Radiative Decay:

Nonperturbative Approaches

Matthias Hübner and Herbert Spohn

Theoretische Physik, Ludwig-Maximilians-Universität
Theresienstraße 37, D-80333 München, Germany
e-mail: spohn@stat.physik.uni-muenchen.de

Abstract. In radiative decay the coupled system, electron bound by some external potential plus radiation field, relaxes to the ground state in the long time limit. Our central issue is to prove such a behaviour on the basis of the corresponding Schrödinger equation. We argue that the spin-boson Hamiltonian is a simple, but physically still acceptable test case. We relate radiative decay to scattering theory and prove the existence of wave operators. Other approaches are reviewed and compared. Some challenging open problems are listed.

1. Introduction and Hamiltonians

Radiative decay is *the* basic physical mechanism of how excited atoms lose energy and relax to the radiationless ground state. Within atomic physics one has developed a very powerful, and successful, perturbational scheme to handle such processes [1,2]. ¿From a less computational viewpoint it would be nice to prove radiative decay as a rigorous consequence of the Schrödinger equation. Over the past years there have been several attempts in this direction. The present paper intends to put the various approaches in perspective, adds a few novel results, and indicates open problems. It seems useful to us - and hopefully to the reader - to collect some background material. However no claims on completeness are made and we will not provide a grand review on rigorous results for "Photons and Atoms".

Radiative decay involves a scale of energy small compared to the rest mass of the electron. Thus we may ignore relativistic effects and base our investigation on nonrelativistic quantum mechanics of matter coupled to the photon field. To be concrete we consider a single electron in the force field of an infinitely heavy nucleus located at the origin. On the level of our undertaking the spin of the electron plays no role. Thus the Hilbert space for the electron is $\mathcal{H}_e = L^2(R^3, d^3x)$ and its Hamiltonian is given by

$$H_e = \frac{p^2}{2m} - \frac{e^2}{|x|}. \tag{1.1}$$

Here x is the position, $p = -i\nabla$ the momentum, m the mass, and e the charge of the electron. (We set $\hbar = 1$.) As standard [1-3], the electromagnetic field is quantized in the Coulomb gauge, $\nabla \cdot A = 0$. The dynamical variables are then the two transverse components of the vector potential. Thus the one particle space equals $C^2 \otimes L^2(R^3, d^3k)$ and $\mathcal{F}$ denotes the corresponding bosonic Fock space. On $\mathcal{F}$ we introduce, in momentum representation, the two component Bose field $a_j(k), a_j^*(k), j = 1, 2$, with the commutation relations $[a_j(k), a_{j'}^*(k')] = \delta_{jj'}\delta(k-k')$. The Hamiltonian of the photon field is

$$H_p = \sum_{j=1}^{2} \int d^3k\omega(k)a_j^*(k)a_j(k) \tag{1.2}$$

with dispersion relation $\omega(k) = |k|$. (We set $c = 1$.) H_p is self-adjoint with domain $D(H_p) \subset \mathcal{F}$. The electron is minimally coupled to the photon field. The full Hamiltonian reads then

$$H = \frac{1}{2m}(p - \alpha A(x))^2 + V(x) + \sum_{j=1}^{2} \int d^3k\omega(k)a_j^*(k)a_j(k). \tag{1.3}$$

Here α is the coupling constant and A the vector potential,

$$A(x) = \sum_{j=1}^{2} \int d^3k \frac{1}{\sqrt{2\omega(k)}} e_j(k)[\hat{\rho}(k)e^{-ikx}a_j^*(k) + \hat{\rho}^*(k)e^{ikx}a_j(k)] \tag{1.4}$$

with $e_1(k), e_2(k), k/|k|$ forming a left-handed dreibein. We introduced the ultraviolet cut-off function $\hat{\rho}(k)$ which is rotational invariant, equal to 1 for $|k| < k_c$ and decays to

0 for large $|k|$. More precisely, the integral $\int d^3k|\hat{\rho}(k)|^2/\omega(k)$ has to be finite in order to arrive at a closable operator $A(x)$. Physically, $\rho(x) = (2\pi)^{-3/2}\int d^3k\hat{\rho}(k)e^{ikx}$ corresponds to the smooth but sharply localized charge distribution of the electron. For consistency the Coulomb potential in (1.1) is then replaced by a smeared out potential, $V(x) = -\int d^3y\rho(y)e^2/|x-y|$. We require additionally $\int d^3k|\hat{\rho}|^2\omega < \infty$ and $\int d^3k|\hat{\rho}|^2/\omega^2 < \infty$.

H acts on $\mathcal{H} = L^2(R^3, d^3x) \otimes \mathcal{F}$. Under our conditions, H is essentially self-adjoint on $D(H_e \otimes I + I \otimes H_p)$, the domain of $H_e \otimes I + I \otimes H_p$ [4,5]. It should be noted that in a situation where the electron scatters off the nucleus, $\mathcal{H}$ is not the appropriate Hilbert space. As the electron is accelerated it produces very many infrared photons of bounded total energy. The physical state has an infinite number of photons and does not live in $\mathcal{H}$, where the number of photons is finite by definition. The infrared problem is discussed at depth by Fröhlich [6]. We refer the reader also to the literature cited there.

Besides scattering, physically, one expects also states where the electron is tightly bound to the nucleus and loses energy through radiation. In such a case it seems reasonable to suppress the variation of the vector potential over a length comparable to the dispersion of the electron. This leads to the dipole approximation

$$H = \frac{1}{2m}(p - \alpha A(0))^2 + V(x) + \sum_{j=1}^{2}\int d^3k\omega(k)a_j^*(k)a_j(k). \tag{1.5}$$

If one does not restrict the set of allowed electronic wave functions, physically, such an approximation makes sense only if $V(x) \to \infty$ sufficiently fast as $|x| \to \infty$.

We perform the unitary transformation

$$T = \exp(-i\alpha x \cdot A(0)), \tag{1.6}$$

with the result

$$TpT^* = p + \alpha A(0), \quad TxT^* = x, \quad Ta_j(k)T^* = a_j(k) + i\alpha(e_j(k)\cdot x)\hat{\rho}(k)/\sqrt{2\omega(k)}. \tag{1.7}$$

The transformed H reads then

$$\begin{aligned} THT^* = \frac{1}{2m}p^2 &+ V(x) + (\frac{2}{3}\alpha^2\int d^3k|\hat{\rho}(k)|^2)x^2 + \sum_{j=1}^{2}\int d^3k\omega(k)a_j^*(k)a_j(k) \\ &+\alpha\sum_{j=1}^{2}\int d^3k\sqrt{\omega(k)/2}\hat{\rho}(k)(e_j(k)\cdot x)ia_j^*(k) + h.c. \quad . \end{aligned} \tag{1.8}$$

At this stage, for mathematical convenience, we suppress the vector character of the photon field and of the position of the electron. Then

$$\begin{aligned} H = \frac{1}{2m}p^2 + V(x) &+ (\alpha^2\int dk|\lambda_e(k)|^2/\omega(k))x^2 + \int dk\omega(k)a^*(k)a(k) \\ &+\alpha x\int dk(i\lambda_e(k)a^*(k) - i\lambda_e(k)^*a(k)). \end{aligned} \tag{1.9}$$

Here $x \in R, p = -i\partial/\partial x$, and $a(k), a^*(k)$ are a one component Bose field over R. λ_e is an effective coupling function and belongs to the one particle space because of our condition $\int d^3k|\hat{\rho}|^2\omega < \infty$. The quadratic "counter" potential appears as in (1.8), namely as the quadratic completion of the last two terms in (1.9). We note that if one chooses $V(x) = \frac{1}{2}\kappa x^2$ and transforms to the extra bosonic mode $a = \frac{1}{\sqrt{2}}(x + ip), a^* = \frac{1}{\sqrt{2}}(x - ip)$, then the Hamiltonian H (in fact also THT^* of (1.8)) becomes quadratic in the field operators. It suffices then to study the appropriate one-particle problem. This program was carried out by Arai [7], cf. also [8]. He proves the properties of H and THT^* as expected on a physical basis: H has a unique ground state in $L^2(R) \otimes \mathcal{F}$ with energy E_0 and otherwise the purely absolutely continuous spectrum $[E_0, \infty)$. The wave operators exist and are asymptotically complete. It is tempting to study next a potential which is a small perturbation of $\frac{1}{2}\kappa x^2$. In the context of quantum Langevin equations Maassen [9] investigates such a model, however in a singular coupling limit, cf. also [10]. Attempts to generalize his techniques to a fixed coupling have failed so far.

In atomic physics, a standard procedure to avoid the complexity of possible electron transitions is to consider only two specific energy levels [2]. Such a two level atom can be written in terms of the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ and is therefore also referred to as "spin". For this approximation we return to the dipole Hamiltonian (1.8) and consider the two eigenstates ψ_1, ψ_2 with energies E_1, E_2 of the bare atom with Hamiltonian $p^2/2m + V(x) + (\frac{2}{3}\alpha^2 \int d^3k|\hat{\rho}(k)|^2)x^2$. Let P_{12} be the projection onto the subspace spanned by ψ_1, ψ_2. The two level approximation is then $(P_{12} \otimes I)(H - E_1)(P_{12} \otimes I)$. In the ψ_1, ψ_2 basis the energy of the two level atom is $\mu(1 + \sigma_z)/2$ with level splitting $E_2 - E_1 = \mu \geq 0$, i.e. if the eigenstates of σ_z are denoted by $\sigma_z u = u, \sigma_z v = -v$, then $\psi_1 = v, \psi_2 = u$. Again we ignore the vector character of the photon field. We arrive then at the spin-boson Hamiltonian

$$H = \mu\frac{1 + \sigma_z}{2} + \int d^3k\omega(k)a^*(k)a(k) + \sigma_x \int d^3k(\lambda(k)a^*(k) + \lambda(k)^*a(k)). \tag{1.10}$$

It is assumed here implicitely that $(\psi_1, x\psi_2) = 0 = (\psi_2, x\psi_1)$. More generally the coupling operator σ_x has also diagonal terms and should be replaced by a symmetric 2×2 matrix M. The specific choice of M is not so important since, for instance, terms proportional to the identity in M can be gauged away by a shift in the bosonic creation and annihilation operators. It only matters that $[\sigma_z, M] \neq 0$, otherwise the photon field does not couple. Therefore we retained the conventional choice $M = \sigma_x$.

One further popular approximation, the rotating wave approximation, is to substitute $\sigma_x = \sigma^+ + \sigma^-$ and to ignore the counter rotating terms $\sigma^+a^*(k), \sigma^-a(k)$. This introduces the spurious conservation law $(1 + \sigma_z)/2 + N_B$ with $N_B = \int d^3ka^*(k)a(k)$ the number of photons. Even worse, the vacuum polarization is lost: the ground state in the rotating wave approximation is just the Fock vacuum and the atom in the lower level v. (Further eigenstates may appear at strong coupling.) ¿From this point of view we regard rotating wave as a nonadmissible oversimplification. On the other hand the mathematical difficulties of the more realistic Hamiltonians as (1.8) or even (1.3) are already present in (1.10). So along the road the spin-boson Hamiltonian cannot be avoided. For radiative decay the spin-boson Hamiltonian is the simplest, yet physically still acceptable model.

In the remainder of the paper we study properties of the Hamiltonian (1.10), in particular spectral properties, long time behaviour, and dilation analyticity. But before, let us mention a few basic facts. $a(k), a^*(k)$ is a one component Bose field over R^3. Thus H acts on $C^2 \otimes \mathcal{F}$ with $\mathcal{F}$ the bosonic Fock space over $L^2(R^3, d^3k)$. The dispersion relation is $\omega(k) = |k|$. λ is the coupling function. (For notational convenience we absorbed the coupling constant α into $\lambda(k)$.) By (1.8), $\lambda(k) = i\alpha\hat{\rho}(k)\sqrt{\omega(k)}$. If $\int |\lambda|^2 < \infty$, then $a(\lambda), a^*(\lambda)$ are densely defined and closed. If $\int |\lambda(k)|^2/\omega < \infty$, then the interaction is operator bounded with relative H_B bound zero, where $H_B = \int d^3k\omega(k)a^*(k)a(k)$. By the Kato-Rellich theorem, H is bounded from below and self-adjoint on $D(I \otimes H_B)$. A sufficient condition for H to have a unique ground state is in addition $\int d^3k|\lambda(k)|^2/\omega(k)^2 < \infty$ [11]. In fact, if $\psi_0 \in C^2 \otimes \mathcal{F}$ denotes the unique ground state of H, we have $\langle\psi_0|e^{\delta N_B}\psi_0\rangle < \infty$ for any $\delta > 0$. If $\lim_{t\to\infty} t^2 \int d^3k|\lambda(k)|^2 e^{-\omega(k)t} = \alpha_0 > 0$, then for sufficiently strong coupling the true physical ground state acquires an infinite number of bosons and lies no longer in $C^2 \otimes \mathcal{F}$ [11]. We do not consider such singular cases here. To summarize: we require

(C1) $\int d^3k|\lambda(k)|^2(1 + \omega(k)^{-2}) < \infty$.

(C2) $\lambda(k) \neq 0$ a.e. such that all field modes couple. If λ would vanish on the set $\Lambda \subset R^3$ of positive measure, then (1.10) would split into two commuting pieces by taking the k-integration over Λ and $R^3 \setminus \Lambda$. On $R^3 \setminus \Lambda$ our condition is satisfied. The photons with $k \in \Lambda$ do not couple to the atom.

(C3) $\int dt \mid \int d^3k|\lambda(k)|^2 e^{-i\omega(k)t}|(1 + |t|)^\delta < \infty$ for some $\delta > 0$. Implicitely, this is a smoothness condition on λ.

H commutes with the parity operator $P = (-1)^{(1+\sigma_z)/2+N_B}$, which induces the symmetry $a(k) \to -a(k), \sigma_x \to -\sigma_x, \sigma_y \to -\sigma_y, \sigma_z \to \sigma_z$. If σ_x is substituted by a more general coupling operator, this parity is broken.

We are now in a position to formulate with more precision the problem of radiative decay. The most basic issue is to prove that after a long time the system approaches its radiationless ground state. Let $\mathcal{A}_R$ be the algebra of bounded operators generated by $I, \vec{\sigma}$, and the Weyl operators $W(f) = \exp[\int d^3x(f(x)a^*(x) - f(x)^*a(x)]$ with the support of f contained in the ball $B_R = \{x \mid |x| \leq R\}$ in position space. An observable is strictly local if $A = A^* \in \mathcal{A}_R$ for some R. We should then prove that

$$\lim_{t\to\infty} \langle e^{-iHt}\psi|Ae^{-iHt}\psi\rangle = \langle\psi_0|A\psi_0\rangle \tag{1.11}$$

for every $\psi \in \mathcal{H}$ and at least every strictly local A. The restriction to local A's is essential: e.g. for a bounded function of H (1.11) does not hold.

More specifically one would like to know how the limit in (1.11) is approached. To pose this question in generality makes little sense and one considers specific physical situations. In atomic physics [2] a natural initial condition is to have the two level atom in the upper state, field in Fock vacuum, i.e. $\psi_u = u \otimes |vac\rangle$ and one studies the probability to remain in the upper level, $\langle e^{-iHt}\psi_u|\frac{1+\sigma_z}{2}e^{-iHt}\psi_u\rangle$. Its Fourier transform yields the experimentally measured line shape. On the other hand, in the study of dissipative quantum systems [12,13] one starts with the Hamiltonian (1.9) and regards x as a quantum mechanical degree of freedom in a symmetric double well potential. ψ_1 is then the ground state, ψ_2 the first excited state. (Indeed, in this case $(\psi_1, x\psi_1) = 0 = (\psi_2, x\psi_2)$.) A natural initial condition is then to have the quantum degree of freedom in the right well, $\psi_+ = \frac{1}{\sqrt{2}}(u+v) \otimes |vac\rangle$, and to investigate the tunneling probability, $\langle e^{-iHt}\psi_+|\sigma_x e^{-iHt}\psi_+\rangle$, which is damped because of the coupling to the boson field. In fact, in both examples it is arguable that a conditional ground state for the boson field would be a physically more realistic initial condition.

In the case of an initially excited atom, beyond the decay time, there are a few photons which carry the energy off to infinity. More generally, one could also have some incoming photons which scatter off and possibly excite the atom. ¿From this point of view we are investigating the wave operators and the S-matrix of the coupled system, radiative decay being a particular initial state. Thus the problem is to understand the long time behaviour of a general solution

$$e^{-iHt}\psi \qquad \text{as} \qquad t \to \infty. \tag{1.12}$$

Clearly (1.11) and (1.12) are closely related. We will come back to them once we have explained the general scattering theory (Section 2). Section 3 covers perturbation theory, the standard machinery of atomic physics, and the weak coupling theory as its rigorous version. Dilation analyticity is discussed in Section 4. We close with some challenging open problems.

2. Scattering Theory

2.1 Wave operator, asymptotic completeness

After a long time the atom plus surrounding photon field should be in their ground state and several photons should be travelling outwards with the speed of light. Except for symmetrization such a state is essentially a product wave function. If there is a single photon travelling outwards with wave function ϕ far away from the origin, then the state is approximately $\int d^3x\phi(x)a^*(x)\psi_0 = a^*(\phi)\psi_0$, ψ_0 being the unique ground state. More generally for n photons far away from the origin with wave function $\phi^{(n)}$ the appropriate state is given by

$$\frac{1}{\sqrt{n!}}\int d^3x_1 \dots d^3x_n \phi^{(n)}(x_1,\dots,x_n)a^*(x_n)\dots a^*(x_1)\psi_0. \tag{2.1}$$

This defines a map J from $\mathcal{F}$ (photon states) to $\mathcal{H} = C^2 \otimes \mathcal{F}$. We write out J more explicitly. Let $\psi_0 = (\psi_0^{(0)}, \psi_0^{(1)}, \dots)$ be the ground state with norm $1 = ||\psi_0||^2 = \sum_{\sigma=\pm 1}\sum_{n=0}^{\infty}\int\prod_{j=1}^{n} d^3x_j|\psi_0^{(n)}(x_1,\dots,x_n,\sigma)|^2$ and let $\phi \in \mathcal{F}$. Then $J : \mathcal{F} \to C^2 \otimes \mathcal{F}$ and $J\phi = \psi_0 \otimes_s \phi$ where

$$(\psi_0 \otimes_s \phi)^{(n)}(x_1,\dots,x_n,\sigma) = \sum_{j=0}^{n}\binom{n}{j}^{1/2} S(\psi_0^{(j)}\phi^{(n-j)})(x_1,\dots,x_n,\sigma). \tag{2.2}$$

Here S is the symmetrizer,

$$(Sf)(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\pi} f(x_{\pi(1)},\dots,x_{\pi(n)}), \tag{2.3}$$

where the sum is over all permutations π of $\{1,\dots,n\}$.

Clearly J is linear, however unbounded. Let $D_e = \{\phi \in \mathcal{F} \mid \exists c > 0, 0 < a < 1, ||\phi^{(n)}|| \leq ca^n\}$.

Lemma 1. *D_e is in the domain of J.*

Proof.

$$\begin{aligned}
||J\phi||^2 =& \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}^{1/2}\binom{n}{j}^{1/2}\sum_{\sigma}\int d^3x_1\ldots d^3x_n(S\psi_0^{(i)}\phi^{(n-i)})^*(S\psi_0^{(j)}\phi^{(n-j)}) \\
\leq& \sum_{n=0}^{\infty}(\sum_{j=0}^{n}\binom{n}{j}^{1/2}||\psi_0^{(j)}||\ ||\phi^{(n-j)}||)^2 \\
\leq& c\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}a^{2(n-i)}e^{-\delta i}\sum_{j=0}^{n}||\psi_0^{(j)}||^2\ e^{\delta j} \\
\leq& c\sum_{n=0}^{\infty}(a^2+e^{-\delta})^n\sum_{j=0}^{\infty}||\psi_0^{(j)}||^2\ e^{\delta j} < \infty
\end{aligned} \tag{2.4}$$

for sufficiently large δ. $\square$

In order to verify our normalization we prove that $||J\phi|| = ||\phi||$ for ϕ very far away from the origin.

Lemma 2. *Let τ_w be the spatial shift by $w \in R^3$. Then for $\phi \in D_e$*

$$\lim_{|w|\to\infty}||\psi_0 \otimes_s \tau_w\phi|| = ||\psi_0||\ ||\phi|| = ||\phi||. \tag{2.5}$$

Proof. Since $||\tau_w\phi|| = ||\phi||$, $\tau_w\phi \in D_e$ and the sum in (2.4) is uniformly bounded. Thus dominated convergence applies. The n-th term of the sum reads

$$\begin{aligned}
\lim_{|w|\to\infty}&\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}^{1/2}\binom{n}{j}^{1/2}\sum_{\sigma}\int d^3x_1\ldots d^3x_n(S\psi_0^{(i)}\tau_w\phi^{(n-i)})^*(S\psi_0^{(j)}\tau_w\phi^{(n-j)}) \\
&=\sum_{j=0}^{n}||\psi_0^{(j)}||^2\ ||\phi^{(n-j)}||^2.
\end{aligned} \tag{2.6}$$

If in the integrand the arguments of the two factors differ, then there is at least one factor of the form $\int d^3x\tilde{\psi}(x)^*\tilde{\phi}(x+w)$ with both $\tilde{\psi}, \tilde{\phi} \in L^2(R^3, d^3x)$. Such a scalar product vanishes as $|w| \to \infty$. Thus

$$\lim_{|w|\to\infty}||\psi_0 \otimes_s \tau_w\phi||^2 = \sum_{n=0}^{\infty}\sum_{j=0}^{n}||\psi_0^{(j)}||^2\ ||\phi^{(n-j)}||^2 = ||\psi_0||^2\ ||\phi||^2. \quad \square \tag{2.7}$$

In Lemma 1 and 2 we needed only that $\langle\psi_0|e^{\delta N_B}\psi_0\rangle < \infty$. Thus we can define the linear map $\phi \mapsto \psi \otimes_s \phi$ for any $\psi \in \mathcal{H}$ which satisfies $\langle\psi|e^{\delta N_B}\psi\rangle < \infty$ for every $\delta > 0$. We reserve the linear operator J for the particular choice where $\psi = \psi_0$.

A photon state ϕ can be shifted to infinity also by applying the free evolution $\exp(-iH_Bt)$.

Lemma 3. *Let $\psi_i \in \mathcal{H}$ such that $\langle\psi_i|e^{\delta N_B}\psi_i\rangle < \infty$ for every $\delta > 0$ and let $\phi \in D_e, i = 1, 2$. Then*

$$\lim_{t\to\infty} \langle\psi_2 \otimes_s e^{-iH_Bt}\phi_2|\psi_1 \otimes_s e^{-iH_Bt}\phi_1\rangle = \langle\psi_2|\psi_1\rangle\,\langle\phi_2|\phi_1\rangle. \tag{2.8}$$

Proof. The computation is as in Lemma 2. One picks up now at least one factor of the form $\int d^3k\hat{\psi}(k)^* e^{-i\omega(k)t}\hat{\phi}(k) = \int_0^\infty d\omega h(\omega)e^{-i\omega t}$ with $h \in L^1(R_+, d\omega)$, which vanishes as $t \to \infty$ by the Riemann-Lebesgue lemma. □

Physically we expect that for every $\psi \in \mathcal{H}$ we can find a comparison photon state ϕ such that

$$e^{-iHt}\psi \approx e^{-iE_0t}\psi_0 \otimes_s e^{-iH_Bt}\phi \tag{2.9}$$

for $t \to \infty$, where $H\psi_0 = E_0\psi_0$ with E_0 the ground state energy. Thus it is natural to define the wave operator

$$\Omega^\mp\phi = s - \lim_{t\to\pm\infty} e^{i(H-E_0)t}Je^{-iH_Bt}\phi. \tag{2.10}$$

We will prove the existence of this limit for all $\phi \in D_e$ by a Cook estimate in the following subsection.

There is a subtlety here. In the standard two Hilbert space scattering theory J is bounded. The existence of the limit (2.10) on a dense set implies then also existence on the closure. In our case we expect $||Je^{-iH_Bt}\phi||$ to be uniformly bounded for any $\phi \in \mathcal{F}$ and sufficiently large t. But this remains to be shown and we can establish the existence of the limit (2.10) only for $\phi \in D_e$.

We prove the standard properties of the wave operator.

Proposition 4. *$\Omega^\mp$ is an isometry from $\mathcal{F}$ to $\mathcal{H}$.*

Proof. For $\phi \in D_e$ also $e^{-iH_Bt}\phi \in D_e$ and

$$\langle\Omega^-\phi|\Omega^-\phi\rangle = \lim_{t\to\infty}\langle Je^{-iH_Bt}\phi|Je^{-iH_Bt}\phi\rangle = \langle\phi|\phi\rangle \tag{2.11}$$

by Lemma 3. By continuity we extend Ω^- to $\mathcal{F}$. □

Proposition 5. *(Intertwining)*

$$e^{-i(H-E_0)t}\Omega^\mp = \Omega^\mp e^{-iH_Bt}. \tag{2.12}$$

Proof. For $\phi \in D_e$ also $e^{-iH_Bt}\phi \in D_e$ and

$$\begin{aligned}\Omega^-\phi = \lim_{s\to\infty} e^{i(H-E_0)(t+s)}Je^{-iH_B(t+s)}\phi = \lim_{s\to\infty} e^{i(H-E_0)t}e^{i(H-E_0)s}Je^{-iH_Bs}e^{-iH_Bt}\phi \\ = e^{i(H-E_0)t}\Omega^- e^{-iH_Bt}\phi.\end{aligned} \tag{2.13}$$

Since D_e is dense in $\mathcal{F}$, the result follows. □

As a consequence, $\mathrm{Ran}\Omega^{\mp}$ are reducing subspaces for H, and $H - E_0$ restricted to $\mathrm{Ran}\Omega^{\mp}$ is unitarily equivalent to H_B on $\mathcal{F}$. Note that $\Omega^{\mp}|vac\rangle = \psi_0$ and projecting out bound states is not needed.

Definition 6. $\Omega^{\mp}$ is called asymptotically complete if

$$\mathrm{Ran}\Omega^{\mp} = C^2 \otimes \mathcal{F}. \tag{2.14}$$

If $\Omega^{\mp}$ is asymptotically complete, then the asymptotics in (2.9) is valid for all $\psi \in \mathcal{H}$ with $\phi = (\Omega^{-})^*\psi$. In addition $H - E_0$ is unitarily equivalent to H_B. Thus H has purely absolutely continuous spectrum $[E_0, \infty)$ and a unique bound state at E_0. In our perspective, the central issue of scattering theory is to prove asymptotic completeness.

Remark. Following [14, Chapter 5] the wave operators are called weakly asymptotically complete if

$$\mathrm{Ran}\Omega^{-} = \mathrm{Ran}\Omega^{+},$$

complete if

$$\mathrm{Ran}\Omega^{-} = \mathrm{Ran}\Omega^{+} = P_{ac}(H)\mathcal{H},$$

and asymptotically complete if

$$\mathrm{Ran}\Omega^{-} = \mathrm{Ran}\Omega^{+} = [P_{pp}(H)\mathcal{H}]^{\perp}.$$

Our notion here is stronger, since it requires in addition that $P_{pp}(H)\mathcal{H} = \{c\psi_0 \mid c \in C\}$.

Clearly, the adjoint $(\Omega^{\mp})^*$ is well defined. We would expect then

$$(\Omega^{\mp})^*\psi = \lim_{t\to\infty} e^{iH_Bt} J^* e^{-i(H-E_0)t}\psi. \tag{2.15}$$

The adjoint $J^* : C^2 \otimes \mathcal{F} \to \mathcal{F}$ and is given by

$$\begin{aligned}(J^*\psi)^{(n)}(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} \binom{m+n}{n}^{1/2} \sum_{\sigma} \int d^3x_{n+1} \ldots d^3x_{n+m} \\ \psi_0^{(n+m)}(x_1, \ldots, x_{n+m}, \sigma)^* \psi^{(m)}(x_{n+1}, \ldots, x_{n+m}, \sigma).\end{aligned} \tag{2.16}$$

As in Lemma 1, J^* is well defined for $\psi \in C^2 \otimes D_e$. The difficulty with (2.15) is that even if $\psi \in C^2 \otimes D_e$ we do not know how to guarantee that also $e^{-iHt}\psi \in C^2 \otimes D_e$. By purely energetic considerations it is not forbidden to have very many infrared photons. A control on the number of photons uniformly in time is one of the deep problems of our subject.

As standard, we define the S-matrix by

$$S = (\Omega^{-})^*\Omega^{+}. \tag{2.17}$$

If the wave operators are weakly asymptotically complete, then $S : \mathcal{F} \to \mathcal{F}$ is unitary. By Proposition 5

$$e^{iH_Bt} S e^{-iH_Bt} = S, \tag{2.18}$$

which reflects the energy conservation for in- and outgoing states. In constructive quantum field theory the analytic structure of the kernel of the S-matrix has been studied in fair detail. It would be of interest to understand how much of this survives in our nonrelativistic model.

2.2 Cook estimate

In this subsection we prove the existence of the wave operator Ω^-. We first establish two identities for $\phi \in D_e$,

$$a(\lambda)[\psi_0 \otimes_s \phi] = a(\lambda)\psi_0 \otimes_s \phi + \psi_0 \otimes_s a(\lambda)\phi, \tag{2.19}$$

and

$$a^*(\lambda)[\psi_0 \otimes_s \phi] = a^*(\lambda)\psi_0 \otimes_s \phi. \tag{2.20}$$

The cumbersome way is to use directly the definition (2.1). A short cut goes as follows. Let

$$\phi = (0, \ldots, 0, \phi^{(n)}, 0, \ldots).$$

Then $\psi_0 \otimes_s \phi$ is equal to

$$\frac{1}{\sqrt{n!}} \int d^3x_1 \ldots d^3x_n \phi^{(n)}(x_1, \ldots, x_n) a^*(x_n) \ldots a^*(x_1)\psi_0, \tag{2.21}$$

and

$$\begin{aligned} a(\lambda)[\psi_0 \otimes_s \phi] = & \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots d^3x_n \phi^{(n)}(x_1, \ldots, x_n) a(\lambda) a^*(x_n) \ldots a^*(x_1)\psi_0 \\ =& a(\lambda)\psi_0 \otimes_s \phi \\ &+ \frac{1}{\sqrt{(n-1)!}}\sqrt{n} \int d^3x_1 \ldots d^3x_n \phi^{(n)}(x_1, \ldots, x_n) \lambda^*(x_n) a^*(x_{n-1}) \ldots a^*(x_1)\psi_0, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} a^*(\lambda)[\psi_0 \otimes_s \phi] =& \frac{1}{\sqrt{n!}} \int d^3x_1 \ldots d^3x_n \phi^{(n)}(x_1, \ldots, x_n) a^*(x_n) \ldots a^*(x_1) a^*(\lambda)\psi_0 \\ =& a^*(\lambda)\psi_0 \otimes_s \phi. \end{aligned} \tag{2.23}$$

Proposition 7. *Let $\int_0^\infty dt |(\lambda, e^{-i\omega t}\lambda)| < \infty$. For all $\phi \in D_e$ the limit*

$$s - \lim_{t\to\infty} e^{i(H-E_0)t} J e^{-iH_B t}\phi = \Omega^-\phi \tag{2.24}$$

exists.

Proof. If $\phi = |vac\rangle$ then $J\phi = \psi_0$ and the limit exists. Let then $\langle\phi|vac\rangle = 0$ and $\phi \in D_e \cap D(H_B)$. Then $Je^{-iH_B t}\phi \in D(H)$ and, using (2.19) and (2.20),

$$\begin{aligned} \frac{d}{dt} e^{i(H-E_0)t} J e^{-iH_B t}\phi =& ie^{i(H-E_0)t}(HJ - E_0 J - JH_B) e^{-iH_B t}\phi \\ =& ie^{i(H-E_0)t}\sigma_x \psi_0 \otimes_s (a(\lambda) e^{-iH_B t}\phi)). \end{aligned} \tag{2.25}$$

Thus

$$e^{i(H-E_0)t}Je^{-iH_Bt}\phi = J\phi + i\int_0^t dse^{i(H-E_0)t}\sigma_x\psi_0 \otimes_s (a(\lambda)e^{-iH_Bt}\phi) \tag{2.26}$$

and we have to show that $t \to ||\sigma_x\psi_0 \otimes_S (a(\lambda)e^{-iH_Bt}\phi)||$ is integrable for a dense set of ϕ's. For this purpose, we define $\mathcal{M}_\lambda \subset L^2(R^3)$ to be the linear subspace spanned by the set $\{e^{-i\omega t}\lambda \mid t \in R\}$. We choose the ϕ's of product form, $\phi = (0,\ldots,0,\phi^{(n)},0,\ldots)$, $\phi^{(n)}(x_1,\ldots,x_n) = \prod_{j=1}^n \phi_j(x_j)$ with $\phi_j \in \mathcal{M}_\lambda \oplus \mathcal{M}_\lambda^\perp$, i.e.

$$\phi_j = \sum_{l=1}^{l_j} \alpha_{jl}e^{-i\omega t_{jl}}\lambda + \phi_j^\perp, \quad j = 1,\ldots,n, \qquad \text{with} \quad \alpha_{jl} \in C, \quad \phi_j^\perp \in \mathcal{M}_\lambda^\perp. \tag{2.27}$$

Then

$$||(\sigma_x\psi_0 \otimes_s (a(\lambda)e^{-iH_Bt}\phi)|| \le const. \sum_{j=1}^n |\int d^3k\lambda(k)^* e^{-i\omega(k)t}\hat{\phi}_j(k)|, \tag{2.28}$$

Integrating over t gives a finite sum of integrals of the form

$$\int_0^\infty dt\, |\int d^3k\lambda(k)^* e^{-i\omega(k)(t+t_{jl})}\lambda(k)|, \tag{2.29}$$

which are bounded by assumption.

The existence of the limit (2.24) for all finite particle number states follows by equicontinuity in the one particle functions, and for states in D_e similarly. □

Since the operators $\Omega^\mp$ are bounded by Proposition 4, they can be uniquely extended to the whole Fock space by taking closures. The finiteness of the time integral in Proposition 7 is implied by condition (C3).

2.3 Long time decay

Let $\mathcal{W}$ be the Weyl algebra defined as the uniform closure of the Weyl operators $W(f) = \exp(a^*(f) - a(f))$, $f \in L^2(R^3)$ on $\mathcal{F}$. $\mathcal{W}$ is also the uniform closure of the local algebras $\mathcal{A}_R$, $\mathcal{W} = \overline{\cup_R \mathcal{A}_R}$ [15].

Proposition 8. *Let A be local in the sense that $A \in B(C^2) \otimes \mathcal{W}$. Then for every $\psi \in Ran\Omega^-$ with $||\psi|| = 1$*

$$\lim_{t\to\infty} \langle e^{-iHt}\psi|Ae^{-iHt}\psi\rangle = \langle\psi_0|A\psi_0\rangle. \tag{2.30}$$

Remark. If Ω^- is asymptotically complete, then the limit (2.30) holds for all $\psi \in \mathcal{H}$.

Proof. By Proposition 7 we have for all ψ such that $\psi = \Omega^-\phi$ with $\phi \in D_e$

$$\lim_{t\to\infty} \langle e^{-iHt}\psi|Ae^{-iHt}\psi\rangle = \lim_{t\to\infty} \langle Je^{-iH_Bt}\phi|AJe^{-iH_Bt}\phi\rangle. \tag{2.31}$$

We now choose $A = M \otimes W(f) \in B(C^2) \otimes \mathcal{W}$ and an n-particle product state $\phi = \frac{1}{\sqrt{n!}} \prod_{j=1}^n a^*(\phi_j)\psi_0$. Then $Je^{-iH_B t}\phi = \frac{1}{\sqrt{n!}} \prod_{j=1}^n a^*(\phi_j(t))\psi_0$ with $\phi_j(t) = e^{-i\omega t}\phi_j, \phi_j \in L^2(R^3)$. Since $W(f)a^*(\phi) = a^*(\phi)W(f) - (f,\phi)W(f)$, we have

$$\begin{aligned}\langle Je^{-iH_B t}\phi | AJe^{-iH_B t}\phi\rangle &= \frac{1}{n!}\langle \prod_{j=1}^n a^*(\phi_j(t))\psi_0 | M \otimes W(f) \prod_{j=1}^n a^*(\phi_j(t))\psi_0\rangle \\ =&\langle \prod_{j=1}^n a^*(\phi_j(t))\psi_0 | \prod_{j=1}^n a^*(\phi_j(t)) M \otimes W(f)\psi_0\rangle \\ &+ \frac{1}{n!}\sum_{b\subset\{1,\ldots,n\}, b\neq\emptyset} sign(b) \prod_{j\in b}^n (f,\phi_j(t))\langle \prod_{j=1}^n a^*(\phi_j(t))\psi_0 | \prod_{j\in b^c}^n a^*(\phi_j(t)) M\psi_0\rangle\end{aligned} \tag{2.32}$$

with $sign(b)$ depending on the number of contractions. Taking the limit $t \to \infty$, the second term vanishes because $\lim_{t\to\infty}(f,\phi_j(t)) = 0$ by Riemann-Lebesgue. The first term converges to $||\phi||^2\langle\psi_0|M \otimes W(f)\psi_0\rangle$ using Lemma 3 and the estimate

$$\langle W(f)\psi_0 | e^{\delta N_B} W(f)\psi_0\rangle < \infty, \tag{2.33}$$

which follows from [11]. Thus the limit (2.30) holds for a particular choice of A and ψ. By taking linear combinations and strong limits for ψ, uniform limits for A, one reaches all $A \in B(C^2) \otimes \mathcal{W}$ and all $\psi \in \mathrm{Ran}\Omega^-$. □

3. Perturbation and weak coupling theories

3.1 Level shift and lifetime

Physically the coupling α to the photon field is weak and it is natural to gain some understanding through perturbation theory, which, however, is singular, since we perturb around the $\alpha = 0$ Hamiltonian, which exhibits no radiative decay at all.

The basic problem is to consider the excited state $\psi_u = u \otimes |vac\rangle$, and to investigate the survival amplitude $\langle\psi_u|e^{-iHt}\psi_u\rangle$. We do this through the resolvent expansion

$$\langle\psi_u|(H-z)^{-1}\psi_u\rangle = \sum_{n=0}^{\infty}\langle\psi_u|(H_0-z)^{-1}[-\alpha V(H_0-z)^{-1}]^n\psi_u\rangle, \tag{3.1}$$

with $V = \sigma_x \otimes (a^*(\lambda) + a(\lambda))$. In this expansion we sum now only those terms where $a^*(\lambda)$ and $a(\lambda)$ alternate. Then

$$\begin{aligned}\langle\psi_u|(H-z)^{-1}\psi_u\rangle &\approx \sum_{n=0}^{\infty}(\mu-z)^{-1}[\alpha^2\int d^3k|\lambda(k)|^2(\omega(k)-z)^{-1}(\mu-z)^{-1}]^n \\ &= (\mu - \alpha^2 R(z) - z)^{-1}.\end{aligned} \tag{3.2}$$

For the limit $t \to +\infty$ we need $R(z)$ just above the real axis. We have

$$\lim_{\eta\downarrow 0} R(E+i\eta) = \lim_{\eta\downarrow 0}\int d^3k \frac{|\lambda(k)|^2}{\omega(k)-E-i\eta} = \lim_{\eta\downarrow 0}\int_0^\infty \frac{\rho(\omega)d\omega}{\omega - E - i\eta} = \frac{i}{2}\gamma(E) - \beta(E). \tag{3.3}$$

We adopted here the conventions

$$\rho(\omega) = \int d^3k |\lambda(k)|^2 \delta(\omega - \omega(k)), \tag{3.4}$$

where $\rho(\omega) > 0$ for $\omega > 0$, and

$$\begin{aligned} \int_0^\infty dte^{iEt} \int_0^\infty d\omega \rho(\omega) e^{-i\omega t} =& \frac{1}{2}\gamma(E) + i\beta(E) \\ =& \pi\rho(E) - i\mathcal{P}\int d\omega \frac{\rho(\omega)}{\omega - E}. \end{aligned} \tag{3.5}$$

Therefore

$$\lim_{\eta\downarrow 0} \langle \psi_u | (H - E - i\eta)^{-1} \psi_u \rangle \approx [\mu - E + \alpha^2\beta(E) - i\alpha^2\gamma(E)/2]^{-1}. \tag{3.6}$$

Since α is small and β, γ are smooth, the dominant part in (3.6) comes from $E = \mu$. Setting $\beta = \beta(\mu), \gamma = \gamma(\mu)$ we have

$$\lim_{\eta\downarrow 0} \langle \psi_u | (H - E - i\eta)^{-1} \psi_u \rangle \approx [\mu - E + \alpha^2\beta - i\alpha^2\gamma/2]^{-1}. \tag{3.7}$$

In this approximation, the resolvent has a pole on the second Riemann sheet. The eigenvalue at μ for $\alpha = 0$ is shifted to $\mu + \alpha^2(\beta - i\gamma/2)$. Such a resonance can be unfolded through complex scaling, cf. Section 4.

Translated to real time, (3.7) yields an exponential decay as

$$\langle \psi_u | e^{-iHt} \psi_u \rangle \approx \exp[-i(\mu + \alpha^2\beta)t - \alpha^2(\gamma/2)t]. \tag{3.8}$$

$\alpha^2\gamma$ is the decay rate for the excited state of the atom.

3.2 Reduced dynamics in the weak coupling limit

In Section 3.1 we considered a particular atomic state and observable. More generally, we could consider an arbitrary atomic observable $M \in B(C^2)$ and its expectation. We write $M \mapsto \langle e^{-iHt}\psi | M \otimes I e^{-iHt}\psi \rangle = \mathrm{tr} M \rho_t$ as a linear functional on $B(C^2)$, defining thereby the reduced (atomic) density matrix ρ_t. Since ρ_t is a statistical state, we might as well take the initial state to be a density matrix, i.e. $\rho \geq 0, \mathrm{tr}\rho = 1$. Usually, one assumes that the atom and photon field are initially uncorrelated and that the field is in the vacuum. Then the reduced dynamics, T_t^α, is defined by

$$\mathrm{tr}[M\ T_t^\alpha \rho] = \mathrm{tr}[M \otimes I\ e^{-iHt} \rho \otimes P_{vac} e^{iHt}] \tag{3.9}$$

with P_{vac} the projection onto $|vac\rangle$. The reduced dynamics carries all information about atomic observables. T_t^α is a linear map on $B(C^2)$, which is (completely) positive and preserves normalization, i.e. $T_t^\alpha \rho \geq 0, \mathrm{tr} T_t^\alpha \rho = \mathrm{tr}\rho$. However, T_t^α does not have the semigroup property. Only at small coupling the loss of energy to the photon field can be approximated by a dissipative generator.

Theorem 9 (Davies [16]). *Let $[\sigma_z,\cdot], K$ be linear operators on $B(C^2)$ defined by*

$$[\sigma_z,\cdot]\rho = [\sigma_z,\rho], \qquad K\rho = \frac{1}{2}\gamma([\sigma^-\rho,\sigma^+] + [\sigma^-,\rho\sigma^+]) \tag{3.10}$$

with the spin raising and lowering operators σ^+,σ^-, $\sigma^+ v = u, \sigma^+ u = 0, \sigma^- = (\sigma^+)^*$. Then for every $\tau > 0$

$$\lim_{\alpha\to 0} \sup_{0\le t\le \alpha^{-2}\tau} ||T_t^\alpha \rho - \exp[-i(\mu/2 + \alpha^2\beta)[\sigma_z,\cdot]t + \alpha^2 K t]\rho|| = 0. \tag{3.11}$$

Remark. Since dissipation is in effect on the time scale $\alpha^{-2}\tau$, the limit in (3.11) is meaningful only if the comparison is on that time scale. $-i[\sigma_z,\cdot]$ and $\alpha^2 K$ differ in order α^2. As a consequence K is not uniquely fixed by the limit (3.11) [17,18].

Proof. The only condition to be checked is that

$$\int_0^\infty dt \,|\int_0^\infty d\omega \rho(\omega) e^{-i\omega t}|(1+t)^\delta < \infty \tag{3.12}$$

which is implied by condition (C3). $\square$

Davies also investigates the structure of the photon field at times of order α^{-2}. As before the initial state is $\psi_u = u\otimes|vac\rangle$. At weak coupling only one photon is emitted. Its wave function is described in

Theorem 10. *Let $\phi_\lambda(t) = (0, e^{-i\omega(k)t}\lambda(k), 0, \ldots) \in \mathcal{F}$. Then for every $\tau > 0$*

$$\begin{aligned}\lim_{\alpha\to 0} \sup_{0\le t\le \alpha^{-2}\tau} ||e^{-iHt}\psi_u - \Big(& e^{-i(\mu+\alpha^2\beta)t - \alpha^2(\gamma/2)t}\psi_u \\ - i\int_0^t & ds e^{i(\mu(t-s)} v\otimes \phi_\lambda(t-s) e^{-i(\mu+\alpha^2\beta)s - \alpha^2(\gamma/2)s}\Big)|| = 0.\end{aligned} \tag{3.13}$$

Proof. The result is a consequence of [17, Theorem 3.1] with $\mathcal{H} = C^2\otimes\mathcal{F}$ and P_0 projection onto ψ_u. $\square$

(3.13) expresses that the excited atom acts as a source with exponentially decaying amplitude and source function λ.

3.3 Scattering

The S-matrix relates in- and outgoing photon states, cf. (2.9), the atom being in the ground state in the remote past and future. To investigate the structure of the S-matrix for small coupling we follow a while formal scattering theory [19]. One defines the T-matrix through $S = I - 2\pi i T$. Let $\psi,\phi\in\mathcal{F}$. Then

$$\begin{aligned}-2\pi i\langle\phi|T\psi\rangle =& \langle\phi|(S-I)\psi\rangle = \langle\phi|(\Omega^- - \Omega^+)^*\Omega^+\psi\rangle = \langle(\Omega^- - \Omega^+)\phi|\Omega^+\psi\rangle \\ =& \lim_{\eta\downarrow 0}(-i)\int_0^\infty dt e^{-\eta t}\langle e^{i(H-E_0)t}\sigma_x\psi_0\otimes_s a(\lambda)e^{-iH_B t}\phi \\ &+ e^{-i(H-E_0)t}\sigma_x\psi_0\otimes_s a(\lambda)e^{iH_B t}\phi|\Omega^+\psi\rangle \\ =& \lim_{\eta\downarrow 0}(-i)\int_0^\infty dt e^{-\eta t}\Big(\langle\sigma_x\psi_0\otimes_s a(\lambda)e^{-iH_B t}\phi|\Omega^+ e^{-iH_B t}\psi\rangle \\ &+ \langle\sigma_x\psi_0\otimes_s a(\lambda)e^{iH_B t}\phi|\Omega^+ e^{iH_B t}\psi\rangle\Big).\end{aligned} \tag{3.14}$$

We used (2.25) to write $(\Omega^- - \Omega^+)\phi$ as an integral, and the intertwining relation (2.12).

It is convenient to work in the momentum representation. To avoid too many indices we set $k \in \bigcup_{n\geq 0}(R^3)^n, \psi(k) = (\psi^{(0)}, \psi^{(1)}(k_1), \ldots)$ and dk stands for $\sum_{n=0}^{\infty}\prod_{j=1}^{n} d^3k_j$. H_B becomes multiplication by $E(k) = (0, \omega(k_1), \ldots, \sum_{j=1}^{n}\omega(k_j), \ldots)$. In (3.14) we have then the factor

$$\lim_{\eta\downarrow 0} -i\int_0^\infty dt e^{-\eta t}(e^{i[E(k')-E(k)]t} + e^{-i[E(k')-E(k)]t}) = -2\pi i\delta(E(k') - E(k)) \tag{3.15}$$

as a distribution. If the kernel of the T-matrix in momentum space is sufficiently smooth, then

$$\langle\phi|T\psi\rangle = \int dkdk'\phi(k)^*\langle k|\mathcal{T}|k'\rangle\delta(E(k') - E(k))\psi(k') \tag{3.16}$$

with the off energy shell $\mathcal{T}$ matrix

$$\langle\phi|\mathcal{T}\psi\rangle = \int dkdk'\phi(k)^*\langle k|\mathcal{T}|k'\rangle\psi(k') = \langle\sigma_x\psi_0 \otimes_s a(\lambda)\phi|\Omega^+\psi\rangle. \tag{3.17}$$

If we use again equation (2.25) and (2.19), we obtain

$$\begin{aligned}\sqrt{n!}\sqrt{m!}\langle k_1, \ldots, k_m|\mathcal{T}|k_1', \ldots, k_n'\rangle &= \langle\sigma_x[a(\lambda), \prod_{j=1}^{m} a^*(k_j)]\psi_0|\prod_{j=1}^{n} a^*(k_j')\psi_0\rangle \\ &+ \lim_{\eta\downarrow 0}\langle\sigma_x[a(\lambda), \prod_{j=1}^{m} a^*(k_j)]\psi_0|(E_0 + E(k') - H + i\eta)^{-1}\sigma_x[a(\lambda), \prod_{j=1}^{n} a^*(k_j')]\psi_0\rangle.\end{aligned} \tag{3.18}$$

The rate of scattering from k into $k' + dk'$ equals

$$\Gamma(k \to k')dk' = 2\pi\delta(E(k) - E(k'))|\langle k|\mathcal{T}|k'\rangle|^2 dk'. \tag{3.19}$$

The δ-function reflects energy conservation in the sense that the kinetic energies of in- and outgoing states are the same.

Of particular interest is the one photon scattering for small coupling. Then (3.18) simplifies to

$$\langle k|\mathcal{T}|k'\rangle = \alpha\lambda(k)\langle\sigma_x\psi_0|a^*(k')\psi_0\rangle + \lim_{\eta\downarrow 0}\alpha\lambda(k)\alpha\lambda(k')^*\langle\sigma_x\psi_0|(E_0 + \omega(k') - H + i\eta)^{-1}\sigma_x\psi_0\rangle, \tag{3.20}$$

where we introduced explicitly the coupling α. For the first term the perturbed vacuum ψ_0 is expanded to first order in α yielding

$$-\alpha^2\lambda(k)\lambda(k')^*(\omega(k') + \mu)^{-1}, \tag{3.21}$$

while the second term equals

$$\alpha^2\lambda(k)\lambda(k')^*\lim_{\eta\downarrow 0}\langle\psi_u|(E_0+\omega(k')-H+i\eta)^{-1}\psi_u\rangle = \alpha^2\lambda(k)\lambda(k')^*(\omega(k')-\mu-\alpha^2\beta+i\alpha^2\gamma/2)^{-1} \tag{3.22}$$

in the approximation (3.7). If k is nonresonant, i.e. $\omega(k) \neq \mu$, then

$$\langle k|\mathcal{T}|k'\rangle = \alpha^2 \lambda(k)\lambda(k')^* 2\mu(\omega(k')^2 - \mu^2)^{-1} \tag{3.23}$$

to second order in α and the scattering rate $\Gamma(k \to k')$ results as

$$\Gamma(k \to k')d^3k' = 2\pi\delta(\omega(k) - \omega(k'))\alpha^4 |\lambda(k)|^2 |\lambda(k')|^2 4\mu^2(\omega(k')^2 - \mu^2)^{-2} d^3k'. \tag{3.24}$$

It would be of interest to establish rigorously the limit $\alpha \to 0$ of the $\mathcal{T}$ matrix. By (3.21) and (3.23) the lowest nonvanishing contribution is of order α^2. Physically, it is more informative to take $\omega(k)$ close to the resonance as $\omega(k) - \mu = O(\alpha^2)$. Then, as can be seen from (3.22), the limit is of order one and depends on β, γ, whereas the contribution (3.21) is negligible. This is the situation envisioned in Theorem 10.

It is only a small step to compute the full second order T-matrix. Using the energy conservation, $\delta(E(k) - E(k'))$ in (3.16), one obtains

$$\begin{aligned}\langle k_1, \ldots, k_m|T|k_1', \ldots, k_n'\rangle &= \alpha^2 \delta_{nm} \frac{1}{n!} \sum_{j=1}^{n} \sum_{i=1}^{m} \lambda(k_j)\lambda(k_i')^* \delta(\omega(k_j) - \omega(k_i')) \\ &\quad [(\omega(k_j) - \mu)^{-1} - (\omega(k_j) + \mu)^{-1}] \langle vac| \prod_{l=1, l\neq j}^{n} a(k_l) \prod_{l=1, l\neq i}^{n} a^*(k_l')]vac\rangle\end{aligned} \tag{3.25}$$

provided $\omega(k_j) \neq \mu$ for $j = 1, \ldots, n$. As expected, except for symmetrization, photons scatter independently. For resonant scattering μ has to be shifted to $\mu + \alpha^2\beta - i\alpha^2\gamma/2$, as in (3.22).

4. Complex scaling

Decaying states also occur on the level of a one particle Schrödinger equation. The standard examples are a metastable local minimum as in the potential $V(x) = x(1 - x^2)$ locally with $V(x) \to 0$ as $x \to \infty$ or a potential with term $E \cdot x$ added (Stark effect). A very powerful technique to analyze the time-dependent behaviour is complex scaling. As the spectrum is rotated one uncovers eigenvalues of the dilated Schrödinger operator just below the real axis. These resonances govern the bulk part of the dynamics.

We explain how the dilation technique applies to the spin-boson Hamiltonian [20,21]. For this purpose we first have to define the analytic dilation H_θ of H. For real θ let

$$U(\theta)\phi(k) = e^{-3\theta/2}\phi(e^{-\theta}k) \tag{4.1}$$

be the unitary dilation acting on the one-particle functions. Lifting $U(\theta)$ to Fock space yields

$$\begin{aligned} U(\theta)\psi^{(n)}(k_1, \ldots, k_n) &= e^{-3n\theta/2}\psi^{(n)}(e^{-\theta}k_1, \ldots, e^{-\theta}k_n), \\ U(\theta)|vac\rangle &= |vac\rangle, \end{aligned} \tag{4.2}$$

and

$$H_{B,\theta} = U(\theta)H_B U(-\theta) = \int d^3k e^{-\theta}|k|a^*(k)a(k). \tag{4.3}$$

We extend now the definition (4.3) to complex θ with $|\mathrm{Im}\theta| < \pi/2$.

Let C_a be the open strip $\{z \in C \mid |\mathrm{Im}z| < a\}$ of the complex plane. For real θ let $\lambda_\theta = U(\theta)\lambda$. We assume that λ_θ can be extended to C_a as an analytic vector valued function, with a suitable choice of a, and that

$$\sup_{\theta \in C_a} \int d^3k |\lambda_\theta(k)|^2 (1 + \omega(k)^{-2}) < \infty. \tag{4.4}$$

The analytically dilated Hamiltonian reads then

$$H_\theta = \mu \frac{1+\sigma_z}{2} \otimes I + e^{-\theta} I \otimes \int d^3k |k| a^*(k) a(k) + \alpha \sigma_x \otimes \int d^3k (\lambda_\theta(k) a^*(k) + \lambda_\theta(-k) a(k)). \tag{4.5}$$

Note that we exploit here the property $\lambda(k)^* = -\lambda(-k)$ and analytically continue $U(\theta)\lambda(-k)$. We emphasize that the perturbation $\alpha\sigma_x \otimes (a^*(\lambda) + a(\lambda))$ is not relatively compact.

Lemma 11. *Let H_θ with $\theta \in C_a$ be defined by (4.5). Then there exists a constant α_0 such that for $|\alpha| < \alpha_0$ H_θ is a holomorphic family of type A separately in α, θ.*

For zero coupling, $\alpha = 0$, $(\mu/2)(1+\sigma_z) \otimes I + I \otimes H_{B,\theta}$ is a normal operator. It has the two eigenvalues $0, \mu$ and, for $\mathrm{Im}\theta > 0$, its essential (purely absolutely continuous) spectrum is rotated downwards to the two halflines $\{z \mid z = re^{-i\mathrm{Im}\theta}, z = \mu + re^{-i\mathrm{Im}\theta}, r > 0\}$ compare with Figure 1. If H_B would have a spectral gap, e.g. by considering massive photons $\omega(k) = \sqrt{k^2 + M_0^2}$ or by merly shifting as $\omega(k) = |k| + M_0$, then the resonant eigenvalue at μ would become isolated for the dilated Hamiltonian. One can then use analytic perturbation theory in the coupling α. $H_\theta(\alpha)$ has the discrete eigenvalue $E(\alpha)$ such that $\lim_{\alpha\to 0} E(\alpha) = \mu$. Furthermore for $|\alpha| < \alpha_0$, $E(\alpha)$ is given through the convergent power series

$$E(\alpha) = \mu + a_1\alpha + a_2\alpha^2 + \cdots. \tag{4.6}$$

The coefficients can be computed through standard perturbation theory. a_1 vanishes since $\langle \psi_u | \sigma_x (a^*(\lambda) + a(\lambda)) \psi_u \rangle = 0$. The second coefficient is given by

$$a_2 = \lim_{\eta \downarrow 0} \langle \psi_u | \sigma_x (a^*(\lambda) + a(\lambda)) (\mu - H_B + i\eta)^{-1} \sigma_x (a^*(\lambda) + a(\lambda)) \psi_u \rangle = \beta - i\gamma/2, \tag{4.7}$$

compare with (3.7).

For the true dispersion relation $\omega(k) = |k|$ the eigenvalues $0, \mu$ are no longer isolated. Still Bach, Fröhlich and Sigal succeed to provide a sufficiently precise estimate on the resolvent set of H_θ for small α. It is convenient to discuss their result at the hand of Figure 1. By turning on the coupling the spectrum of H_θ shifts downward to order α^2 at μ and to the left by the same order at 0. In addition the sharp line for $\alpha = 0$ opens to a cone with angle $O(\alpha^2)$. Since the eigenvalues are not isolated, one has to allow for an additional layer of $O(\alpha^3)$. But then every point away from the shaded region is guaranteed to be in the resolvent set of H_θ.

The resonant decay can now be understood as coming mostly from the spectral part of the dilated Hamiltonian located at distance α^2 below μ. One has no information on the fate of the eigenvalue inside the cone. But the time behaviour has the appearance as if there sits an eigenvalue computed from perturbation theory.

In fact [21] provides estimates on the resolvent. Thereby one can obtain information on the time dependence of physical observables . As one typical result we quote

Theorem 12. [21] *Let λ satisfy the condition (4.4). Then for $|\alpha| < \alpha_0$*

$$\langle \psi_u | e^{-itH} \psi_u \rangle = \exp[-i(\mu + \alpha^2 \beta)t - \alpha^2(\gamma/2)t] + O(\alpha^2) \tag{4.8}$$

uniformly in t.

Remark. Under more stringent conditions on λ the same result is proved by King [20].

The error in (4.8) comes from the contributions $O(\alpha^2)$ close to the ground state energy, on which analytic dilation provides no information. If in (4.8) we would localize somewhat in energy as $\langle \psi_u | g(H) e^{-itH} \psi_u \rangle$ with some smooth g supported, say, on $[\mu/2, 3\mu/2]$, then this amplitude has only a uniform error which decays as $C_n(1+t)^{-n}$, arbitrary $n > 0$ [22]. Similar arguments [23] imply that H has purely absolutely continuous spectrum in $[E_0 + O(\alpha^2), \infty)$. We expect that estimates of the form (4.8) could be established also for local observables.

If applicable, analytic dilation provides an explicit dynamical information. One further case is [24], where the weak coupling limit $\alpha \to 0, t \to \infty, \alpha^2 t = \tau$, is proved for the dipole approximation with massive photons and without cut-off in the interaction. On the other hand it seems difficult to go beyond small coupling and to extract information on infrared photons. E.g. if one just wants to prove that the survival amplitude (4.8) decays as $t \to \infty$, other methods, such as scattering theory, are required.

5. Cut-offs

5.1 Finite photon approximation

Considering the foregoing it seems unlikely that the long time asymptotics (2.30) could be proved, so to speak, in one blow. One step towards a partial solution is to restrict by fiat the number of photons to be less than or equal to N with N arbitrary. I.e. if $P_{\leq N}$ projects onto the subspace of $\mathcal{F}$ spanned by $(\psi^{(0)}, \ldots, \psi^{(N)}, 0, \ldots)$ then one studies $P_{\leq N} H P_{\leq N}$. To have a structure close to the standard N-body Schrödinger operator it is convenient to regard in addition the photons as distinguishable [25]. The states of a single photon are of the form $(c, \psi) \in C \oplus L^2(R^3)$, where c is the amplitude for the photon being dead and ψ is the wave function for the alive photon. The Hilbert space for N distinguishable photons is then $\mathcal{H}_N = \otimes_{j=1}^N \mathcal{H}_j$ with each $\mathcal{H}_j$ a copy of $C \oplus L^2(R^3)$ and the Hilbert space for the coupled system is $C^2 \otimes \mathcal{H}_N$. The Hamiltonian, H_N, is fixed by the requirement that H_N restricted to the symmetric subspace $C^2 \otimes P_{\leq N}\mathcal{F}$ is unitarily equivalent to $P_{\leq N} H P_{\leq N}$. Clearly, the kinetic energy of the photons is

$$T = \sum_{j=1}^{N} T_j \tag{5.1}$$

with

$$T_j = I \otimes \cdots \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}_j \otimes \cdots \otimes I, \tag{5.2}$$

which reflects that photons move independently. The energy of the atom is $\mu(1+\sigma_z)/2$, as before. Somewhat surprisingly, the interaction between photons and the atom is not strictly of two-body type. Let $Q_j = I \otimes \cdots \otimes \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}_j \otimes \cdots \otimes I$ be the projection onto the j-th photon being alive and let $Y_j = \sum_{i=1, i\neq j}^{N} (1 - Q_j)$ be the number of dead photons except for photon j. Then the interaction reads

$$\alpha\sigma_x \otimes V = \sum_{j=1}^{N} \alpha(1+Y_j)^{-1/2}\sigma_x \otimes V_j, \tag{5.3}$$

where $V_j = I \otimes \cdots \otimes V_j \otimes \cdots \otimes I$ and

$$V_j \begin{pmatrix} c \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & (\lambda, \cdot) \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} c \\ \psi \end{pmatrix} = \begin{pmatrix} (\lambda, \psi) \\ c\lambda \end{pmatrix} \tag{5.4}$$

on factor j. From (5.3) we conclude that the interaction strength depends on the number of dead photons around. A dead photon becomes alive with wave function λ, whereas an alive photon at x dies with amplitude $\lambda(x)$. Since λ decays as $|x| \to \infty$, photons which have escaped from the atom can no longer disappear. Combining the three pieces, the full Hamiltonian is given by

$$H_N = \mu \frac{1+\sigma_z}{2} \otimes I + I \otimes T + \alpha\sigma_x \otimes V. \tag{5.5}$$

For $N = 1$ we decompose our Hilbert space into $\mathcal{H}_+ \oplus \mathcal{H}_-$ according to $(-1)^{(1+\sigma_z)/2+Q} = \mp 1$. $\mathcal{H}_+$ and $\mathcal{H}_-$ are reducing subspaces for H_1. H_1 restricted to $\mathcal{H}_+$ reads

$$\begin{pmatrix} 0 & (\lambda, \cdot) \\ \lambda & \omega \end{pmatrix} \tag{5.6}$$

and H_1 restricted to $\mathcal{H}_-$ reads

$$\begin{pmatrix} \mu & (\lambda, \cdot) \\ \lambda & \omega \end{pmatrix}. \tag{5.7}$$

Thus the one-photon problem reduces to the Friedrichs model [26]. In the context of the theory of elementary particles related versions are known as Lee model. The Friedrichs model has been studied very extensively, amongst other approaches also through dilation analyticity [27]. The decay amplitude of the state $(1, 0)$ can be estimated in the form of Eq. (4.8) [28,29]. The case $N = 2$ corresponds to the three-body problem, for which asymptotic completeness is proved for small coupling [30].

For general N we establish [25] the analogue of the HVZ theorem

Theorem 13. *Let E_N be the ground state energy and Σ_N be the edge of the essential spectrum of H_N. Then*

$$E_{N+1} < E_N, \qquad \Sigma_N = E_{N-1}. \tag{5.8}$$

We note that the finite photon approximation introduces some unphysical features. H_N has now a spectral gap. In addition, if there are precisely n outgoing photons, then the remaining $N-n$ photons are bound. Thus as $t \to \infty$ the local state depends on the number of outgoing photons.

A more ambitious goal is asymptotic completeness. This remains as an open problem, but it seems likely that the methods of Graf [31] and Yafaev [32] will succeed. One important ingredient will be a Mourre estimate. A natural candidate for the conjugate operator is

$$I \otimes \tilde{A} = \sum_{j=1}^{N} I \otimes \tilde{A}_j \tag{5.9}$$

where

$$\tilde{A}_j = I \otimes \cdots \otimes \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}_j \otimes \cdots \otimes I. \tag{5.10}$$

Here D is the operator which in momentum space is given by

$$D = \frac{1}{2}\left(\frac{1}{|\nabla_k \omega|^2} \nabla_k \omega \cdot \nabla_k + \nabla_k \cdot \nabla_k \omega \frac{1}{|\nabla_k \omega|^2} \right) \tag{5.11}$$

and it is chosen such that formally $[D, \omega] = 1$. We note that D generates only a strongly continuous semigroup of isometries and iD is not self-adjoint.

Theorem 14. *Let λ be in the domain of D and D^2 and let $||D\lambda|| < 1/2$. Then $I \otimes \tilde{A}$ is a conjugate operator to H_N and provides a strong Mourre estimate in the form*

$$[I \otimes \tilde{A}, H_N] \geq \kappa I - C, \qquad \kappa > 0, \qquad C \text{ compact}. \tag{5.12}$$

Remark. The theorem uses the extension of Mourre estimates to isometric semigroups [33].

If ψ_n is an eigenvector of H_N, then by the virial theorem $\langle \psi_n | [I \otimes \tilde{A}, H_N] \psi_n \rangle = 0 \geq \kappa - \langle \psi_n | C \psi_n \rangle$. Thus through control over κ and C we can count the number of eigenvalues. With the $I \otimes \tilde{A}$ as given in (5.9) the estimates never fall below 2. ¿From an improvement of the conjugate operator we infer

Theorem 15. [33] *Let λ be in the domain of D^2. Then for every $\mu > 0$ there exists an α_0 such that for $0 < \alpha < \alpha_0$ the spectrum of H_N decomposes as $\sigma_{pp} = \{E_N\}, \sigma_{sc} = \emptyset, \sigma_{ac} = [E_{N-1}, \infty)$.*

5.2 Finite mode approximation

A widely studied problem [34] is the finite mode approximation to (1.10) which in the extreme case of a single mode becomes

$$H = \mu \frac{1 + \sigma_z}{2} \otimes I + I \otimes \omega a^* a + \alpha \sigma_x \otimes (a^* + a). \tag{5.13}$$

This Hamiltonian describes the physics of an atom placed in a cavity of suitable geometry so that the interaction is dominated by a single mode of the electromagnetic field. In particular the radiation emitted by the atom will be reflected at the walls of the cavity and subsequently reexcite the atom. If one ignores dissipative losses, the back and forth bouncing goes on indefinitely and there is no radiative decay. Correspondingly, the spectrum of H is purely discrete and we have a quasiperiodic time evolution. Since a nonlinear oscillator (σ_z) is coupled to a linear one, questions of nonlinear dynamics as irregular recurrences and level statistics come into focus.

If one writes in (5.13) the interaction as $(\sigma^- + \sigma^+) \otimes (a^* + a)$ and neglects the rotating terms $\sigma^- \otimes a, \sigma^+ \otimes a^*$, then one obtains the Jaynes-Cummings model. It can be solved exactly, with the amplitude to be in the upper state exhibiting quasi-random revivals [35]. Three level models and their experimental realizations are surveyed in [36]. The hydrogen atom coupled to a single radiation mode was studied through analytic dilation in [37]. [38-40] discuss the quality of the rotating wave approximation.

6. Perspectives

We indicate some open problems.

(i) Asymptotic completeness

As discussed in Section 2, a central problem is asymptotic completeness of the wave operators. Even with a cut-off in the photon number N this property remains to be established.

(ii) Absence of embedded eigenvalues

Even if (i) holds, there could be bound states embedded in the continuum (by the Mourre estimate there cannot be too many). It would be of interest to understand whether methods as in [14] would allow one to exclude embedded eigenvalues. For finite N we can exclude embedded eigenvalues only for small coupling [33].

(iii) Bounds on the number of photons

For the sequence of ground states one knows that $\psi_0^{(N)} \to \psi_0$ as $N \to \infty$. In fact, with $p_0(n)$ denoting the probability to have n photons in the state $\psi_0^{(N)}$, the bound $\sum_{n\geq 0} p_0(n) e^{\delta n} < \infty$ for all $\delta > 0$ holds uniformly in N. No such bound is known to us for time-dependent states. If $p(t,n)$ is the probability to have n photons in the state $e^{-iHt}\psi$ and if $p(0,n)$ has compact support initially, then we would expect an exponential bound of $p(t,n)$ uniformly in time. The need for such bounds reappears at various stages. E.g. if possible eigenvectors of H (with dispersion $\omega(k) = |k|$) would satisfy such a bound, we could get complete spectral information for small coupling [33].

(iv) More realistic atoms

Already in the dipole approximation with no cut-off in the coupling operator x, cf. (1.8),(1.9), very little is known on a rigorous level. The uniqueness of the ground state is established only under restrictive conditions. ¿From a Mourre estimate with the conjugate operator $I \otimes \tilde{A}$, $\tilde{A}$ given by the second quantization of D, cf. (5.11), one cannot even conclude that $\dim P_{pp}(H) < \infty$ for small α. The technical difficulty results here from the unboundedness of the coupling operator x.

A further step towards realistic atoms is the minimally coupled Hamiltonian (1.3) with a smeared out Coulomb potential. In such a model the atom can ionize. This opens a new scattering channel, where the electron travels off to infinity.

(v) A closed theory of point charges interacting with the radiation field

Point charges interacting through the electrostatic Coulomb potential give rise to a self-adjoint Hamiltonian, thus to a mathematically well defined theory. If one includes the radiation field as in the Pauli-Fierz Hamiltonian (1.3), then the point charge limit can be reached only through a suitable renormalization which removes the ultraviolet cut-off in (1.3). For a particle coupled to a scalar Bose field such a program is carried through by Fröhlich [41]. At present, it is not clear, at least to us, whether a mathematically well defined theory of point charges plus radiation field can be achieved. Constructive quantum field theory is heading for much more ambitious goals. ¿From this perspective mathematical physics seems to have completely overlooked that the Pauli-Fierz Hamiltonian (generalized to many electrons and nuclei) covers essentially all of low energy physics.

Acknowledgements. We are grateful to V.Bach for explaining us the results in Ref. [21] and for comments on the paper. This work is supported by a grant of Deutsche Forschungsgemeinschaft.

Figure captions

Figure 1. The resolvent set of H_θ. The dashed lines are the spectrum of H_θ for $\alpha = 0$. $\bullet$ are eigenvalues. For small α the resonant sector shifts downwards and the ground state sector to the left by $O(\alpha^2)$. The half lines open up to cones of width $O(\alpha^2)$ and an error zone of $O(\alpha^3)$ must be allowed. Points away from the shaded regions are in the resolvent set of H_θ.

References

1. C.Cohen-Tannoudji, J.Dupont-Roc, G.Grynberg, *Photons and Atoms - Introduction to Quantum Electrodynamics*, John Wiley, New York 1991
2. C.Cohen-Tannoudji, J.Dupont-Roc, G.Grynberg, *Atom-Photon Interactions* John Wiley, New York 1992
3. J.J.Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley, Reading 1978
4. A.Arai, "Selfadjointness and spectrum of Hamiltonians in nonrelativistic quantum electrodynamics", *J. Math. Phys.* **22**, 534-537 (1981)
5. T.Okamoto, K.Yajima, "Complex scaling technique in non-relativistic massive QED", *Ann. Inst. Henri Poincaré* **42**, 311-327 (1985)
6. J.Fröhlich, "On the infrared problem in a model of scalar electrons and massless, scalar bosons", *Ann. Inst. Henri Poincaré* **19**, 1-103 (1973)
7. A.Arai, "On a model of a harmonic oscillator coupled to a quantized, massless scalar field", *J. Math. Phys.* **22**, 2539-2548 and 2549-2552 (1981)
8. A.Arai, "Spectral analysis of a quantum harmonic oscillator coupled to infinitely many scalar bosons", *J. Math. Anal. Appl.* **140**, 270-288 (1989)
9. H.Maassen, "Return to thermal equilibrium by the solution of a quantum Langevin equation", *J. Stat. Phys.* **34**, 239-261 (1984)
10. P.Smedt, D.Dürr, J.L.Lebowitz, C.Liverani, "Quantum system in contact with a thermal environment: rigorous treatment of a simple model", *Comm. Math. Phys.* **120**, 195-231 (1988)
11. H.Spohn, "Ground state(s) of the spin-boson Hamiltonian", *Comm. Math. Phys.* **123**, 277-304 (1989)
12. A.J.Leggett, S.Chakravarty, A.T.Dorsey, M.P.A.Fisher, A.Garg, W.Zwerger, "Dynamics of the dissipative two-state system", *Rev. Mod. Phys.* **59**, 1-85 (1987)
13. U.Weiss, *Quantum dissipative systems*, World Scienctific, Singapore 1993
14. H.L.Cycon, R.G.Froese, W.Kirsch, B.Simon, *Schrödinger operators*, Springer-Verlag, Berlin 1987
15. O.Bratteli, D.Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer-Verlag, New York 1979
16. E.B.Davies, "Markovian master equations", *Comm. Math. Phys.* **39**, 91-110 (1974)
17. E.B.Davies, "Markovian master equations II", *Math. Ann.* **219**, 147-158 (1976)
18. R.Dümcke, H.Spohn, "The proper form of the generator in the weak coupling limit" *Z. Phys.* **B 34**, 419-422 (1979)
19. M.L.Goldberger, K.M.Watson, *Collision Theory*, John Wiley, New York 1964
20. C.King, "Resonant decay of a two state atom interacting with a massless non-relativistic quantised scalar field", preprint 1993
21. V.Bach, J.Fröhlich, I.M.Sigal, to be published
22. W.Hunziker, "Resonances, metastable states and exponential decay laws in perturbation theory", *Comm. Math. Phys.* **132**, 177-188 (1990)
23. V.Bach, private communication
24. V.Jaksic, C.A.Pillet, "On a model for quantum friction I. Fermi's golden rule and dynamics at zero temperature", preprint 1993
25. M.Hübner, H.Spohn, "Atom interacting with photons: an N-body Schrödinger

problem", preprint 1994
26. K.O.Friedrichs, *Perturbations of Spectra in Hilbert Space*, AMS, Providence 1965
27. R.A.Weder, "On the Lee model with dilatation analytic cutoff function", *J. Math. Phys.* **15**, 20-24 (1974)
28. E.B.Davies, "The harmonic oscillator in a heat bath", *Comm. Math. Phys.* **33**, 171-186 (1973)
29. C.King, "Exponential decay near resonance, without analyticity", *Lett. Math. Phys.* **23**, 215-222 (1991)
30. R.A.Minlos, H.Spohn, "The three body problem in radiative decay", preprint 1993
31. G.M.Graf, "Asymptotic completeness for N-body short range quantum systems: a new proof", *Comm. Math. Phys.* **132**, 73-101 (1990)
32. D.R.Yafaev, "Radiation conditions and scattering theory for N-particle Hamiltonians", *Comm. Math. Phys.* **154**, 523-554 (1993)
33. M.Hübner, H.Spohn, "Spectral properties of the spin-boson Hamiltonian", preprint 1994
34. S.Stenholm, "Quantum theory of electromagnetic fields interacting with atoms and molecules", *Phys. Rep.* **6**, 1-122 (1973)
35. F.T.Hioe, "On the mean motion and some statistical properties of a quasiperiodic observable in a fermion-boson model", *J. Math. Phys.* **23**, 1430-1438 (1982)
36. H.-I.Yoo, J.H.Eberly, "Dynamical theory of an atom with two or three levels interacting with quantized cavity fields", *Phys. Rep.* **118**, 239-337 (1985)
37. A.Grossmann, A.Tip, "Hydrogen atoms interacting with a quantised radiation mode", *J. Phys.* **A 13**, 3381-3397 (1980)
38. D.F.Walls, "Higher order effects in the single atom field mode approximation", *Phys. Lett.* **42A**, 217-218 (1972)
39. R.Graham, M.Höhnerbach, "Quantum chaos of the two-level atom", *Phys. Lett.* **101A**, 61-65 (1984)
40. F.Leyvraz, P.Pfeifer, "A one-fermion-one-boson model for a magnetic elastic system", *Helv. Phys. Acta* **50**, 857-870 (1978)
41. J.Fröhlich, "Existence of dressed one electron states in a class of persistent models", *Fortschr. Phys.* **22**, 159-198 (1974)