PERTURBATIVE QUANTUM FIELD THEORY AT POSITIVE TEMPERATURES: AN AXIOMATIC APPROACH

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Abstract

It is shown that the perturbative expansions of the correlation functions of a relativistic quantum field theory at finite temperature are uniquely determined by the equations of motion and standard axiomatic requirements, including the KMS condition. An explicit expression as a sum over generalized Feynman graphs is derived. The canonical formalism is not used, and the derivation proceeds from the beginning in the thermodynamic limit. No doubling of fields is invoked. An unsolved problem concerning existence of these perturbative expressions is pointed out.

1 INTRODUCTION

The traditional way of describing thermal equilibrium states of an infinitely extended quantum system, in particular of a quantum field theory, begins by restricting the system to a finite volume V, defining the canonical or grand canonical equilibrium by means of the familiar density matrices, and then going to the limit $V \to \infty$ (the "thermodynamic limit") for the quantities for which this limit can be expected to exist [1,2]. This applies especially to the correlation functions of the fields and closely related objects like the expectation values of time ordered field products. Up to now most actual calculations of such functions have been based on this approach, using a Hamiltonian or Lagrangian formalism at finite V.

Another description of equilibria and their local disturbances, which can be used directly in the thermodynamic limit, has been developed in the framework of the algebras of local observables [3]. In this approach equilibrium states are characterized through an analyticity requirement for correlation functions, the so-called KMS condition. In the present paper we intend to show that this axiomatic method, suitably adapted to a field theoretical context, is perfectly capable of handling dynamical problems. More exactly, it will be shown that perturbative expansions for the correlation functions of a relativisitic field theory, and related functions, can be derived directly in the thermodynamic limit, not making use of the canonical formalism, but using as only inputs the equations of motion and the axiomatic requirements that the correlation functions must satisfy. The result is represented as a sum over generalized Feynman graphs. For the special case of time ordered functions it agrees with the well-known result of the canonical approach.

Dispensing with the canonical formalism is also a major difference between our approach and thermo field dynamics [2,4], a Fock space method developed by H. Umezawa and coworkers. We differ also from thermo field dynamics by not invoking a doubling of fields, and by not assigning a basic role to particles, including quasi-particles. We hold particles to be secondary objects of the theory, of great phenomenological importance, but little fundamental significance. In this respect we differ also from the views put forward by Landsman in ref. [5].

We consider only the Φ^4_4-model, i.e. the theory of a scalar hermitian field

$\Phi(x)$ satisfying the equation of motion

$$(\Box + m^2)\Phi = -\frac{g}{6}N(\Phi^3), \tag{1}$$

where N stands for a normal-product prescription taking care of renormalization. The restriction to Φ_4^4 is merely a matter of convenience. The generalization of the method to other models, including gauge theories in local gauges, is straightforward.

We are interested in the correlation functions

$$W(x_1, \ldots, x_n) = \langle \Phi(x_1) \ldots \Phi(x_n) \rangle, \tag{2}$$

where $\langle \cdot \rangle$ denotes the expectation value in a thermal equilibrium state with temperature $T \geq 0$. These correlation functions describe the full physical content of the theory: all observable quantities can in principle be derived from them (for examples see e.g. [1,2]). This is so because knowledge of the W allows the reconstruction of the full representation of the field algebra by means of the GNS construction [2,3], which yields a Hilbert space representation of the field algebra with a cyclic vector $|\rangle$, such that $\langle A \rangle = \langle | R(A) | \rangle$ for any sufficiently regular function $A(\Phi)$, with $R(A)$ its representative. All observables of the theory are supposed to be of this form, and local disturbances of the equilibrium are created by applying suitable functions $F(\Phi)$ to $|\rangle$.

More generally, we consider the set of functions (or rather, distributions)

$$\mathcal{W}(X_1, s_1 | \ldots | X_N, s_N) = \langle T^{s_1}(X_1) \ldots T^{s_N}(X_N) \rangle. \tag{3}$$

Here the X_α are non-overlapping sets of 4-vectors x_i, the s_α are signs, and $T^{\pm}(X)$ denotes respectively the time-ordered or anti-time-ordered product of the fields $\Phi(x_i), x_i \in X$. If each X_α contains only one variable, then $\mathcal{W}$ is the correlation function $W(x_1, \ldots, x_N)$ irrespective of the choice of the signs s_α. For $N = 1$ we obtain the usual time-ordered and anti-time-ordered functions (Green's functions) of the theory. In the sequel the signs s_α will be frequently suppressed when they are not essential to understanding.

In a previous work [6], henceforth quoted as V, we have derived perturbative expressions for the functions $\mathcal{W}$ in terms of generalized Feynman graphs for the case $T = 0$, in which case $\langle \cdot \rangle$ denotes the vacuum expectation value. This derivation uses neither the Hamiltonian nor the Lagrangian formalism, but is instead relying on the Wightman axioms as an essential input. We propose to generalize this method to the case $T > 0$. Our way of proceeding is closely modeled on that taken in V. The ideas and results of that paper will be freely used. Equation (n.m.) of V will be referred to as eq. (V.n.m.). The method conists in solving the differential equations for W which follow from the field equation (1.1), by a power series expansion in the coupling constant g, using the axiomatic properties of W as subsidiary conditions.

The paper is organized as follows. The assumptions on which the formalism is based will be stated and discussed in section 2. An explicit formal expression for the $\mathcal{W}$'s as sums over generalized Feynman graphs will be stated in section 3 and shown to possess the required properties. In section 4 the ultraviolet (UV) divergences of these graphs will be removed by renormalization. It will be pointed out, however, that renormalization does not guarantee the existence of the resulting finite-order expressions, on account of the local singularities of the integrands. This problem remains unsolved. Finally we will show in section 5 that the expressions of sections 3 and 4, provided they exist, are the only ones satisfying the assumptions stated in section 2.

2 ASSUMPTIONS

In this section the assumptions on which our derivations are based, in addition to the field equation (1.1), will be enumerated.

The following conditions for W and $\mathcal{W}$ are taken over unchanged from V.

a) The W and $\mathcal{W}$ are *invariant* under space-time translations and under space rotations. Invariance under Lorentz boosts cannot be demanded.

b) *Locality* holds, i.e. $W(X)$ is invariant under the exchange of two neighbouring variables x_i, x_{i+1}, if $x_i - x_{i+1}$ is space-like.

c) The *reality condition*

$$\mathcal{W}(X_1, s_1|\dots|X_N, s_N)^* = \mathcal{W}(X_N, -s_N|\dots|X_1, -s_1) \tag{4}$$

holds.

d) The *cluster property* (V.1.7) holds: the truncated correlation function $W^T(X)$ vanishes if two complementary non-empty subsets of X are removed from each other to infinity in a space-like direction. W^T is recursively defined through the cluster expansion

$$W(X) = \sum W^T(X_1)\dots W^T(X_N), \qquad W^T(x_1, x_2) = W(x_1, x_2), \tag{5}$$

the sum running over all partitions of $X = \{x_1, \ldots, x_n\}$ into an arbitrary number of complementary subsets with an even number of elements. The cluster condition implies that the state $\langle\cdot\rangle$ describes a pure phase [3]. Perturbation theory is not the proper vehicle for studying phase transitions.

e) The functions $\mathcal{W}$ are *permutation invariant* within each sector X_α, and they satisfy the *splitting property*

$$\mathcal{W}(\ldots|X_1 \cup X_2, +|\ldots) = \mathcal{W}(\ldots|X_1, +|X_2, +|\ldots) \tag{6}$$

if $x_i^0 > x_j^0$ for all $x_i \in X_1, x_j \in X_2$. This property shall hold in every Lorentz frame, not only in the rest frame of the infinite system under consideration. To that extent we retain Lorentz invariance. These conditions are not merely a definition of time ordering. That they can be satisfied is intimately connected with locality.

The spectral condition of the vacuum representation does not hold at positive temperatures. It is replaced by the *KMS condition*, which we use in its p-space form (see [3], lemma 1.1.1 of chapter V). Define the Fourier transform $\tilde{\Phi}(p)$ of Φ by

$$\tilde{\Phi}(p) = (2\pi)^{-5/2} \int dx e^{ipx} \Phi(x), \tag{7}$$

and the Fourier transform $\tilde{\mathcal{W}}(P_1|\ldots|P_N)$ of $\mathcal{W}(X_1|\ldots|X_N)$ accordingly. Let $P_1, \ldots, P_N, Q_{N+1}, \ldots, Q_M, M > N$, be finite, non-empty sets of 4-momenta. Define

$$P^0 = \sum_{p_i \in \cup P_\alpha} p_i^0. \tag{8}$$

Then the KMS condition states that

$$\tilde{\mathcal{W}}(P_1|\ldots|P_N|Q_{N+1}|\ldots|Q_M) = e^{\beta P^0} \tilde{\mathcal{W}}(Q_{N+1}|\ldots|Q_M|P_1|\ldots|P_N), \tag{9}$$

where $\beta = \frac{1}{kT}$ is the inverse temperature.

The normalization conditions stipulated in V are taken over unchanged for the case $T = 0$. They are the standard conditions demanding that m be the physical mass of the particles associated with the field Φ at $T = 0$, and defining the coupling constant g and the field normalization in terms of the Green's functions. In addition we demand that Φ, considered as an element of an abstract, representation independent, field algebra, is T-independent.

This implies that the field equation is T-independent, meaning that the parameters m and g and the subtraction prescription N do not depend on the temperature. In perturbation theory the T-independence of N means that the UV subtractions and the subsequent finite renormalizations have the values used at $T = 0$ for all T. Note that in the BPHZ method [7], which we will be using, the subtractions do not involve prior regularizations or any formal juggling with divergent quantities. The procedure is therefore well defined. For more details we refer to section 4. A nonperturbative definition of N can possibly be given with the help of a Wilson expansion, as explained in section IV.2. of ref. [7], i.e. by a point-splitting method with suitable singular coefficient functions. T-independence of N could then be defined as T-independence of these coefficients. We will not explore this possibility any further.

Note that the parameter m is for $T > 0$ not the physical mass of a particle. Quasi-particle masses are determined by the positions of the singularities of the clothed propagator, which are T-dependent.

3 UNRENORMALIZED SOLUTION

The coefficient $W_\sigma(X)$ of order σ in the power series expansion

$$W(X) = \sum_{\sigma=0}^{\infty} g^\sigma W_\sigma(X) \tag{10}$$

is determined as a solution of the system of differential equations

$$(\Box_i + m^2)\langle \ldots \Phi(x_i) \ldots \rangle_\sigma = -\frac{1}{6}\langle \ldots N(\Phi(x_i)^3) \ldots \rangle_{\sigma-1} \tag{11}$$

satisfying the subsidiary conditions described in section 2. For $\sigma = 0$ the right-hand side is zero, for $\sigma > 0$ it can be calculated if the problem has already been solved in order $\sigma - 1$.

We first state a formal, unrenormalized, graphical representation for $\mathcal{W}_\sigma(X_1|\ldots|X_N)$ and then show that it does indeed satisfy all the requirements.

For the free propagators we use the following notations. We define first the p-space expressions

$$\tilde{\Delta}_+(p) = \frac{1}{(2\pi)^3}\delta_+(p), \quad \tilde{\Delta}_F(p) = \frac{i}{(2\pi)^4}\frac{1}{p^2 - m^2 + i\epsilon}, \tag{12}$$

which are the usual vacuum propagators, and their thermal extensions

$$\tilde{D}_+(p) = \tilde{\Delta}_+(p) + \tilde{C}(p), \qquad \tilde{D}_F(p) = \tilde{\Delta}_F(p) + \tilde{C}(p) \tag{13}$$

with the T-dependent correction term

$$\tilde{C}(p) = \frac{1}{(2\pi)^3}\frac{1}{e^{\beta\omega}-1}\delta(p^2-m^2). \tag{14}$$

Here $\delta_+(p) = \theta(p_0)\delta(p^2-m^2)$ is the δ-function of the positive mass shell, and $\omega = (\mathbf{p}^2+m^2)^{1/2}$. Note that the additional term $\tilde{C}$ is the same for $\tilde{D}_+$ and $\tilde{D}_F$.

The x-space versions of these propagators are defined by

$$\Delta_{\cdot}(x) = i\int d^4p\tilde{\Delta}_{\cdot}(p)e^{-ipx}, \qquad D_{\cdot}(x) = i\int d^4p\tilde{D}_{\cdot}(p)e^{-ipx}. \tag{15}$$

The thermal propagators (3.4-5) need not be taken over from the traditional formalism. They can be derived in our framework by the methods that will be used in section 5 to establish uniqueness of our solution. But in the present section we rely on an a posteriori justification, by showing that these forms give rise to expressions with all the required properties.

$\mathcal{W}_\sigma$ is represented as a sum over generalized Feynman graphs which are defined as follows. Draw first an ordinary Feynman graph of the Φ^4 theory with $|X| = \sum_\alpha |X_\alpha|$ external and σ internal vertices. Here $|X_\alpha|$ is the number of points in the set X_α. The graph need not be connected, but must not contain any components without external points. This graph is called the "scaffolding" of the generalized graph. Next, it is partitioned into non-overlapping subgraphs, called "sectors", such that the external points of a set X_α belong all to the same sector, but variables in different X_α to different sectors. There may also exist "internal sectors" not containing external points. The sectors are either of type T^+ or T^-. For external sectors this sign is given by s_α. To each sector S we affix its number $\nu(S)$ according to the following rules.

i) $\nu(S) = \alpha$ for the external X_α-sector.

ii) If $s_\alpha = s_{\alpha+1}$ there may be an internal sector with number $\alpha+\frac{1}{2}$. Its type is the reverse of the adjacent external sectors: $s_{\alpha+\frac{1}{2}} = -s_\alpha$. If $s_\alpha = -s_{\alpha+1}$ no such intermediate internal sector exists.

iii) If $s_1 = s_N$ there may be an internal sector with number $N + \frac{1}{2}$ and $s_{N+\frac{1}{2}} = -s_N$. Equivalently we could give this additional sector the number $\frac{1}{2}$, but we must choose one of these possibilities and use it consistently.

With a partitioned graph we associate a Feynman integrand, which we state first in x-space. To the external points correspond the external variables x_i. To each internal vertex we assign an integration variable $u_j, \quad j = 1, \ldots, \sigma$. z_i denotes a variable which may be either external or internal. Within a T^+ sector the usual Feynman rules hold: each internal vertex carries a factor $-ig$, a line connecting the points z_i and z_j carries the propagator $-iD_F(z_i - z_j)$. In a T^- sector the complex-conjugates of these rules apply. A line connecting points z_i, z_j in different sectors, with z_i lying in the lower-numbered sector, carries the propagator $-iD_+(z_i - z_j)$. The graph must be divided by the usual symmetry number if it is invariant under certain permutations of points and lines.

In p-space, integration variables are assigned to the lines. The vertex factors in $T^\pm$ factors are $\mp(2\pi)^4 g$ and the propagators are $\tilde{D}_F(p)$ or $(\tilde{D}_F(p))^*$ respectively. Lines connecting different sectors carry propagators $\tilde{D}_+(p)$, the momentum p flowing from the lower to the higher sector. For each external point there is a factor $(2\pi)^{3/2}$. Momentum is conserved at each internal vertex.

For $T = 0$ the rule ii) governing internal sectors seems to differ from the corresponding rule in V, where instead of one (possibly empty) intermediate sector of different type we had chains of intermediate sectors of the same type as the bracketing external sectors. However, it can be shown with the help of Lemma V.3.1 that the two formulations are equivalent. Graphs containing non-empty internal sectors according to iii) vanish for $T = 0$, wherefore they do not occur in V.

For the Green's functions ($N = 1$) our rules agree with those of the conventional real-time formalism. Our graphs are equal to those of the Keldysh formulation [8].

The first point to be checked is that the above prescription gives an unambiguous result for the correlation functions W. The problem is that in the definition $W(x_1, \ldots, x_n) = \mathcal{W}(x_1, s_1 | \ldots | x_n, s_n)$ the signs s_α can be chosen arbitrarily. But it can be shown that the sum over all graphs with the same scaffolding does not depend on the choice of these signs. The proof is modeled closely on the corresponding proof in V and will only be briefly indicated. Consider a given scaffolding. Let S denote an internal T^+ sector, S_x an external T^+ sector with external variable $x, \bar{S}$ and $\bar{S}_x$ T^- sectors. A product S_xS or the like denotes the sum over all partitions of a given subgraph into two adjacent sectors S_x, S. The propagators connecting these two sectors are included. We will first show that the choice of s_N is irrelevant. The graphs belonging to $s_N = +$ or $s_N = -$ respectively differ only in the sectors with number $\nu(S) > N-1$. In the case $s_{N-1} = +, s_1 = -$ these variable sectors are (we put $x_N = x$) $S_x + \bar{S}S_x$ for $s_N = +$, and $\bar{S}_x + \bar{S}_xS$ for $s_N = -$. But

$$S_x + \bar{S}S_x = \bar{S}_x + \bar{S}_xS$$

is a special case of Lemma V.3.1, which remains valid for our new propagators. Similarly, if $s_1 = s_{N-1} = +$ we must show that

$$S_x + S_x\bar{S} + \bar{S}S_x + \bar{S}S_x\bar{S} = \bar{S}_x,$$

or

$$(S_x + S_x\bar{S} - \bar{S}_x - \bar{S}S_x) + \bar{S}(S_x + S_x\bar{S} - \bar{S}_x - \bar{S}S_x) + (S + \bar{S} + \bar{S}S)\bar{S}_x = 0,$$

which is correct because all three brackets vanish as a result of Lemma V.3.1. The proof of independence on the other s_α follows similar lines. The case $1 < \alpha < N$ is considerably simplified compared to V by the new formulation of the rules concerning internal sectors.

Of the conditions stated in section 2 invariance under translations and rotations, and the symmetry of $\mathcal{W}$ within $T^\pm$ factors are trivially satisfied. That the equation of motion (3.2) is satisfied, is shown exactly as in V. At the present formal level the product $N(\Phi^3)$ denotes Wick ordering $:\Phi^3(x):$ with respect to the free vacuum field.

The proof of the reality condition (2.1) and Ostendorf's proof [9] of the splitting relation (2.3) can also been taken over unchanged from V. Since the non-invariant part $C(x-y)$ is the same in D_+ and in D_F, the crucial relation $D_F(x-y) = D_+(x-y)$ if $x^0 > y^0$ holds in every orthochronous Lorentz frame. Hence the same is true for the splitting relation. From this and the symmetry within sectors we can prove locality. Let $(x-y)^2 < 0$.

Then there exist two Lorentz frames such that in one $x^0 > y^0$, in the other $x^0 < y^0$. Hence

$$\begin{aligned} W_\sigma(\ldots, x, y, \ldots) &= \mathcal{W}_\sigma(\ldots|x, y, +|\ldots) = \\ &= \mathcal{W}_\sigma(\ldots|y, x, +|\ldots) = W_\sigma(\ldots, y, x, \ldots). \end{aligned}$$

Finally, the decay properties of $D.(\xi)$ for $\xi \to \infty$ in space-like directions are the same as those of $\Delta.(\xi)$, so that the cluster property is also unaffected by the change of propagators.

Yet to be proved remains the KMS condition (2.6). It is easy to see that a graph contributing to the left-hand side of (2.6) becomes a graph contributing to the right-hand side, if the directions of all lines connecting sectors with $\nu(S) < N+1$ to sectors with $\nu(S) \geq N+1$ are reversed, and vice versa. This means that the corresponding propagators $\tilde{D}_+(k)$ are replaced by $\tilde{D}_+(-k)$. From the definitions (3.3-5) we find $\tilde{D}_+(k) = e^{\beta k_0}\tilde{D}_+(-k)$. Hence the two variants of the considered graph differ by the factor $\exp(\beta \sum_\alpha k_0^\alpha)$, the sum extending over the momenta of the lines in question. But $K = \sum k^\alpha$ is the total momentum flowing from the P-part of the graph to the Q-part, so that $K^0 = P^0$ as defined in (2.5). As a side remark we note that a similar argument can be used to show that it is immaterial whether extremal internal sectors are assigned the number $\frac{1}{2}$ or $N + \frac{1}{2}$.

4 RENORMALIZATION AND THE EXISTENCE PROBLEM

As yet, the expressions derived in the preceding section have only a formal meaning. There remains the question of the existence of the integrals symbolized by the graphs.

The UV problem is concerned with the behaviour of the integrands at infinity in momentum space. It can be handled exactly as was done in V for the vacuum case. We note first that loop integrals over loops extending over more than one sector are finite because of the strong decrease of $\tilde{D}_+(k)$ for $k_0 \to -\infty$ and momentum conservation: primitively divergent subgraphs exist only within sectors, where they can be treated with conventional methods. We choose BPHZ renormalization (see [7]), which introduces suitable subtractions in the integrand, before integrating, thereby avoiding the need for regularization. The subtractions are found by expanding the integrands of potentially dangerous subgraphs, the "renormalization parts", into power series of sufficiently high degree in their external momenta. The temperature dependent part $\tilde{C}$ of the propagator $\tilde{D}_F$ decreases exponentially fast at large momenta and is innocent of any UV problems. We can therefore define the mentioned subtraction terms using $\tilde{\Delta}_F$ instead of $\tilde{D}_F$ inside the renormalization parts, without destroying the UV convergence achieved by the subtraction. We also define the finite renormalizations of the BPHZ prescription to be T-independent, giving them the values needed to satisfy the normalization conditions at $T = 0$ stated in V. These rules are the expression of the required T-independence of the renormalization prescription N in the field equation (1.1). An important effect of this limited subtraction is the emergence for $T > 0$, of lines connecting a vertex to itself. But they carry the propagator $\tilde{C}(k)$, hence the loop integral over k exists, and is the same in T^+ and in T^- sectors.

But the UV problem is not the only existence problem we are faced with. There is also the problem of the local singularities of the integrand, again in p-space. The integrand is a product of distributions in variables which are quadratic functions of a complete set of independent external and internal momenta. The variables in a given subset of propagators may be dependent (i.e. their gradients in momentum space may be linearly dependent) on certain manifolds, in which case the product of propagators does not necessarily define a distribution. It is no longer clearly integrable even in the sense of distributions. Contrary to assertions found in the literature, the problem is not restricted to the case of two propagators separated by a self energy insertion, and thus both depending on the same variable. As an example of a more complex situation, consider the integral $\int dk \tilde{D}.(k)\tilde{D} \cdot (p-k)$ over a two-line loop, the dots standing for either $+$ or F. The mass shell $\delta' s$ in the two factors coalesce at $p = 0$, hence the integral diverges at that point, and this singularity in the external variable p (external to the considered 2-line subgraph) is not removed by renormalization. Closer inspection shows that the singularity is of first order. A chain of n two-line bubbles will then produce a singularity of order $|p|^{-n}$, which is not integrable for $n \geq 4$. Nor is it defined in another way as a distribution. The remaining integration over p, which may be an internal or external variable of the full graph, is therefore not defined: individual graphs containing such chains diverge. These divergences may cancel between graphs with the same scaffolding but different sector assignments of the vertices in the chain. But we are not aware of a proof to this effect, even for this still relatively simple example.

How is this problem solved in the vacuum case? This is easiest for the fully time ordered function $\tilde{\tau}(P) = \tilde{\mathcal{W}}(P,+)$. If $m > 0$ this function is everywhere the boundary value of an analytic function in complexified variables p_i. The same is true for the Feynman propagator $(k^2 - m^2 + i\epsilon)^{-1}$. For variables $\{p_i\}$ in the domain of analyticity of $\tilde{\tau}$ we can deform the integration contours for the internal variables k_j into the complex in such a way that they never meet the singularities at $k_j^2 = m^2$. The integrand is then a smooth function, and there are no problems with the local existence of the integral. But the presence of a δ-term in $\tilde{D}_F$ destroys analyticity, so that the method does not work for positive T. For $m = 0$, $\tilde{\Delta}_F(k)$ is singular at $k = 0$ and is there not a boundary value of an analytic function. The same holds for $\tilde{\tau}$ at points where a partial sum of p_i's vanishes. These singularities can lead to infrared divergences, which need special attention. An x-space method for achieving this, and also for proving existence of the general $\mathcal{W}_\sigma$, has been described in V. It relies heavily on the fact that $\Delta_+(\xi)$ is analytic in $Im\xi^0 < 0$ and decreases at least of second order for $Im\xi^0 \to -\infty$. But $D_+(\xi)$ is only analytic in the strip $-\beta < Im\xi^0 < 0$, so that this method can also not be extended to positive T.

At present the problem of existence of $\mathcal{W}_\sigma$ remains unsolved.

5 UNIQUENESS

It will be shown that the expansion described in sections 3 and 4, assuming its existence in finite orders, is the only expression satisfying all the requirements.

Assume that uniqueness has been established up to order $\sigma - 1$. Let W_σ^1, W_σ^2, be two solutions of the equations (3.2) plus subsidiary conditions, W_σ^{T1} and W_σ^{T2} their truncated (= connected) parts defined by eq. (2.2). Then their difference

$$h_\sigma(x_1, \ldots, x_n) = W_\sigma^{T1}(x_1, \ldots) - W_\sigma^{T2}(x_1, \ldots) \tag{16}$$

satisfies the homogeneous equations

$$(\Box_i + m^2) h_\sigma(\ldots, x_i, \ldots) = 0 \tag{17}$$

and all the subsidiary conditions.

For $n > 2$ we prove exactly as in V that h_σ is invariant under permutations of its arguments, in particular that it satisfies in p-space

$$\tilde{h}_\sigma(p_1, \ldots, p_{n-1}, p_n) = \tilde{h}_\sigma(p_n, p_1, \ldots, p_{n-1}), \tag{18}$$

and that it is different from zero only for $p_i^2 = m^2$ for all i. But from the KMS condition (2.6) we find that the two sides of (5.3) differ by a factor $\exp(\beta p_n^0)$. Hence the support of $\tilde{h}_\sigma(\ldots, p_n)$ is restricted to $p_n = 0$. But a factor $\delta^4(p)$ in $\tilde{h}_\sigma$ violates the cluster property, so that we must have $\tilde{h}_\sigma = 0$.

The Fourier transform of the 2-point function $h_\sigma(x, y)$ is of the form

$$\tilde{h}_\sigma(p, q) = \delta^4(p + q)\hat{h}_\sigma(p) \tag{19}$$

by translation invariance. The equation (5.3) and rotation invariance imply

$$\hat{h}_\sigma(p) = H_\sigma(p_0)\delta(p^2 - m^2), \tag{20}$$

with H_σ a tempered distribution. Next we use the KMS condition (2.6), from which we find $H_\sigma(-p_0) = e^{-\beta p_0}H(p_0)$. This relation is solved by the ansatz

$$H_\sigma(p_0) = \frac{1}{1 + e^{-\beta p_0}} G_\sigma(|p_0|). \tag{21}$$

Because of the δ-factor in (5.5) we can replace $|p_0|$ by $\omega = (|\mathbf{p}|^2 + m^2)^{\frac{1}{2}}$ as the argument of G_σ.

Define $K_\sigma(x - y) = h_\sigma(x, y) - h_\sigma(y, x)$. Locality demands that $K_\sigma^{ret}(\xi) = \theta(\xi_0)K_\sigma(\xi)$ has its support in the closed forward cone. Its Fourier transform

$$K_\sigma^{ret}(p) = \frac{i}{2\pi} G_\sigma(\omega) \frac{e^{\beta\omega} - 1}{e^{\beta\omega} + 1} \frac{1}{(p_0 + i\epsilon)^2 - \omega^2} \tag{22}$$

is then analytic in the forward tube $\{Im(\mathrm{p}) \in V_+\}$, and is polynomially bounded in the slightly smaller tube $\{(Im(\mathrm{p}) - a) \in V_+\}$ for any $a \in V_+$ [10]. The last factor in (5.7) has the correct analyticity. In order for this to be the case for the full function K_σ^{ret}, the product $g_\sigma(\omega)$ of the ω-dependent factors in (5.7) must be an entire function of $\mathbf{p}$, and thus of ω^2. Furthermore g_σ must be polynomially bounded, hence it is a polynomial in ω^2.

Combining all the information hitherto obtained we find

$$\hat{h}_\sigma(p) = \gamma_\sigma(\omega^2)[\delta_+(p) + (e^{\beta\omega} - 1)^{-1}\delta(p^2 - m^2)] \tag{23}$$

with γ_σ a polynomial. We want to show that $\gamma_\sigma = 0$. To that end we show first that γ_σ must be a constant, because otherwise the inductive construction could not proceed to the next higher order. The trouble comes from the fact that the time-ordered 2-point function $\gamma_\sigma(-\partial_0^2)D_F(\xi)$ is not a multiple of a Green's function of the Klein-Gordon operator, if γ_σ is not a constant. To see the problem explicitly we consider the function $\mathcal{W}_{\sigma+1}(x, y, +|z_1|z_2)$. It satisfies the equation

$$\begin{aligned}(\Box_x + m^2)\mathcal{W}_{\sigma+1}(x, y|\ldots) &= -\frac{1}{6} < T(N(x)\Phi(y))\Phi(z_1)\phi(z_2) >_\sigma \\ &\quad + R_\sigma(x, y, z_1, z_2) =: I_\sigma, \end{aligned} \tag{24}$$

where we have abbreviated $N(x) = N(\Phi^3(x))$. R_σ is a term with support at $x = y$, stemming from the ambiguity of the T-product at that point. Let p, q, k_i, be the momenta corresponding to x, y, z_i. For p, q, off the mass shell, the solution of (5.9) reads in p-space:

$$\tilde{\mathcal{W}}_{\sigma+1}(p, q|k_1|k_2) = \frac{1}{6}\frac{1}{p^2 - m^2}\hat{I}_\sigma(p, \ldots, k_2). \tag{25}$$

We develop I_σ in a cluster expansion (see (2.2)), treating $N(x)$ as a product of three individual fields. We transform this expansion into p-space and consider it for p, q, not on the mass shell, $p + q \neq 0$. There the only term affected by the addition of an h_σ (remember that $h_\sigma = 0$ for the 4- and 6-point functions) is

$$\begin{aligned}\Delta_{\sigma+1}(p, \ldots, k_2) &= \frac{i}{6(2\pi)^4}\delta^4(p + \ldots + k_2)\{\frac{1}{p^2 - m^2}\frac{1}{q^2 - m^2} \\ &\quad \times[\gamma_\sigma(q_0^2) + \gamma_\sigma(k_{10}^2) + \gamma_\sigma(k_{20}^2)]\tilde{D}_+(k_1)\tilde{D}_+(k_2) \\ &\quad + \frac{1}{p^2 - m^2}B_p(q, k_1, k_2)\}, \end{aligned} \tag{26}$$

with B_p being a polynomial describing a possible h-dependence of R_σ. Renormalization terms in N do not contribute to $\Delta_{\sigma+1}$ at the points considered. By applying to $\mathcal{W}_{\sigma+1}$ a Klein-Gordon operator in y instead of x, we obtain in the same way

$$\begin{aligned}\Delta_{\sigma+1} &= \frac{i}{6(2\pi)^4}\delta^4(p + \ldots)\{\frac{1}{p^2 - m^2}\frac{1}{q^2 - m^2} \\ &\quad \times[\gamma_\sigma((q_0 + k_{10} + k_{20})^2) + \gamma_\sigma(k_{10}^2) + \gamma_\sigma(k_{20}^2)]\tilde{D}_+(k_1)\tilde{D}_+(k_2) \\ &\quad + \frac{1}{q^2 - m^2}B_q(p, k_1, k_2)\}, \end{aligned} \tag{27}$$

with B_q again a polynomial in the first variable $p = -q - k_1 - k_2$. Equality of the expressions (5.11) and (5.12) means

$$\begin{aligned} & [\gamma_\sigma(q_0^2) - \gamma_\sigma((q_0 + k_{10} + k_{20})^2)]\tilde{D}_+(k_1)\tilde{D}_+(k_2) = \\ = \;\; & (p^2 - m^2)B_q(p, \ldots) - (q^2 - m^2)B_p(q, \ldots) \end{aligned} \tag{28}$$

for $p = -q-k_1-k_2$. We choose an open neighbourhood $\mathcal{N}$ in the (q, k_i) space, intersecting the mass shell $p^2 = q^2 = k_i^2 = m^2$, in which the components of k_i and p^2, q^2, q_0, q_1, are independent variables. Such neighbourhoods clearly exist. We write $B_{p,q}$ as functions of the new variables. Then we integrate (5.13) over a test function $\varphi(k_1, k_2)$ with support in $\mathcal{N}$, chosen such that the left-hand side does not give a vanishing result. The left-hand side is then a bounded, non-vanishing, function of q_0 only, independent of p^2 and q^2. The integrals $\int dk_1 dk_2 \varphi(k_1, k_2)B_{p,q}(\ldots)$ are bounded functions of p^2, q^2, q_0, q_1, as long as we remain inside $\mathcal{N}$. Hence the right-hand side of (5.13) converges to zero if we let p^2 and q^2 tend towards zero, in contrast to the behaviour of the left-hand side. Hence (5.13) cannot be satisfied unless γ_σ is a constant, in which case the left-hand side vanishes identically.

But a non-vanishing constant γ_σ leads to an addition to the 2-point function of the same form as $W_0(x, y)$: it changes the normalization of the field, hence, if m, g and N remain unchanged, the new field would no longer satisfy the non-linear equation (1.1).

It is a curious aspect of this argumentation that it does not work in the free case $g = 0$, which should, however, not be confused with the zero-order terms in our expansion of an interacting theory. For the free theory one could arrive at the same result faster by postulating the usual canonical commutation relations. For interacting fields this method is rather less convincing, because in that case the CCR's have no rigorous meaning, since interacting fields cannot be restricted to sharp times.

As has been mentioned in section 3, the methods explained here can also be used to derive the forms of W_0 used there.

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