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ANOMALOUS ANGULAR MOMENTUM FLUCTUATIONS FOR AN IDEAL BOSE GAS IN A ROTATING BUCKET

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Abstract

An equilibrium superfluid is usually modelled by a Bose gas below its transition temperature. Here we compute rigorously the deviation from normality of the angular momentum fluctuations, and prove that the extremal phases, or the extremal equilibrium state components, show normal behaviour, except for special boundary conditions.

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Blatt and Butler [1] showed that an ideal Bose - Einstein gas below its transition point is an equilibrium superfluid in the sense that it is a substance whose moment of inertia in thermal equilibrium is less than its classical value (see also [2]).

Blatt, Butler and Schafroth [3] considered the rotating bucket model and discussed the question whether the effective moment of inertia is the same for classical as well as for quantum statistical mechanics. By heuristic computations they showed that this holds above the transition point, but not below. This result got a rigorous derivation in [4] for quantum statistical interacting systems with an equilibrium state satisfying suitable clusterproperties, i.e. at temperatures high enough.

Now the moment of inertia is directly related to the fluctuation of the angular momentum (see e.g. [3,4]).

In this letter we report on rigorous results about the moment of inertia, its deviations from its classical behaviour below the transition point in the model of the rotating bucket. We compute rigorously the angular momentum fluctuations. We start from the rigorous study of the average angular momentum in the equilibrium state of this model given by Lewis and Pulé [5]. They study in much detail the thermodynamic limit of the equilibrium state of the free Boson gas in a rotating bucket given by the following Hamiltonian. Let Λ^1 be the cylindrical region of unit volume in $\mathbb{R}^3$:

$$\Lambda^1 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < a^2; \mid x_3 \mid < (2\pi a^2)^{-1}\} \tag{1}$$

and for each $L > 0$, let Λ^L be the region :

$$\Lambda^L = \{x \in \mathbb{R}^3 : L^{-1}x \in \Lambda^1\} \tag{2}$$

then :

$$H_L - \Omega_L J_L - \mu_L N_L = \int\limits_{\Lambda^L} dx \frac{1}{2} \nabla a^+(x) \nabla a(x) - \Omega_L J_L - \mu_L N_L \tag{3}$$

where J_L is the third component of the angular momentum, Ω_L the angular velocity of the system, μ_L the chemical potential and N_L the number operator, of course $a^{\pm}(x)$ are boson creation and annihilation operators.

The main contribution of [5] consists in proving rigorously that the thermodynamic limit $L \to \infty$ makes sense requiring the constraints of fixed particle density $\overline{\rho}$ and average angular momentum $\overline{\lambda}$ such that for all volumes Λ^L :

$$\langle N_L \rangle_L = \overline{\rho} L^3, \quad \langle J_L \rangle_L = \overline{\lambda} L^3 \tag{4}$$

where :

$$\langle A \rangle_L = \frac{Tr\left(e^{-\beta(H_L - \Omega_L J_L - \mu_L N_L)} A\right)}{Tr\left(e^{-\beta(H_L - \Omega_L J_L - \mu_L N_L)}\right)} \tag{5}$$

These constraints determine the L-dependence of the chemical potential μ_L and the angular velocity Ω_L. The constraints (4) are written as follows in the limit $L \to \infty$:

$$\overline{\rho} = \lim_{L \to \infty} \frac{\langle N_L \rangle_L}{L^3}; \quad \overline{\lambda} = \lim_{L \to \infty} \frac{\langle J_L \rangle_L}{L^3} \tag{6}$$

In [5], using analogous techniques as in [6], it is proved that these limits exist and they analyse the condensation problem. Let

$$z \to g_\alpha(z) = \sum_{n=1}^{\infty} n^{-\alpha} z^n; \quad \alpha > 1, \quad z \in [0,1] \tag{7}$$

The function $z \to (2\pi\beta)^{-\frac{3}{2}} g_{\frac{3}{2}}(z)$ is continuous on $[0,1]$ and increases monotonically to a maximum ρ_c at $z = 1$ so that for $\overline{\rho} \leq \rho_c$ the equation :

$$\overline{\rho} = (2\pi\beta)^{-\frac{3}{2}} g_{\frac{3}{2}}(z) \tag{8}$$

has a unique root $z(\overline{\rho})$. If $\overline{\rho} > \rho_c$ then :

$$\overline{\rho} = (\overline{\rho} - \rho_c) + \rho_c \tag{9}$$

The quantity $\overline{\rho} - \rho_c$ represents the condensate density. It is also proved [5], that for the rotating bucket the condensation takes place in the two lowest energy levels which we denote simply by ϵ_1 and ϵ_2, such that :

$$\overline{\rho} - \rho_c = \rho_{0,1} + \rho_{0,2} \tag{10}$$

where

$$\rho_{0,i} = \lim_{L\to\infty} \frac{\langle n_i \rangle}{L^3}; \quad i = 1,2 \quad n_i = \text{ the number operator for the level } \epsilon_i \tag{11}$$

Consequently the formula (6) for the angular momentum can be written as follows :

$$\overline{\lambda} - \lambda_c = \lambda_{0,1} + \lambda_{0,2} \tag{12}$$

where $\lambda_{0,i}$ represents the contribution to the angular momentum from the condensate in the level i; depending on the values of $\overline{\lambda}$ and $\overline{\rho}$, one or both of the $\lambda_{0,i}$ are different from zero. In fact $\overline{\lambda}$ and $\overline{\rho}$ are coupled through the formulae (6).

Now we turn to the fluctuation of the angular momentum I_δ. Define the fluctuation by:

$$I_\delta = \lim_{L\to\infty} \frac{1}{L^{6\delta}} \langle \frac{(J_L - \overline{\lambda} L^3)^2}{L^3} \rangle_L \tag{13}$$

Now we solve the following problem : Find the value of $\delta \in [0, \frac{1}{2}]$ such that I_δ exists and is not trivial, i.e. such that $0 < I_\delta < \infty$. The parameter δ indicates at which level the fluctuations appear.

If $\overline{\rho} < \rho_c$, i.e. in the absence of condensate or under good clusterconditions, or short range correlations it is given in the thermodynamic limit [3,4] by

$$I_{\frac{1}{3}} = \lim_{L\to\infty} \frac{1}{L^2} \langle \frac{(J_L - \overline{\lambda} L^3)^2}{L^3} \rangle_L \tag{14}$$

i.e. $\delta = \frac{1}{3}$, this value of δ refers to the normal situation coinciding with the classical case. Any other value of δ , this is $\delta \neq \frac{1}{3}$, which can be found in particular if $\overline{\rho} > \rho_c$, refers to an abnormality of angular momentum fluctuations. Let us now examine the case $\overline{\rho} > \rho_c$, i.e. in the presence of condensate, in the so-called superfluid region.

If one computes for $\overline{\rho} > \rho_c$ straightforwardly the angular momentum fluctuation in the equilibrium state $\langle . \rangle_L$, then one finds that $\delta = \frac{1}{2}$ is the solution of the problem, posed in (13). We get a discontinuous jump in δ from $\delta = \frac{1}{3}$ for $\overline{\rho} < \rho_c$ to $\delta = \frac{1}{2}$ for $\overline{\rho} > \rho_c$. This is due to the abrupt appearance of long range order. Also, $\delta = \frac{1}{2}$ is the rigorous expression for the heuristic discussions given in [3] and [4].

On the other hand, the abnormality $\delta = \frac{1}{2}$, is a typical value corresponding to the computation of a mean square deviation in a mixed state i.e. a convex combination of extremal equilibrium states. In this free Boson gas model, of course, we are in this situation. If there is condensation it is well known that there is spontaneous breaking of the gauge symmetry, and the equilibrium state is the integral over the equilibrium states with fixed gauge. The intrinsic properties of the equilibrium states are situated in these gauge breaking extremal states. Therefore also the study of the deviation from normality of the angular momentum fluctuation should refer to these gauge breaking states. That is exactly what we report on in the following.

In particular we use the standard technique to recover the gauge breaking extremal equilibrium states. One considers the Hamiltonian :

$$H_L - \Omega_L J_L - \mu_L N_L + \overline{h}\sqrt{L^3} a(f_L) + h\sqrt{L^3} a^+(f_L) \tag{15}$$

where H_L is given by (3),

$$a^{\pm}(f_L) = \int_{\Lambda} dx f_L(x) a^{\pm}(x)$$

and f_L is any element of $L^2(\Lambda)$ which has nonvanishing components for the condensate levels (10). The presence of the factor $\sqrt{L^3}$ is to make the perturbation extensive and the complex number h is an external field which tends to zero after having taken the thermodynamic limit. Here we differ from this procedure in the sense that we let h tend to zero following a power law in the volume, we take $h = h_L$

$$h_L = \frac{h_1}{L^{3\alpha}} \quad \text{with} \quad \alpha > 0, \quad h_1 \text{ fixed.} \tag{16}$$

This way of proceeding is treating the perturbation as a boundary condition. Now we compute the parameter δ using the state :

$$\langle A \rangle_L^{h_L} = \frac{Tr(e^{-\beta(H_L - \Omega_L J_L - \mu_L N_L + (\overline{h}\sqrt{L^3} a(f_L) + h\sqrt{L^3} a^+(f_L))} A)}{Tr\left(e^{-\beta(H_L - \Omega_L J_L - \mu_L N_L + \overline{h}\sqrt{L^3} a(f_L) + h\sqrt{L^3} a^+(f_L))}\right)} \tag{17}$$

instead of (5). One finds the following remarkable results which hold true in both situations of condensation in one or two levels :

$$\begin{cases} \text{if} \quad 0 < \alpha < 1 & \text{then} \quad \delta = \max\{\frac{1}{3}, \frac{\alpha}{2}\} \\ \text{if} \quad \alpha > 1 & \text{then} \quad \delta = \frac{1}{2} \end{cases}$$

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There are a number of remarks to be made, explaining this result. In general the parameter δ depends on the boundary conditions i.e. on the parameter α; this phenomenon is found also in other models [7,8,9], as well in quantal as classical ones; this effect is seen only in the case when the external field drops off quickly enough i.e. if $1 > \alpha \geq \frac{1}{3}$. However if the field tends to zero too fast with the volume i.e. if $\alpha \geq 1$, the effect of the boundary condition is nonexistent and the limit state

$$\lim_{L \to \infty} \langle . \rangle_L^{h_L} \tag{18}$$

behaves as the mixed, nonextremal, equilibrium state, yielding the parameter value $\delta = \frac{1}{2}$. On the other hand, if α is small, i.e. $\alpha < \frac{2}{3}$, or the external field vanishes very slowly, then the field forces the system into an extremal phase; from the thermodynamic point of view the state of the system is in a minimum of an harmonic potential, the distribution of the angular momentum fluctuation is Gaussian and normal (for details about the distribution see [10]), yielding $\delta = \frac{1}{3}$, the same value as in the case of low density $\overline{\rho} < \rho_c$, or as in the classical systems.

So far the discussions of the results. It is clear that *non - classical* behaviour appears only in the case, $\frac{2}{3} < \alpha < 1$, and that it is solely due to the boundary conditions. Disregarding the effect of this external field, the angular momentum fluctuations do not show any abnormal behaviour passing the critical line $\overline{\rho} = \rho_c$. These rigorous results cast serious doubts on the ideal Bose - Einstein gas being a good model for superfluidity under *all circumstances*. This confirms other reservations in this direction which were made in the literature (e.g. already in [1]). The detailed proofs of the above results will appear in a more elaborate paper [10].

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