ON THE INVARIANCE PRINCIPLE FOR NON-UNIFORMLY EXPANDING TRANSFORMATIONS OF $[0,1]$

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Abstract. We consider a class of maps of $[0,1]$ with an indifferent fixed point at 0 and expanding everywhere else. Using a suitable uniformly expanding induced map we prove a functional central limit theorem (invariance principle) with anomalous scaling $n/\log n$ for the random stationary process generated by this dynamical system.

Keywords: Donsker theorem, infinite invariant measure, Markov partition, induced transformation.

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We first introduce the basic setting. Let f be the map of the unit interval $[0,1]$ defined as follows:

(i) $f(0)=0,\ f(1)=1$;

(ii) f is monotone and non decreasing on $I_0=[0,\frac{1}{2}[$ and $I_1=]\frac{1}{2},1]$;

(iii) For each $i=0,1$, $f_{|I_i}$ extends to a C^2 function f_i on its closure which is onto $[0,1]$.

(iv) There are two numbers $\alpha>1$ and $L>0$ such that:

$$f'_{|I_1}\geq\alpha,\ \ f'(0)=1,\ \ f'_{|]0,1/2[}\geq 1,\quad \sup_{x\in[0,1]}|f''(x)/f'(x)|\leq L.$$

(v) $f''(0)\neq 0$ which implies $f''(0)>0$.

In [CI2] we proved that under the above assumptions the following ergodic theorem holds: there exist an increasing sequence $c_n=\kappa\, n/\log n$, where κ is a positive constant depending on f, such that for any real function u compactly supported on $]0,1]$

$$\frac{1}{c_n}\sum_{k=0}^{n-1}u(f^kx)\to\nu(u)\qquad\text{in probability}$$

where ν is a σ-finite absolutely continuous f-invariant measure whose density e satisfies $C_1/x\leq e(x)\leq C_2/x$ for any $x\in[0,1]$ and C_1,C_2 suitable positive constants (for related results see [T1],[T2],[CF]).

In this paper we want to study the fluctuations of the finite sums $\frac{1}{c_n}\sum_{k=0}^{n-1}u(f^kx)$. In particular, we shall prove a functional central limit theorem for the random variables $\frac{1}{\sqrt{c_n}}\sum_{k=0}^{n-1}u(f^kx)$ (see below, Theorem 1). A central limit theorem for the case of a finite measure has been proved in [LSV] and, in a slightly different context, in [ADU] (see also [HK] for a more general situation).

Let us consider the sequence of points c_k, $k\geq 0$, given by

$$c_0=1,\quad c_k=f_0^{-1}(c_{k-1}),\quad k\geq 1.$$

This sequence generates a countable partition of $[0,1]$ into the intervals $A_k = [c_k, c_{k-1}]$, $k \geq 1$, which is a Markov partition. In particular, $f(A_k) = A_{k-1}$, $k \geq 1$.

Let $\Omega_{\geq}$ be the set of one-sided sequences $\omega = (\omega_0, \omega_1, \ldots)$, $\omega_i \in \{1, 2, \ldots\}$ satisfying the compatibility condition: given ω_i then either $\omega_{i-1} = \omega_i + 1$ or $\omega_{i-1} = 1$. Then, the map

$$\phi : \omega \to \phi(\omega) = x \quad \text{according to} \quad f^i(x) \in A_{\omega_i}, i \geq 1$$

is a bijection between $\Omega_{\geq}$ and the points of $[0,1]$ which are not preimages of the origin. Moreover, ϕ conjugates the map f with the shift τ on $\Omega_{\geq}$.

For every integer $i \geq 1$ we denote by x_i the projection on the i^{th} symbol, i.e. $x_i(\omega) = \omega_i$, and define the "free" probability measure μ by

$$\mu(\omega_i) = |A_{\omega_i}|, \quad i \geq 1 \tag{1}$$

With slight abuse of language we shall again denote by μ the measure $\mu \circ \phi^{-1}$, i.e. the Lebesgue measure on $[0,1]$.

We now introduce the infinite sequence τ_j, $j \geq 1$, of successive entrance times in the state 1: $\tau_1(\omega) = \inf\{i \geq 0 \ : \ x_i(\omega) = 1\}$ and, for $j \geq 2$, $\tau_j(\omega) = \inf\{i > \tau_{i-1} \ : \ x_i(\omega) = 1\}$. Furthermore, we define a sequence of integer valued random variables by

$$\sigma_j(\omega) = \tau_{j+1} - \tau_j, \quad j \geq 0 \tag{2}$$

with the convention that $\tau_0 = -1$.

Definition 1. *The 'first passage' map (on the interval A_1), is the map $g : [0,1] \to [0,1]$ induced by f in the following way:*

$$x \to g(x) = f^{n(x)}(x) \quad \text{where} \quad n(x) = 1 + \min\{n \geq 0 \ : \ f^n(x) \in A_1 \} \tag{3}$$

Definition 2. *Let* $u :]0,1] \to I\!R$ *be any real function. Its* induced *version* $\tilde{u}$ *is defined by*

$$\tilde{u}(x) = \sum_{s=0}^{n(x)-1} u(f^s x) \tag{4}$$

Remark 1. The map g is uniformly expanding and surjective on each A_k, and enjoys the following property: let $x = \phi(\omega)$, where $\omega \in \Omega_{\geq}$, then $g^j(x) \in A_{\sigma_j}$, $j \geq 1$, where the integers $\sigma_j = \sigma_j(\omega)$ are defined in (2). It has been proved in [CI1] that the dynamical system $([0,1], g)$ leaves invariant an ergodic absolutely continuous probability measure $d\rho = h\, d\mu$, such that h is Hölder continuous and satisfies $d^{-1} \leq h \leq d$ for some $d > 0$. Furthermore, ρ satisfies the exponential uniform mixing property [R]. The g-invariant probability measure ρ is related to the f-invariant infinite measure ν by the identity

$$\nu(u) = \rho(\tilde{u}),$$

valid for any u of compact support in $]0,1]$ (see [CI2]).

We now introduce a space of locally Hölder continuous functions. Let $x, x' \in I \subseteq A_k$, for some $k \geq 1$, and write

$$\mathrm{var}_I\, u = \sup \{\, |u(x) - u(x')| \, : \, x, x' \in I \,\}$$

Let $\mathcal{F}_\gamma$ be the space of bounded continuous functions $u : [0,1] \to I\!R$ with compact support in $]0,1]$ such that

$$\sup_k \sup_{I \subseteq A_k} \left(\frac{\mathrm{var}_I\, u}{|I|^\gamma} \right) \leq M < \infty$$

for some $0 < \gamma \leq 1$ and $M > 0$. Notice that, since u is of compact support over $]0,1]$ the first sup above is actually taken over a finite set of k's.

Lemma 0.

1) If $u \in \mathcal{F}_\gamma$ *then* $\tilde{u} \in \mathcal{F}_\gamma$.

2) $u : [0,1] \to I\!R$ *is a cocycle with respect to the map* f*, i.e.* $u(x) = v(f(x)) - v(x)$ *for some* v*, if and only if* $\tilde{u}$ *is a cocycle with respect to the map* g.

Proof. Let $u \in \mathcal{F}_\gamma$. If $x, x' \in I \subseteq A_k$ we have $n(x) = n(x') = k$ and

$$|\tilde{u}(x) - \tilde{u}(x')| \leq \sum_{0 \leq j < k} |u(f^j(x)) - u(f^j(x'))| \leq \sum_{0 \leq j < k} \operatorname{var}_{f^j(I)} u$$

On the other hand, for the class of transformations considered here one has the following property of uniform distorsion (see [CI1], Lemma 2.1):

$$\frac{|f^j(I)|}{|f^j(A_k)|} \leq R \frac{|I|}{|A_k|}, \quad \text{for any} \quad 1 \leq j \leq k$$

where $R > 0$ is a constant independent of j and k. Hence

$$\frac{\operatorname{var}_I \tilde{u}}{|I|^\gamma} \leq M \sum_{0 \leq j < k} \left(\frac{|f^j(I)|}{|I|} \right)^\gamma \leq M R \sum_{0 \leq j < k} \left(\frac{|f^j(A_k)|}{|A_k|} \right)^\gamma$$

so that, taking the sup over k and recalling that u is of compact support over $]0,1]$, it follows that $\tilde{u} \in \mathcal{F}_\gamma$.

To show the last assertion notice first that Definition 2 implies at once that if $u = v(f(x)) - v(x)$ then $\tilde{u} = v(g(x)) - v(x)$.

To see the converse, observe that, again from Definition 2, one has

$$u(x) = \begin{cases} \tilde{u}(x), & \text{if } x \in A_1 \\ \tilde{u}(x) - \tilde{u}(f(x)), & \text{if } x \notin A_1. \end{cases}$$

Suppose now that $u(x) \equiv 0$ only for $x \notin A_1$ and assume that $\tilde{u}(x) = V(g(x)) - V(x)$. Then, since $f(x) = g(x)$ for $x \in A_1$, one also has $u(x) = V(f(x)) - V(x)$. On the other hand, if $x \notin A_1$ then $\tilde{u}(x) = \tilde{u}(f(x))$ and $g(f(x)) = g(x)$. Hence

$$\tilde{u}(x) = \tilde{u}(f(x)) = V(g(x)) - V(x) = V(g(f(x))) - V(x) = V(g(f(x))) - V(f(x))$$

so that

$$V(f(x)) = V(x)$$

and this implies that $u(x) = V(f(x)) - V(x)$ for any $x \in [0,1]$.

Now, for any u compactly supported in $]0,1]$, one can reduce to the previous case by observing that using an induction procedure u can be always decomposed as $u' + \phi$ where $u'(x) \equiv 0$ for $x \notin A_1$ and ϕ is a cocycle with respect to the map

f. Moreover, if $\tilde{u}$ is a cocycle then $\tilde{u'}$ is a cocycle as well and the argument above can be applied. Q.E.D.

Let $\Sigma_\geq$ be the set of *all* one-sided sequences σ of the form $\sigma = (\sigma_0, \sigma_1, \ldots)$, $\sigma_j \in \{1, 2, \ldots\}$. Then, the map

$$\pi : \sigma \to \pi(\sigma) = x \quad \text{according to} \quad g^j(x) \in A_{\sigma_j}, \;\; j \geq 1 \tag{6}$$

is a bijection between $\Sigma_\geq$ and the points of $[0, 1]$ which are not preimages of zero. Moreover, π conjugates the map g with the shift τ on $\Sigma_\geq$. Notice also (cf. (4)) that $n(g^k(x)) = \sigma_k$ where $\sigma = (\sigma_0, \sigma_1, \ldots) = \pi^{-1}(x)$.

Let us now consider an orbit $\{f^k x\}_{k=0}^{n-1}$, for some $x \in]0, 1]$ and denote by $N(n, x)$ the number of its passages in A_1, or, in other terms, the number of symbols in its (truncated) σ-coding $(\sigma_0, \sigma_1, \ldots \sigma_{N(n,x)-1})$.

We have

$$\sum_{k=0}^{n-1} u(f^k x) = \sum_{s=0}^{N(n,x)-1} \tilde{u}(g^s x) + R_n(x, u) \tag{7}$$

where the remainder is given by

$$R_n(x, u) = \sum_{s=m(n,x)}^{n-1} u(f^s x) \quad \text{with} \quad m(n, x) = \sum_{k=0}^{N(n,x)-1} n(g^k x) \tag{8}$$

Consider now a continuous function u compactly supported on $]0, 1]$. The remainder $R_n(x, u)$ is then uniformly bounded in n and x.

For such an u, we define

$$S_n(x) = \sum_{k=0}^{n-1} u(f^k x), \qquad \tilde{S}_n(x) = \sum_{s=0}^{n-1} \tilde{u}(g^s x) \tag{9}$$

In order to deal with a continuous process we define for $t \geq 0$

$$S(x, t) = \begin{cases} S_{n-1}(x) + (t - n + 1)(S_n(x) - S_{n-1}(x)), & \text{if } n - 1 \leq t < n. \\ S_n(x), & \text{if } t = n \end{cases} \tag{10}$$

with the convention $S_0(x) = 0$. An identical definition with $\tilde{S}_n$ in place of S_n yields $\tilde{S}(x, t)$.

We are now in the position to state the main result.

Theorem 1. *Let $u \in \mathcal{F}_\gamma$ be such that $\nu(u) = 0$ and not a cocycle. Let moreover $a_n = n/\log n$.*

Then, there exist a positive constant D such that the random element X_n of $C([0,1])$ defined on the probability space $([0,1], d\rho)$ by

$$X_n(x,t) = \frac{1}{\sqrt{D\,a_n}} S(x, nt), \qquad 0 \le t \le 1$$

converges in law to the Brownian motion $B(t)$.

We shall prove Theorem 1 through a sequence intermediate results.

Theorem 1'. *Let $\tilde{u} \in \mathcal{F}_\gamma$ be such that $\rho(\tilde{u}) = 0$ and not a cocycle.*

Then, there is a positive constant $\tilde{D}$ such that the random element $\tilde{X}_n$ of $C([0,1])$ defined on the probability space $([0,1], d\rho)$ by

$$\tilde{X}_n(x,t) = \frac{1}{\sqrt{\tilde{D}\,n}} \tilde{S}(x, nt), \qquad 0 \le t \le 1$$

converges in law to the Brownian motion $B(t)$.

Proof. Let $\{\xi_i\}$ be the sequence of random variables defined by $\xi_i = \tilde{u}(\pi(\tau^i\sigma))$ where $\pi(\sigma) = x$ and τ is the shift on $\Sigma_\geq$ (see (6)). Set

$$\tilde{X}_n(\sigma,t) = \frac{1}{\sqrt{\tilde{D}\,n}} \tilde{S}(\sigma, nt), \qquad 0 \le t \le 1$$

with the identification $\tilde{S}(\sigma, nt) = \tilde{S}(\pi(\sigma), nt)$ and

$$\tilde{D} = \rho\{\xi_0^2\} + 2\sum_{j=0}^{\infty} \rho\{\xi_0\xi_j\}$$

where $\rho(\xi_i) = 0$ by assumption. The exponential uniform mixing property for the random variables $\{\sigma_i\}$, proved in [CI1], and the smoothness hypothesis on $\tilde{u}$ entail that the above series converges absolutely. Moreover the assumption that $\tilde{u}$ is not a cocycle implies that $\tilde{D} > 0$ (see, e.g., [Bo]). Now the result follows from the functional central limit theorem (Donsker's theorem) for dependent variables proved, e.g., in [Bi] page 174. Q.E.D.

This result enables us to prove limit results for various functions of the partial sums $\tilde{S}_n$. In particular, we have the following result:

Lemma 1. *Set $b_n = [\kappa\, a_n]$ where again $a_n = n/\log n$. Then, there are two positve constants C, c such that for any $\epsilon > 0$ and n large enough*

$$\rho\{\, x \in [0,1] \,:\, \max_{|m-b_n|\le \epsilon\, a_n} \left| \tilde{S}_m(x) - \tilde{S}_{b_n}(x) \right| \,<\, \epsilon^{1/4}\sqrt{a_n}\,\} \,\geq\, 1 - C\, \exp\left(-c\,\epsilon^{-1/2}\right)$$

Proof. The proof is a trivial adaptation to the present situation of the argument given in [Bi], Section 10. Q.E.D.

Lemma 2. *There is a constant $\kappa > 0$ such that for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \rho\left(\left\{ x : 1-\epsilon < \frac{N(n,x)}{\kappa\, a_n} < 1+\epsilon \right\}\right) = 1$$

where $a_n = n/\log n$.

Proof. See [CI2], Lemma 3.3. Q.E.D.

Lemma 3.

$$\frac{1}{\sqrt{a_n}} \left(S_n(x) - \tilde{S}_{b_n}(x) \right) \to 0 \quad \text{in } \rho\text{-probability}$$

and therefore, for any $0 \le t \le 1$,

$$\frac{1}{\sqrt{a_n}} \left(S(x, nt) - \tilde{S}(x, b_n t) \right) \to 0 \quad \text{in } \rho\text{-probability}$$

Proof. Let us write (7) in the form

$$S_n(x) = \tilde{S}_{N(n,x)}(x) + R_n(x) \tag{12}$$

with the obvious identifications. Recall that under our assumptions the remainder $R_n(x)$ is uniformly bounded in n and x. Now, from Lemma 2 we have that for any $\epsilon > 0$ and for n sufficently large

$$\rho\{\, x \in [0,1] \,:\, |\, N(n,x) - b_n \,| \,<\, \epsilon\, a_n \,\} \,\geq\, 1 - \epsilon \tag{13}$$

and the statement follows by putting together (12), (13) and Lemma 1. Q.E.D.

An easy consequence of Theorem 1' and Lemma 3 is the following

Lemma 4. *Let* $D = \kappa \tilde{D}$ *and*

$$X_n(x,t) = \frac{1}{\sqrt{D\, a_n}} S(x, nt), \qquad 0 \le t \le 1$$

Then, for any finite sequence $0 \le t_1 \le \ldots \le t_\ell \le 1$ *the random vector*

$$(X_n(x,t_1), \ldots, X_n(x,t_\ell))$$

converge in law to $(B(t_1), \ldots, B(t_\ell))$ *as* $n \to \infty$.

Remark 3. We have proved so far that the finite dimensional distributions of the random element $X_n(x,t)$ converge to those of the Brownian motion $B(t)$. In particular, this implies the validity of the central limit theorem for the random variables $\frac{1}{\sqrt{D\, a_n}} S_n$, that is

$$\lim_{n\to\infty} \rho \{ \frac{1}{\sqrt{D\, a_n}} S_n \le \alpha \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-y^2/2} dy$$

To complete the proof of Theorem 1 it remains to show that the sequence $\{X_n\}$ satisfies a tightness condition [Bi].

Lemma 5. *The sequence of random functions* $\{X_n\}$ *is tight.*

Proof. We shall exploit the tightness of the sequence $\{\tilde{X}_n\}$. Let $\epsilon > 0$ and $\eta > 0$ be fixed. Then, there exist a $\delta > 0$ and an integer n_0 such that

$$\rho \{x : \sup_{0\le t\le 1} \sup_{t \le s \le t+\delta} |\tilde{X}_n(x,t) - \tilde{X}_n(x,s)| \ge \epsilon\} \le \eta \tag{14}$$

for all $n > n_0$.

Let M be a fixed positive integer such that $1/M < \delta$ and let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$ and $|t_1 - t_2| < 1/M$. Define $N(t,x)$ for non integer t as $N(t,x) = N([t],x)$.

Using (12) and the boundedness of the remainders we then find

$$\begin{aligned}|X_n(x,t_1)-X_n(x,t_2)| =& \frac{1}{\sqrt{Da_n}}|S(x,nt_1)-S(x,nt_2)| \\ \leq& \frac{1}{\sqrt{Da_n}}\left(2C+|\tilde{S}(x,N(nt_1,x))-S(x,N(nt_2,x)|\right)\end{aligned} \tag{15}$$

for some constant $C>0$. Let now $1 \leq k_1, k_2 \in \mathbb{Z}_+$ be such that

$$0<\frac{k_1}{M}-t_1 \leq \frac{1}{M}, \qquad 0<\frac{k_2}{M}-t_2 \leq \frac{1}{M}.$$

Clearly $|k_1-k_2| \leq 1$ and

$$N(nt_1,x) \geq N\left(\frac{n(k_1-1)}{M},x\right) \quad \text{and} \quad N(nt_2,x) \leq N\left(\frac{nk_2}{M},x\right)$$

so that

$$N(nt_2,x)-N(nt_1,x) \leq N\left(\frac{nk_2}{M},x\right)-N\left(\frac{n(k_1-1)}{M},x\right)$$

On the other hand, if n is large enough, using Lemma 2 we can estimate the r.h.s. as

$$\begin{aligned}&N\left(\frac{nk_2}{M},x\right)-N\left(\frac{n(k_1-1)}{M},x\right) \leq \\ &(1+\frac{\delta}{4})\left(\frac{\kappa\frac{nk_2}{M}}{\log\frac{nk_2}{M}}\right)-(1-\frac{\delta}{4})\left(\frac{\kappa\frac{n(k_1-1)}{M}}{\log\frac{n(k_1-1)}{M}}\right) \leq \delta\,\kappa n/\log n\end{aligned}$$

Finally, using the above inequality and the tightness of $\tilde{X}_{b_n}$ we get

$$|X_n(x,t_1)-X_n(x,t_2)| \leq \frac{2C}{\sqrt{Da_n}}+C\sup_{0\leq t\leq 1}\sup_{t\leq s\leq t+\delta}|\tilde{X}_{b_n}(x,t)-\tilde{X}_{b_n}(x,s)|$$

which ends the proof. Q.E.D.

Proof of Theorem 1. The proof now follows putting together Lemma 0, Lemma 4 and Lemma 5. Q.E.D.

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