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LAGRANGIAN FORMALISM OVER GRADED ALGEBRAS

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Abstract. This paper provides a description of an algebraic setting for the Lagrangian formalism over graded algebras and is intended as the necessary first step towards the noncommutative $\mathcal{C}$-spectral sequence (variational bicomplex). A noncommutative version of integration procedure, the notion of adjoint operator, Green's formula, the connection between integral and differential forms, conservation laws, Euler operator, Noether's theorem is considered.

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Introduction

An outstanding progress which has in last years been made by noncommutative geometry stems out of shifting of the interests away from the original idea of geometrizing noncommutative rings using the language of noncommutative schemes. Now it has become clear that, bypassing the difficult problem of glueing noncommutative spectra, one can define directly differential geometric objects on a hypothetical "noncommutative space". This is based on two essential points: the possibility of reformulating the classical notions of analysis and differential geometry in pure algebraic terms, so that differential calculus becomes an extension of the language of commutative algebra (for a very enlightening discussion see [1]), and the existence of several quite important cases in which one is able to go beyond the commutative case (see [2]). This ideas have proven to be very successful and gives an impetus to new researches whose aim is to transplant the classical tools of analysis and geometry into a noncommutative setting. An instance of this comes from the current work on noncommutative (and first of all supercommutative) theory of integration, conservation laws, and the Lagrangian formalism (see [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and others). A key problem arises here is the description of integration procedure. The first question is the following: Given a commutative algebra, how to define the module of volume forms? An answer to this question must, in particular, give the possibility to define the Berezin volume forms on a supermanifold in usual fashion, i.e., by using the rule of signs. The well-known peculiarities of the Berezin integration show the character of the problem and result in the loss of a clear algebraic setting for integration procedure and related things.

The subject of this paper is an algebraic picture underlying the Lagrangian formalism and giving a solution of the problems discussed above. We work here over an arbitrary graded-commutative (with respect to a commutation factor) algebra to show that our constructions of volume forms, adjoint operators, the Euler operator, algebraic Green's formula, Noether's theorem, etc. can be extended to such an algebra in a simple and straightforward manner. On the other hand, the class of graded-commutative algebras is important in itself because it includes supercommutative, colour-commutative (see [14],[15],[16] and references therein), and "quantum" algebras (see [17], [16], [18] and others). We have chosen here not to accumulate formulas for specific algebras, but to present a general scheme, using examples for illustrations only. The applications will be described separately. Our ultimate goal, which is outside the present paper, is to develop the super- and non-commutative generalizations of the $\mathcal{C}$-spectral sequence (variational bicomplex) (see [19], [20]), which is a means for studying all aspects of the Lagrangian formalism: the inverse problem, the description of conservation laws, characteristic classes and so on.

The paper is organized as follows. In Section 1 we sketch the necessary definitions and facts from calculus over graded algebras. In Section 2 the main objects of this paper — adjoint operators and the Berezinian[1] — are defined. We want to emphasize that these definitions, which are the deus ex machina from that everything else follows, arose from an interplay between ideas and constructions of works [21], where the Berezin forms have been explained in terms of $\mathcal{D}$-modules, and [19, part I], where structure of Lagrangian formalism for a smooth commutative algebra has been clarified. In Section 3 we consider the Spencer complexes, algebraic Green's formula, and related staff. The main difference from non-graded case (see [19]) is the appearance of a new complex dual to the de Rham complex: complex of integral forms (the name borrowed from the supergeometry). The Lagrangian formalism, theory of conservation laws, and the Noether theorem are developed in Section 4. In the Appendix we briefly describe, in a pure algebraic manner, the formalism of right connections (see [22]), which is closely related with our subject.

1. A Sketch of Differential Calculus over Graded Algebras

1.1. We start with definitions of graded objects (see, e.g., [23]).

Let G be an Abelian group written additively and K a commutative ring with unit.

[1] We use this name for the module of volume forms.

Denote by K^* the multiplicative group of invertible elements of K. Let us fix a *commutation factor* $\{\cdot,\cdot\}\colon G\times G\to K^*$, $g_1\times g_2\mapsto\{g_1,g_2\}$, i.e., a map satisfying two properties:

1. $\{g_1,g_2\}^{-1}=\{g_2,g_1\}$
2. $\{g_1+g_2,g_3\}=\{g_1,g_3\}\{g_2,g_3\}$

We have also $\{g_1,g_2+g_3\}=\{g_1,g_2\}\{g_1,g_3\}$ as a consequence of the definition.

Example 1. Let $G=\mathbb{Z}^n=\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}$. Then it is easily shown that any commutation factor has the form:

$$\{g,h\}=\prod_{i=1}^{n}q_i^{g_ih_i}\cdot\prod_{i<j}q_{ij}^{(g_ih_j-h_ig_j)},$$

where $g=(g_1,\cdots,g_n)\,,h=(h_1,\cdots,h_n)\in\mathbb{Z}^n, q_i=\pm1, q_{ij}\in K^*$

Example 2. In particular, if $G=\mathbb{Z}$ there exist two commutation factors: the trivial one $\{g_1,g_2\}=1$ and the standard superfactor $\{g_1,g_2\}=(-1)^{g_1g_2}$.

Example 3. If $G=\mathbb{Z}^n$ and $K=\mathbb{C}$ then $\{g,h\}=\prod_{i=1}^{n}(\pm1)^{g_ih_i}\cdot e^{\theta(g,h)}$, where $\theta\colon G\times G\to\mathbb{C}$ is an antisymmetric bilinear form.

Suppose $A=\oplus_{g\in G}A_g$ is a G-graded associative K-algebra with unit. A is called *graded commutative* if

$$ab=\{a,b\}ba\qquad\forall a,b\in A.$$

For this type of notation we always assume that a and b are homogeneous and that the symbol of graded object used as argument of the commutative factor denotes the grading of this object.

Example 4. A commutative algebra (graded or not) is graded commutative with respect to the trivial commutation factor $\{g_1,g_2\}=1$.

Example 5. The algebra $C^\infty(M)$ of smooth functions on a supermanifolds M is graded commutative with respect to the superfactor $\{g_1,g_2\}=(-1)^{g_1g_2}\,,G=\mathbb{Z}_2$.

Example 6. Quantum superplane ([24], [2]). Let K be a field and K^n the n-dimensional coordinate vector space over K. Picking up an arbitrary commutation factor over $\mathbb{Z}^n$ (see Examples 1–3), the quadratic algebra $A = \mathrm{T}(K^n)/R$, where $\mathrm{T}(K^n)$ is the tensor algebra over K^n with the natural $\mathbb{Z}^n$-grading and R is the ideal in $\mathrm{T}(K^n)$ generated by the elements $x_1 \otimes x_2 - \{x_1, x_2\} x_2 \otimes x_1$, $x_1, x_2 \in K^n$, is called the algebra of polynomial functions on the quantum superplane $\mathbb{A}_q$.

Example 7. Non-commutative torus ([25], [2]). Let $K = \mathbb{C}$, $G = \mathbb{Z}^n$, and A be the space of all complex valued functions on $\mathbb{Z}^n$ that decay faster than any polynomial. A has the natural $\mathbb{Z}^n$-grading. Denote by $e(k)$ the function which is equal to 1 at $k \in \mathbb{Z}^n$ and zero at all other points (a basic harmonic on a non-commutative torus). Then any element of A can be written as $\sum_k a_k e(k)$. Given a skew bilinear form $\theta\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, we define a multiplication on A by setting $e(k)\, e(l) = e^{2\pi i \theta(k,l)} e(k+l)$. It is straightforward to check that the algebra A, called the algebra of (smooth) functions on the non-commutative torus $\mathbb{T}_\theta$, is graded commutative with respect to the commutation factor from Example 3.

Remark. In [26], [27] Lychagin has shown that one can also define commutation factors over a non-commutative group G and introduce all notions of the G-graded differential calculus in these circumstances.

In this article we work in the category $\mathcal{M}od_A$ of all G-graded modules over a graded commutative algebra A. Clearly any left module P can be transformed canonically into a right module, $pa = \{p, a\} ap$ for $a \in A$, $p \in P$, so that we may consider P as a bimodule.

Remark. The category $\mathcal{M}od_A$ is a *closed tensor category* (see, e.g., [2]) with respect to the commutativity constraint

$$S_{P,Q}\colon P \otimes Q \to Q \otimes P, \qquad p \otimes q \mapsto \{p, q\} q \otimes p.$$

A generalization of our constructions to an arbitrary Abelian closed tensor category appears to be very interesting. Meaningful results along this line, dealing with basic concepts of the differential calculus, may be found in [28], [29].

1.2. We now introduce basic objects of the differential calculus(for details see [30], [31]).

Let $\Delta \in \mathrm{Hom}_K(P, Q)$ be a K-homomorphism, P and Q being A-modules. For every element $a \in A$ define a K-homomorphism $\delta_a(\Delta) : P \to Q$ by setting $\delta_a(\Delta)(p) = \{a, \Delta\}\Delta(ap) - a\Delta(p)$, $p \in P$. Obviously, $\delta_a \circ \delta_b = \delta_b \circ \delta_a \quad \forall a, b \in A$. Put $\delta_{a_0,\dots,a_k} = \delta_{a_0} \circ \ldots \circ \delta_{a_k}$.

Definition. A K-homomorphism $\Delta \in \mathrm{Hom}_K(P, Q)$ is called a *differential operator* (d.o.) of order $\leqslant k$, if for all $a_0, \dots, a_k \in A$ we have $\delta_{a_0,\dots,a_k}(\Delta) = 0$.

The set of all d.0.'s of order $\leqslant k$, from P to Q, may be endowed with two A-module structures by putting $a\Delta = a \circ \Delta$ or $a\Delta = \Delta \circ a$, where $a \in A$ is understood as the operator of multiplying by a. The modules that arise in this way are denoted by $\mathrm{Diff}_k(P, Q)$ and $\mathrm{Diff}_k^+(P, Q)$ respectively. Clearly, $\mathrm{Diff}_k^{(+)}(P, Q) \subset \mathrm{Diff}_l^{(+)}(P, Q)$ for $k \leqslant l$, so that we may consider the union $\mathrm{Diff}^{(+)}(P, Q) = \bigcup_{k \geqslant 0} \mathrm{Diff}_k^{(+)}(P, Q)$.

Proposition.

(1) *If* $\Delta_1 \in \mathrm{Diff}_k(P, Q)$, $\Delta_2 \in \mathrm{Diff}_l(Q, R)$, P, Q, R *A-modules, then* $\Delta_2 \circ \Delta_1 \in \mathrm{Diff}_{k+l}(P, R)$.

(2) *The maps* $\mathrm{Diff}_k(P, Q) \to \mathrm{Diff}_k^+(P, Q)$ *and* $\mathrm{Diff}_k^+(P, Q) \to \mathrm{Diff}_k(P, Q)$ *generated by the identity map are d.o.'s of order* $\leqslant k$.

(3) *There exists a canonical isomorphism*

$$\mathrm{Diff}_k^+(P, Q) \to \mathrm{Hom}_A\left(P, \mathrm{Diff}_k^+(Q)\right),$$

where $\mathrm{Diff}_k^+(Q) = \mathrm{Diff}_k^+(A, Q)$. *To every d.o.* $\Delta\colon P \to Q$ *corresponds the homomorphism* $\varphi_\Delta \in \mathrm{Hom}_A\left(P, \mathrm{Diff}_k^+(Q)\right)$, $\varphi_\Delta(p)(a) = \Delta(pa)$ *under this isomorphism. The inverse mapping takes a homomorphism* $\varphi\colon P \to \mathrm{Diff}_k^+(Q)$ *to an operator* $\mathcal{D}_k \circ \varphi$, *where* $\mathcal{D}_k\colon \mathrm{Diff}_k^+(Q) \to Q$ *is a d.o. of order* $\leqslant k$ *defined by the formula* $\mathcal{D}_k(\nabla) = \nabla(1)$, $\nabla \in \mathrm{Diff}_k^+(Q)$.

Proof consists of a series of automatic verifications. □

This proposition has the following:

Corollary. *The commutative diagram*

$$\begin{array}{ccc} \mathrm{Diff}_k^+\left(\mathrm{Diff}_l^+(P)\right) & \xrightarrow{\mathcal{D}_k} & \mathrm{Diff}_l^+(P) \\ {\scriptstyle c_{k,l}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_l} \\ \mathrm{Diff}_{k+l}^+(P) & \xrightarrow{\mathcal{D}_{k+l}} & P \end{array}$$

uniquely defines the operator $c_{k,l}$*, which is said to be glueing operator.*

Definition. A k-multilinear over K mapping $\nabla\colon A\times\ldots\times A\to P$, P being an A-module, is said to be a *skew multiderivation* if the following conditions hold:

1. $\nabla(a_1,\ldots,a_i,a_{i+1},\ldots,a_k) = -\{a_i,a_{i+1}\}\nabla(a_1,\ldots,a_{i+1},a_i,\ldots,a_k)$
2. $\nabla(a_1,\ldots,a_{i-1},ab,a_{i+1},\ldots,a_k) =$
   $\{\nabla,a\}\{a_1\ldots a_{i-1},a\}a\nabla(a_1,\ldots,b,\ldots,a_k)+$
   $\{\nabla,b\}\{a_1\ldots a_{i-1}a,b\}b\nabla(a_1,\ldots,a,\ldots,a_k)$

The set of all skew multiderivations is an A-module denoted by $\mathrm{D}_k(P)$. Obviously $\mathrm{D}_1(P)$ is a submodule of $\mathrm{Diff}_1(A,P)$.

The Spencer Diff*-operator* $S\colon \mathrm{D}_k\left(\mathrm{Diff}_l^+(P)\right)\to \mathrm{D}_{k-1}\left(\mathrm{Diff}_{l+1}^+(P)\right)$ is defined by the formula

$$S(\nabla)(a_1,\ldots,a_{k-1})(a) = \nabla(a_1,\ldots,a_{k-1},a)(1),$$

$\nabla\in\mathrm{D}_k\left(\mathrm{Diff}^+(P)\right)$.

Proposition.

(1) *S is a d.o. of order $\leqslant 1$;*

(2) $S^2=0$.

Proof is straightforward. □

The complex $0\leftarrow P\xleftarrow{\mathcal{D}}\mathrm{Diff}^+(P)\xleftarrow{S}\mathrm{D}_1\left(\mathrm{Diff}^+(P)\right)\xleftarrow{S}\mathrm{D}_2\left(\mathrm{Diff}^+(P)\right)\leftarrow\ldots$ is said to be *the Spencer* Diff*-complex.*

1.3. The operations $Q\mapsto \mathrm{D}_k(Q)$, $Q\mapsto\mathrm{Diff}(P,Q)$, and $Q\mapsto\mathrm{Diff}^+(Q,P)$ are functors from the category of all A-modules $\mathcal{M}od_A$ to itself. We have seen that the latter functor is representable. The following proposition shows that two others functors are also representable.

Proposition (see [32], [30], [31]). *There exists a module Λ^k (resp. $\mathcal{J}^k(P)$), which is called the module of k-form (resp. k-jets) over A, such that the functor $Q \mapsto \mathrm{D}_k(Q)$ (resp. $Q \mapsto \mathrm{Diff}_k(P, Q)$) is isomorfic to the functor $Q \mapsto \mathrm{Hom}_A(\Lambda_k, Q)$ (resp. $Q \mapsto \mathrm{Hom}_A\left(\mathcal{J}^k(P), Q\right)$).*

Let us denote by $j_k = j_k(P) : P \to \mathcal{J}^k(P)$ the d.o. of order $\leqslant k$ that corresponds under the isomorphism $\mathrm{Diff}_k\left(P, \mathcal{J}^k(P)\right) = \mathrm{Hom}_A\left(\mathcal{J}^k(P), \mathcal{J}^k(P)\right)$ to the identical map $id_{\mathcal{J}^k(P)}$. Proposition 1.3 implies that the operator j_k is universal, i.e., for every d.o. $\Delta \in \mathrm{Diff}_k(P, Q)$ there exists a unique homomorphism $\psi_\Delta \colon \mathcal{J}^k(P) \to Q$ such that $\Delta = \psi_\Delta \circ j_k$.

Now let $\mathcal{M}$ be a subcategory of category $\mathcal{M}od_A$ of all A-modules that is closed under the action of the above-discussed functors. Then it is quite natural to ask if these functors are representable in $\mathcal{M}$.

Example. The most important example of the closed (in the above sense) category is that of geometrical modules $\mathcal{M}od_A^g$. A module P is called geometrical if $\widetilde{P} = \bigcap_{\wp,k} \wp^k P = 0$, where intersection is taken over all powers of prime ideals of A. There is the geometrization functor $P \mapsto P/\widetilde{P}$ from $\mathcal{M}od_A$ to $\mathcal{M}od_A^g$. It can easily be checked that the functors D_k and $\mathrm{Diff}_k(P, \cdot)$ are representable in $\mathcal{M}od_A^g$, the geometrization functor transforming the representing objects in $\mathcal{M}od_A$ into the corresponding representing objects in $\mathcal{M}od_A^g$.

1.4. A natural transformation of the functors D_k, Diff_k, and their compositions by duality gives rise to operators between the corresponding representing objects.

Example 1. The natural inclusion $\mathrm{D}_{k+l}(P) \to \mathrm{D}_k(\mathrm{D}_l(P))$ defines the wedge product of forms over A: $\Lambda^k \otimes \Lambda^l \to \Lambda^{k+l}$.

Example 2. The Spencer operator $S\colon \mathrm{D}_k(P) \to \dot{\mathrm{D}}_{k-1}\left(\mathrm{Diff}_1^+(P)\right)$, where the superscribed dot means that the K-module $\mathrm{D}_{k-1}\left(\mathrm{Diff}_1^+(P)\right)$ is supplied with A-module structure by putting $a\theta = \mathrm{D}_{k-1}\left(\mathrm{Diff}_1^+(a)\right)\theta$, $\theta \in \mathrm{D}_{k-1}\left(\mathrm{Diff}_1^+(P)\right)$, induces the homomorphism $\mathcal{J}^1\left(\Lambda^{k-1}\right) \to \Lambda^k$. The composition of $\Lambda^{k-1} \xrightarrow{j_1} \mathcal{J}^1\left(\Lambda^{k-1}\right) \to \Lambda^k$ is called *the exterior differentiation* operator and is denoted by $d\colon \Lambda^{k-1} \to \Lambda^k$. Using the fact that $S^2 = 0$, one can easily prove that $d^2 = 0$.

The complex $0 \to A \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \dots$ is said to be the *de Rham complex* of the algebra A.

1.5. The above described algebraic formalism can be realized geometrically, the algebra A being the algebra $C^\infty(M)$ of smooth real functions on a supermanifold M (for the supergeometry see, e.g., [33], [34], [22], [35]). In this situation it may be shown in the same way as in non-graded case that the standard notions of differential operator, forms, jets, etc. coincide with the ones introduced above. Having this in mind, in the next section we shall illustrate constructions under consideration by giving their local coordinate description.

2. Adjoint Operators and Berezinian

2.1. Given an A-module P, consider the complex of homomorphisms

$$0 \to \mathrm{Diff}^+(P, A) \xrightarrow{w} \mathrm{Diff}^+\left(P, \Lambda^1\right) \xrightarrow{w} \mathrm{Diff}^+\left(P, \Lambda^2\right) \xrightarrow{w} \dots, \tag{1}$$

where $w(\nabla) = d \circ \nabla \in \mathrm{Diff}^+\left(P, \Lambda^k\right)$, $\nabla \in \mathrm{Diff}^+\left(P, \Lambda^{k-1}\right)$. Let us denote the cohomology module in a term $\mathrm{Diff}^+(P, \Lambda^n)$ by $\widehat{P}_n$, $n \geqslant 0$. Every d.o. $\Delta\colon P \to Q$ generates the natural map of the complexes:

$$\begin{array}{ccccccc}
\dots & \longrightarrow & \mathrm{Diff}^+\left(Q, \Lambda^{k-1}\right) & \xrightarrow{w} & \mathrm{Diff}^+\left(Q, \Lambda^{k}\right) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle\widetilde{\Delta}} & & \downarrow{\scriptstyle\widetilde{\Delta}} & & \\
\dots & \longrightarrow & \mathrm{Diff}^+\left(P, \Lambda^{k-1}\right) & \xrightarrow{w} & \mathrm{Diff}^+\left(P, \Lambda^{k}\right) & \longrightarrow & \dots,
\end{array}$$

where $\widetilde{\Delta}(\nabla) = \{\Delta, \nabla\}\nabla \circ \Delta \in \mathrm{Diff}^+\left(P, \Lambda^k\right)$, $\nabla \in \mathrm{Diff}^+\left(Q, \Lambda^k\right)$.

Definition. The operator $\Delta^*_n\colon \widehat{Q}_n \to \widehat{P}_n$ induced by $\widetilde{\Delta}$ is called the *adjoint operator*.

Below we assume that an integer n is fixed and omit the corresponding index.

Proposition.

(1) *Δ^* has the same grading as Δ.*
(2) *If $\Delta \in \mathrm{Diff}_k(P, Q)$ then $\Delta^* \in \mathrm{Diff}_k\left(\widehat{Q}, \widehat{P}\right)$.*
(3) *For all $\Delta_1 \in \mathrm{Diff}(P, Q)$ and $\Delta_2 \in \mathrm{Diff}(Q, R)$ we have*

$$(\Delta_2 \circ \Delta_1)^* = \{\Delta_2, \Delta_1\}\Delta_1^* \circ \Delta_2^*.$$

Proof. (1) Obvious.

(2) Denote by $[\nabla]$ the cohomologous class of an operator $\nabla \in \mathrm{Diff}^+(P, \Lambda^n)$, $w(\nabla) = 0$. Then $\Delta^*([\nabla]) = \{\Delta, \nabla\}[\nabla \circ \Delta]$ and we have
$\delta_a(\Delta^*)([\nabla]) = \{a, \Delta^*\}\Delta^*(a[\nabla]) - a\Delta^*([\nabla]) =$
$\{a, \Delta^*\}\{a, \nabla\}\{\Delta, \nabla \circ a\}[\nabla \circ a \circ \Delta] - \{\Delta, \nabla\}\{a, \nabla \circ \Delta\}[\nabla \circ \Delta \circ a] =$
$(a \circ \Delta)^*([\nabla]) - \{a, \Delta\}(\Delta \circ a)^*([\nabla]) = -\delta_a(\Delta)^*([\nabla])$,
i.e., $\delta_a(\Delta^*) = -\delta_a(\Delta)^*$. Thus $\delta_{a_0,\dots,a_k}(\Delta^*) = (-1)^{k+1}\delta_{a_0,\dots,a_k}(\Delta)^* = 0$.

(3) $(\Delta_2 \circ \Delta_1)^*([\nabla]) = \{\Delta_2 \circ \Delta_1, \nabla\}[\nabla \circ \Delta_2 \circ \Delta_1] =$
$\{\Delta_2, \nabla\}\{\Delta_2, \Delta_1\}\Delta_1^*([\nabla \circ \Delta_2]) = \{\Delta_2, \Delta_1\}\Delta_1^*(\Delta_2^*([\nabla]))$.

□

Let us consider some examples of adjoint operators.

Example 1. Let $a\colon P \to P$ be the operator of multiplying by $a \in A$. Then $a^*([\nabla]) = \{a, \nabla\}[\nabla \circ a] = a[\nabla]$, i.e., $a^* = a$.

Example 2. Let $p\colon A \to P$ be the operator $p(a) = pa$, $a \in A$, $p \in P$. Then $p^*([\nabla]) = \{p, \nabla\}[\nabla \circ p]$, $p^* \in \mathrm{Hom}_A\left(\widehat{P}, \widehat{A}\right)$. Thus we have a natural pairing $\langle\cdot,\cdot\rangle : P \otimes \widehat{P} \to \widehat{A}$, $\langle p, \widehat{p}\rangle = p^*(\widehat{p})$, $\widehat{p} \in \widehat{P}$.

Example 3. Berezinian and Integral forms. Let $\dots \to P_{k-1} \xrightarrow{\Delta_k} P_k \to \dots$ be a complex of d.o.'s. Since $\Delta_k^* \circ \Delta_{k+1}^* = \{\Delta_k, \Delta_{k+1}\}(\Delta_{k+1} \circ \Delta_k)^* = 0$, we get a complex $\dots \leftarrow \widehat{P}_{k-1} \xleftarrow{\Delta_k^*} \widehat{P}_k \leftarrow \dots$, which is called dual to given one. The complex dual to the de Rham complex is said to be the *complex of integral forms* and denote by

$$0 \leftarrow \Sigma_0 \xleftarrow{\delta} \Sigma_1 \xleftarrow{\delta} \dots,$$

where $\Sigma_i = \widehat{\Lambda^i}$, $\delta = d^*$. The module $\Sigma_0 = \widehat{A}$ is called the *Berezinian* (*or the module of volume forms*) and will be denoted by B.

The d.o.'s $\mathcal{D}\colon \mathrm{Diff}^+\left(\Lambda^k\right) \to \Lambda^k$ induce the cohomology map: $\int\colon B \to \mathrm{H}^n(\Lambda^*)$, so that to any element $\omega \in B$ (volume form) corresponds the n-th de Rham cohomology class $\int\omega$. This is an algebraic version of the integration operation. Clearly, $\int[\nabla] = \nabla(1) \mod d\Lambda^{n-1}$, $\nabla \in \mathrm{Diff}(A, \Lambda^n)$.

2.2. Proposition. *$\int \delta\omega = 0$, where $\omega \in \Sigma_1$.*

Proof. Suppose $\omega = [\nabla]$, then $\delta\omega = [\nabla \circ d]$. Therefore $\int \delta\omega = [\nabla \circ d(1)] = 0$. $\square$

2.3. Proposition ("Integration by parts"). *For any d.o. $\Delta \colon P \to Q$ and $p \in P$, $\widehat{q} \in \widehat{Q}$, we have*

$$\int \langle \Delta(p), \widehat{q} \rangle = \{\Delta, p\} \int \langle p, \Delta^*(\widehat{q}) \rangle$$

Proof. Suppose $\widehat{q} = [\nabla]$, $\nabla \colon Q \to \Lambda^n$. Then $\int \langle \Delta(p), \widehat{q} \rangle = \int \{\Delta(p), \nabla\} [\nabla \circ \Delta(p)]$ $= \int \{\Delta(p), \nabla\} [\nabla \circ \Delta \circ p] = \int \{\Delta, \nabla\} \{\Delta, p\} \langle p, [\nabla \circ \Delta] \rangle = \{\Delta, p\} \int \langle p, \Delta^*(\widehat{q}) \rangle$. $\square$

2.4 Coordinates. Let M be a smooth supermanifold of dimension $s|t$, $G = \mathbb{Z}_2$, $K = \mathbb{R}$ or $\mathbb{C}$, $A = C^\infty_K(M)$, $P = \Gamma(\alpha)$ and $Q = \Gamma(\beta)$ the modules of smooth sections of vector bundles over M. Suppose $x = (y_i, \xi_j)$, $i = 1, \dots, s$, $j = 1, \dots, t$, $x_1 = y_1, \dots, x_s = y_s$, $x_{s+1} = \xi_1, \dots, x_{s+t} = \xi_t$ is a coordinate system on a domain $\mathcal{U} \subset M$.

First of all let us calculate the Berezinian, i.e., the cohomology of complex (1) for $P = A$.

Theorem ([21]). (1) $\widehat{A}_n = 0$ *for $n \neq s$.*
(2) *$\widehat{A}_s$ is the module of sections for the bundle of volume forms* $\mathrm{Ber}(M)$[2].

Proof. The assertion is local, so we can consider the domain $\mathcal{U}$ and split the complex $\mathrm{Diff}^+(\Lambda^*)$ in a tensor product of complexes $\mathrm{Diff}^+(\Lambda^*)_{even} \otimes \mathrm{Diff}^+(\Lambda^*)_{odd}$, where $\mathrm{Diff}^+(\Lambda^*)_{even}$ is complex (1) on the underlying even domain of $\mathcal{U}$ and $\mathrm{Diff}^+(\Lambda^*)_{odd}$ is complex (1) for the Grassmann algebra in variables $\xi_1, \dots, \xi_t$.

It is known that $\mathrm{H}^i\left(\mathrm{Diff}^+(\Lambda^*)_{even}\right) = 0$ for $i \neq s$ and $\mathrm{H}^i\left(\mathrm{Diff}^+(\Lambda^*)_{even}\right) = \Lambda^s_u$, where Λ^s_u is the module of s-form on the underlying even domain of $\mathcal{U}$ (see [19,

[2] Recall that locally sections of $\mathrm{Ber}(M)$ are written in the form $f(x)\,\mathbf{D}(x)$, where $f \in C^\infty(\mathcal{U})$ and $\mathbf{D}$ is a basis local section that is multiplied by the Berezin determinant of the Jacobi matrix under the change of coordinates. The Berezin determinant of an even matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is equal to $\det\left(A - BD^{-1}C\right)(\det D)^{-1}$.

section 2]). To compute the cohomology of $\mathrm{Diff}^+(\Lambda^*)_{odd}$ consider the quotient complexes

$$0 \to \mathrm{Smbl}_k(A)_{odd} \to \mathrm{Smbl}_{k+1}\left(\Lambda^1\right)_{odd} \to \dots,$$

where $\mathrm{Smbl}_k(P)_{odd} = \mathrm{Diff}_k^+(P)_{odd} / \mathrm{Diff}_{k-1}^+(P)_{odd}$. An easy calculation shows that these complexes are the Koszul complexes, hence $\mathrm{H}^i\left(\mathrm{Diff}^+(\Lambda^*)\right)_{odd} = 0$ for $i > 0$ and $\mathrm{H}^0\left(\mathrm{Diff}^+(\Lambda^*)\right)$ is a module of rank 1. Therefore $\widehat{A}_i = \mathrm{H}^i\left(\mathrm{Diff}^+(\Lambda^*)\right) = 0$ for $i \neq s$ and the only operators that represent non-trivial cocycles have the form $f(y,\xi)\, dy_1 \wedge \dots \wedge dy_s \frac{\partial^t}{\partial \xi_1 \cdots \partial \xi_t}$.

To complete the proof it remains to check that $\widehat{A}_s$ is precisely $\mathrm{Ber}(M)$, i.e., that changing coordinates we obtain:

$$f\, dy_1 \wedge \dots \wedge dy_s \frac{\partial t}{\partial \xi_1 \dots \xi_t} = f\, \mathrm{Ber}\, J\left(\frac{x}{z}\right) dv_1 \wedge \dots \wedge dv_s \frac{\partial t}{\partial \eta_1 \dots \eta_t} + T,$$

where $z = (v_i, \eta_j)$ is a new coordinate system on $\mathcal{U}$, Ber denotes the Berezin determinant, $J\left(\frac{x}{z}\right)$ is the Jacobi matrix, T is cohomologous to zero. This is an immediate consequence of the following:

Claim. If $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $X^{-1} = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix}$ are mutually inverse matrices written in the standard format, then $\mathrm{Ber}\, X = \det A \cdot \det \widetilde{D}$.

Proof. Obviously, $A\widetilde{B} + B\widetilde{D} = 0$ and $C\widetilde{B} + D\widetilde{D} = 1$, whence $D = \widetilde{D}^{-1} + CA^{-1}B$. Therefore

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & \widetilde{D}^{-1} + CA^{-1}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & \widetilde{D}^{-1} \end{pmatrix}$$

and we get

$$\mathrm{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Ber} \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \mathrm{Ber} \begin{pmatrix} A & B \\ 0 & \widetilde{D}^{-1} \end{pmatrix} = \det A \cdot \det \widetilde{D}.$$

□

Now let us consider the coordinate expression for adjoint operator. Suppose $\Delta \in \mathrm{Diff}_k(A, A)$ is a scalar operator: $\Delta = \sum_\sigma a_\sigma \frac{\partial^{|\sigma|}}{\partial x_\sigma}$, where $\sigma = (i_1, \dots, i_r)$ is an ordered set of integers $1 \leqslant i_j \leqslant s + t$, $|\sigma| = r$, $\frac{\partial^{|\sigma|}}{\partial x_\sigma} = \frac{\partial^{|\sigma|}}{\partial x_{i_1} \dots \partial x_{i_r}}$. Then we have

$$\Delta^* = \sum_\sigma (-1)^\Theta \frac{\partial^{|\sigma|}}{\partial x_\sigma} \circ a_\sigma,$$

where $\Theta = |\sigma| + \widetilde{a_\sigma} \sum_{j=1}^{|\sigma|} \widetilde{x_{i_j}} + \sum_{1\leqslant j<k\leqslant|\sigma|} \widetilde{x_{i_j}}\widetilde{x_{i_k}}$ and tilde over an object denotes the parity of the object. For a matrix d.o. $\Delta = \|\Delta_{ij}\|$ one has $(\Delta^*)_{ij} = (-1)^{\widetilde{ij}} (\Delta_{ji})^*$.

3. Spencer Complexes and Green's Formula

3.1. ¿From now onwards we assume that the module Λ^1 is of finite type and projective. This implies that the same true for the modules Λ^k and $\mathcal{J}^k(A)$ and, also, that for any projective module the Spencer Diff-complex is exact (see[30]). In this case it is expedient to consider another variant of construction of adjoint operator.

Set $P^\circ = \mathrm{Hom}(P, B)$. Obviously, $\Sigma_i = \widehat{\Lambda^i} = (\Lambda^i)^\circ = \mathrm{D}_i(B)$, $\widehat{\mathcal{J}^k(A)} = \mathcal{J}^k(A)^\circ = \mathrm{Diff}_k(A, B)$.

Definition. For an operator $\Delta \in \mathrm{Diff}_k(A, B)$ we define the operator $\Delta^\circ \colon A \to B$, $\Delta^\circ(a) = \{\Delta, a\} j_k^*(a\Delta)$, $a \in A$, which, for economy of language, will be called *adjoint*.

Proposition. (1) *The grading of Δ° is equal to the one of Δ.*

(2) *If $\Delta \in \mathrm{Diff}_k(A, B)$ then $\Delta^\circ \in \mathrm{Diff}_k(A, B)$.*

(3) $\omega^\circ = \omega$, $\omega \in B = \mathrm{Diff}_0(A, B)$.

(4) *If $X \in \mathrm{D}_1(B)$ then $X + X^\circ \in \mathrm{Diff}_0(A, B) = B$ and $X + X^\circ = \delta(X)$.*

(5) $(a\Delta)^\circ = \{a, \Delta\}\Delta^\circ \circ a$, $\Delta \in \mathrm{Diff}(A, B)$, $a \in A$.

(6) *If $\Delta(a) = [\nabla_a]$, $\Delta \in \mathrm{Diff}(A, B)$, $a \in A$, $\nabla_a \in \mathrm{Diff}^+(\Lambda^n)$, then $\Delta^\circ(a) = [\Box_a]$, where $\Box_a(a') = \{a, a'\}\nabla_{a'}(a)$.*

(7) $(\Delta^\circ)^\circ = \Delta$.

Proof. (1) Obvious.

(2) $\delta_a(\Delta^\circ)(a') = \{a, \Delta^\circ\}\Delta^\circ(aa') - a\Delta^\circ(a') =$
$\{\Delta^\circ, a'\} j_k^*(aa'\Delta) - \{\Delta^\circ, a'\} a j_k^*(a'\Delta) =$
$\{\Delta^\circ, a'\}\delta_a(j_k^*)(a'\Delta)$, so $\delta_{a_0,\dots,a_k}(\Delta^\circ)(a') =$
$\{\Delta^\circ, a'\}\delta_{a_0,\dots,a_k}(j_k^*)(a'\Delta) = 0.$

(3) Obvious.

(4) Clearly, $\delta_a(j_1) = j_1(a) - aj_1(1) \in \mathcal{J}^1(A)$, hence

$$(\delta_a(j_1))^*(\Delta) = \{a, \Delta\}\Delta(a) - a\Delta(1) = \delta_a(\Delta)(1), \qquad \Delta \in \mathrm{Diff}_1(A, A).$$

Thus $\delta_a(X + X^\circ)(1) = \delta_a(X)(1) + \delta_a(j_1^*)(X) = \delta_a(X)(1) - \delta_a(j_1)^*(X) = 0$. Further, $\delta(X) = j_1^*(\gamma^*(X))$, where $\gamma\colon \mathcal{J}^1(A) \to \Lambda^1$ is the natural projection. Obviously, $\gamma^*\colon \mathrm{D}_1(B) \to \mathrm{Diff}_1(A, B)$ is the natural inclusion, so that $\delta(X) = X^\circ(1) = X + X^\circ$.

(5) $a\Delta^\circ(a') = \{a\Delta, a'\}j_k^*(a'a\Delta) = \{a\Delta, a'\}\{a'a, \Delta\}\Delta^\circ(a'a) = \{a, \Delta\}(\Delta^\circ \circ a)(a')$.

(6) Obvious.

(7) Let $\Delta(a) = [\nabla_a]$ and $\Delta^\circ(a) = [\Box_a]$. Then $(\Delta^\circ)^\circ(a) = \left[\widetilde{\Box_a}\right]$, where $\widetilde{\Box_a}(a') = \{a, a'\}\Box_{a'}(a) = \nabla_a(a')$, i.e., $(\Delta^\circ)^\circ(a) = [\nabla_a] = \Delta(a)$.

$\Box$

Now let us define the adjoint operator Δ° in the general case when $\Delta \in \mathrm{Diff}(P, Q^\circ)$, the P and Q being A-modules. As in non-graded case consider the family of operators $\Delta(p, q) \in \mathrm{Diff}(A, B)$, $p \in P$, $q \in Q$, $\Delta(p, q)(a) = \{p, a\}\{q, a\}\langle\Delta(ap), q\rangle$, $a \in A$; the brackets $\langle\cdot, \cdot\rangle$ denote the natural pairing $Q^\circ \otimes Q \to B$. Then set $\langle\Delta^\circ(q), p\rangle = \{q, p\}\Delta(p, q)^\circ(1)$, $\Delta^\circ \in \mathrm{Diff}(Q, P^\circ)$.

Proposition. (1) *For any $\Delta \in \mathrm{Diff}_k(P, Q^\circ)$ the operator Δ° is well-defined and of order $\leqslant k$.*

(2) *For every $\Delta \in \mathrm{Diff}(P, Q^\circ)$ we have $(\Delta^\circ)^\circ = \Delta$.*

(3) *If the modules P, Q, Q° are projective, then for $\Delta \in \mathrm{Diff}(P, Q^\circ)$ the adjoint operator $\Delta^\circ\colon Q \to P^\circ$ coincides with the composition of $Q \to Q^{\circ\circ} \xrightarrow{\Delta^*} P^\circ$, where $Q \to Q^{\circ\circ}$ is the natural homomorphism.*

Proof. Statements (1) and (2) are straightforward.

(3) Take $\Delta\colon P \to Q^\circ$, $p \in P$, $q \in Q$. We have $\Delta = \psi_\Delta j_k$, so $\langle\Delta^*(q), p\rangle = \langle j_k^*\psi_\Delta^*(q), p\rangle = j_k^*(\psi_\Delta^*(q) \circ p) = (\psi_\Delta^*(q) \circ p)^\circ(1) = \{q, p\}\Delta(p, q)^\circ(1)$

$\Box$

Now let us consider one more example of adjoint operators.

3.2 Example: The Spencer Operators. The complex

$$0 \to P \xrightarrow{j} \mathcal{J}^{\infty}(P) \xrightarrow{s} \mathcal{J}^{\infty}(P) \otimes \Lambda^1 \xrightarrow{s} \mathcal{J}^{\infty}(P) \otimes \Lambda^2 \xrightarrow{s} \dots,$$

where $s(j(p) \otimes \omega) = j(p) \otimes d\omega$, is called the Spencer $\mathcal{J}$-complex (for details see [30]). Note that s is a d.o. of order $\leqslant 1$. Let us compute the dual complex. Clearly, for the projective module P we have $\widehat{\mathcal{J}^{\infty}(P) \otimes} \Lambda^i = \left(\mathcal{J}^{\infty}(P) \otimes \Lambda^i\right)^{\circ} = \mathrm{D}_i(\mathrm{Hom}(\mathcal{J}^{\infty}(P), B)) = \mathrm{D}_i(\mathrm{Diff}(P, B)) = \mathrm{D}_i\left(\mathrm{Diff}^{+}(P^{\circ})\right)$. To describe s^* we need the following remark.

Let $\Delta(P) : \mathrm{F}(P) \to \mathrm{G}(P)$ be a *natural* d.o., F and G functors on a category of projective modules over A. For any functors F denote by $\dot{\mathrm{F}}(P)$ the abelian group $\mathrm{F}(P)$ supplied with A-module structure induced by the A-module structure in P. Then Δ generates the natural homomorphism $\dot{\Delta} : \dot{\mathrm{F}} \to \dot{\mathrm{G}}$.

Lemma. $(\Delta^*)^{\cdot} = \left(\dot{\Delta}\right)^*$

Proof is trivial. □

The lemma immediately implies that the complex dual to the Spencer $\mathcal{J}$-complex for module P is the Spencer Diff-complex for module P°. On the other hand we have $\widehat{\mathcal{J}^{\infty}(P) \otimes} \Lambda^k = \mathrm{Hom}(\mathcal{J}^{\infty}(P), \mathrm{D}_k(B)) = \mathrm{Diff}(P, \Sigma_k)$ and the operator $s^* : \mathrm{Diff}(P, \Sigma_k) \to \mathrm{Diff}(P, \Sigma_{k-1})$ has the form $s^*(\nabla) = \delta \circ \nabla$, $\nabla \in \mathrm{Diff}(P, \Sigma_k)$; $j^*(\nabla) = \nabla^{\circ}(1)$. Bringing all this together we can state the following:

Theorem. *For a projective module P there is an isomorphism of complexes:*

$$\begin{tikzcd}
0 & P^{\circ} \arrow[l] \arrow[d, equal] & \mathrm{Diff}^{+}(P^{\circ}) \arrow[l, "\mathcal{D}"'] \arrow[d, "\psi"'] & \mathrm{D}_1\left(\mathrm{Diff}^{+}(P^{\circ})\right) \arrow[l, "S"'] \arrow[d, "\psi"'] & \mathrm{D}_1\left(\mathrm{Diff}^{+}(P^{\circ})\right) \arrow[l, "S"'] \arrow[d, "\psi"'] & \dots \arrow[l] \\
0 & P^{\circ} \arrow[l] & \mathrm{Diff}(P, B) \arrow[l, "\mu"'] & \mathrm{Diff}(P, \Sigma_1) \arrow[l, "\omega"'] & \mathrm{Diff}(P, \Sigma_2) \arrow[l, "\omega"'] & \dots, \arrow[l]
\end{tikzcd}$$

where $\omega(\nabla) = \delta \circ \nabla$, $\nabla \in \mathrm{Diff}(P, \Sigma_k)$, $\mu(\nabla) = \nabla^{\circ}(1)$, the maps ψ are the composition of isomorphisms:

$$\mathrm{D}_k\left(\mathrm{Diff}^{+}(P^{\circ})\right) \xrightarrow{\mathrm{D}_k(\circ)} \mathrm{D}_k(\mathrm{Diff}(P, B)) \to \mathrm{Diff}(P, \mathrm{D}_k(B)) \to \mathrm{Diff}(P, \Sigma_k).$$

This theorem has the following:

Corollary (Green's Formula). *If the Spencer* Diff*-cohomology of the Berezinian B in the term* $\mathrm{Diff}^+(B)$ *is trivial, then for any $\Delta \in \mathrm{Diff}(P, Q^\circ)$, $p \in P$, $q \in Q$, we have*

$$\langle \Delta(p), q\rangle - \{\Delta, p\}\langle p, \Delta^\circ(q)\rangle = \delta G,$$

where $G \in \Sigma_1$ is an integral 1-form.

Proof. Suppose $\nabla \in \mathrm{Diff}^+(B)$; then $(\nabla - \nabla(1)) \in \ker \mathcal{D}$, hence there exists $\square \in \mathrm{D}_1\left(\mathrm{Diff}^+(B)\right)$ such that $S(\square) = \nabla - \nabla(1)$. It follows from the theorem that $\nabla^*(1) - \nabla(1) = \psi \circ S(\square) = \omega(\psi(\square))$. Therefore $\nabla^*(1) - \nabla(1) = \delta G$, where $G = \psi(\square)(1)$. Letting $\nabla = \Delta(p, q)$, we get the Green formula. $\square$

Remark. If B is projective then there exists an A-homomorphism $\kappa\colon \ker \mathcal{D} \to \mathrm{D}_1\left(\mathrm{Diff}^+(B)\right)$, such that $S \circ \kappa = id$, and we can take $\square = \kappa(\Delta - \Delta(1))$ and, therefore, $G = G_\kappa = \psi(\kappa(\Delta(p, q) - \langle \Delta(p), q\rangle))(1)$. Since the Spencer complex of B is exact, for any two homomorphisms κ and κ' one can find a homomorphism $f\colon \ker \mathcal{D} \to \mathrm{D}_2\left(\mathrm{Diff}^+(B)\right)$ such that $\kappa - \kappa' = S \circ f$. Hence $G_\kappa - G_{\kappa'} = \delta F$, where $F = \psi(f(\Delta(p, q) - \langle \Delta(p), q\rangle))(1)$.

3.3. In this subsection we describe a spectral sequence, which establishes the relationship between the de Rham cohomology and the homology of the complex of integral forms.

Proposition (Poincaré duality). *There is a spectral sequence, $\left(E^r_{*,*}, d^r_*\right)$, with*

$$E^2_{p,q} = \mathrm{H}_p\left((\Sigma_*)_{-q}\right),$$

the homology of complexes of integral forms, and converging to the de Rham cohomology $\mathrm{H}(\Lambda^*)$.

Proof. Let $K_{p,q} = \mathrm{D}_p\left(\mathrm{Diff}^+(\Lambda^{-q})\right)$, d' be the Spencer operator $d'\colon K_{p,q} \to K_{p-1,q}$, and $d'' = (-1)^p\,\mathrm{D}_p\left(\mathrm{Diff}^+(d)\right)$, $d''\colon K_{p,q} \to K_{p,q-1}$. Then $\{K_{*,*}, d', d''\}$ is a double complex. Obviously, $\mathrm{H}^I(K) = \mathrm{H}(K_{*,*}, d') = \Lambda^{-q}$ for $p = 0$ and 0 for $p \neq 0$. Therefore in the second spectral sequence ${}^{II}E^2_{p,q} = {}^{II}\mathrm{H}_{p,q}\left({}^{I}\mathrm{H}(K)\right) = \mathrm{H}_{p,q}\left({}^{I}\mathrm{H}_{*,*}(K), d''\right) = \mathrm{H}^{-q}(\Lambda^*)$ for $p = 0$ and ${}^{II}E^2_{p,q} = 0$ for $p \neq 0$. Thus the second spectral sequence converge to the de Rham cohomology. From the first spectral sequence we get ${}^{I}E^{2}_{p,q} = {}^{I}\mathrm{H}_{p,q}\left({}^{II}\mathrm{H}\left(K\right)\right) = {}^{I}\mathrm{H}_{p,q}\left(\mathrm{H}\left(K_{*,*}, d''\right)\right) = \mathrm{H}_{p,q}\left(\mathrm{D}_{*}\left(B\right), d'\right) = \mathrm{H}_{p}\left(\left(\Sigma_{*}\right)_{-q}\right)$. $\square$

Corollary. *If $\widehat{A_i} = 0$ for all $i \neq n$, then $\mathrm{H}_i\left(\Sigma_*\right) = \mathrm{H}^{n-i}\left(\Lambda^*\right)$.*

4. Quadratic Lagrangians, the Euler Operator, and the Noether theorem

In this section P and Q are projective modules.

4.1. Consider the complex

$$0 \leftarrow \mathrm{Diff}^{sym}_{(2)}\left(P, P^{\circ}\right) \xleftarrow{\mu} \mathrm{Diff}^{sym}_{(2)}\left(P, B\right) \xleftarrow{\omega} \mathrm{Diff}^{sym}_{(2)}\left(P, \Sigma_1\right) \xleftarrow{\omega} \dots, \tag{2}$$

where $\mathrm{Diff}^{sym}_{(2)}\left(P, Q\right)$ denotes the submodule of $\mathrm{Diff}\left(P, \mathrm{Diff}\left(P, Q\right)\right)$ consisting of all symmetric bidifferential Q-valued operators $\nabla\left(p_1\right)\left(p_2\right) = \{p_1, p_2\}\nabla\left(p_2\right)\left(p_1\right)$, $\nabla \in \mathrm{Diff}\left(P, \mathrm{Diff}\left(P, Q\right)\right)$, $\mathrm{Diff}^{sym}\left(P, P^{\circ}\right)$ is the module of all self-adjoint operators from $\mathrm{Diff}\left(P, P^{\circ}\right)$, $\omega\left(\nabla\right) = \delta \circ \nabla$, $\nabla \in \mathrm{Diff}^{sym}_{(2)}\left(P, \Sigma_k\right)$, $k > 0$, and $\mu\left(\nabla\right)\left(p\right) = \left(\nabla\left(p\right)\right)^{\circ}\left(1\right)$, $p \in P$.

Theorem. *This complex is acyclic.*

Proof. From theorem 3.2 it follows easily that the complex

$$0 \leftarrow \mathrm{Diff}\left(P, P^{\circ}\right) \xleftarrow{\widetilde{\mu}} \mathrm{Diff}\left(P, \mathrm{Diff}\left(P, B\right)\right) \xleftarrow{\widetilde{\omega}} \mathrm{Diff}\left(P, \mathrm{Diff}\left(P, \Sigma_1\right)\right) \xleftarrow{\widetilde{\omega}} \dots, \tag{3}$$

where $\widetilde{\omega}\left(\nabla\right) = \delta \circ \nabla$, $\widetilde{\mu}\left(\nabla\right)\left(p\right) = \left(\nabla\left(p\right)\right)^{\circ}\left(1\right)$, is acyclic. To prove the theorem it is sufficient to show that this complex split into the symmetric and the skew-symmetric parts. To do this let us check that the involution ρ of complex (3), $\rho\left(\nabla\right)\left(p_1\right)\left(p_2\right) = \{p_1, p_2\}\nabla\left(p_2\right)\left(p_1\right)$, $\nabla \in \mathrm{Diff}\left(P, \mathrm{Diff}\left(P, \Sigma_k\right)\right)$ and $\rho\left(\nabla\right) = \nabla^{\circ}$, $\nabla \in \mathrm{Diff}\left(P, P^{\circ}\right)$, is an authormorphism of this complex. The fact that $\widetilde{\omega} \circ \rho = \rho \circ \widetilde{\omega}$ is obvious. Let us verify that $\widetilde{\mu} \circ \rho = \rho \circ \widetilde{\mu}$. Take $\Delta \in \mathrm{Diff}\left(P, \mathrm{Diff}\left(P, B\right)\right)$ and let $\delta\left(p_1\right)\left(p_2\right) = \left[\nabla_{p_1,p_2}\right]$, $\nabla_{p_1,p_2} \in \mathrm{Diff}^{+}\left(\Lambda^n\right)$. It follows from proposition 3.1 that $\left\langle \widetilde{\mu}\left(\Delta\right)\left(p_1\right), p_2\right\rangle = \left[\square\right]$, where $\square \in \mathrm{Diff}^{+}\left(\Lambda^n\right)$, $\square\left(a\right) = \{p_2, a\}\nabla_{p_1, ap_2}\left(1\right)$, $p_1, p_2 \in P$. Therefore $\left\langle \rho\widetilde{\mu}\left(\Delta\right)\left(p_1\right), p_2\right\rangle = \left\langle \widetilde{\mu}\left(\Delta\right)^{\circ}\left(p_1\right), p_2\right\rangle = \{p_1, p_2\}\left[\square'\right]$, where

$\square'(a) = \{p_1, a\}\{p_2, a\}\nabla_{ap_2,p_1}(1)$. On the other hand $\langle\widetilde{\mu}\rho(\Delta)(p_1), p_2\rangle = [\square'']$, where $\square''(a) = \{p_2, a\}\nabla'_{p_1,ap_2}(1)$, $\langle\rho(\Delta)(p_1), p_2\rangle = \left[\nabla'_{p_1,p_2}\right]$. Clearly, $\nabla'_{p_1,p_2} = \{p_1, p_2\}\nabla_{p_2,p_1}$, hence $\square''(a) = \{p_2, a\}\{p_1, ap_2\}\nabla_{ap_2,p_1}(1)$. $\square$

4.2 Lagrangian Formalism.

Definition. The space $\mathcal{L}ag(P)$ of *quadratic Lagrangians* on P is defined as the cokernel of the operator $\omega\colon \operatorname{Diff}^{sym}_{(2)}(P, \Sigma_1) \to \operatorname{Diff}^{sym}_{(2)}(P, B)$. An operator $L \in \operatorname{Diff}^{sym}_{(2)}(P, B)$ is called the *density* of quadratic Lagrangian $\mathcal{L}$ if $\mathcal{L} = L \mod \operatorname{im}\omega$.

¿From theorem 4.1 we see that the operator μ gives rise to an isomorphism of $\mathcal{L}ag(P)$ to the submodule of the module $\operatorname{Diff}(P, P^\circ)$ consisting of self-adjoint operators. This isomorphism is said to be the *Euler operator* and denoted by $\mathcal{E}$.

4.3 Conservation laws. Let $\Delta \in \operatorname{Diff}_k(P, Q)$ and $E = \{p \in P | \Delta(p) = 0\}$ is the corresponding equation. The operator Δ generates the chain map Ω_Δ of the complexes (2):

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \operatorname{Diff}(Q,B) & \xleftarrow{\omega} & \operatorname{Diff}(Q,\Sigma_1) & \xleftarrow{\omega} & \operatorname{Diff}(Q,\Sigma_2) & \xleftarrow{\omega} & \dots \\
 & & \Big\downarrow{\scriptstyle\Omega_\Delta} & & \Big\downarrow{\scriptstyle\Omega_\Delta} & & \Big\downarrow{\scriptstyle\Omega_\Delta} & & \\
0 & \longleftarrow & \operatorname{Diff}(P,B) & \xleftarrow{\omega} & \operatorname{Diff}(P,\Sigma_1) & \xleftarrow{\omega} & \operatorname{Diff}(P,\Sigma_2) & \xleftarrow{\omega} & \dots
\end{array}$$

(*Linear*) *conservation laws* for the equation E are defined by analogy with non-graded case (see [19]) as classes of 1-dimensional homology of the complex $\operatorname{coker}\Omega_\Delta$. Let us denote the group of linear conservation laws for the equation $\Delta = 0$ by $\operatorname{Cl}(\Delta) = \operatorname{H}_1(\operatorname{coker}\Omega_\Delta)$. The following theorem and the corollary describe the group $\operatorname{Cl}(\Delta)$.

Theorem. *There exists an exact sequence*

$$0 \to \operatorname{H}_1(\operatorname{im}\Omega_\Delta) \to \operatorname{H}_0(\ker\Omega_\Delta) \to \ker\Delta^* \to \operatorname{Cl}(\Delta) \to 0.$$

Proof. It follows from theorem 4.1 that exact homology sequences corresponding to the short exact sequences of complexes

$$0 \to \ker\Omega_\Delta \to \operatorname{Diff}(Q, \Sigma_*) \to \operatorname{im}\Omega_\Delta \to 0$$

$$0 \to \operatorname{im}\Omega_\Delta \to \mathrm{Diff}\,(p, \Sigma_*) \to \operatorname{coker}\Omega_\Delta \to 0$$

have the form

$$0 \to \mathrm{H}_1\,(\operatorname{im}\Omega_\Delta) \to \mathrm{H}_0\,(\ker\Omega_\Delta) \xrightarrow{i_1} Q^\circ \xrightarrow{i} \mathrm{H}_0\,(\operatorname{im}\Omega_\Delta) \to 0$$

$$0 \to \mathrm{H}_1\,(\operatorname{coker}\Omega_\Delta) \to \mathrm{H}_0\,(\operatorname{im}\Omega_\Delta) \xrightarrow{j} P^\circ \to \mathrm{H}_0\,(\operatorname{coker}\Omega_\Delta) \to 0.$$

It is straightforward to check that the composition $j \circ i \colon Q^\circ \to P^\circ$ coincides with the adjoint operator $\Delta^* \colon Q^\circ \to P^\circ$. Hence $\ker\Delta^* / \operatorname{im} i_1$ is isomorphic to $\ker j = \mathrm{Cl}\,(\Delta)$ and we get the desired exact sequence. □

Corollary. *If* $\ker\Omega_\Delta = 0$ *then the group of linear conservation laws* $\mathrm{Cl}\,(\Delta)$ *is isomorphic to* $\ker\Delta^*$*.*

Let us give an explicit expression for the map $\ker\Delta^* \to \mathrm{Cl}\,(\Delta)$. Suppose $q^\circ \in \ker\Delta^* \subset Q^\circ$. Then, choosing a homomorphism κ (see Remark 3.2), we have a d.o. from P to Σ_1, $p \mapsto G_\kappa\,(\Delta\,(p, q^\circ))$. The Green formula yields $\langle\Delta\,(p)\,, q^\circ\rangle = \delta G_\kappa\,(\Delta\,(p, q^\circ))$. Hence the operator $p \mapsto G_\kappa\,(\Delta\,(p, q^\circ))$ is an 1-cocycle of the complex $\operatorname{coker}\Omega_\Delta$ and we obtain a map $\chi\colon \ker\Delta^* \to \mathrm{Cl}\,(\Delta)$, where $\chi\,(q^\circ)$ is the conservation law corresponding to the operator $p \mapsto G_\kappa\,(\Delta\,(p, q^\circ))$. Let us show that this is the map under consideration. If ∇ is an 1-cocycle of $\operatorname{coker}\Omega_\Delta$, the element q° from $\ker\Delta^*$ corresponding to it according to the proof of theorem 4.3 can be found as $q^\circ = \mu\,(\Box)$, where $\Box \in \mathrm{Diff}\,(Q, B)$ satisfy the relation $\Box \circ \Delta = \delta \circ \nabla$. If ∇ is the operator $p \mapsto G_\kappa\,(\Delta\,(p, q^\circ))$ then $\Box = q^\circ \in \mathrm{Diff}\,(P, B)$ and $\mu\,(\Box) = \mu\,(q^\circ) = q^\circ$.

4.4 The Noether Theorem. We start with a description of transformations of the objects that Noether's theorem includes.

Let $\mathrm{Der}\,(P) = \{(X_P, X) \,|\, X \in \mathrm{D}_1\,(A)\,, X_P \in \mathrm{Diff}_1\,(P, P)\,, \text{and } \forall a \in A, \forall p \in P \quad X_P\,(ap) = \{X_P, a\}aX_P\,(p) + X\,(a)\,p\,\}$. If $X \in \mathrm{D}_1\,(A)$ then $(X_B, X) \in \mathrm{Der}\,(B)$, where $X_B = -X^*$ (more generally, if $(X_P, X) \in \mathrm{Der}\,(P)$ then one can define $(X_{P^\circ}, X) \in \mathrm{Der}\,(P^\circ)$, $X_{P^\circ} = -\,(X_P)^*)$. Given $(X_P, X) \in \mathrm{Der}\,(P)$, define $(X_{\mathrm{Diff}}, X) \in \mathrm{Der}\,(\mathrm{Diff}\,(P, B))$ by the formula $X_{\mathrm{Diff}}\,(\Delta) = [X, \Delta] = X_B \circ \Delta - \{X, \Delta\}\Delta \circ X_P$. For $L \in \mathrm{Diff}\,(P, \mathrm{Diff}\,(P, B))$ we put $X_P\,(L) = X_{\mathrm{Diff}} \circ L - \{X, L\}L \circ$

X_P. If $L \in \mathrm{Diff}^{sym}_{(2)}(P, B)$ then $X_P(L) \in \mathrm{Diff}^{sym}_{(2)}(P, B)$ and is called the *variation* of L under the infinitesimal transformation X_P. Clearly, X_P generates the map of Lagrangians on P: $X_P \colon \mathcal{L}ag(P) \to \mathcal{L}ag(P)$.

¿From the definition of variation it follows that
$X_P(L)(p_1)(p_2) = X_B(L(p_1)(p_2)) - \{X, L\}\{X, p_1\}L(p_1)(X_P(p_2)) -$
$\{X, L\}L(X_P(p_1))(p_2)$.
Further, using proposition 3.1(4) we get

$$X_B(L(p_1)(p_2)) = -\{X, L\}\{X, p_1\}\{X, p_2\}\delta(L(p_1)(p_2) \circ X).$$

It follows from Green's formula that $\forall p_1, p_2 \in P$

$$L(p_1, p_2) = \langle \mathcal{E}_L(p_1), p_2 \rangle + \delta G_\kappa(L(p_1)(p_2, 1)).$$

Combining these formulae, we obtain the following *formula for the first variation*:
$X_P(L)(p_1)(p_2) = -\{L, p_2\}\{p_1, p_2\}\langle X_P(p_2), \mathcal{E}_L(p_1)\rangle -$
$\{L, p_1\}\langle X_P(p_1), \mathcal{E}(p_2)\rangle - \delta n_\kappa(p_1, p_2)$,
where $n_\kappa(p_1, p_2) =$
$\{X, L\}\{X, p_1\}\{X, p_2\}L(p_1)(p_2) \circ X + \{X, L\}\{X, p_1\}G_\kappa(p_1, X_P(p_2)) +$
$\{X, L\}\{X, p_2\}\{p_1, p_2\}G_\kappa(p_2, X_P(p_1))$.

Definition. $X_P \in \mathrm{Der}(P)$ is said to be a *symmetry* of a Lagrangian $\mathcal{L}$ if $X_P(\mathcal{L}) = 0$.

¿From theorem 4.1 it follows that a symmetry of $\mathcal{L}$ is a symmetry of the operator $\mathcal{E}_\mathcal{L} = \mathcal{E}(\mathcal{L})$, i.e., $X_P(\mathcal{E}_\mathcal{L}) = X_{P^\circ} \circ \mathcal{E}_\mathcal{L} - \{X, \mathcal{E}_\mathcal{L}\}\mathcal{E}_\mathcal{L} \circ X_P = 0$, and conversely.

If X_P is a symmetry of $\mathcal{L}$, then $X_P(L) = \omega(L')$, where L is a density of $\mathcal{L}$, $L' \in \mathrm{Diff}^{sym}_{(2)}(P, \Sigma_1)$. Consider the integral 1-form $n_\kappa(p_1, p_2) + L'(p_1)(p_2)$. The first variation formula implies that this integral form is closed whenever $p_1, p_2 \in \ker \mathcal{E}_\mathcal{L}$. Clearly, its homological class does not depend on the choice of κ and L'. Thus we have proved the following:

Theorem (Noether). *If X_P is a symmetry of the Lagrangian $\mathcal{L} = L \mod \operatorname{im}\omega$ and the module P is projective, then the map $p \mapsto n_\kappa(p, p) + L'(p)(p)$, $p \in P$, gives rise to a conservation law of the equation $\mathcal{E}_\mathcal{L} = 0$.*

Appendix: Right Connections

This appendix is devoted to a general algebraic setting for the formalism of right connections.

1. We begin with a collection of a few facts about (usual) linear connections.

One can consider a *connection* on an A-module P as an A-homomorphism of grading zero $\gamma\colon P \mapsto \mathcal{J}^1(P)$ satisfying $\nu \circ \gamma = id_P$, where $\nu\colon \mathcal{J}^1(P) \to P$ is the natural projection.

The composition $\Lambda^i \otimes P \xrightarrow{id\otimes\gamma} \Lambda^i \otimes \mathcal{J}^1(P) \xrightarrow{s} \Lambda^{i+1} \otimes P$ of γ and the Spencer operator s is called the *de Rham operator* associated with γ and denoted by $d\colon \Lambda^i \otimes P \to \Lambda^{i+1} \otimes P$. The first de Rham operator $d\colon P \to \Lambda^1 \otimes P$ gives rise in a natural way to a *covarient derivative*, i.e., a homomorphism $\nabla\colon \mathrm{D}_1(A) \to \mathrm{Diff}_1(P,P)$, $X \mapsto \nabla_X$, such that $\nabla_X(ap) = \{X,a\}a\nabla_X(p) + X(a)\,p$, $X \in \mathrm{D}_1(A)$. The de Rham sequence with values in P

$$0 \to P \xrightarrow{d} \Lambda^1 \otimes P \xrightarrow{d} \Lambda^2 \otimes P \xrightarrow{d} \dots$$

is not a complex in general. The operator $d^2\colon \Lambda^i \otimes P \to \Lambda^{i+2} \otimes P$ is said to be *curvature* of the connection γ. It is straightforward to check that d^2 is a Λ^*-linear operator and, therefore, for P projective the curvature d^2 is multiplication by a $R_\gamma \in \Lambda^2 \otimes \mathrm{Hom}_A(P,P)$.

2. We define a right connection in a dual way.

Definition. A *right connection* on P is defined to be an A-homomorphism of grading zero $\lambda\colon \mathrm{Diff}_1^+(P) \to P$ satisfying $\lambda \circ \iota = id_P$, where $\iota\colon P \to \mathrm{Diff}_1^+(P)$ is the natural inclusion.

Given a module P with a right connection λ, one can carry out the construction of the *sequence of integral form with values in* P by letting the operator $\delta\colon \mathrm{D}_{i+1}(P) \to \mathrm{D}_i(P)$ be the composition $\mathrm{D}_{i+1}(P) \xrightarrow{S} \mathrm{D}_i\left(\mathrm{Diff}_1^+(P)\right) \xrightarrow{\mathrm{D}_i(\lambda)} \mathrm{D}_i(P)$ of the Spencer operator S and $\mathrm{D}_i(\lambda)$:

$$0 \leftarrow P \xleftarrow{\delta} \mathrm{D}_1(P) \xleftarrow{\delta} \mathrm{D}_2(P) \xleftarrow{\delta} \dots$$

The first operator $\delta\colon \mathrm{D}_1(P) \to P$ provides a *right covariant derivative*, i.e., a homomorphism $\nabla\colon \mathrm{D}_1(A) \to \mathrm{Diff}_1^+(P,P)$, $X \mapsto \nabla_X$, such that

$$\nabla_X(ap) = \{X,a\}a\nabla_X(p) - X(a)\,p, \qquad X \in \mathrm{D}_1(A). \tag{1}$$

Remark. This "right Leibniz rule" may be reformulated by the following way. The operator ∇ can be extended to an A-homomorphism $\nabla\colon \mathrm{Diff}_1(A,A) \to \mathrm{Diff}_1^+(P,P)$ by putting $\nabla_{id_A} = id_P$. Then (1) means that the map $\nabla\colon \mathrm{Diff}_1^+(A,A) \to \mathrm{Diff}_1(P,P)$ is also an A-homomorphism.

The operator $\delta^2\colon \mathrm{D}_{i+2}(P) \to \mathrm{D}_i(P)$ is called the *curvature* of the right connection λ. A direct calculation shows that this is a $\mathrm{D}_*(A)$-linear operator and, so, for P projective the curvature δ^2 can be interpreted as inner product with a $R_\lambda \in \Lambda^2 \otimes \mathrm{Hom}_A(P,P)$.

Example. If on a projective module P there is a connection $\gamma\colon P \to \mathcal{J}^1(P)$, then on P° there is a right connection $\lambda = \gamma^*\colon \left(\mathcal{J}^1(P)\right)^\circ = \mathrm{Diff}_1^+(P^\circ) \to P^\circ$. In particular, the obvious flat connection on A $\gamma\colon A \to \mathcal{J}^1(A)$, $\gamma(a) = aj_1(1)$, provides the canonical flat right connection on the Berezinian B. The complex of integral forms with values in P° is dual to the de Rham complex with coefficients in P.

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