Singular Continuous Spectrum for Palindromic Schrödinger Operators

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Abstract

We give new examples of discrete Schrödinger operators with potentials taking finitely many values that have purely singular continuous spectrum. If the hull X of the potential is strictly ergodic, then the existence of just one potential x in X for which the operator has no eigenvalues implies that there is a generic set in X for which the operator has purely singular continuous spectrum. A sufficient condition for the existence of such an x is that there is a $z \in X$ that contains arbitrarily long palindromes. Thus we can define a large class of primitive substitutions for which the operators are purely singularly continuous for a generic subset in X. The class includes well-known substitutions like Fibonacci, Thue-Morse, Period Doubling, binary non-Pisot and ternary non-Pisot. We also show that the operator has no absolutely continuous spectrum for all $x \in X$ if X derives from a primitive substitution. For potentials defined by circle maps, $x_n = 1_J(\theta_0 + n\alpha)$, we show that the operator has purely singular continuous spectrum for a generic subset in X for all irrational α and every half-open interval J.

1 Introduction

Discrete Schrödinger operators with potentials taking values in a finite set $A \subset \mathbf{R}$ have interesting spectral properties. Topological spaces of such operators are obtained by choosing a compact shift-invariant subset X of the compact metric space $A^{\mathbf{Z}}$. If T denotes the left shift on X, the dynamical system (X, T) is called a subshift. Every point $x \in X$ defines an operator on $l^2(\mathbf{Z})$ by

$$(L(x)u)_n = u_{n+1} + u_{n-1} + x_n u_n.$$

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In $X = A^{\mathbf{Z}}$, any spectral type can occur: the dense set of periodic operators in $A^{\mathbf{Z}}$ have purely absolutely continuous spectrum; for almost all x with respect to any non-trivial product measure on $X = A^{\mathbf{Z}}$ the spectrum is pure point [7]. Hence, by the wonderland theorem [22], there exists a generic set in $A^{\mathbf{Z}}$ for which $L(x)$ has purely singular continuous spectrum. The shift on $A^{\mathbf{Z}}$ has many invariant measures and there are many orbits which are not dense. It is therefore convenient to consider the case of a compact shift-invariant $X \subset A^{\mathbf{Z}}$ that is minimal (i.e., the set of translates of every $x \in X$ is dense in X) and uniquely ergodic (i.e., there exists only one T-invariant measure). A system (X, T) that is both minimal and uniquely ergodic is called strictly ergodic. Two common methods for generating strictly ergodic subshifts are the following:

• A *primitive substitution*, which is a map S from the alphabet A to the set of finite words A^*, defines a fixed point $z^+ \in A^{\mathbf{N}}$. Taking any $z \in A^{\mathbf{Z}}$ satisfying $z_n = z_n^+$ for $n \in \mathbf{N}$ and defining X_S as the set of accumulation points of $\{T^n z \mid n \in \mathbf{N}\}$ gives a strictly ergodic system.

• A *circle map* $\theta \mapsto \theta + \alpha$ with irrational α together with a countable union J of half-open intervals and an initial point $\theta_0 \in \mathbf{T}^1$ defines a sequence $z_n = 1_J(\theta_0 + n\alpha)$. The orbit closure $X_J = X(J, \theta_0, \alpha)$ is independent of θ_0 and is a strictly ergodic dynamical system (see [13] Proposition A.1.)

We do not know of any strictly ergodic aperiodic $X \subset A^{\mathbf{Z}}$ for which some $L(x)$ has a spectral type different from purely singular continuous. The aim of this paper is to extend the set of examples that have purely singular continuous spectrum. We therefore need to exclude absolutely continuous spectrum and eigenvalues.

The belief that singular continuous spectrum is the rule for such aperiodic operators is supported by Kotani's theorem [17] which says that almost all $L(x)$ have no absolutely continuous spectrum. Although the spectrum of $L(x)$ does not depend on $x \in X$ if X is strictly ergodic [13], it does not follow that there is no absolutely continuous spectrum for all $x \in X$. If, however, the Lyapunov exponent exists for all $x \in X$ and is independent of x, then indeed no $L(x)$ has absolutely continuous spectrum as we will see below. In [13] it is shown that the Lyapunov exponent exists uniformly if X is defined by a primitive substitution. Thus we exclude absolutely continuous spectrum for all operators $L(x)$ generated by substitutions.

In order to get singular continuous spectrum one has also to exclude eigenvalues. This was done by a Gordon-type criterion for circle maps [10] and by using the so-called trace map in [8, 24, 25, 4, 1, 11, 2, 5] for circle maps and substitutions. In this paper we note that the criterion of Jitomirskaya and Simon [15] can be used for sequences that we call *strongly palindromic*. A strongly palindromic sequence contains palindromes w_i of length l_i centered at $m_i \to \infty$ such that l_i grows exponentially fast with respect to m_i. We show that if a sequence $x \in X$

contains arbitrary large palindromes (we call this palindromic), then there is an uncountable set in X which is strongly palindromic. Many kinds of sequences defined by circle maps and substitutions are palindromic. In addition, we show that the existence of just one $x \in X$ for which $L(x)$ has no eigenvalues implies that there is a generic set in X for which $L(x)$ has purely singular continuous spectrum.

Our work gives new examples of operators with purely singular continuous spectrum for both substitutions and circle maps.

For circle maps, Delyon and Petritis [10] have shown absence of eigenvalues for almost all α, all intervals $[0, \beta)$ and almost every θ_0. They do not exclude eigenvalues for all irrational α. Eigenvalues have been excluded for all irrational α in the case $\beta = \alpha$ by Bellissard et al. [4], but only for $\theta_0 = 0$. This was proved independently by Sütő [25] in the "golden case" $\alpha = (\sqrt{5}-1)/2$. We prove here that for all irrational α and all intervals $J = [0, \beta)$, there is a generic set in X_J for which $L(x)$ has purely singular continuous spectrum.

All papers on Schrödinger operators with potentials defined by primitive substitutions [1, 2, 11, 5] exclude eigenvalues for just one x; [2, 5] claim that the spectrum is a Cantor set of Lebesgue measure 0. We get purely singular continuous spectrum for a generic set in X_S for a large class of substitutions. Only Bovier and Ghez [5] claim to exclude eigenvalues for a class of substitutions. Our class of substitutions contains all but one of the examples (the "circle sequence") given in [5] and the Bovier-Ghez class does not contain the "period doubling" and the "Binary-Non-Pisot" substitutions, which are contained in ours. (It should be noted, however, that the results in [2, 5] do not apply to the sequences stated in those papers, but to different ones (see the erratum [6]) that are not necessarily in X_S.)

2 Palindromic sequences

Let A be a finite set called *alphabet* and A^* the set of (finite) *words* $w_1 w_2 \cdots w_n$ with $w_i \in A$. A word $w \in A^*$ is called a *palindrome* if it is the same when read backwards. The empty word is considered a palindrome. An element $x \in A^{\mathbf{Z}}$ is called *palindromic* if x contains arbitrarily long palindromes. We say that a word w is *centered* at $(n+m)/2$ in x if $w = x_n \cdots x_m$.

The *shift* T on $A^{\mathbf{Z}}$ is defined by $T(z)_n = z_{n+1}$. The *orbit* $\mathrm{Orb}(z)$ of $z \in A^{\mathbf{Z}}$ is the set $\{T^n z\}_{n \in \mathbf{Z}}$. The closure of $\mathrm{Orb}(z)$ in the product topology is called the *orbit closure* of z and is denoted by X_z. A compact shift-invariant subset X of $A^{\mathbf{Z}}$ is called *minimal* if $X_z = X$ for all $z \in X$.

If X is minimal then every word occuring in some $y \in X$ occurs in all $x \in X$. Thus it makes sense to speak of *palindromic sets* $X \subset A^{\mathbf{Z}}$, compact shift-invariant minimal sets containing one (and therefore only) palindromic sequences. Similarly, X is called aperiodic if the elements in X are aperiodic.

Define $x \in A^{\mathbf{Z}}$ to be *strongly palindromic*, if there exists for every $B > 0$ a sequence w_i of palindromes of length l_i centered at $m_i \to \infty$ such that $e^{Bm_i}/l_i \to 0$. If x is strongly palindromic then it is clearly palindromic.

Proposition 2.1 *If an aperiodic minimal set $X \subset A^{\mathbf{Z}}$ is palindromic, then it contains uncountably many strongly palindromic sequences.*

Proof. Let $\{w^i\}_{i=1}^{\infty}$ be a sequence of palindromes of length $l_i \to \infty$ that occur in the sequences in X. We will construct sequences $y_n \in A^{\mathbf{Z}}$ such that
1) y_n is equal to one of the w^i on an interval I_n containing 0;
2) $I_n \subset I_{n+1}$ and every $m \in \mathbf{Z}$ is eventually in these intervals;
3) y_{n+1} is equal to y_n on I_n;
4) for every y_n, there are two possibilities for y_{n+1} that are different on the intersection of the intervals where they are equal to one of the w^i.
Clearly, every such sequence y_n converges to a different $y \in X$. Since there are infinitely many choices to be made, there are uncountably many different limit points. Conditions will be imposed to make sure that the limit points are strongly palindromic.

Choose B and a sequence of integers m_i such that $m_i \to \infty$ and $e^{Bm_i}/l_i \to 0$ as $i \to \infty$. For every word w there is a $d(w) \in \mathbf{N}$ such that in every $x \in X$ there occurs a copy of w in every interval of length $d(w)$ (see, e.g., [21], p. 71).

Let y_n be equal to w^{j_n} on an interval I_n of length l_{j_n} containing 0. We show how two choices arise for y_{n+1}. There exists an integer i_{n+1} such that $d(w^{j_n}) < m_{i_{n+1}}/2$. This means there is a copy of w^{j_n} in each of the first two subwords of length $m_{i_{n+1}}/2$ to the left of the center of each w^j with $j \geq i_{n+1}$. There exists a $j_{n+1} \geq i_{n+1}$ such that $w^{j_{n+1}}$ is not periodic with any period less than or equal to $m_{i_{n+1}}$ over an interval of length $l_{j_{n+1}} - m_{i_{n+1}}$, for otherwise X would contain a periodic sequence by minimality. The two occurrences of w^{j_n} in those first two subwords to the left of the center of $w^{j_{n+1}}$ give the two possibilities for y_{n+1}. Denote the two possibilities by z and z'. Let I and I' denote the intervals on which z and z' are equal to $w^{j_{n+1}}$, respectively.

We are left to show that z and z' differ on $I \cap I'$. Note that the length of $I \cap I'$ is at least $l_{j_{n+1}} - m_{i_{n+1}}$ and that z' on $I \cap I'$ is equal to z translated by at most $m_{i_{n+1}}$. Now suppose z and z' are equal on $I \cap I'$. Then $w^{j_{n+1}}$ would be periodic with period at most $m_{i_{n+1}}$ over a length of at least $l_{j_{n+1}} - m_{i_{n+1}}$. But j_{n+1} was chosen such that this does not happen.

Note that each limit point of the y_n must be strongly palindromic because $l_{j_n} \geq l_{i_n}$ and because the center of w^{j_n} is within distance m_{j_n} of 0. $\square$

3 Substitution sequences

A *substitution* is a map $S : A \mapsto A^*$; it will be extended to a map $A^* \mapsto A^*$ and $A^{\mathbf{N}} \mapsto A^{\mathbf{N}}$ by concatenation, that is, $S(a_1 a_2 \cdots a_n) = S(a_1)S(a_2)\cdots S(a_n)$. A substitution S is called *primitive* if there exists a k such that for all $a \in A$ the word $S^k(a)$ contains at least one copy of every symbol. Without loss of generality, one can assume that there exists $a \in A$ such that $S(a)$ starts with a (see [13]). Iterating S on a then gives a fixed point $z^+ \in A^{\mathbf{N}}$. Let z be any element in $A^{\mathbf{Z}}$ satisfying $z_n = z_n^+$ for $n \in \mathbf{N}$. The substitution dynamical system (X_S, T) is now defined by

$$X_S = \{x \in A^{\mathbf{Z}} \mid x = \lim_{j \to \infty} T^{n_j} z \quad \text{and} \quad n_j \to \infty\} .$$

It is possible to generate the sequences in X_S directly from the substitution (see, e.g., [9, 12]). The system (X_S, T) is strictly ergodic (see, e.g., [21]). A substitution S is called palindromic if X_S is palindromic.

We say that a primitive substitution S is in the *class* P if there exists a palindrome p and for each $b \in A$ a palindrome q_b such that $S(b) = pq_b$ for all $b \in A$. We allow p to be the empty word; if p is not the empty word then q_b is allowed to be the empty word.

Lemma 3.1 *Class P substitutions are palindromic.*

Proof. We show first by induction that if w is a palindrome, then $S(w) = pu$ with a palindrome u. We can write $w = bvb$ with some $b \in A$ and some palindrome v. We have $S(w) = S(b)S(v)S(b) = pq_bS(v)pq_b$. By induction, $S(v)$ is of the form pu with some palindrome u. We get $S(w) = pq_bpupq_b$ which is of the form $p\tilde{u}$ with palindrome $\tilde{u}$. All words $S^k(a)$ occur in all $x \in X_S$, so the sequences contain arbitrarily long palindromes. $\square$

Corollary 3.2 *If S is class P, then there are uncountably many elements in X_S that are strongly palindromic.*

Proof. Apply 2.1 and 3.1. $\square$

Remarks. 1) It is sufficient that some power of S is of class P in order that S is palindromic.
2) A subclass of class P are substitutions for which $S(b)$ is a palindrome for all $b \in A$. Just take for p the empty palindrome.
3) Clearly, we could include into class P substitutions of the form $S(b) = q_bp$. We do not know whether all palindromic X_S arise from substitutions that are in this extended class P.

4) Not all substitutions are palindromic, as the following example shows:

$$S : a \mapsto abbaaabbba, b \mapsto ab \; .$$

The word *bbaaabbb* occurs in the sequences but the word *bbbaaabb* does not occur.
5) The set of strongly palindromic sequences is invariant and therefore has either measure 0 or 1; but which we do not know. We cannot even exclude that all x are strongly palindromic.

Here are some examples of substitutions in class P that have appeared in the literature on Schrödinger operators:

Thue-Morse: $S^2 : a \mapsto abba, b \mapsto baab$ [18, 1, 11].
Fibonacci: $S : a \mapsto ab, b \mapsto a$ [16, 20, 8, 24, 25, 4, 3, 5].
Period doubling: $S : a \mapsto ab, b \mapsto aa$ [18, 2, 3, 5].
Binary non-Pisot: $S : a \mapsto ab, b \mapsto aaa$ [3, 5].
Ternary non-Pisot: $S : a \mapsto c, b \mapsto a, c \mapsto bab$ [3, 5].

Remarks. 6) The papers [2, 3, 5] implicitly consider the set X_z where z is the symmetric extension of z^+: for $n \in \mathbf{N}$, $z_{-n-1} = z_n^+$ and $z_n = z_n^+$. This is not natural. The system (X_z, T) is, in general, neither uniquely ergodic nor minimal.
7) We do not know if the Rudin-Shapiro sequence ($r_n = (-1)^{f_n}$, where f_n is the number of pairs 11 in the binary expansion of n) is palindromic. Computer experiments suggest it is. A Schrödinger operator with potential based on the Rudin-Shapiro sequence has appeared in [18].

4 Sequences defined by circle maps

Consider the irrational rotation $\theta \mapsto \theta + \alpha$ on the circle $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$. For a half-open interval $J = [0, \beta) \subset \mathbf{T}^1$ and a point θ_0 in $\mathbf{T}^1$, we consider the sequence $z = z(\theta_0)$ given by

$$z_n = 1_J(n\alpha + \theta_0) \in \{0, 1\}^{\mathbf{Z}} \; .$$

We denote by X_J the compact subset of $A^{\mathbf{Z}}$ obtained by taking the closure of the orbit of $z(\theta_0)$. It is independent of θ_0 and the system (X_J, T) is strictly ergodic. This remains true if the half-open interval J is replaced by a countable union of half-open intervals [13].

Lemma 4.1 *For all irrational α and every half-open interval J there exists $\theta_0 \in \mathbf{T}^1$ such that $z(\theta_0)$ is palindromic.*

Proof. If $\beta \neq 2k\alpha$, then the orbit of $\theta_0 = \beta/2$ is disjoint from $\{0, \beta\}$. If $\beta = 2k\alpha \pmod 1$, the orbit of $\theta_0 = (\beta + 1)/2$ is disjoint from $\{0, \beta\}$. By symmetry, these orbits are even and therefore palindromic if they are disjoint from $\{0, \beta\}$. □

Remarks. 1) The lemma can be generalized. Suppose J is a symmetric countable union of half-open intervals with a point of symmetry θ_0. Then if one of the orbits of the points $\theta_0 + (i + j\alpha)/2$ $(i, j \in \{0, 1\})$ does not hit the boundary of J, then that orbit is even, and X_J is palindromic.
2) Another generalization of the lemma can be obtained by taking an ergodic group translation $\theta \mapsto \theta + \alpha$ on $\mathbf{T}^\nu$ and taking $J = [0, \beta_1) \times \ldots \times [0, \beta_\nu)$. The sequence $z_n = 1_J(\theta + n\alpha)$ generates a strictly ergodic set X independent of θ as in the case $\nu = 1$. This set X is palindromic: it can be shown (by induction in ν) that one of the orbits starting at $\theta_0 = (\beta + v)/2$, where $v_i \in \{0, 1\}$, does not hit the boundary of J.
3) There are (in cardinality) more sequences generated by circle maps than sequences generated by substitutions: the length of the interval $[0, \beta)$ measures the average frequency of occurrences of 1 in the sequence z_n. In this way one obtains uncountably many different sequences for different β. On the other hand, there are only countably many substitutions.
4) The circle $\mathbf{T}^1$ is embedded in X_J by $z(\theta) = 1_J(n\alpha + \theta)$ but not every sequence $z \in X_J$ is of this form.

5 The point spectrum

In the preceding three sections A was a finite set of symbols. In the remainder of the paper we map the elements to A to real numbers and again denote the image by A. So from now on sequences $x \in A^{\mathbf{Z}}$ take values in $\mathbf{R}$. Every $x \in A^{\mathbf{Z}}$ defines a discrete Schrödinger operator on $l^2(\mathbf{Z})$ by

$$(L(x)u)_n = u_{n+1} + u_{n-1} + x_n u_n \ .$$

Theorem 5.1 (Jitomirskaya-Simon) *If x is strongly palindromic, then $L(x)$ has no eigenvalues.*

Proof. The proof is essentially [15]. We give a proof in the appendix. □

Remarks. 1) Most (with respect to the product measure) sequences in $A^{\mathbf{Z}}$ are palindromic since almost all sequences contain all possible finite words. But most sequences in $A^{\mathbf{Z}}$ are not strongly palindromic since most sequences give Schrödinger operators with Anderson localization [7] and strongly palindromic sequences give operators without eigenvalues.
2) In the golden case treated in [25, 4] the sequence x is symmetric around 0. The sequence in [11] is antisymmetric around 0. We do not know whether these sequences are strongly palindromic, so we can not exclude eigenvalues for those particular sequences, although in both cases X is palindromic.

6 The absolutely continuous spectrum

Theorem 6.1 *If the dynamical system (X,T) has the property that the Lyapunov exponent $\lambda_x(E) = \lambda(E)$ of $L(x)$ exists for all $x \in X$ and is independent of $x \in X$, then $L(x)$ has no absolutely continuous spectrum for all $x \in X$.*

Proof. For all $E \notin \sigma(L)$ or $E \in \sigma(L(x))$ with $\lambda(E) > 0$, there exist two solutions $u^{\pm} \in R^{\mathbf{Z}}$ of $Lu = Eu$ which are in $l^2(\pm\mathbf{N})$. By Kotani's result [17] and the Lyapunov assumption, this is the case for a set $Y \subset \mathbf{R}$ of full measure that is independent of x. The vectors $\Phi^{\pm}(n) = (u^{\pm}(n+1), u^{\pm}(n))$ satisfy $A_E(n)\Phi^{\pm}(n) = \Phi^{\pm}(n+1)$. If $\Phi^+ \neq \Phi^-$, the Titchmarsh-Weyl functions $m_E^{\pm}(n) = u^{\pm}(n+1)/u^{\pm}(n)$ are different and real (but they may be infinite). If $\Phi^+ = \Phi^-$, E is an eigenvalue. The resolvent $(L-E)^{-1}_{nn} = G_E(n,n)$ of L satisfies $G_E(n,n) = 1/(m_E^+(n) - m_E^-(n))$ and is the Borel transform of the spectral measure dk_{e_n}. The absolutely continuous part of dk_{e_n} is by Fatou's theorem given by $\mathrm{Im}(G_{E+i0}(n,n)) = \mathrm{Im}(1/(m^+_{E+i0}(n) - m^-_{E+i0}(n)))$, which is zero for all $E \in Y$ which are not eigenvalues. $\square$

Remark. A different proof can be obtained by using a result of Berezanskii (see [23]): if μ_ϕ is any spectral measure of any discrete Schrödinger operator L then for μ_ϕ-almost every $E \in \mathbf{R}$ there exist polynomially bounded solutions of $Lu = Eu$. If there is an $x \in X$ such that $L(x)$ has absolutely continuous spectrum then it has an absolutely continuous spectral measure μ_ϕ supported on a set A of positive Lebesgue measure. For μ_ϕ-a.e. $E \in A$ — which is equivalent to Lebesgue a.e. E — the equation $L(x)u = Eu$ has polynomially bounded solutions. So $\lambda_x(E) = 0$ for those E. But we assumed that $\lambda_x(E)$ does not depend on x and it is strictly positive Lebesgue almost everywhere by Kotani [17]. Thus the assumption that $L(x)$ has absolutely continuous spectrum leads to a contradiction.

Corollary 6.2 *If (X_S,T) is a substitution dynamical system, then $L(x)$ has no absolutely continuous spectrum for all $x \in X_S$.*

Proof. ¿From [13] we know that substitution sequences give operators with Lyapunov exponents independent of $x \in X_S$. $\square$

Remarks. 1) Another way to exclude absolutely continuous spectrum for all x is to show that the spectrum has Lebesgue measure zero and is independent of x. This was done in [4] for circle maps with $\alpha = \beta$. Circle maps give spectrum independent of x for all values of α and β [13]. Kotani's theorem [17] does not imply that the spectrum has Lebesgue measure 0; a singular spectrum can have positive Lebesgue measure. We do not know whether the Lyapunov exponent is independent of x for sequences generated by circle maps.

2) One might guess that if (X,T) is a strictly ergodic dynamical system and $V : X \to \mathbf{R}$ is continuous, then the absolutely continuous spectrum of the operator $(L(x)u)_n = u_{n+1} + u_{n-1} + V(T^n x)u_n$ is independent of $x \in X$. We have shown that this is the case for dynamical systems (X_S, T) obtained by substitutions if $V(T^n x) = x_n$. It is also true for substitution dynamical systems if $V(x)$ only depends on $x_{-l} \cdots x_l$, because then the Lyapunov exponent remains independent of $V(x)$ by Proposition 5.2 in [13]. It was already known for circle maps with $\alpha = \beta$ [4] and $V(T^n x) = x_n$.
3) Assume that (X,T) is strictly ergodic, that $\lambda_x(E)$ exists for all $x \in X$ and is independent of x and that there exists $x \in X$ such that $L(x)$ has no eigenvalues. Then it follows for example from the theorem of de la Vallée Poussin that the spectrum of $L(x)$ is the closure of $\{E \mid \lambda_x(E) = 0\}$. We expect (but can't prove) that the spectrum of $L(x)$ is equal to $\{E \mid \lambda_x(E) = 0\}$ from which it would follow that it is a Cantor set of zero Lebesgue measure. The spectrum has been shown to be equal to the set $\{E \mid \int d\mu(x)\lambda_x(E) = 0\}$, where the integration is with respect to the invariant measure on X, for circle maps with $\alpha = \beta$ in [4]. It has been shown that spec(L(v)) $= \{E \mid \lambda_v(E) = 0\}$ for a sequence v defined by the Period Doubling substitution in [2] and this result was extended to a class of substitutions including Period Doubling in [5].

7 The singular continuous spectrum

Theorem 7.1 (Simon) *Let (X,d) be a complete metric space of selfadjoint operators on a Hilbert space H such that the metric d is stronger than the strong resolvent convergence. Then both of the sets $\{A \in X \mid \sigma_{pp} = \emptyset\}$, $\{A \in X \mid \sigma_{ac} = \emptyset\}$ are G_δ's, countable intersections of open sets.*

See [22].

Corollary 7.2 *Let $X \subset A^{\mathbf{Z}}$ be strictly ergodic. If there is one $x \in X$ such that $L(x)$ has no eigenvalues, then there exists a generic subset in X on which L is purely singular continuous.*

Proof. By Kotani [17] there is a set of full measure with no absolutely continuous spectrum. By strict ergodicity, this set is dense and by Theorem 7.1 it is a dense G_δ. Since Orb(x) is dense by minimality, there is dense set on which L has no eigenvalues. Again, this set is a dense G_δ by Theorem 7.1. Since X is a complete metric space, the intersection of two dense G_δ's is a dense G_δ by the Baire category theorem. □

By 5.1, 2.1, 3.1 and 4.1 we get:

Corollary 7.3 *If X is strictly ergodic and palindromic, then there is a generic $Y \subset X$ such that for $x \in Y$ the operator $L(x)$ has purely singular continuous spectrum.*

Corollary 7.4 (Case of substitutions) *For substitutions of class P, there exists a generic set Y in X_S such that for $x \in Y$ the operator $L(x)$ has purely singular continuous spectrum.*

Corollary 7.5 (Case of circle maps) *For all irrational α and every half-open interval J, there exists a generic set Y in X_J such that for $x \in Y$ the operator $L(x)$ has purely singular continuous spectrum.*

Remark. Strictly ergodic palindromic systems other then those defined by circle maps and class P substitutions can be constructed from so-called Toeplitz sequences [14, 19].

8 Appendix: The criterion of Jitomirskaya and Simon

For the convenience of the reader we repeat the proof of the theorem of Jitomirskaya and Simon [15] for the case of potentials taking finitely many values.

Theorem 8.1 (Jitomirskaya-Simon) *Let $x \in A^Z$, where $A \subset \mathbf{R}$ is finite. Consider the operator on $l^2(Z)$ given by $(Lu)(n) = u(n+1)+u(n-1)+x_n u(n)$. Suppose there is a sequence of intervals I_i of length $|I_i|$ centered at m_i such that x is a palindrome on I_i, $0 \in I_i$ and $|I_i| \to \infty$ as $i \to \infty$. Then there exists a constant C depending only on A such that L has no eigenvalues if $C^{m_i}/|I_i| \to 0$ as $i \to \infty$.*

Proof. We can assume that $|I_i|$ is either odd or even for all i. We prove the theorem for the case that $|I_i|$ is odd and then indicate what changes if it is even. So first let $I_i = \{m_i - l_i, m_i + l_i\}$.

Assume $Lu = Eu$ with $||u|| = 1$. Define $u_i(n) = u(2m_i - n)$, the vector reflected at m_i. Let $W(u, u_i)(n) = u(n+1)u_i(n) - u_i(n+1)u(n)$ be the Wronskian of u and u_i. Let $\Phi(n) = (u(n+1), u(n))$ and $\Phi_i(n) = (u_i(n+1), u_i(n))$ and $u_i^{\pm} = u \pm u_i$ and $\Phi_i^{\pm} = \Phi \pm \Phi_i$.

Step 1. $W(n) := W(u, u_i)(n)$ is constant for $n \in I_i$.
Proof. Compute using $u(n+1) = -u(n-1) + (E - x_n)u(n)$ and $u_i(n+1) = -u_i(n-1) + (E - x_n)u_i(n)$

$$W(n) - W(n-1) = (x_{2m_i - n} - x_n)u(2m_i - n)u(n) = 0 .$$

Step 2. $|W(m_i)| \leq 2/|I_i| = 2/(2l_i+1)$.
Proof. $\sum_n |W(n)| \leq 2$ by Schwarz inequality and $||u|| = ||u_i|| = 1$. Use Step 1.

Step 3. Either (i) $|u_i^+(m_i)| \leq 2|I_i|^{-1/2}$ or (ii) $|u_i^-(m_i+1)| \leq 2|I_i|^{-1/2}$.
Proof. Since $u_i^-(m_i) = 0$, one has

$$u_i^+(m_i)u_i^-(m_i+1) = W(u_i^-, u_i^+)(m_i) = 2 \cdot W(u, u_i)(m_i) = 2 \cdot W(m_i)$$

Use Step 2.

Step 4. There exists a constant C_1 such that $\|\Phi_i^\pm(m_i)\| \leq C_1|I_i|^{-1/2}$ holds for all i, with either the sign $+$ or $-$.
Proof. In case (ii), we have $||\Phi_i^-(m_i)|| = ||(u_i^-(m_i+1), 0)|| \leq 2|I_i|^{-1/2}$. Assume now case (i). We have $u_i(m_i) = u(m_i)$ and $u_i(m_i-1) = u(m_i+1)$, rsp. $u_i(m_i+1) = u(m_i-1)$ so that $u_i^+(m_i-1) = u_i^+(m_i+1)$. Therefore $u_i^+(m_i+1) = \frac{1}{2}(E - x_{m_i})u_i^+(m_i)$ so that $\|\Phi^+(m_i)\| \leq C_1'|u^+(m_i)| \leq C_1 \cdot |I_i|^{-1/2}$.

Step 5. For some $C > 0$, $\|\Phi_i^\pm(0)\| \leq C^{m_i} C_1 \cdot |2l_1+1|^{-1/2}$, for either sign $+$ or sign $-$.
Proof. There exists C such that $||A(n)|| \leq C$ for all transfer matrices $A(n)$, for all E in the spectrum of L. Denote by A_i the transfer matrix that maps vectors at m_i to vectors at 0. In particular, because x is exactly symmetric on I_i,

$$A_i\Phi(m_i) = \Phi(0), \qquad A_i\Phi_i(m_i) = \Phi_i(0)\ .$$

Then also

$$\Phi_i^\pm(0) = A_i\Phi_i^\pm(m_i)$$

so that

$$||\Phi_i^\pm(0)|| \leq C^{m_i} C_1 \cdot |2l_1+1|^{-1/2}\ .$$

Step 6. Proof of the theorem.
By Step 5,

$$\big|\|\Phi(0)\| - \|\Phi(2m_i)\|\big| \leq \|\Phi^\pm(0)\| \to 0$$

for suitable choices of $+$ and $-$ as $i \to \infty$. Since $\|\Phi(m_i)\| \to 0$ as $m_i \to \infty$ because $u \in l^2$, it follows that $\Phi(0) = 0$. But then $u = 0$, contradicting $\|u\| = 1$.

If I_i is even for all i we can put $I_i := \{m_i - l_i, \dots, m_i, m_i+1, \dots, m_i + l_i + 1\}$. Define $u_i(n) = u(2m_i - n + 1)$, the at $m_i + 1/2$ reflected vector. We have $x_{2m_i-n+1} = x_n$ on I_i.

Step 1 and Step 2 do not change.
In Step 3, the claim stays the same. Note that $u_i^-(m_i) = 0$ is no longer true in general. However, the claim follows from

$$W(u_i^-, u_i^+) = 2u_i^+(m_i)u_i^-(m_i+1)$$

(even with a factor 2 deleted), which is a consequence of $u_i^+(m_i)u_i^-(m_i+1) = -u_i^+(m_i+1)u_i^-(m_i)$.
Step 4. In case (ii) we take

$$\|\Phi_i^+(m_i)\| = ||(u_i^+(m_i+1), u_i^+(m_i))|| \leq \sqrt{2} \cdot |u_i^+(m_i)|$$

since $u_i^+(m_i) = u_i^+(m_i+1)$. Case (i) with a different C_1 follows from $u_i^-(m_i) = -u_i^-(m_i+1)$

Step 5. Define A_i by $A_i\Phi(m_i) = \Phi(0)$. Because x is symmetric on I_i, we have

$$A_iA(m_i)\Phi(m_i) = A_i\Phi(m_i+1) = \Phi(2m_i+1) = \Phi_i(0) \ .$$

Together

$$\|\Phi_i^{\pm}(0)\| = \|\Phi(0) \pm \Phi_i(0)\| \leq 2C^{m_i+1}\|\Phi^{\pm}(m_i)\|$$

for either the sign $+$ or the sign $-$.

Step 6 is the same (replace $\Phi(2m_i)$ by $\Phi(2m_1+1)$). □

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