t|\lambda_{N+1}|} \|(1-\chi_\ell) e^{(1-\alpha)t\mathcal{L}_0} v_>\|_{H_r^{q,2}} \\ \leq& C e^{\frac{R^2}{6}} e^{-\alpha t|\lambda_{N+1}|} e^{\frac{(1-\alpha)t}{p-1}} e^{\frac{1}{2}(1-\alpha)t(s-\frac{d}{2})} \|v_>\|_{H_r^{s,2}} \\ \leq& C e^{\frac{R^2}{6}} e^{-t(\alpha|\lambda_{N+1}| - (1-\alpha)(\frac{1}{p-1} + \frac{s}{2} - \frac{d}{4}))} \|v\|_{H_r^{s,2}} \quad . \end{aligned}$$

To control the first term on the RHS of (5.3), use Corollory 5.5 and Proposition 5.1 to obtain

$$\begin{aligned}
\|e^{\alpha t\mathcal{L}_0}Q_N\chi_\ell e^{(1-\alpha)t\mathcal{L}_0}v_>\|_{H_r^{s,2}} &\le Ce^{\alpha t(\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4})}\|\chi_\ell e^{(1-\alpha)t\mathcal{L}_0}v_>\|_{H_r^{s,2}} \\
&\le Ce^{\alpha t(\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4})}e^{(1-\alpha)t(\frac{1}{p-1}-\frac{d}{4})}\{e^{\frac{(1-\alpha)(s-r)t}{2}}+e^{\frac{(1-\alpha)st}{2}}e^{-\frac{3}{16}R^2}\}\|v_>\|_{H_r^{s,2}} \\
&\le \quad Ce^{t(\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4})}\{e^{-\frac{(1-\alpha)rt}{2}}+e^{-\frac{3}{16}R^2}\}\|v\|_{H_r^{s,2}} \quad .
\end{aligned}$$

If one fixes $\alpha=1/2$, and combines these two estimates one obtains

Lemma 5.6. *If $q=s$ and $r\ge q$, then there exists $C>0$ such that*

$$\begin{aligned}
&\|e^{t\mathcal{L}_0}Q_Nv\|_{H_r^{s,2}} \\
&\le C\{e^{\frac{R^2}{6}}e^{-t[\frac{|\lambda_{N+1}|}{2}-\frac{1}{2}(\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4})]}+e^{t[\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4}-\frac{r}{4}]}+e^{t[\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4}]}e^{-\frac{3}{16}R^2}\}\|v\|_{H_r^{s,2}} \ .
\end{aligned}$$

Now fix $s>1+d/2$, an integer. Choose r so large that $\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4}-\frac{r}{4}<-|\lambda_{N+1}|/4$. Choose R so that

$$e^{t[\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4}]}e^{-\frac{3R^2}{16}}=e^{-\frac{1}{4}t|\lambda_{N+1}|} \quad .$$

(*i.e.* $R=\{\frac{16}{3}t(\frac{|\lambda_{N+1}|}{4}+\frac{1}{p-1}+\frac{s}{2}-\frac{d}{2})\}^{1/2}$.) Then,

$$e^{\frac{R^2}{6}}e^{-t[\frac{|\lambda_{N+1}|}{2}-\frac{1}{2}(\frac{1}{p-1}+\frac{s}{2}-\frac{d}{4})]}\le e^{-\frac{5}{18}t|\lambda_{N+1}|}e^{2t[\frac{1}{p-1}+\frac{s}{2}-\frac{d}{2}]} \quad .$$

Finally, choose N sufficiently large that this expression is bounded by $\exp(-t|\lambda_{N+1}|/4)$. Thus, one has

Corollary 5.7. *If $q=s$, and N and r are sufficiently large, there exists $C>0$ such that*

$$\|e^{t\mathcal{L}_0}Q_Nv\|_{H_r^{s,2}}\le Ce^{-\frac{1}{4}t|\lambda_{N+1}|}\|v\|_{H_r^{s,2}} \quad .$$

To complete the proof of Proposition 3.2, we now suppose that $n\ge 0$ is fixed. Choose N large enough that it satisfies the hypotheses of Corollary 5.7 and so that $|\lambda_{N+1}|>4|\lambda_n|$. Consider the projection operators P_n, Q_n and P_N, Q_N. We rewrite

$$e^{t\mathcal{L}_0}Q_nv=e^{t\mathcal{L}_0}(Q_N+P_N)Q_nv=e^{t\mathcal{L}_0}Q_NQ_nv+e^{t\mathcal{L}_0}P_NQ_nv \quad .$$

By Corollary 5.7 we can bound $\|e^{t\mathcal{L}_0} Q_N Q_n v\|_{H_r^{s,2}} \leq C \exp(-t|\lambda_{N+1}|/4) \|v\|_{H_r^{s,2}} \leq C \exp(-t|\lambda_{n+1}|) \|v\|_{H_r^{s,2}}$. On the other hand, by the orthonormality of the $\psi_j{}'s$, $P_N(Q_n v) = P_N(1 - P_n)v = \sum_{j=n+1}^{N} \psi_j \langle\langle \psi_j, v \rangle\rangle_s$, and $\exp(t\mathcal{L}_0)\psi_j = \exp(t\lambda_j)\psi_j$, so,

$$\|e^{t\mathcal{L}_0} P_N Q_n v\|_{H_r^{s,2}} \leq \sum_{j=n+1}^{N} \|e^{t\mathcal{L}_0} \psi_j\|_{H_r^{s,2}} |\langle\langle \psi_j, v \rangle\rangle_s| \leq C e^{-t|\lambda_{n+1}|} \|v\|_{H_r^{s,2}} \quad .$$

Combining these two estimates we obtain Proposition 3.2

■

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