Scattering on a finite chain of vortices *

PAVEL ŠŤOVÍČEK

Abstract

This problem is related to the Aharonov–Bohm effect. The Hamiltonian $\hat{H}$ in $L^2(\mathbb{R}^2)$ is defined as a self–adjoint extension of the symmetric operator $X = -\Delta$, with $D(X) = C_0^\infty(\mathbb{R}_+^2) \oplus C_0^\infty(\mathbb{R}_-^2)$, determined by boundary conditions on the first coordinate axis. There is proven a perturbative formula for the inverted operator $[(-\partial_u^2+1)^{1/2} + \exp(-2\pi i\nu(u))\,(-\partial_u^2+1)^{1/2}\exp(2\pi i\nu(u))]^{-1}$ in $L^2(\mathbb{R}, du)$, with $\nu(u)$ being a piecewise constant function related to the boundary conditions. This result jointly with the Krein's formula enables one to construct generalized eigen–functions of $\hat{H}$ starting from the explicit form of the unitary mapping between the deficiency subspaces of X defining the self–adjoint extension. These formulae also make it possible to prove existence and completeness of the wave operators using the trace class methods.

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1 Introduction

The vortices mentioned in the title should be interpreted as intersections of the plane with magnetic fluxes concentrated in parallel lines. So the problem is related to the Aharonov–Bohm effect [1]. While the configuration with one vortex enjoys the rotational symmetry and enables an explicit solution by separation of variables (a very brief summery is given in the beginning of Sec. 5.1) the case of two and more vortices is considerably more complicated. Another feature worth of mentioning is that this problem differs somewhat from the usual potential scattering as the Hamiltonian involves only gauge field and no potential. As explained in [2] instead of the formulation employing a vector potential the Hamiltonian can be defined by boundary conditions on a cut in the plane. Here, the first coordinate axis serves as the cut since it contains, by our choice, all the vortices.

The aim of this paper is to study this problem in the framework of the mathematical scattering theory. This means to compare quantum time evolution governed by the given Hamiltonian with that governed by the free Hamiltonian. The two Hamiltonians don't differ by a potential but, on the other hand, they are self–adjoint extensions of the same symmetric operator. It is clear that the methods known for potential scattering should be modified or supplied by additional approaches. In view of the relation between the Hamiltonians, the Krein's formula suggests itself as a powerful tool, particularly in connection with the Kato–Birman theory. To be able to apply this approach effectively one has to determine the deficiency subspaces $\mathcal{N}(z)$ and to transcribe the boundary conditions into the unitary mapping $V(z):\mathcal{N}(z)\to\mathcal{N}(\bar{z})$ defining the self–adjoint extension.

Evaluating $V(z)$ one is confronted with the following problem. Denote temporarily by $\hat{X}$ and $\hat{P}$ the coordinate and momentum operators in $L^2(\mathbb{R})$, $[\hat{X},\hat{P}]=i\mathbf{I}$. Let $\nu:\mathbb{R}\to\mathbb{R}$ be a measurable function. The operator

$$\Xi^{-1}=[(\hat{P}^2+\mathbf{I})^{1/2}+\exp(-2\pi i\,\nu(\hat{X}))\,(\hat{P}^2+\mathbf{I})^{1/2}\exp(2\pi i\,\nu(\hat{X}))]^{-1}$$

is manifestly positive and bounded. If $\nu\equiv 0$ then $\Xi_0^{-1}=(1/2)\,(\hat{P}^2+\mathbf{I})^{-1/2}$. It is desirable to have some information about $\Xi^{-1}-\Xi_0^{-1}$. Perhaps the simplest and most straightforward way is to assume that $\nu\approx 0$ and hence $\exp(2\pi i\,\nu(\hat{X}))$ is close to $\mathbf{I}$ and to attempt a perturbative expansion. Most probably, this naive approach would be possible provided ν is, say, smooth and compactly supported. Unfortunately, this is not the case we are interested in. Our ν is related to the boundary conditions and is piecewise constant with finitely many discontinuities. Nevertheless, it turns out that a perturbative formula exists even in this case though in a much less trivial manner. One has to tensor the Hilbert space $L^2(\mathbb{R})$ with the finite–dimensional space $\mathbb{C}^N\otimes\mathbb{C}^2$, with N being the number of discontinuities, and to introduce some auxiliary operators $R^{(j)}$ in $\mathbb{C}^N\otimes\mathbb{C}^2\otimes L^2(\mathbb{R})$, one for each discontinuity.

Concerning the history, the author derived an expression for Ξ^{-1} in [3], but in a bit heuristic manner and in a form of an infinite series. The approach was based on some geometric ideas going back to Schulman [4] and making use of the existence of the universal covering space for $\mathbb{R}^2\setminus\{a_1,\ldots,a_N\}$. Afterwards he learned that the same formula can be extracted with some effort also from the results due to Sato, Miwa and Jimbo and thus obtained by a completely different method. These authors developed a powerful machinery applicable to various problems involving some holonomy condition. After having studied rotations in finite–dimensional Clifford algebras [5] they transferred the results formally also to infinite–dimensional spaces. In connection to our problem, a formula is of interest expressing the Green function of the operator $-\Delta + w^2$, $w > 0$, [6]. The Laplacian is defined on $\mathbb{R}^2 \setminus \{a_1, \ldots, a_N\}$ and its definition involves some holonomy condition. By comparing with the Krein's formula one can extract from this result the expression for Ξ^{-1}, this time in a closed form.

In the current paper, there is given a direct verification of this perturbative formula. The advantage is that this way one avoids completely the heuristic ideas present in any of the two approaches mentioned above though, on the other hand, the origin of the formula remains somewhat dimmed. Furthermore, the analysis given in [6] is not fully appropriate in the situation considered here and is done from the very beginning and based on different principles.

Let us point out another difference if compared with the potential scattering. Instead of using the Lippman-Schwinger equation, one can gain generalized eigen–functions of the Hamiltonian $\hat{H}$ as follows. Suppose that one can choose $\varphi(z) \in \mathcal{N}(z)$ depending on the spectral parameter $z \in \mathbb{C} \setminus [0, +\infty)$ in such a way that the function $\psi_+(z) = \varphi(z) + V(z)\varphi(z)$ admits the limit $z = k^2 - i\sigma$, $\sigma \downarrow 0$. Since $\hat{H}(\varphi(z) + V(z)\varphi(z)) = z\,\varphi(z) + \bar{z}\,V(z)\varphi(z)$ the limiting function $\psi_+(k^2 - i0)$ is expected to be a generalized eigen–function. The perturbative formula plays a key role in this construction of generalized eigen–functions, announced in a less formal manner in [7]. Besides, this formula also enables one to apply the trace class methods to verify existence and completeness of the wave operators.

Let us summarize that the main goals of the paper are a precise verification of the perturbative formula and a presentation of a construction of generalized eigen–functions of $\hat{H}$ giving a stationary picture of the scattering. The paper is organized as follows. In Section 2, the Hamiltonian $\hat{H}$ acting in $L^2(\mathbb{R}^2)$ is defined and its point spectrum is shown to be empty. Section 3 is devoted to the transcription of the boundary conditions into the unitary mapping $V(z) : \mathcal{N}(z) \to \mathcal{N}(\bar{z})$ between the deficiency subspaces. Section 4 is concerned with the perturbative formula for Ξ^{-1}. It is divided into two subsections. In the first one, some auxiliary operators are introduced and studied, in the second one, the formula is stated and proven. An application of the perturbative formula jointly with the Krein's formula to the investigation of the scattering problem itself is presented in Section 5, also divided into two subsections. Existence and completeness of the wave operators for the considered scattering problem are proven in the framework of the Kato–Birman theory in Subsection 5.1. Subsection 5.2 presents a construction of the generalized eigen–functions $\psi_+(k^2 - i0)$. It is also shown that the wave operator W_- sends the plane wave ψ_0 to $\psi_+(k^2 - i0)$.

2 The Hamiltonian

Let us begin with a short comment about notation and conventions. The scalar product $\langle\cdot,\cdot\rangle$ in a complex Hilbert space is linear in the second variable. Throughout the paper we exploit frequently the fact that, by the Schwartz's kernel theorem, at least all bounded operators in $L^2(I,dx)$, with $I \subset \mathbb{R}$ being an interval, are determined by their generalized kernels. In accordance with the physical convention we denote by $\langle x|A|y\rangle$ the kernel corresponding to an operator A. The Fourier transform is normalized to a unitary operator. We set, as usual, $\hat{\psi}(x) = \mathcal{F}\psi(x) = (2\pi)^{-1/2}\int_{\mathbb{R}} dy\, e^{-ixy}\,\psi(y)$ and $\check{\psi} = \mathcal{F}^{-1}\psi$. The symbol ϑ stands for the Heaviside step function.

Now let us introduce the Hamiltonian $\hat{H}$ acting in $L^2(\mathbb{R}^2)$. Suppose we are given a measurable real function $\nu(u)$ defined on $\mathbb{R}$. Denote by $\mathbb{R}^2_\pm$ the upper (resp. lower) open halfplane in $\mathbb{R}^2$. Let $\mathcal{Q}_\nu$ be the closed subspace in the orthogonal sum of Sobolev spaces $\mathcal{H}^1(\mathbb{R}^2_+)\oplus\mathcal{H}^1(\mathbb{R}^2_-)\subset L^2(\mathbb{R}^2)$ determined by the boundary condition

$$\psi(u,0_-) = e^{2\pi i\,\nu(u)}\psi(u,0_+)\,. \tag{2.1a}$$

The restriction to $\mathcal{Q}_\nu$ of the scalar product in the sum of Sobolev spaces yields a closed form $\mathfrak{q}_\nu$ in $L^2(\mathbb{R}^2)$ fulfilling $\mathfrak{q}_\nu(\psi,\psi) \geq \|\psi\|^2$. Hence by a standard construction, one can relate to $\mathfrak{q}_\nu$ a positive self–adjoint operator $\hat{H}_\nu$; $\mathfrak{q}_\nu(\psi,\psi) = \langle\psi,\hat{H}_\nu\psi\rangle + \langle\psi,\psi\rangle$, for $\psi\in D(\hat{H}_\nu)$. It is easy to see that the intersection of $D(\hat{H}_\nu)$ with the subspace $\mathcal{H}^2(\mathbb{R}^2_+)\oplus\mathcal{H}^2(\mathbb{R}^2_-)\subset L^2(\mathbb{R}^2)$ is formed by those functions ψ which fulfill in addition to (2.1a) also

$$\partial_2\psi(u,0_-) = e^{2\pi i\,\nu(u)}\partial_2\psi(u,0_+)\,. \tag{2.1b}$$

In this case, $\hat{H}_\nu\psi = -\Delta\psi$ where the generalized derivative on the RHS is taken in $\mathbb{R}^2_+\cup\mathbb{R}^2_-$. In this sense, $\hat{H}_\nu$ is determined by the boundary conditions on the first coordinate axis (2.1a,b). It is also clear that for $\nu(u)\equiv 0$ one obtains the free Hamiltonian $\hat{H}_0 = -\Delta$ in $L^2(\mathbb{R}^2)$.

Proposition 2.1. *The point spectrum of $\hat{H}_\nu$ is empty.*

Proof. Suppose that there exists $\psi\in D(\hat{H}_\nu)$ such that $\hat{H}_\nu\psi = \lambda\psi$. From the equality

$$\mathfrak{q}_\nu(\psi,\psi) - \langle\psi,\psi\rangle = \langle\psi,\hat{H}_\nu\psi\rangle = \lambda\langle\psi,\psi\rangle \tag{2.2}$$

it follows immediately that, provided $\psi\neq 0$, λ is non–negative. But the value $\lambda = 0$ is excluded, too, since in that case

$$\int_{\mathbb{R}^2_-\cup\mathbb{R}^2_+} d^2x\,\left(|\partial_1\psi|^2 + |\partial_2\psi|^2\right) = \mathfrak{q}_\nu(\psi,\psi) - \langle\psi,\psi\rangle = 0\,.$$

So ψ would be constant almost everywhere in $\mathbb{R}^2_\pm$ and, at the same time, L^2–integrable. Consequently $\psi = 0$, a contradiction.

Set $\lambda = a^2,\ a > 0$. Let us consider ψ as a tempered distribution from $\mathcal{S}'(\mathbb{R}^2)$ and denote by $(\cdot,\cdot)$ the pairing between $\mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^2)$. Any φ from $\mathcal{S}(\mathbb{R}^2)$ fulfilling $\varphi(u,0) \equiv 0,\ \partial_2\varphi(u,0) \equiv 0$, belongs to $D(\hat{H}_\nu)$ and it holds

$$\begin{aligned} - & \Big((\partial_1^2 + \partial_2^2 + a^2)\psi, \varphi\Big) = \int_{\mathbb{R}^2} d^2x\ \psi\,(-\partial_1^2 - \partial_2^2 - a^2)\varphi \\ = & \ \langle \hat{H}_\nu \bar{\varphi}, \psi\rangle - a^2\langle\bar{\varphi},\psi\rangle = \langle\bar{\varphi}, \hat{H}_\nu\psi\rangle - a^2\langle\bar{\varphi},\psi\rangle = 0\,. \end{aligned}$$

Fix $\eta \in \mathcal{S}(\mathbb{R})$ such that $\eta \equiv 1$ in some neighbourhood of 0. Any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ can be written as

$$\begin{aligned} \varphi(x_1,x_2) &= \varphi(x_1,0) + \partial_2\varphi(x_1,0)\,x_2 + \tilde{\varphi}(x_1,x_2)\,, \quad \text{where} \\ \tilde{\varphi}(x_1,x_2) &= \left(\int_0^1 ds\ (1-s)\,\frac{\partial^2}{\partial x_2^2}\varphi(x_1,x_2 s)\right) x_2^2\,. \end{aligned}$$

Since $\varphi(x_1,x_2) - \varphi(x_1,x_2)\eta(x_2) \equiv 0$ in some neighbourhood of the first coordinate axis and $\tilde{\varphi}(x_1,0) = \partial_2\tilde{\varphi}(x_1,0) = 0$ we can replace, by the above observation, $\varphi(x_1,x_2)$ first by $\varphi(x_1,x_2)\eta(x_2)$ and afterwards by $\varphi(x_1,0)\eta(x_2) + \partial_2\varphi(x_1,0)\,x_2\eta(x_2)$ to get

$$\begin{aligned} & \Big((\partial_1^2 + \partial_2^2 + a^2)\psi(x_1,x_2), \varphi(x_1,x_2)\Big) \\ & = \Big((\partial_1^2 + \partial_2^2 + a^2)\psi(x_1,x_2), \varphi(x_1,0)\eta(x_2) + \partial_2\varphi(x_1,0)\,x_2\eta(x_2)\Big)\,. \end{aligned}$$

Define $f_0, f_1 \in \mathcal{S}'(\mathbb{R})$ by $(\xi \in \mathcal{S}(\mathbb{R}))$,

$$\begin{aligned} \Big(f_0(u),\xi(u)\Big) &:= -\Big((\partial_1^2 + \partial_2^2 + a^2)\psi(x_1,x_2), \xi(x_1)\,x_2\eta(x_2)\Big)\,, \\ \Big(f_1(u),\xi(u)\Big) &:= -\Big((\partial_1^2 + \partial_2^2 + a^2)\psi(x_1,x_2), \xi(x_1)\eta(x_2)\Big)\,. \end{aligned}$$

Hence it holds

$$-\Big((\partial_1^2 + \partial_2^2 + a^2)\psi, \varphi\Big) = \Big(f_1(u),\varphi(u,0)\Big) + \Big(f_0(u),\partial_2\varphi(u,0)\Big)$$

or, in other words,

$$-(\partial_1^2 + \partial_2^2 + a^2)\psi = f_0 \otimes \delta' + f_1 \otimes \delta\,.$$

Using the Fourier transform we get for $|k| \neq a$,

$$(2\pi)^{1/2}\hat{\psi}(k) = (|k|^2 - a^2)^{-1}(\hat{f}_1(k_1) + ik_2\hat{f}_0(k_1))\,.$$

Since $\hat{\psi}(\cdot,k_2) \in L^2(\mathbb{R})$ for almost all k_2, f_0, f_1 must be represented by classical functions. Conversely, $\hat{\psi}(k_1,\cdot) \in L^2(\mathbb{R})$ for almost all k_1. But if $|k_1| < a$ then the singularities in the points $k_2 = \pm(a^2 - {k_1}^2)^{1/2}$ are non–integrable. Consequently, one can suppose that $\hat{f}_0(u) = \hat{f}_1(u) = 0$ for all $u,\ |u| < a$. Set

$$g_\pm(u) = \frac{i}{2}\,(\hat{f}_0(u) \mp (u^2 - a^2)^{-1/2}\hat{f}_1(u))\,.$$

Then

$$(2\pi)^{1/2}\hat{\psi}(k) = (k_2 - i\sqrt{k_1^2 - a^2})^{-1} g_+(k_1) + (k_2 + i\sqrt{k_1^2 - a^2})^{-1} g_-(k_1)\,.$$

Making now use of the Plancherel–Parseval identity for the functions $\psi, \partial_1\psi, \partial_2\psi \in L^2(\mathbb{R}^2)$ (the generalized derivatives are taken outside the first coordinate axis) one finds that (2.2) means

$$\int_{\mathbb{R}} du\, \sqrt{u^2 - a^2}\, (|g_+(u)|^2 + |g_-(u)|^2) = 0\,.$$

Hence $g_+(u) = 0,\ g_-(u) = 0$, almost everywhere and $\psi = 0$ in $L^2(\mathbb{R}^2)$. ∎

In the sequel, the function ν is piecewise constant with finitely many discontinuities. Each point of discontinuity is interpreted as a vortex and the value of the jump is related to the magnitude of the corresponding magnetic flux. So let $a_1, a_2, \ldots, a_N$ be a finite sequence of reals ordered increasingly and $\alpha_1, \alpha_2, \ldots, \alpha_N$ be some numbers from the interval $(0, 1)$. Denote $I_0 = (-\infty, a_1),\ I_j = (a_j, a_{j+1})$ for $j = 1, \ldots, N-1,\ I_N = (a_N, +\infty)$, and set

$$\nu(u) = \nu_k := \sum_{j=1}^{k} \alpha_j \quad \text{for } u \in I_k\,. \tag{2.3}$$

We shall write simply $\hat{H}$ instead of $\hat{H}_\nu$ and $\mathfrak{q}$ instead of $\mathfrak{q}_\nu$.

3 Krein's Formula Versus Boundary Conditions

Both Hamiltonians $\hat{H}$ and $\hat{H}_0$ are self–adjoint extensions of the same symmetric operator $X = -\Delta$ with the domain $D(X) = C_0^\infty(\mathbb{R}^2_+) \oplus C_0^\infty(\mathbb{R}^2_-)$. The resolvents are related by the Krein's formula. Denote by $\mathcal{N}(z)$ the deficiency subspace, with $z \in \mathbb{C} \setminus \mathbb{R}$, and let $P(z) : L^2(\mathbb{R}^2) \to \mathcal{N}(z)$ be the orthogonal projector. Furthermore, $V(z), V_0(z) : \mathcal{N}(z) \to \mathcal{N}(\bar{z})$ designate the unitary operators determining these self–adjoint extensions. The symbol $\mathcal{N}(z)$ makes sense also for $z < 0$. The Krein's formula then reads

$$(\hat{H} - z)^{-1} = (\hat{H}_0 - z)^{-1} + P(z)^* \, \frac{V(\bar{z}) - V_0(\bar{z})}{\bar{z} - z}\, P(\bar{z}). \tag{3.1}$$

Our first task is to determine the operator $V(z)$ explicitly. To this end let us introduce some operators entering the formula for $V(z)$. Denote by $\Lambda(z)$ the closed operator in $L^2(\mathbb{R}, d\kappa)$ acting by multiplication with $(\kappa^2 - z)^{1/2},\ z \in \mathbb{C} \setminus [0, +\infty)$. Ω stands for an operator in $L^2(\mathbb{R})$ defined by

$$\Omega\hat{\psi} := (e^{2\pi i \nu}\psi)^\wedge\,. \tag{3.2}$$

So $\Lambda(z)^{-1}$ is bounded and Ω is unitary. Denote by $\mathfrak{p}_z$, with $z \notin [0, +\infty)$, the following closed sectorial form in $L^2(\mathbb{R})$,

$$\mathfrak{p}_z(\psi,\psi) = \int_{I\!R} d\kappa\,\sqrt{\kappa^2 - z}\,(|\psi(\kappa)|^2 + |\Omega\psi(\kappa)|^2)\,,$$

with the natural form domain and by $\Xi(z)$ the associated m-sectorial operator [8]. One easily finds that

$$(\mathrm{Re}\,\sqrt{\kappa^2 - z})^2 = \frac{1}{2}(\kappa^2 - \mathrm{Re}\,z + |\kappa^2 - z|) \geq \frac{1}{2}(|z| - \mathrm{Re}\,z)\,,$$

and thus for $\psi \in D(\Xi(z))$,

$$\mathrm{Re}\,\langle\psi, \Xi(z)\psi\rangle \geq (2(|z| - \mathrm{Re}\,z))^{1/2}\langle\psi,\psi\rangle\,.$$

Consequently, $\Xi(z)^{-1}$ exists and is bounded. Since $\mathfrak{p}_z^* = \mathfrak{p}_{\bar{z}}$ it holds $\Xi(z)^* = \Xi(\bar{z})$. It follows that $\mathrm{Ran}\,\Xi(z) = D(\Xi(z)^{-1})$ is dense in $L^2(\mathbb{R})$ and so coincides with $L^2(\mathbb{R})$.

Lemma 3.1. *$\Xi(z)$ is the closure of the operator $\Lambda(z) + \Omega^{-1}\Lambda(z)\Omega$. Consequently, $\mathrm{Ran}\,(\Lambda(z) + \Omega^{-1}\Lambda(z)\Omega)$ is dense in $L^2(\mathbb{R})$ and $\Xi(z)^{-1}$ is the closure of $(\Lambda(z) + \Omega^{-1}\Lambda(z)\Omega)^{-1}$.*

Proof. We have to show that $D(\Lambda + \Omega^{-1}\Lambda\Omega)$ is dense in the form domain $\mathcal{Q}(\mathfrak{p}_z)$ with respect to the norm [8]

$$\psi \mapsto \left(\int_{I\!R} d\kappa\,(1 + |\kappa|)\,(|\psi(\kappa)|^2 + |\Omega\psi(\kappa)|^2)\right)^{1/2}\,. \tag{3.3}$$

First observe that, provided $\psi \in \mathcal{H}^{1/2}(\mathbb{R})$ can be decomposed as $\psi = \psi_1 + \psi_2$, with $\mathrm{supp}\,\psi_1 \subset (-\infty, a]$, $\mathrm{supp}\,\psi_2 \subset [b, +\infty)$, and $a < b$, then $\psi_1, \psi_2 \in \mathcal{H}^{1/2}(\mathbb{R})$.This fact can be checked with the help of an equivalent norm in $\mathcal{H}^{1/2}(\mathbb{R})$, namely [9]

$$\left(\|\psi\|_2^2 + \int_{I\!R^2} du\,dv\left|\frac{\psi(u) - \psi(v)}{u - v}\right|^2\right)^{1/2}\,.$$

A similar assertion is valid also for the spaces $\mathcal{H}^0(\mathbb{R})$, $\mathcal{H}^1(\mathbb{R})$ but here it is quite obvious.

Clearly, $\psi \in D(\Lambda + \Omega^{-1}\Lambda\Omega) = D(\Lambda) \cap D(\Omega^{-1}\Lambda\Omega)$ iff both $\check{\psi}$ and $e^{2\pi i\nu}\check{\psi}$ belong to $\mathcal{H}^1(\mathbb{R})$. But as $e^{2\pi i\nu}$ is piecewise constant and discontinuous in the points $a_1, \ldots, a_N$, this is the same as the condition $\chi_j\check{\psi} \in \mathcal{H}^1(\mathbb{R})$ for all $j = 1, \ldots, N$, with χ_j being the characteristic function of I_j. Similarly, $\psi \in \mathcal{Q}(\mathfrak{p}_z)$ iff the norm (3.3) is finite and this means that $\check{\psi}, e^{2\pi i\nu}\check{\psi} \in \mathcal{H}^{1/2}(\mathbb{R})$. By the above observation, this is equivalent to the condition $\chi_j\check{\psi} \in \mathcal{H}^{1/2}(\mathbb{R})$ for all $j = 1, \ldots, N$. Thus we have shown that $D(\Lambda + \Omega^{-1}\Lambda\Omega) = \mathcal{F}(\mathcal{H}_0^1(I_0) \oplus \ldots \oplus \mathcal{H}_0^1(I_N))$ and $\mathcal{Q}(\mathfrak{p}_z) = \mathcal{F}(\mathcal{H}_0^{1/2}(I_0) \oplus \ldots \oplus \mathcal{H}_0^{1/2}(I_N))$. But $\mathcal{H}_0^1(I)$ is dense in $\mathcal{H}_0^{1/2}(I)$. ∎

The deficiency subspaces are easy to describe. $\mathcal{N}(z)$ is the range of the unitary mapping

$$U(z) : \mathbb{C}^2 \otimes L^2(\mathbb{R}) \to \mathcal{N}(z) \tag{3.4}$$

defined by $\psi = U(z)\varphi$,

$$
\begin{aligned}
&\psi(x) = \mathcal{F}^{-1}_{\kappa\to x_1}\,\tilde{\varphi}(\kappa, x_2)\ \ , \text{ where}\\
&\tilde{\varphi}(\kappa,u) = \left(2\mathrm{Re}\,\sqrt{\kappa^2 - z}\right)^{1/2} \exp(-\sqrt{\kappa^2 - z}\,|u|)\,(\varphi_+(\kappa)\vartheta(u) + \varphi_-(\kappa)\vartheta(-u))\ .
\end{aligned}
$$

We have ascribed the symbols $\{e_+, e_-\}$ to the standard basis in $\mathbb{C}^2$ (ϑ stands for the Heaviside step function). We are going to determine $\tilde{V}(z) := U(\bar{z})^{-1}V(z)U(z)$. With respect to the chosen basis in $\mathbb{C}^2$, $\tilde{V}(z)$ splits into a 2×2 matrix $(V_{\varepsilon\varrho}(z))$, $\varepsilon,\ \varrho = \pm$.

Proposition 3.2. *It holds*

$$
\begin{aligned}
\mathbf{I} + V_{++} &= (z-\bar{z})\,(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\Xi(\bar{z})^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\,,\\
V_{+-} &= (z-\bar{z})\,(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\Xi(\bar{z})^{-1}\Omega^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\,,\\
V_{-+} &= (z-\bar{z})\,(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\Omega\,\Xi(\bar{z})^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\,,\\
\mathbf{I} + V_{--} &= (z-\bar{z})\,(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\Omega\,\Xi(\bar{z})^{-1}\Omega^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\,.
\end{aligned}
\tag{3.5}
$$

Proof. It is clear that the operator $\tilde{V}(z) = (V_{\varepsilon\varrho}(z))$ defined in (3.5) is bounded and $\tilde{V}(z)^* = \tilde{V}(\bar{z})$. Let us check that it is also unitary. It is enough to verify only one equality, say, $\tilde{V}(z)\tilde{V}(z)^* = \mathbf{I}$. A straightforward calculation shows that this means

$$
(z-\bar{z})^{-1}(\Xi(z)^{-1} - \Xi(\bar{z})^{-1}) = \Xi(\bar{z})^{-1}((2\,\mathrm{Re}\,\Lambda(z))^{-1} + \Omega^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1}\Omega)\,\Xi(z)^{-1}\,.
$$

But the last equality follows immediately from Lemma 3.1.

Thus $V'(z) := U(\bar{z})\tilde{V}(z)U(z)^{-1}$ defines some self–adjoint extension $\hat{H}'$ of X. We have to show that $\hat{H}' = \hat{H}$. But it suffices to verify that $\hat{H}' \subset \hat{H}$. As $D(\hat{H}') = D(\bar{X}) + (\mathbf{I} + V'(z))\mathcal{N}(z)$, one has to check that $g = U(z)\varphi + U(\bar{z})\tilde{V}(z)\varphi$ belongs to $D(\hat{H})$ for any $\varphi \in \mathbb{C}^2\otimes L^2(\mathbb{R})$. To avoid too cumbersome formulas let us consider the case when $\varphi_- = 0$. The complementary case of $\varphi_+ = 0$ can be treated similarly. We have

$$
g(x) = \mathcal{F}^{-1}_{\kappa\to x_1} f(\kappa, x_2), \text{ where}
$$

$$
\begin{aligned}
f(\kappa,u) &= (2\,\mathrm{Re}\,\sqrt{\kappa^2 - z\,})^{1/2}(\exp(-\sqrt{\kappa^2 - z}\,u) - \exp(-\sqrt{\kappa^2 - \bar{z}}\,u))\,\varphi_+(\kappa)\\
&\quad + (z-\bar{z})\ \exp(-\sqrt{\kappa^2 - \bar{z}}\,u)\,\Xi(\bar{z})^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\varphi_+(\kappa)\,,\\
&\qquad\qquad \text{for } u > 0\,,\\
&= (z-\bar{z})\ \exp(\sqrt{\kappa^2 - \bar{z}}\,u)\,\Omega\,\Xi(\bar{z})^{-1}(2\,\mathrm{Re}\,\Lambda(z))^{-1/2}\varphi_+(\kappa)\,,\\
&\qquad\qquad \text{for } u < 0\,.
\end{aligned}
\tag{3.6}
$$

Observe that the operators $\Lambda^{1/2}\Xi^{-1}$ and $\Omega^{-1}\Lambda^{1/2}\Omega\,\Xi^{-1}$ are everywhere defined and closed and hence bounded. It follows that $\kappa\,f(\kappa,u)$ and $\partial_2 f(\kappa,u)$ and consequently both $\partial_1 g$ and $\partial_2 g$ (the generalized derivatives are taken in $\mathbb{R}^2_- \cup \mathbb{R}^2_+$) belong to $L^2(\mathbb{R}^2)$. Recalling (3.2) and letting $u \downarrow 0$ resp. $u \uparrow 0$ in (3.6) one can also check that g satisfies the boundary condition (2.1a). Hence $g \in \mathcal{Q}(\mathfrak{q})$.

It remains to show that

$$\int_{\mathbb{R}^2} d^2x\,(\partial_1\bar{\psi}\,\partial_1 g + \partial_2\bar{\psi}\,\partial_2 g) = \int_{\mathbb{R}^2_- \cup \mathbb{R}^2_+} d^2x\,\bar{\psi}\,(-\Delta g) \tag{3.7}$$

is fulfilled for all $\psi \in \mathcal{Q}(\mathfrak{q})$. As a matter of fact, it suffices to verify (3.7) only for ψ's from some subspace in $\mathcal{Q}(\mathfrak{q})$ which is dense with respect to the norm induced by $\mathfrak{q}(\cdot,\cdot)$. Since the restriction morphisms $\mathcal{H}^s(\mathbb{R}^2_\pm) \to L^2(\mathbb{R})$ are retractions (they have right bounded inverses) and since $C_0^\infty(\mathbb{R}^2_\pm)$ is dense in $\mathcal{H}_0^s(\mathbb{R}^2_\pm)$ [9] , functions compactly supported in $\mathbb{R}^2 \setminus \{a_1,\ldots,a_N\}$ and smooth on the closure of both the upper and the lower halfplane are dense in $\mathcal{Q}(\mathfrak{q})$. But if ψ possesses these properties and g is given as above then the equality (3.7) is valid provided g satisfies the second boundary condition (2.1b) in the sense of distributions. This is actually the case as one can easily see, again with the help of Lemma 3.1. ∎

4 A Perturbative Formula for $\Xi(z)^{-1}$

4.1 Auxiliary results

In this subsection we introduce some operators occurring in the formula for $\Xi(z)^{-1}$ and investigate their main properties. Let $\tilde{R} = (\tilde{R}_{\varepsilon\varrho})$, $\varepsilon, \varrho = \pm$, be a 2×2 matrix of operators in $L^2(\mathbb{R})$ depending on two parameters, $\alpha \in (0,1)$ and $w \in \mathbb{C}$, $\operatorname{Re} w > 0$, and determined by the kernels

$$\begin{aligned}&\langle\kappa|\tilde{R}_{\varepsilon\varrho}(\alpha;w)|\lambda\rangle \\ &:= \frac{1}{\pi}\,(\kappa^2+w^2)^{-1/4}\,\frac{\exp[-\alpha(\omega(\kappa)-\omega(\lambda)) - i\frac{\pi}{2}\alpha(\varepsilon-\varrho)]}{1+\exp[-\omega(\kappa)+\omega(\lambda)-i(\frac{\pi}{2}-0)(\varepsilon-\varrho)]}\,(\lambda^2+w^2)^{-1/4}\,,\end{aligned} \tag{4.1}$$

where

$$\omega(\kappa) \equiv \omega(w;\kappa) := \operatorname{argsh}(w^{-1}\kappa)\,.$$

Dependence on α will be mostly suppressed in notation. The kernel (4.1) can be interpreted without difficulties only for $w > 0$ or $(\varepsilon\varrho) = (++),(--)$. In the former case, after the substitution $\kappa = w\sinh x$ we get an involution operator and hence diagonalizable by the Fourier transform. The appearance of $(\frac{\pi}{2}-0)$ in the denominator indicates a regularization. More precisely, it holds

$$\langle x|\mathcal{F}L_w\,\tilde{R}_{\varepsilon\varrho}L_w^{-1}\mathcal{F}^{-1}|y\rangle = r_{\varepsilon\varrho}(x)\,\delta(x-y)\,,$$

$$r_{\varepsilon\varrho}(x) = (\sin\pi(\alpha+ix))^{-1}\exp(-\frac{\pi}{2}(\varepsilon-\varrho)x)\,, \tag{4.2}$$

with L_w being the unitary operator in $L^2(\mathbb{R})$ defined by

$$L_w\psi(x) := (w\cosh x)^{1/2}\psi(w\sinh x), \quad w > 0\,. \tag{4.3}$$

Notice also the scaling property of $\tilde{R}$. Let $T(c)$, $c \in \mathbb{R}_+$, be the one–parameter unitary group of scaling operators, $T(c)\psi(\kappa) := c^{1/2}\psi(c\kappa)$. Then $T(c)^{-1}\tilde{R}(w)\,T(c) = \tilde{R}(cw)$. This is true verbatimly for $w > 0$ and, by now, only symbolically for w non–real. But with the below given definition of $\tilde{R}_{\varepsilon\varrho}(w)$ as a bounded operator it turns out that whatever w is, the norm $\|\tilde{R}_{\varepsilon\varrho}(w)\|$ depends only on $\arg w$. For $w > 0$,

$$\tilde{R}(w) = T(w)^{-1}\tilde{R}(1)\,T(w)\,. \tag{4.4}$$

The definition of $\tilde{R}_{\varepsilon\varrho}(w)$ for w non–real imitates this property. The operator $T(w)$, $0 < \arg w < \pi/2$, is well defined on functions ψ which are sectorial analytic, i.e., analytic on the union of two sectors $0 < \arg z < \arg w$ and $\pi < \arg z < \pi + \arg w$, continuous on the closure and, moreover, fulfilling $\psi(e^{i\phi}\kappa) \in L^2(\mathbb{R})$ for all $\phi \in [0, \arg w]$. A similar situation takes place also for $-\pi/2 < \arg w < 0$. Let us first construct a sufficiently rich family of functions possessing this property. Set $(\omega' = \partial_\kappa\omega)$

$$\Phi_\zeta[h](\kappa) := (2\pi)^{-1/2}\,\omega'(\zeta;\kappa)^{1/2}\int_{\mathbb{R}} dx\ \exp(ix\,\omega(\zeta;\kappa))\,h(x)\,. \tag{4.5}$$

Lemma 4.1. *Suppose that $e^{\theta|x|}h \in L^2(\mathbb{R})$, for some $\theta \in (0, \frac{\pi}{2}]$, and $\zeta \in \mathbb{C}\setminus\{0\}$, $|\arg\zeta| < \theta$. Then the function $\Phi_\zeta[h]$ is L^2–integrable and it holds*

$$\|\Phi_\zeta[h]\| \le C\,\|e^{\theta|x|}h\|\,, \tag{4.6}$$

where the bound C can be chosen uniformly provided ζ runs over some compact subset in the sector. For a fixed ζ, functions $\Phi_\zeta[h]$, with $h \in C_0^\infty(\mathbb{R})$, are dense in $L^2(\mathbb{R})$.

Proof. L^2–integrability follows from the asymptotic behaviour

$$\operatorname{argsh}(\pm z) = \pm(\ln 2z + \gamma(z)),\ \text{with } |\gamma(z)| \le C'|z|^{-2}\,. \tag{4.7}$$

valid in the sector $|\arg z| \le \theta_0 < \pi/2$.

To show the density observe first that for any ζ, $\operatorname{Re}\zeta > 0$, the functions $\kappa^n\operatorname{sech}(\zeta\kappa)$, $n \in \mathbb{N}_0$, form a total subset in $L^2(\mathbb{R},d\kappa)$. Actually, suppose that $f \in L^2(\mathbb{R})$ is orthogonal to all these functions. But the Fourier image of $\operatorname{sech}(\zeta\kappa)\bar{f}(\kappa)$ has an analytic extension to the strip $|\operatorname{Im} z| < \operatorname{Re}\zeta$ and the orthogonality means that all its derivatives vanish at the point $z = 0$. Whence $f = 0$ in $L^2(\mathbb{R})$.

¿From the analycity and decay properties of the function $(\cosh x)^{1/2}(\sinh x)^n \operatorname{sech}(\sinh x)$ one can conclude that its Fourier image h_n fulfills $e^{\theta|x|}h_n \in L^2(\mathbb{R})$ for any $\theta \in [0, \frac{\pi}{2})$. Since $\Phi_\zeta[h_n] = \zeta^{-n-1/2}\kappa^n\operatorname{sech}(\zeta^{-1}\kappa)$ and owing to the above observation, functions $\Phi_\zeta[h]$, with $e^{\theta|x|}h \in L^2(\mathbb{R})$ for some $\theta > |\arg\zeta|$, are dense in $L^2(\mathbb{R})$. The inequality (4.6) then implies that the functions $h \in C_0^\infty(\mathbb{R})$ will do as well. ∎

Assuming again $e^{\theta|x|}h \in L^2(\mathbb{R})$ and $|\arg\zeta| < \theta$, the function $\Phi_\zeta[h]$ has an analytic extension to the sector $|\arg z - \arg\zeta| < \theta$. Particularly, if $|\arg w - \arg\zeta| < \theta$ then

$$T(w)\,\Phi_\zeta[h] = \Phi_{w^{-1}\zeta}[h]\,. \tag{4.8}$$

Respecting the relation (4.4), we define

$$\tilde{R}_{\varepsilon\varrho}\,\Phi_w[h] := \Phi_w[r_{\varepsilon\varrho}h]\,. \tag{4.9}$$

By Lemma 4.1, the operators $\tilde{R}_{\varepsilon\varrho}$ are densely defined. We have to show that they are bounded. Moreover, it will turn out that they depend on w analytically.

Let us consider shortly a general operator Q of this kind, that means defined by

$$Q\,\Phi_w[h] := \Phi_w[qh]\,,$$

with q being a bounded measurable function. Let us also suppose that the function $\check{q}$ can be extended analytically to the strip $|\mathrm{Im}\, z| < 2|\arg w|$. Notice that (4.5) means $\Phi[h](\kappa) = \omega'(\kappa)^{1/2}\check{h}(\omega(\kappa))$. Expressing $\mathcal{F}^{-1}(qh)$ as a convolution and assuming that the integration curve can be deformed in the complex plane, from the real axis to the curve $z = \omega(\lambda)$, $\lambda \in \mathbb{R}$, one gets the generalized kernel of Q written as

$$\langle\kappa|Q|\lambda\rangle = (2\pi)^{-1/2}\,\omega'(\kappa)^{1/2}\,\check{q}(\omega(\kappa) - \omega(\lambda))\,\omega'(\lambda)^{1/2}\,. \tag{4.10}$$

It is easy to see that this procedure works provided $|q(x)| \le C\,e^{-\pi|x|}$. Another important example allowing this treatment is $q(x) = \vartheta(\pm x)$. Then $\check{q}(z) = \mp(\sqrt{2\pi}\,iz)^{-1}$, with $\mathrm{Im}\, z > 0$ resp. $\mathrm{Im}\, z < 0$. Though $\check{q}(z)$ has a pole at $z = 0$, the integration curve can be deformed provided one obeys the regularization indicated in the following formula

$$\langle\kappa|Q|\lambda\rangle = \mp\frac{1}{2\pi i}\,\omega'(\kappa)^{1/2}\,\frac{1}{\omega(\kappa) - \omega(\lambda) \pm i0}\,\omega'(\lambda)^{1/2}\,.$$

Returning to the operators $\tilde{R}_{\varepsilon\varrho}$, one finds immediately that the kernel of $\tilde{R}_{++} = \tilde{R}_{--}$ is given by the formula (4.1). Furthermore, one can write

$$r_{\pm\mp}(x) = p_\pm(x) \pm 2i\,e^{\mp i\pi\alpha}\vartheta(\mp x)\,, \quad \text{where} \tag{4.11}$$

$$\sin\pi(\alpha + ix)\;p_\pm(x) = e^{\mp\pi x}\vartheta(\pm x) + e^{\mp 2\pi i\alpha}\,e^{\pm\pi x}\vartheta(\mp x)\,,$$

and so $e^{2\pi|x|}p_\pm$ is bounded. It follows that

$$\langle\kappa|\tilde{R}_{\pm\mp}|\lambda\rangle = \tag{4.12}$$
$$\omega'(\kappa)^{1/2}\left((2\pi)^{-1/2}\check{p}_\pm(\omega(\kappa) - \omega(\lambda)) + e^{\mp i\pi\alpha}\,\frac{1}{\pi(\omega(\kappa) - \omega(\lambda) \mp i0)}\right)\omega'(\lambda)^{1/2}\,.$$

Notice also that for $-2\pi < \mathrm{Im}\, z < 0$ resp. $0 < \mathrm{Im}\, z < 2\pi$, it holds

$$(\pi/2)^{1/2}\,\check{p}_\pm(z) + e^{\mp i\pi\alpha}z^{-1} = e^{-\alpha z \mp i\pi\alpha}(1 - e^{-z})^{-1}\,. \tag{4.13}$$

Lemma 4.2. *Suppose that* $\operatorname{supp} q \subset [0,+\infty)$, *the restriction of* q *to the positive half–axis belongs to* $C^2([0,+\infty))$ *and* $e^{\pi x}q$, $e^{\pi x}q'$, $e^{\pi x}q''$ *are bounded. Then the operator* Q *with the kernel (4.10) is bounded. The bound is uniform provided* w *runs over some compact subset in the halfplane* $\operatorname{Re} w > 0$.

Proof. Let us consider $L_1 Q\, L_1^{-1}$ rather than Q; L_1 is given in (4.3). Since $|\cosh(x)\,\omega'(\sinh x)|$ is bounded we can restrict ourselves to the kernel $\check{q}(\omega(\sinh x) - \omega(\sinh y))$. Fix some θ, $|\arg w| < \theta < \pi/2$. By integrating by parts one finds that

$$(2\pi)^{1/2}\,\check{q}(z) = q(0)\,(2\theta - iz)^{-1} + g(z), \quad \text{with } |g(z)| \le C_1\,|2\theta - iz|^{-2}\,,$$

provided $|\operatorname{Im} z| \le 2|\arg w|$. It is elementary to check that there exists $C_2 > 0$ such that

$$|\operatorname{Re}\,\operatorname{argsh}(w^{-1}\sinh x) - \operatorname{Re}\,\operatorname{argsh}(w^{-1}\sinh y)| \ge C_2|x-y|\,. \tag{4.14}$$

Consequently,

$$|g(\omega(\sinh x) - \omega(\sinh y))| \le C_3\,(1 + (x-y)^2)^{-1}\,.$$

Owing to the Young inequality, the RHS is a kernel of some bounded convolution operator. Furthermore, the Fourier transform shows that the kernel $[2\theta + i\ln w\,(\operatorname{sgn} x - \operatorname{sgn} y) - i(x-y)]^{-1}$ corresponds to a bounded operator. As one can verify with the help of (4.7), subtracting this kernel from $[2\theta - i(\omega(\sinh x) - \omega(\sinh y))]^{-1}$ one gets a Hilbert–Schmidt kernel. ∎

The following Lemma is an adaptation of Proposition 2.3.8 in [10].

Lemma 4.3. *For any* $\sigma \in \mathbb{R} \setminus \{0\}$, *the operator* Q *corresponding to the kernel*

$$\omega'(\kappa)^{1/2}\,(\omega(\kappa) - \omega(\lambda) + i\sigma)^{-1}\,\omega'(\lambda)^{1/2}$$

is bounded in $L^2(\mathbb{R})$. *There exist strong limits for* $\sigma \downarrow 0$ *and* $\sigma \uparrow 0$.

Proof. To be specific, we shall consider the case $\sigma > 0$. Again rather than on Q we shall concentrate on $L_1 Q\, L_1^{-1}$. Let us denote $\eta(x) = \operatorname{argsh}(w^{-1}\sinh x)$. Since $|\eta'(x)|$ is bounded from below as well as from above, we can switch to the kernel $(\eta(x) - \eta(y) + i\sigma)^{-1}\eta'(y)$. The splitting of the Hilbert space, $L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$, induces a splitting of the operator into four blocks. We shall focus just on the case of operator acting in $L^2(\mathbb{R}_+)$. The remaining cases can be treated similarly. The idea is to compare the kernel with another one, namely $(x - y + i\sigma)^{-1}$. The latter one corresponds to a bounded operator and the strong limit $\sigma \downarrow 0$ exists, as manifested by the Fourier transform. The asymptotic behaviour (4.7) means that $\eta(x) = -\ln w + x + \gamma_1(x)$, $e^{2x}\gamma_1(x)$ is bounded, and the same is true for all derivatives of γ_1. One can write

$$\frac{1}{x-y+i\sigma} - \frac{1}{\eta(x) - \eta(y) + i\sigma}\,\eta'(y) = \frac{x-y+i\sigma}{\eta(x)-\eta(y)+i\sigma} \tag{4.15}$$

$$\times \left\{ \frac{(x-y)^2}{(x-y+i\sigma)^2} \, \frac{\eta(x)-\eta(y)-\eta'(y)\,(x-y)}{(x-y)^2} + \frac{i\sigma}{(x-y+i\sigma)^2}\,(1-\eta'(y)) \right\} .$$

Next we are going to estimate the absolute value of the two summands on the RHS of (4.15) by kernels corresponding to manifestly bounded operators. One can easily derive from (4.14) that the factor standing in front of the composite bracket is bounded independently of σ. To treat the first summand, let us rewrite

$$\frac{\eta(x)-\eta(y)-\eta'(y)\,(x-y)}{(x-y)^2} = \int_0^1 ds\,(1-s)\;\eta''(sx+(1-s)y)\,.$$

¿From the asymptotic behaviour of η one can conclude that $\eta''(sx+(1-s)y)$ is a Hilbert–Schmidt kernel and the norm is bounded by $C\,s^{-1/2}(1-s)^{-1/2}$. Whence the first summand corresponds to a Hilbert–Schmidt operator. Since the prefactor depends on σ continuously and is bounded uniformly, the limit $\sigma\downarrow 0$ exists even in the Hilbert–Schmidt norm.

Let us turn to the second summand. By the Young inequality, the kernel $\sigma|x-y+i\sigma|^{-2}$ corresponds to a bounded operator. Moreover, the bound can be chosen independently of σ since the L^1–norm of the function $\sigma|x+i\sigma|^{-2}$ does not depend on σ. To get the strong limit $\sigma\downarrow 0$ for the kernel $\sigma(x-y+i\sigma)^{-2}$, it is enough to apply the Fourier transform sending a convolution operator to an operator acting by multiplication. It turns out that the limit is the zero operator. ∎

Proposition 4.4. *The operators $\tilde{R}_{\varepsilon\varrho}(w)$ are bounded and so, after having been closed, defined everywhere on $L^2(\mathbb{R})$. The bound can be chosen uniformly with respect to w running over some compact subset in the halfplane $\mathrm{Re}\,w>0$. Moreover, they depend analytically on w in this halfplane.*

Proof. The boundedness follows immediately from (4.12), Lemma 4.2 and Lemma 4.3. Let us turn to the analycity. We claim that to any $\theta\in(0,\pi/2)$ one can find $\sigma>0$ such that the equation

$$\Phi_\zeta[g]=\Phi_1[h]$$

has a unique solution g, $e^{\theta|x|}g\in L^2(\mathbb{R})$, whenever $|\zeta-1|<\sigma$ and $h\in C_0^\infty(\mathbb{R})$. Actually, using the analycity of both functions and some elementary estimates one can check that, provided ζ is close enough to 1, the equation allows a substitution $\kappa=\zeta\,\kappa'$. Hence it can be rewritten as

$$(1+\kappa^2)^{-1/4}\,\breve{g}(\operatorname{argsh}\kappa)\equiv\Phi_1[g](\kappa)=\Phi_{\zeta^{-1}}[h](\kappa)\,.$$

Observe also that provided ψ_1, ψ_2 belong to the domain of the scaling operator $T(w^{-1})$, as specified after the formula (4.4), then

$$\langle\psi_1,T(w^{-1})\psi_2\rangle=\langle T(\bar{w})\psi_1,\psi_2\rangle\,.$$

To get this equality one has to rotate the integration path in the complex plane.

Suppose that we are given some $w \in \mathbb{C}$, Re $w > 0$. By the above observations and owing to (4.8), there exists $\sigma > 0$ such that for any couple $h_1, h_2 \in C_0^\infty(\mathbb{R})$, it holds

$$\langle \Phi_{\bar{w}}[h_1], \tilde{R}_{\varepsilon\varrho}(\zeta)\, \Phi_w[h_2] \rangle = \langle \Phi_{\bar{\zeta}^{-1}\bar{w}}[h_1], \tilde{R}_{\varepsilon\varrho}(1)\, \Phi_{\zeta^{-1}w}[h_2] \rangle$$

and this expression depends analytically on ζ in the neighbourhood $|\zeta^{-1}w - 1| < \sigma$. Taking into account the uniform boundedness of $\tilde{R}_{\varepsilon\varrho}(\zeta)$ and the density of functions $\Phi_w[h],\ h \in C_0^\infty$, one obtains the weak analycity. But the weak analycity is equivalent to the strong analycity [11]. ∎

4.2 Statement and proof of the formula

Let Π be the parity operator in $L^2(\mathbb{R})$, $\Pi\psi(\kappa) := \psi(-\kappa)$, and $Y_c,\ c \in \mathbb{R}$, a unitary operator acting by

$$Y_c\psi(\kappa) := e^{-ic\kappa}\psi(\kappa)\,.$$

For $z \in \mathbb{C} \setminus [0, +\infty)$ and $j = 1, 2, \ldots, N$, set

$$R^{(j)}(z) := -\sin(\pi\alpha_j)\, Y_{a_j}\, \mathrm{diag}(\Pi, \mathbf{I})\, \tilde{R}(\alpha_j; \sqrt{-z})\, \mathrm{diag}(\mathbf{I}, \Pi)\, Y_{-a_j}\,. \tag{4.16}$$

Here we identify Y with $\mathbf{I}_2 \otimes Y$. So $R^{(j)}(z) = (R^{(j)}_{\varepsilon\varrho})_{\varepsilon,\varrho=\pm}$ is an operator in $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ related to the j[th] vortex. Where not necessary, the indication of dependence on z will be suppressed. Symbolically,

$$\begin{aligned}
\langle \kappa | R^{(j)}_{\varepsilon\varrho} | \lambda \rangle \;&=\; -\frac{\sin\pi\alpha_j}{\pi}\, (\kappa^2 + w^2)^{-1/4}\, \exp(-ia_j\kappa) \\
&\times\;\; \frac{\exp[\alpha_j(\varepsilon\,\omega(\kappa) + \varrho\,\omega(\lambda)) - i\frac{\pi}{2}\alpha_j(\varepsilon - \varrho)]}{1 + \exp[\varepsilon\,\omega(\kappa) + \varrho\,\omega(\lambda) - i(\frac{\pi}{2} - 0)(\varepsilon - \varrho)]}\, \exp(ia_j\lambda)\, (\lambda^2 + w^2)^{-1/4}\,,
\end{aligned}$$

with $w = \sqrt{-z}$. Clearly, $R^{(j)}(z)^* = R^{(j)}(\bar{z})$. Furthermore, let E_a be the Fourier transform of the orthogonal projector onto $L^2((-\infty, a))$, i.e.,

$$E_a\hat{\psi} := (\vartheta(a - x)\, \psi)^\wedge\,. \tag{4.17}$$

We set also $E_a^c := \mathbf{I} - E_a$ and write E_j instead of E_{a_j}. As an intermediate step leading to the formula, we shall prove some algebraic identities relating $R^{(j)}$, E_k and Λ. Verifications can be simplified provided the following relations and scaling properties, valid for $w > 0$, are used

$$E_a = Y_a E_0\, Y_{-a}\,, \tag{4.18a}$$
$$T(w^{-1})\Lambda(-1)\, T(w) = w^{-1}\Lambda(-w^2)\,, \tag{4.18b}$$
$$T(w)\, E_0\, T(w^{-1}) = E_0\,. \tag{4.18c}$$

Lemma 4.5. *Let $F \in \mathcal{S}'(\mathbb{R})$ be the Fourier image of $\exp(i\tau\, \mathrm{argsh}(\kappa))$, with τ being a real parameter, in the space of tempered distributions. Then the pairing between*

F and some $\varphi \in \mathcal{S}(\mathbb{R})$ reads

$$
\begin{aligned}
(F,\varphi) \;&=\; \left(\sqrt{2\pi} - \sqrt{\frac{2}{\pi}}\,\sinh(\frac{\pi\tau}{2}) \int_{I\!R} ds\; e^{-\cosh s}\,\tanh s\;\sin\tau s\right)\varphi(0) \\
&+\; \sqrt{\frac{2}{\pi}} \int_{I\!R} dx\,\frac{\tau}{x}\,K_{i\tau}(|x|)\,\exp(\frac{1}{2}\,\pi\tau\,\mathrm{sgn}\,x)\,(\varphi(x) - \vartheta(1-|x|)\,\varphi(0))\,,
\end{aligned}
$$

where $K_{i\tau}(z)$ is the Macdonald function. Particularly, restricting F to $\mathbb{R}\setminus\{0\}$ one obtains

$$
F(x) = \sqrt{\frac{2}{\pi}}\,\frac{\tau}{x}\,K_{i\tau}(|x|)\,\exp(\frac{1}{2}\,\pi\tau\,\mathrm{sgn}\,x)\,. \tag{4.19}
$$

Proof. One starts from the equality

$$
\mathcal{F}((\kappa^2+1)^{-1/2}\exp(i\tau\,\mathrm{argsh}\,\kappa))(x) = \sqrt{2/\pi}\,K_{i\tau}(|x|)\,\exp(\tfrac{1}{2}\,\pi\tau\,\mathrm{sgn}\,x)\,. \tag{4.20}
$$

Rewriting 1 as $(\kappa^2+1)(\kappa^2+1)^{-1/2}\cdot(\kappa^2+1)^{-1/2}$, one can express F as $(-d^2/dx^2+1)$ applied to a convolution integral equaling, up to a constant factor,

$$
\begin{aligned}
&\frac{1}{\pi}\int_{I\!R} d\xi\;K_0(|x-\xi|)\,K_{i\tau}(|\xi|)\,e^{\pi\tau\,\vartheta(\xi)} \\
&= \frac{\pi}{2}\,e^{-|x|+\pi\tau\,\vartheta(x)} - \frac{1}{2}\,\mathrm{sgn}\,x\;e^{\pi\tau\,\vartheta(x)}\int_{I\!R} du\;e^{-|x|\,\cosh u}\,\frac{\sin\tau u}{\sinh u}\,.
\end{aligned}
$$

Now it is enough to make use of the simple fact that, provided $\mathrm{supp}\,f \subset \mathbb{R}_+$, $f, f' \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\lim_{x\downarrow 0} x f(x) = 0$, it holds

$$
(\frac{d}{dx}f,\varphi) = f(1)\,\varphi(0) + \int_{I\!R_+} dx\,f'(x)\,(\varphi(x) - \vartheta(1-x)\,\varphi(0))\,.
$$

We omit the details. ∎

Proposition 4.6. *For any $j = 1, 2, \ldots, N$, and $z \in \mathbb{C}\setminus[0,+\infty)$, it holds*

$$
R^{(j)}_{\varepsilon,-}\,\Lambda^{-1/2}(E_j + e^{-2\pi i\alpha_j}E^c_j) - e^{-2\pi i\alpha_j}R^{(j)}_{\varepsilon,+}\,\Lambda^{-1/2} = (e^{-2\pi i\alpha_j}-1)\,\Lambda^{-1/2}E_j\,, \tag{4.21}
$$

$$
R^{(j)}_{\varepsilon,-}\,\Lambda^{1/2}(E_j + e^{-2\pi i\alpha_j}E^c_j) + e^{-2\pi i\alpha_j}R^{(j)}_{\varepsilon,+}\,\Lambda^{1/2} = \varepsilon\,(e^{-2\pi i\alpha_j}-1)\,\Lambda^{1/2}E_j\,, \tag{4.22}
$$

with $\varepsilon = \pm$, and (4.22) should be considered on the functions from $\mathcal{F}(C_0^\infty((-\infty,a_j)) \oplus C_0^\infty((a_j,+\infty)))$.

Proof. Since both sides of (4.21) as well as of (4.22) (after having been applied to some function from the above indicated domain) depend on z analytically, we can restrict ourselves to the case $z < 0$. But owing to the properties (4.18), it is sufficient to consider the case $a_j = 0$ and $z = -1$. Further in this proof, the index j is suppressed. The composition from the right with E and E^c results in a couple of equations equivalent to the original equality, namely

$$
(R_{\varepsilon,-} - e^{-2\pi i\alpha}R_{\varepsilon,+} - (e^{-2\pi i\alpha}-1)\,\mathbf{I})\,\Lambda^{-1/2}E = 0\,, \tag{4.23a}
$$

$$(R_{\varepsilon,-} - R_{\varepsilon,+})\,\Lambda^{-1/2}E^c = 0\,, \tag{4.23b}$$

for (4.21), and

$$(R_{\varepsilon,-} + e^{-2\pi i\alpha}R_{\varepsilon,+} - \varepsilon\,(e^{-2\pi i\alpha} - 1)\,\mathbf{I})\,\Lambda^{1/2}E = 0\,, \tag{4.24a}$$

$$(R_{\varepsilon,-} + R_{\varepsilon,+})\,\Lambda^{1/2}E^c = 0\,, \tag{4.24b}$$

for (4.22). Verification is done by explicit calculations. In the case of (4.23a), expressing the kernel $R_{\varepsilon\varrho}$ with the help of the integral identity

$$\frac{e^{-\alpha u}}{1+e^{-u}} = \frac{1}{2}\int_{I\!R} d\tau\;\frac{e^{iu\tau}}{\sin\pi(\alpha+i\tau)}, \quad |\mathrm{Im}\,u| < \pi\,, \tag{4.25}$$

and applying (4.20), one finds that for $x < 0$ it holds

$$\begin{aligned}&\sqrt{2\pi}\,\langle\kappa|(R_{\varepsilon,-} - e^{-2\pi i\alpha}R_{\varepsilon,+} - (e^{-2\pi i\alpha}-1)\,\mathbf{I})\,\Lambda^{-1/2}\mathfrak{F}|x\rangle\\ &= (1-e^{-2\pi i\alpha})\,(\kappa^2+1)^{-1/4}\left(e^{-i\kappa x} - \frac{1}{\pi}\int_{I\!R} d\tau\;e^{-i\varepsilon\,\omega(\kappa)\,\tau}\,e^{-\varepsilon\pi\tau/2}\,K_{i\tau}(|x|)\right),\end{aligned}$$

where $\omega(\kappa) = \mathrm{argsh}\,\kappa$. But since $K_{i\tau}(|x|)$ is even in τ and

$$\frac{1}{\pi}\int_{I\!R} d\tau\;e^{-iu\tau}\,K_{i\tau}(|x|) = e^{-|x|\,\cosh u}, \quad |\mathrm{Im}\,u| < \frac{\pi}{2}\,, \tag{4.26}$$

the last expression is zero (in the sense of distributions). Now it suffices to recall (4.17). Similarly one can proceed also in the remaining three cases. Verifying (4.24a,b), one has to use (4.19) rather than (4.20). We omit the details. ∎

To get the formula for Ξ^{-1} one has to tensor $L^2(\mathbb{R})$ with the finite–dimensional space $\mathbb{C}^N\otimes\mathbb{C}^2$. We denote, as before, by $\{e_+, e_-\}$ the standard basis in $\mathbb{C}^2$ and $\{\epsilon_1,\ldots,\epsilon_N\}$ stands for the standard basis in $\mathbb{C}^N$. To simplify the notation we identify, where appropriate, an operator Z in $L^2(\mathbb{R})$ with $\mathbf{I}_N\otimes\mathbf{I}_2\otimes Z$ and similarly for other factors. Equipped with the algebraic relations (4.21), (4.22) one can prove

Theorem 4.7. *For any $z \in \mathbb{C}\setminus[0,+\infty)$, it holds*

$$\Xi(z)^{-1} = \frac{1}{2}\,\Lambda(z)^{-1} + \frac{1}{2}\,\Lambda(z)^{-1/2}\,(\mathbf{f}^t\otimes e_-^t\,(\mathbf{I}-\mathcal{D}\mathcal{J})^{-1}\mathcal{D}\,\mathbf{f}\otimes e_-)\,\Lambda(z)^{-1/2}\,, \tag{4.27}$$

where

$$\mathcal{D} = \mathrm{diag}(R^{(1)}(z),\ldots,R^{(N)}(z))$$

is an operator in $\mathbb{C}^N\otimes\mathbb{C}^2\otimes L^2(\mathbb{R})$,

$$\mathcal{J} = \begin{pmatrix} 0 & K & \ldots & K\\ K^t & 0 & \ldots & K\\ \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots\\ K^t & K^t & \ldots & 0\end{pmatrix}, \qquad \textit{with}\quad K = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \tag{4.28}$$

is a $2N \times 2N$ matrix, and $\mathbf{f} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_N$.

In the next Lemma as well as several times in the rest of the paper, we apply the following well–known criterion to show that an operator in $L^2(\mathbb{R})$ belongs to the trace class [12]. Suppose that $(1+|x|)^{\delta/2} f$, $(1+|x|)^{\delta/2} g \in L^2(\mathbb{R})$ for some $\delta > 1$, then the kernel $f(x)\,\hat{g}(x-y)$ corresponds to a trace class operator.

Lemma 4.8. *For $z < 0$, the operator $(\mathcal{D}\mathcal{J})^{2N}$ belongs to the trace class and its norm can be estimated by*

$$\|(\mathcal{D}\mathcal{J})^{2N}\| \leq C\,(\max_j\, \sin(\pi\alpha_j))\; K_0(wd)\,, \tag{4.29}$$

where $w = \sqrt{-z} > 0$ and $d = \min_{j\neq k}\, |a_j - a_k|$.

Proof. First observe that

$$\epsilon_j^t \otimes e_\varepsilon^t\, (\mathcal{D}\mathcal{J})^{M-1} \mathcal{D}\, \epsilon_k \otimes e_\varrho = \sum_{(j_1,\ldots,j_M)} R^{(j_1)}_{\varepsilon,\mu_2}\, R^{(j_2)}_{-\mu_2,\mu_3}\, \ldots\, R^{(j_M)}_{-\mu_M,\varrho}\,, \tag{4.30}$$

where we have put $\mu_m = \mathrm{sgn}(j_m - j_{m-1})$ and the sum runs over all finite sequences $(j_1,\ldots,j_M)$, $j_i \in \{1,\ldots,N\}$, $j_i \neq j_{i+1}$, and $j_1 = j$, $j_M = k$.

If $M = 2N$, one can find in any sequence $(j_1,\ldots,j_{2N})$ three subsequent indices j_m, j_{m+1}, j_{m+2}, such that $j_m < j_{m+1} > j_{m+2}$. Thus any summand in (4.30) contains a neighbouring couple $R^{(m)}_{\mu,+} R^{(n)}_{-,-}$. Set $d_{mn} = |a_m - a_n|$. For $w > 0$, we have (c.f. (4.2))

$$\|R^{(j)}_{\varepsilon\varrho}\| = |\sin\pi\alpha_j|\,\|\tilde{R}_{\varepsilon\varrho}(\alpha_j;w)\| = |\sin\pi\alpha_j|\,\|r_{\varepsilon\varrho}\|_\infty\,,$$

and so $\|R^{(j)}_{\varepsilon\varrho}\| \leq 1$, if $\varepsilon = \varrho$, and ≤ 2, if $\varepsilon \neq \varrho$. From the form of (4.16) it is clear that it suffices to show that the kernel

$$\langle x|\mathcal{F}\, L_w \tilde{R}_{\mu,+}(\alpha_m;w)\, Y_{d_{mn}}\, \tilde{R}_{-,-}(\alpha_n;w)\, L_w^{-1}\mathcal{F}^{-1}|y\rangle$$

corresponds to a trace class operator and to get the sought estimate for it. But this kernel can be expressed as

$$\frac{1}{\pi}\,\sin\pi\alpha_m\,\sin\pi\alpha_n\,\frac{e^{-\mu\pi x/2}}{\sin\pi(\alpha_m + ix)}\; K_{i(x-y)}(w\,d_{mn})\,\frac{e^{\pi y/2}}{\sin\pi(\alpha_n + iy)}\,.$$

By the above criterion, $e^{-\tau|x|} K_{i(x-y)}(wd)$ corresponds to a trace class operator for any $\tau > 0$ (c.f. (4.26)). Besides, one can estimate the Hilbert–Schmidt norm using $|K_{i\tau}(wd)| \leq K_0(wd)$ and

$$\int_{\mathbb{R}} dx\; |\sin\pi(\alpha + ix)|^{-2}\, e^{\pm\pi x} = (\sin\pi\alpha)^{-1}\,. \qquad \blacksquare$$

Remark It is easy to see that if $A(z)$ is a family of bounded operators in some Hilbert space, analytic in z on the disk $|z| < 1$ and such that $A(z)$ is compact for all $z \in (-1,1)$ then $A(z)$ is compact for all z from the disk. Actually, since the norm limit of compact operators is a compact operator, all the derivatives of $A(z)$ at the point $z = 0$ are compact. The assertion then follows from the Taylor expansion. Using repeatedly this argument one concludes that $(\mathcal{D}\mathcal{J})^{2N}$ is compact in the region $\mathbb{C}\setminus[0,+\infty)$. By the analytic Fredholm theorem [11],

$$(\mathbf{I}-\mathcal{D}\mathcal{J})^{-1} = \left(\mathbf{I}+\sum_{k=1}^{2N-1}(\mathcal{D}\mathcal{J})^k\right)(\mathbf{I}-(\mathcal{D}\mathcal{J})^{2N})^{-1}$$

is a meromorphic function in this region. For z negative, the estimate (4.29) tells us that a perturbative expansion into a geometric series is possible if the numbers characterizing the strength of the vortices, $\sin\pi\alpha_j$, are small enough or, on the other hand, if $|z|$ is large enough.

Proof of Theorem 4.7. The following operator identities are thought as being applied to a vector from $\mathcal{F}(C_0^\infty(I_0)\oplus\ldots\oplus C_0^\infty(I_N))$. First note that combining (4.21) and (4.22) one obtains

$$R^{(j)}_{\varepsilon,-}\Lambda^{-1/2}[(E_j+e^{-2\pi i\alpha_j}E^c_j)\,\Lambda+\Lambda(E_j+e^{-2\pi i\alpha_j}E^c_j)] = (e^{-2\pi i\alpha_j}-1)\,\Lambda^{-1/2}(E_j\Lambda+\varepsilon\,\Lambda\,E_j)\,. \tag{4.31}$$

Suppose that $k > j$ and set $\mu_{kj} = \operatorname{sgn}(k-j) = +$. As $E^c_jE^c_k = E^c_k$, one can derive from (4.23b), (4.24b) that

$$R^{(j)}_{\varepsilon,-}\Lambda^{-1/2}(E^c_k\,\Lambda+\Lambda E^c_k) = -R^{(j)}_{\varepsilon,+}\Lambda^{-1/2}(E_k\,\Lambda-\Lambda E_k)\,.$$

Combining this equality with (4.31) one finds that

$$\begin{aligned}&(e^{-2\pi i\alpha_k}-1)\,R^{(j)}_{\varepsilon,-}\Lambda^{-1/2}(E^c_k\,\Lambda+\Lambda E^c_k) \qquad (4.32)\\ &= -R^{(j)}_{\varepsilon,\mu_{kj}}R^{(k)}_{-\mu_{kj},-}\Lambda^{-1/2}[(E_k+e^{-2\pi i\alpha_k}E^c_k)\,\Lambda+\Lambda\,(E_k+e^{-2\pi i\alpha_k}E^c_k)]\,.\end{aligned}$$

Suppose now, on the contrary, that $k < j$ and set $\mu_{kj} = \operatorname{sgn}(k-j) = -$. Owing to (4.31), it holds

$$\begin{aligned}&(e^{-2\pi i\alpha_k}-1)\,R^{(j)}_{\varepsilon,-}\Lambda^{-1/2}(E_k\,\Lambda+\Lambda E_k) \qquad (4.33)\\ &= R^{(j)}_{\varepsilon,\mu_{kj}}R^{(k)}_{-\mu_{kj},-}\Lambda^{-1/2}[(E_k+e^{-2\pi i\alpha_k}E^c_k)\,\Lambda+\Lambda\,(E_k+e^{-2\pi i\alpha_k}E^c_k)]\,.\end{aligned}$$

Further notice that for any fixed j it is true

$$\begin{aligned}\Omega^{-1} \;=\; &-\sum_{k<j}(e^{-2\pi i\,\nu_k}-e^{-2\pi i\,\nu_{k-1}})\,E_k+e^{-2\pi i\,\nu_{j-1}}E_j+e^{-2\pi i\,\nu_j}E^c_j\\ &+\sum_{k>j}(e^{-2\pi i\,\nu_k}-e^{-2\pi i\,\nu_{k-1}})\,E^c_k\,.\end{aligned}$$

Using the notation

$$\mathcal{V} = \mathrm{diag}(e^{-2\pi i\,\nu_1}, e^{-2\pi i\,\nu_2}, \dots, e^{-2\pi i\,\nu_N}), \quad \mathcal{V}' = \mathrm{diag}(1, e^{-2\pi i\,\nu_1}, \dots, e^{-2\pi i\,\nu_{N-1}}),$$

$$\mathcal{E} = \mathrm{diag}(E_1, E_2, \dots, E_N), \quad \mathcal{E}^c = \mathrm{diag}(E_1^c, E_2^c, \dots, E_N^c),$$

one can derive from the expression for Ω^{-1} and from (4.32), (4.33) that

$$\begin{aligned}
&\mathcal{D}\,\Lambda^{-1/2}(\Omega^{-1}\Lambda + \Lambda\,\Omega^{-1})\;\mathbf{f}\otimes e_- \\
&= (\mathbf{I} - \mathcal{D}\mathcal{J})\mathcal{D}\,\Lambda^{-1/2}[(\mathcal{V}'\mathcal{E} + \mathcal{V}\mathcal{E}^c)\,\Lambda + \Lambda\,(\mathcal{V}'\mathcal{E} + \mathcal{V}\mathcal{E}^c)]\;\mathbf{f}\otimes e_-\,.
\end{aligned}$$

As remarked above, $(\mathbf{I} - \mathcal{D}\mathcal{J})^{-1}$ exists, possibly up to some isolated values of the spectral parameter z. Applying once more (4.31) one obtains

$$\begin{aligned}
&(\mathbf{f}^t\otimes e_-^t\;(\mathbf{I} - \mathcal{D}\mathcal{J})^{-1}\mathcal{D}\;\mathbf{f}\otimes e_-)\,\Lambda(z)^{-1/2}\,(\Omega^{-1}\Lambda + \Lambda\,\Omega^{-1}) \\
&= \mathbf{f}^t\otimes e_-^t\;\mathcal{D}\,\Lambda^{-1/2}[(\mathcal{V}'\mathcal{E} + \mathcal{V}\mathcal{E}^c)\,\Lambda + \Lambda\,(\mathcal{V}'\mathcal{E} + \mathcal{V}\mathcal{E}^c)]\;\mathbf{f}\otimes e_- \\
&= \Lambda^{-1/2}(\mathbf{f}^t\;(\mathcal{V} - \mathcal{V}')\,(\mathcal{E}\,\Lambda - \Lambda\,\mathcal{E})\;\mathbf{f})\,.
\end{aligned}$$

Since

$$\mathbf{f}^t\;(\mathcal{V} - \mathcal{V}')\,\mathcal{E}\;\mathbf{f} = e^{-2\pi i\,\nu_N} - \Omega^{-1}\,,$$

one arrives, after some simple manipulations, at

$$\frac{1}{2}\,[\Lambda(z)^{-1} + \Lambda(z)^{-1/2}\,(\mathbf{f}^t\otimes e_-^t\;(\mathbf{I} - \mathcal{D}\mathcal{J})^{-1}\mathcal{D}\;\mathbf{f}\otimes e_-)\,\Lambda(z)^{-1/2}]\,(\Lambda + \Omega^{-1}\Lambda\Omega) = \mathbf{I}\,.$$

Now it is enough to reexamine the proof of Lemma 3.1 to observe that $\mathcal{F}(C_0^\infty(I_0)\oplus \dots \oplus C_0^\infty(I_N)) \subset D(\Lambda + \Omega^{-1}\Lambda\Omega)$ is a core for Ξ (as $C_0^\infty(I)$ is dense in $\mathcal{H}_0^{1/2}(I)$). ∎

5 Wave Operators

5.1 Existence and completeness

Let us begin with recalling some basic known facts about the one–vortex scattering. This problem was solved in the stationary picture already in the original paper [1] and later reconsidered several times from different points of view, e.g., [13, 14, 15]. One can assume that the vortex is sitting in the origin. Respecting the rotational symmetry, it is natural to pass to the polar coordinates, i.e., to identify $L^2(\mathbb{R}^2, d^2x) \simeq L^2(\mathbb{R}_+, r\,dr)\otimes L^2((0,2\pi), d\phi)$. The boundary conditions (2.1a,b), determining the domain of the Hamiltonian $\hat{H} = -(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\phi^2)$, now read

$$\psi(r, 2\pi) = e^{2\pi i\alpha}\psi(r, 0), \quad \partial_\phi\,\psi(r, 2\pi) = e^{2\pi i\alpha}\partial_\phi\,\psi(r, 0)\,.$$

Though in this case much more usual choice of the Hamiltonian is $\hat{H}' = -(\partial_r^2 + r^{-1}\partial_r + r^{-2}(\partial_\phi + i\alpha)^2)$ with the 2π–periodic boundary conditions. The both Hamiltonians are unitarily equivalent, $\hat{H}' = G_\alpha^{-1}\hat{H}\,G_\alpha$, with $G_\alpha\psi(r,\phi) = e^{i\alpha\phi}\psi(r,\phi)$.

$\hat{H}'$ can be diagonalized by the Henkel transformation ($\tau > 0$),

$$\mathcal{H}_\tau : L^2(\mathbb{R}_+, k\,dk) \to L^2(\mathbb{R}_+, r\,dr) \quad , \mathcal{H}_\tau f(r) = \int_{\mathbb{R}_+} dk\; k\; J_\tau(rk)\, f(k)\,.$$

That means ($n \in \mathbb{Z}$)

$$e^{-it\hat{H}'}\; \mathcal{H}_{|n+\alpha|}(f) \otimes e^{in\phi} = \; \mathcal{H}_{|n+\alpha|}(e^{-it\,k^2}\, f) \otimes e^{in\phi}\,.$$

The stationary phase method then yields explicitly both the wave operators $W_\pm(\hat{H}', \hat{H}_0)$ and $W_\pm(\hat{H}_0, \hat{H}')$,

$$W_\pm(\hat{H}', \hat{H}_0) = s - \lim_{t\to\pm\infty} e^{it\,\hat{H}'} e^{-it\,\hat{H}_0}\,,$$

$$W_\pm(\hat{H}', \hat{H}_0)\; f \otimes e^{in\phi} = \exp(\pm\, i\tfrac{\pi}{2}(|n+\alpha| - |n|))\, f \otimes e^{in\phi}\,.$$

It is worth of emphasizing that the existence of the wave operators $W_\pm(\hat{H}, \hat{H}_0)$ does not follow immediately. The point is that while $\hat{H}'$ is unitarily transformed $\hat{H}_0$ is kept fixed. Writing

$$e^{it\,\hat{H}} e^{-it\,\hat{H}_0} = G_\alpha \left(e^{it\,\hat{H}'} e^{-it\,\hat{H}_0}\right) \left(e^{it\,\hat{H}_0} G_\alpha^{-1} e^{-it\,\hat{H}_0}\right)\,,$$

one is lead to considering the limit

$$s - \lim_{t\to\pm\infty} e^{it\,\hat{H}_0} G_\alpha^{-1} e^{-it\,\hat{H}_0}\,.$$

But its existence can be verified by an explicit calculation, again with the help of the stationary phase method [7]. So let us close this summary by pointing out that, in the case $N = 1$, the wave operators $W_\pm(\hat{H}, \hat{H}_0)$ exist and are complete.

Theorem 5.1. *For arbitrary number of vortices N, the wave operators $W_\pm(\hat{H}, \hat{H}_0)$ exist and are complete.*

Proof. We shall show the existence and completeness of $W_\pm(\hat{H}, \hat{H}_1)$, with $\hat{H}_1$ being a properly chosen one–solenoid Hamiltonian. Provided this is true, the theorem follows from the just recalled result and the chain rule. By the Kato–Birman theory [12], it suffices to show that $(\hat{H} + \mathbf{I})^{-1} - (\hat{H}_1 + \mathbf{I})^{-1}$ is a trace class operator. Using the Krein's formula (3.1) and Proposition 3.2, one finds ($z = -1$, $\mathrm{Re}\,\Lambda = \Lambda$ and U was introduced in (3.4))

$$\begin{aligned}
&(\hat{H} + \mathbf{I})^{-1} - (\hat{H}_1 + \mathbf{I})^{-1} \\
&= \frac{1}{2}\, P^* U\, \Lambda^{-1/2} \begin{pmatrix} \Xi^{-1} - \Xi_1^{-1} & \Xi^{-1}\Omega^{-1} - \Xi_1^{-1}\Omega_1^{-1} \\ \Omega\, \Xi^{-1} - \Omega_1 \Xi_1^{-1} & \Omega\, \Xi^{-1}\Omega^{-1} - \Omega_1\Xi_1^{-1}\Omega_1^{-1} \end{pmatrix} \Lambda^{-1/2} U^{-1} P\,.
\end{aligned}$$

We claim that $\Xi^{-1} - \frac{1}{2}\,\Lambda^{-1}$ is a trace class operator. In view of Theorem 4.7 and Lemma 4.8, the statement will hold provided the operators $\Lambda^{-1/2}(\mathbf{f}^t \otimes e^t_-\, (\mathcal{D}\mathcal{J})^{M-1}\mathcal{D}\; \mathbf{f} \otimes e_-)\,\Lambda^{-1/2}$, $M = 1, 2, \ldots$, are shown to belong to the trace class. To this end, let us check the summands on the RHS of (4.30) with $\varepsilon = \varrho = -$. One of the following two possibilities occurs: (i) $j_1 > j_2$ or $j_{M-1} < j_M$, (ii) $j_1 < j_2$ and $j_{M-1} > j_M$. Considering the case (i), let us assume that, say, $j_1 > j_2$. In view of (4.16), it is enough to show that $\Lambda^{-1/2}\tilde{R}_{--}$ is a trace class operator. But this can be done with the help of the criterion preceding Lemma 4.8 since (c.f. (4.3))

$$\langle x|L_1\Lambda^{-1/2}\tilde{R}_{--}\,L_1^{-1}|y\rangle = (\cosh x)^{-1/2}\,\frac{e^{-\alpha(x-y)}}{1+e^{-x+y}}\,.$$

In the case (ii), there exist three subsequent indices j_m, j_{m+1}, j_{m+2}, such that $j_m < j_{m+1} > j_{m+2}$. But this situation was considered already in the course of the proof of Lemma 4.8.

So one can replace, where appropriate, Ξ^{-1} as well as Ξ_1^{-1} by $(1/2)\Lambda^{-1}$ modulo trace class operators. Consequently, $\Xi^{-1}-\Xi_1^{-1}$ is a trace class operator. But the same is true also for $\Omega\,\Xi^{-1}\Omega^{-1}-\Omega_1\Xi_1^{-1}\Omega_1^{-1}$ as can be seen when replacing Ω by Ω^{-1} and Ω_1 by Ω_1^{-1}. The proof will be finished provided one can choose $\hat{H}_1$ and hence Ω_1 in such a way that $(\Omega-\Omega_1)\,\Lambda^{-3/2}$ and $\Lambda^{-3/2}(\Omega^{-1}-\Omega_1^{-1})$ are trace class operators. But applying once more the same criterion as above and recalling (3.2), it is easy to see that this is the case provided $\exp(2\pi i\nu)-\exp(2\pi i\,\nu_1)$ has a bounded support. ∎

5.2 Generalized eigen–functions

As explained already in Introduction, the main idea consists in considering the functions $\psi_+(z)=\varphi(z)+V(z)\varphi(z)$, with $\varphi(z)\in\mathcal{N}(z)$ being properly chosen so that the limit $z=k^2-i\sigma$, $\sigma\downarrow 0$, is possible. To be able to apply this approach one needs some information about the operators $R(z)$ in this limit. We assume that $k>0$. Let us first consider the limiting operators $\tilde{R}_{\varepsilon\varrho}(w)$, with $w=\sqrt{-z}=ik+0$. As $\operatorname{argsh}(ix)=i\arcsin x$, for $|x|\le 1$, and $\operatorname{argsh}(ix\pm 0)=i(\pi/2)\mp\operatorname{argsh}(x^2-1)^{1/2}$, for $x\ge 1$, there exists the limiting curve $\lambda\mapsto\omega(\lambda)\equiv\omega(w;\lambda)$, with $w=ik+0$. It is continuous though it is not smooth as it has two wedges in the points $z=\pm i\pi/2$. The reparameterization operator L_1 given in (4.3) is useless in this situation and must be redefined. Thus we put $L\psi := \mathcal{L}\text{'}(x)^{1/2}\psi(\mathcal{L}(x))$, where

$$\begin{aligned}\mathcal{L}(x) &= \mp k\,\cosh(x\pm\tfrac{\pi}{2})\,, &&\text{for}\quad x\le-\tfrac{\pi}{2}\quad\text{resp.}\quad x\ge\tfrac{\pi}{2}\,,\\ &= k\,\sin x\,, &&\text{for}\ -\tfrac{\pi}{2}\le x\le\tfrac{\pi}{2}\,.\end{aligned}\tag{5.1}$$

Clearly, $\mathcal{L}\in C^1(\mathbb{R})$. Setting $\eta=\omega\circ\mathcal{L}$, we have

$$\begin{aligned}\eta(x) &= x\pm\tfrac{\pi}{2}\pm i\tfrac{\pi}{2}\,, &&\text{for}\quad x\le-\tfrac{\pi}{2}\quad\textit{resp.}\quad x\ge\tfrac{\pi}{2}\,,\\ &= -ix\,, &&\text{for}\ -\tfrac{\pi}{2}\le x\le\tfrac{\pi}{2}\,.\end{aligned}$$

The functions $\Phi[h]\equiv\Phi_w[h]$ (c.f. (4.5)), with $w=ik+0$, are well defined and belong to $L^2(\mathbb{R})$ provided there exists $\theta>\pi/2$ such that $e^{\theta|x|}h\in L^2(\mathbb{R})$. The estimate (4.6) is again valid. This is easy to see from the equality $L\,\Phi[h](x)=\eta'(x)^{1/2}\,\check{h}(\eta(x))$. Hence then definition (4.9) of $\tilde{R}_{\varepsilon\varrho}(w)$ makes sense also in this limiting case.

Lemma 5.2. *The limiting operators $\tilde{R}_{\varepsilon\varrho}(w)$, with $w=ik+0$, are densely defined, bounded and hence, after having been closed, defined everywhere. Moreover, it holds*

$$s-\lim_{\sigma\downarrow 0}\tilde{R}_{\varepsilon\varrho}(ik+\sigma)=\tilde{R}_{\varepsilon\varrho}(ik+0)\,.$$

Proof. Concerning the density, the argument given in Lemma 4.1 is not applicable here. Suppose that $f \in L^2(\mathbb{R})$ is orthogonal to all functions $L\,\Phi[h]$, with h fulfilling $e^{\theta|x|}h \in L^2(\mathbb{R})$ for some $\theta > \pi/2$. Given any $\tau > 0$, we shall show that $f(x) = 0$ for almost all x, $x < -\frac{\pi}{2} - \tau$. Set $\phi = \arctan(\pi\,\tau^{-1}) \in (0, \pi/2)$. To any c positive relate the function $F_c(x) = M_c\,\exp(-e^{i\phi}\,c^{-2}x^2)$. The normalization constant $M_c = c^{-1}M_1$ is chosen so that $\int_{\mathbb{R}} dx\; F_c(x) = 1$. F_c extends as an entire function and $F_c(\cdot + iy) \in L^2(\mathbb{R})$ for all $y \in \mathbb{R}$. It follows that to any $b < -\frac{\pi}{2} - \tau$, there exists a unique function h_{bc} fulfilling $e^{\theta|x|}h_{bc} \in L^2(\mathbb{R})$, for all $\theta > 0$, and $L\,\Phi[h_{bc}](x) = F_c(x - b)$, for all $x < -\pi/2$. Moreover, since $\arg(\eta(x) - \frac{\pi}{2} - i\frac{\pi}{2} - b) \in [-\phi, 0]$, for all $x \in \mathbb{R}$, it holds

$$|L\,\Phi[h_{bc}](x)| \le M_c\,\exp\left(-\cos(\phi)\,c^{-2}\,|\eta(x) - \frac{\pi}{2} - i\frac{\pi}{2} - b|^2\right).$$

Now it is easy to find that $\int dx\,\bar{f}(x)\,\varphi(x) = 0$, for each $\varphi \in C_0^\infty((-\infty, -\frac{\pi}{2} - \tau))$. This is a consequence of the fact that $F_c(x-b)$ converges to $\delta(x-b)$, as $c \downarrow 0$, in the space of distributions and f is orthogonal to $L\,\Phi[h_{bc}]$ and hence also to $\int db\; F_c(x-b)\,\varphi(b)$. Thus the above assertion is verified. But since $\tau > 0$ was arbitrary $f(x) = 0$ for a. a. $x < -\pi/2$. Similarly, $f(x) = 0$ for a. a. $x > \pi/2$. To show that $f(x)$ vanishes also on the interval $|x| < \pi/2$, it is enough to consider the entire functions $F_n(z) = z^n\,\exp(-z^2)$, $n = 0, 1, 2, \ldots$, to relate to them functions h_n so that $\check{h}_n = F_n$ and to apply the orthogonality argument.

Concerning the fact that $\tilde{R}_{\varepsilon\varrho}(w)$ are bounded, note that the formula (4.12) is still true. The only slight change required is to adjust the regularization since the complex number i is tangent to the central part of the curve $\lambda \mapsto \omega(\lambda)$. To be precise one has to write, say, $(\omega(\kappa) - \omega(\lambda) \mp e^{i\pi/4}\epsilon)$, $\epsilon \downarrow 0$. Considering the operators $L\,\tilde{R}_{\varepsilon\varrho}(w)\,L^{-1}$ rather than $\tilde{R}_{\varepsilon\varrho}(w)$ and splitting the Hilbert space as $L^2(\mathbb{R}) = L^2((-\infty, -\pi/2)) \oplus L^2((-\pi/2, \pi/2)) \oplus L^2((\pi/2, +\infty))$, one can proceed exactly the same way as in the proof of Lemmas 4.2 and 4.3. We omit the details.

Let us now turn to the strong limit. Here we focus only on the most troublesome case of the kernel (4.10), with $\omega(\kappa) = \operatorname{argsh}(w^{-1}\kappa)$ and $(2\pi)^{-1/2}\,\check{q}(z) = (z - e^{i\pi/4}\epsilon)$, $\epsilon \downarrow 0$. Owing to the scaling properties we can assume that $k = 1$. So we are going to consider the limit $w^{-1} = (-1 - i\sigma)^{1/2}$, $\sigma \downarrow 0$. The difficulties are caused by the appearance of the two wedge points of the curve $\lambda \mapsto \omega(\lambda)$ in the limiting case. After cutting off some δ–neighbourhoods of the points $\kappa = \pm 1$ in the real line, one can proceed the same way as in Lemma 4.3 not to show only the existence of the strong limit $\epsilon \downarrow 0$ when regularizing the kernel but also to verify even the norm continuity in the parameter σ, $\sigma \downarrow 0$. Since $C_0^\infty(\mathbb{R} \setminus \{-1, 1\})$ is dense in $L^2(\mathbb{R})$ it is enough to show that the operator Q is bounded uniformly in the parameter σ. Furthermore, we can concentrate only on some vicinity of the wedge points. To be specific, let us restrict Q (by means of sandwiching it by orthogonal projectors) to the subspace $L^2((1 - \delta, 1 + \delta))$. The idea is to approximate the curves $\lambda \mapsto \omega(\lambda)$, as σ tends to zero, by hyperbolas for they are comparatively easy to handle.

Set $\omega_A(\kappa) = -(-1 - i\sigma)^{1/2}(\kappa^{-2} - 1 - i\sigma)^{1/2}$. Clearly, ${\omega_A}'(\kappa) = \kappa^{-2}\omega'(\kappa)$. Let us rotate the curves by $\pi/4$,

$$\tilde{\omega}(\kappa) := e^{i\pi/4}(1 + i\sigma)^{-1/2}[\omega(\kappa) - \omega(1)], \quad \tilde{\omega}_A(\kappa) := e^{i\pi/4}(1 + i\sigma)^{-1/2}\omega_A(\kappa)\,.$$

The kernel can be now written as $\langle\kappa|Q|\lambda\rangle = \tilde{\omega}'(\kappa)^{1/2}\,(\tilde{\omega}(\kappa)-\tilde{\omega}(\lambda)-i0)^{-1}\,\tilde{\omega}'(\lambda)^{1/2}$. Let $\tilde{\mathfrak{L}}(x)$ be the inverse to the function $\kappa\mapsto \operatorname{Re}\tilde{\omega}(\kappa)$ and, similarly, $\tilde{\mathfrak{L}}_A(x)$ is the inverted function to $\kappa\mapsto\operatorname{Re}\tilde{\omega}_A(\kappa)$. Set $\beta=\tilde{\omega}\circ\tilde{\mathfrak{L}}$, $\beta_A=\tilde{\omega}_A\circ\tilde{\mathfrak{L}}_A$ and $\varphi=\tilde{\mathfrak{L}}_A^{-1}\circ\tilde{\mathfrak{L}}$. It holds $\varphi(0)=0$, $\varphi'(x)=\tilde{\mathfrak{L}}(x)^{-2}$ and so $|\varphi(x)|\geq C_1\,|x|$, with $C_1>0$. Furthermore, $\operatorname{Re}\beta(x)=\operatorname{Re}\beta_A(x)=x$, $\beta'(x)=\beta_A{}'(\varphi(x))$ and one can check that $\beta_A(x)=x+i\,(\sigma+x^2)^{1/2}$. Reparameterizing and taking onto account that $1\leq|\beta'(x)|\leq\sqrt{2}$, one can switch to the kernel $\langle x|\tilde{Q}|y\rangle=(\beta(x)-\beta(y)-i0)^{-1}\beta'(y)$, with $\tilde{Q}$ acting in $L^2((-\delta_1,\delta_2))$, $(-\delta_1,\delta_2)=\tilde{\mathfrak{L}}^{-1}((1-\delta,1+\delta))$. Now the method of the proof of Lemma 4.3 is again applicable since $\sqrt{2}\,|x-y|\geq|\beta(x)-\beta(y)|\geq|\operatorname{Re}\beta(x)-\operatorname{Re}\beta(y)|=|x-y|$, $|1-\beta'(x)|$ is bounded independently of σ and $|\beta''(x)|^2\leq(1-\delta)^{-2}|\beta_A''(\varphi(x))|^2\leq C_2\,\sigma^2(\sigma+x^2)^{-3}$. ∎

We need also some estimates to show that $(\mathbf{I}-\mathcal{D}\mathcal{J})^{-1}$ exists also in the limiting case.

Lemma 5.3. *Let $d_0,d_1,\ldots,d_n$ and k_0 be some positive constants, $u,v\in\mathbb{R}$ and $k\geq k_0$ be a parameter. Then the absolute value of the integral*

$$\int_{\mathbb{R}^n} d^n\tau\; K_{i(u-\tau_1)}(ik\,d_0)\,\operatorname{sgn}\tau_1\;K_{i(\tau_1-\tau_2)}(ik\,d_1)\ldots\operatorname{sgn}\tau_n\;K_{i(\tau_n-v)}(ik\,d_n) \tag{5.2}$$

can be estimated from above by the expression

$$C\,k^{-1/2}\,(1+k^{-1/2}|v|)^n\,\exp\left(\frac{\pi}{2}\,(|u|+|v|)\right),$$

where the constant $C>0$ depends only on $d_0,\ldots,d_n$ and k_0. The estimate is valid also for $n=0$ provided the integral is replaced by $K_{i(u-v)}(ik\,d_0)$.

Proof. Recalling (4.26) and the fact that the Fourier image of $\operatorname{sgn}\tau$ equals $-i(2/\pi)^{1/2}\,P(1/x)$, with P indicating the regularization in the sense of Cauchy principal value, one can rewrite (5.2) with the help of the Fourier transform as

$$\frac{1}{2}\,(-i)^n\int_{\mathbb{R}^{n+1}} d^{n+1}x\;\exp(iu\,x_0-iv\,x_n)\,\exp\left(-ik\sum_{j=0}^{n}d_j\cosh x_j\right)\prod_{j=1}^{n}P\frac{1}{x_{j-1}-x_j}\,.$$

The substitution $x_j=2\operatorname{argsh}(k^{-1/2}s_j)$ yields

$$\begin{aligned}
&k^{-1/2}\,(-i)^n\,\exp\left(-ik\sum_{j=0}^{n}d_j\right)\int_{\mathbb{R}^{n+1}}d^{n+1}s\\
&\times\exp(2iu\operatorname{argsh}(k^{-1/2}s_0)-2iv\operatorname{argsh}(k^{-1/2}s_n))\,\prod_{j=0}^{n}(1+k^{-1}\,s_j^2)^{-1/2}\\
&\times\prod_{j=1}^{n}\left\{k^{-1/2}(s_{j-1}-s_j)\,[\operatorname{argsh}(k^{-1/2}s_{j-1})-\operatorname{argsh}(k^{-1/2}s_j)]^{-1}\right\}\\
&\times\prod_{j=1}^{n}P\frac{1}{s_{j-1}-s_j}\;\exp\left(-2i\sum_{j=0}^{n}d_j\,s_j^2\right).
\end{aligned}$$

Now it is possible to rotate the variables s_j, each in its own complex plane, by the angle $-\pi/4$. As a result, the Gaussian term $\exp(-2\sum d_j\, s_j^2)$ appears in the integrand. To get rid of the singular terms, perform another substitution $s_m = \sum_{j\le m} t_j$ and recall that, in the sense of distributions, $P(1/t) = d/dt \ln|t|$. Integration by parts produces a polynomial of degree n in the variable $k^{-1/2}v$. Since $|\mathrm{Im}\ \mathrm{argsh}(e^{-i\pi/4}s)| < \pi/4$, for $s \in \mathbb{R}$, it is not difficult to see that the integrand can be estimated by $(1 + k^{-1/2}|v|)^n \exp(\pi(|u| + |v|)/2)\, F(t)$, with $F \in L^1(\mathbb{R}^{n+1})$ being independent of u, v and k. ∎

Lemma 5.4. *In the limiting case $z = k^2 - i0$, $\mathcal{D}(\mathcal{J}\mathcal{D})^{4N-3}$ is a Hilbert–Schmidt operator and its norm decays as $k^{-1/2}$ for $k \to +\infty$.*

Proof. First note that when evaluating the kernel of the composition Q_1Q_2, with Q_1 and Q_2 being of type (4.10), one has to integrate along the curve $z = \omega(\lambda)$, $\lambda \in \mathbb{R}$, in the complex plane. This concerns also the operators $\tilde{R}_{\varepsilon\varrho}(w)$, $w = \sqrt{-z} = ik + 0$. The corresponding kernels are given by (4.1), for $\varepsilon = \varrho$, and by (4.12), for $\varepsilon \neq \varrho$. From the form of (4.16) it is clear that when evaluating the entries of the matrix of operators $\mathcal{D}\mathcal{J}\mathcal{D}$ one is confronted with the expressions of the type

$$\tilde{R}_{\varepsilon\mu}(\alpha_i; w)\, Y_d\, \tilde{R}_{-\mu,\varrho}(\alpha_j; w)\,,$$

where $d = |a_i - a_j|$. The operator Y_d contributes to the integration by the weight $\exp(kd \sinh z)$. Now it is easy to see that the integration curve $z = \omega(\lambda)$, $\lambda \in \mathbb{R}$, can be deformed to $z = -i\frac{\pi}{2} + s$, $s \in \mathbb{R}$. It may happen that while being deformed the curve passes through a pole of the integrand and thus, by the Cauchy theorem, additional summands appear in the final expression. This case occurs when $\varepsilon = +$, $\mu = -$ and $\varrho = +$ but it is avoided when ε is arbitrary, $\mu = +$, $\varrho = -$, and also when $\varepsilon = \mu = -$ and ϱ is arbitrary. Nevertheless, whatever case happens it is important to observe that the kernels resulting in the process of deformation, namely ($\kappa, x \in \mathbb{R}$ are the variables)

$$\omega'(\kappa)^{1/2} \exp(-\alpha(\omega(\kappa) - x - i\varepsilon\tfrac{\pi}{2}))\, [1 + \exp(-\omega(\kappa) + x + i\varepsilon(\tfrac{\pi}{2} - 0))]^{-1}\,,$$

$$\omega'(\kappa)^{1/2}\, \check{p}_+(\omega(\kappa) - x + i\tfrac{\pi}{2}), \quad \text{and} \quad \omega'(\kappa)^{1/2}\, (\omega(\kappa) - x + i\tfrac{\pi}{2} + i0)^{-1}\,,$$

correspond to bounded operators with norms independent of k.

Next let us examine the equality (4.30) with $M = 4N - 2$. It is not difficult to see that in a finite sequence of couples of signs, $(\varepsilon, \mu_2)(-\mu_2, \mu_3)\ldots(-\mu_M, \varrho)$, there exists a subsequence of the type $(--)\,(+-)^m\,(++)\,(-+)^n\,(--)(+\varrho')$, with $m, n \in \mathbb{N}_0$. Consequently, each summand on the RHS of (4.30) contains a subchain corresponding to this subsequence and when evaluating the kernel of this composite operator one can deform integration curves in the complex plane as described above. The most decisive part of the subsequence is $(++)\,(-+)^n\,(--)$. The role of the terms standing to the left and to the right is to enable additional deformation of the integration curves. Bearing in mind that the norm of $R^{(j)}(z)$ does not depend on k, we conclude that is is enough to show that the kernel (x, y are the variables)

$$\int_{\mathbb{R}^{n+1}} d^{n+1}s\, \frac{\exp(-\beta_0(x-s_1))}{1+\exp(-x+s_1)}$$
$$\times \left(\prod_{j=1}^{n} \exp(-ik\, d_j\, \cosh s_j)\, \frac{\exp(-\beta_j(s_j - s_{j+1} - i\pi))}{1+\exp(-s_j + s_{j+1} + i(\pi - 0))} \right)$$
$$\times \exp(-ik\, d_{n+1}\, \cosh s_{n+1})\, \frac{\exp(-\beta_{n+1}(s_{n+1} - y))}{1+\exp(-s_{n+1} + y)}\,,$$

where the β's belong to $(0,1)$ and the d's are positive, corresponds to a Hilbert–Schmidt operator with a norm decaying as $k^{-1/2}$ for $k \to +\infty$.

Applying the Fourier transformation and the relations (4.25) and (4.26), the latter one with u real but with $|x|$ replaced by ik, one obtains another kernel in the variables $u, v \in \mathbb{R}$,

$$\pi\, \frac{1}{\sin\pi(\beta_0 + iu)} \int_{\mathbb{R}^n} d^n\tau\, K_{i(u-\tau_1)}(ik\, d_1)\, \frac{\exp(\pi\tau_1)}{\sin\pi(\beta_1 + i\,\tau_1)}$$
$$\times \left(\prod_{j=2}^{n} K_{i(\tau_{j-1}-\tau_j)}(ik\, d_j)\, \frac{\exp(\pi\tau_j)}{\sin\pi(\beta_j + i\,\tau_j)} \right) K_{i(\tau_n - v)}(ik\, d_{n+1})\, \frac{1}{\sin\pi(\beta_{n+1} + iv)}\,.$$

Next decompose each multiplier $e^{\pi\tau}/\sin\pi(\beta + i\tau)$ as it was done in (4.11) and write $1 + \operatorname{sgn}\tau$ instead of $2\,\vartheta(\tau)$. The identity

$$\int_{\mathbb{R}} d\tau\, K_{i(u-\tau)}(ic_1)\, K_{i(\tau-v)}(ic_2) = \pi\, K_{i(u-v)}(i(c_1+c_2))$$

then implies that the resulting expression is a finite sum of kernels and each of them, by the estimate given in Lemma 5.3, has a finite Hilbert–Schmidt norm decaying as $k^{-1/2}$ for $k \to +\infty$. ∎

As a final auxiliary result we shall need another algebraic identity.

Lemma 5.5. *For any $z \in \mathbb{C} \setminus [0, +\infty)$, it holds*

$$\Omega\,\Xi^{-1} = \frac{1}{2}\, e^{2\pi i\,\nu_N}\, [\Lambda^{-1} + \Lambda^{-1/2}\, (\mathbf{f}^t \otimes e_+^t\, \mathcal{D}(\mathbf{I} - \mathcal{D}\mathcal{J})^{-1}\, \mathbf{f} \otimes e_-)\, \Lambda^{-1/2}]\,, \tag{5.3}$$

Proof. The proof follows the same lines as the proof of Theorem 4.7 and so we omit it. Note only that instead from (4.23a,b) one has to start from the adjoint relations, namely

$$E\,\Lambda^{-1/2}(R_{-,\varepsilon} - e^{2\pi i\alpha} R_{+,\varepsilon} - (e^{2\pi i\alpha} - 1)\,\mathbf{I})\, = 0\,,$$

$$E^c \Lambda^{-1/2}(R_{-,\varepsilon} - R_{+,\varepsilon}) = 0\,. \qquad ∎$$

Let us now turn to the construction of generalized eigen–functions. We assume that $z = k^2 - i\sigma$, $\sigma > 0$, and $k > k_0$ is large enough so that, by Lemma 5.4, $(\mathbf{I} - \mathcal{J}\mathcal{D})^{-1}$

exists also in the limit $\sigma \downarrow 0$. In the relation (3.6) perform the substitution $\varphi_+ = (2\,\mathrm{Re}\,\Lambda)^{-1/2} h$, with $h \in C_0^\infty((-k,k))$, to get

$$
\begin{aligned}
f(z;\kappa,u) &= (\exp(-\sqrt{\kappa^2 - z}\,u) - \exp(-\sqrt{\kappa^2 - \bar z}\,u))\,h(\kappa) && (5.4)\\
&+ \exp(-\sqrt{\kappa^2-\bar z}\,u)\,\Xi(\bar z)^{-1}(\Lambda(\bar z)-\Lambda(z))\,h(\kappa)\,, && \text{for } u>0\,,\\
&= \exp(\sqrt{\kappa^2-\bar z}\,u)\,\Omega\,\Xi(\bar z)^{-1}(\Lambda(\bar z)-\Lambda(z))\,h(\kappa)\,, && \text{for } u<0\,.
\end{aligned}
$$

Redenote

$$
\psi_+(z;x) := \mathcal{F}^{-1}_{\kappa\to x_1} f(z;\kappa,x_2)\,. \tag{5.5}
$$

Hence $\psi_+(z;x) = U(z)\varphi + U(\bar z)\tilde V(z)\varphi$ with $\varphi = (\varphi_+,0) \in \mathbb{C}^2 \otimes L^2(\mathbb{R})$. Let us perform the substitution $\kappa = k\,\mathcal{L}(s)$ (c.f. (5.1)) in the Fourier integral (5.5) and observe that the operator A with the kernel $\langle s|A|\lambda\rangle = k\,\mathcal{L}'(s)\,\langle k\,\mathcal{L}(s)|\Lambda^{-1/2} R^{(j)}_{\varepsilon\varrho}|\lambda\rangle$ is bounded in $L^2(\mathbb{R})$ even in the limiting case $\sigma = 0$. On the basis of identities (4.27), for $\Xi(\bar z)^{-1}$, and (5.3), for $\Omega\,\Xi(\bar z)^{-1}$, one can conclude that the function $\mathcal{L}'(\cdot)\,f(z;k\,\mathcal{L}(\cdot),u)$ belongs to $L^2(\mathbb{R})$, for all $u \in \mathbb{R}$. Owing to the rapid decay in the infinity, this function is L^1–integrable, too. This is true also in the limit $\sigma \downarrow 0$ and so $\psi_+(z;x)$, with $z = k^2 - i0$, makes sense as well. To any density $g \in C_0^\infty((k_0,+\infty))$ relate the integral

$$
F_\sigma(x) := \int_{\mathbb{R}_+} dk\; g(k)\,\psi_+(k^2 - i\sigma;x)\,. \tag{5.6}
$$

Decomposing $\psi_+ = \psi_1 + \psi_2$, with

$$
\psi_2(x) = \frac{1}{\sqrt{2\pi}} \int_{|\kappa|>k} d\kappa\; e^{i\kappa\,x_1}\, f(z;\kappa,x_2)\,,
$$

and using the substitution $\kappa = k\,\sin\theta$, one can rewrite

$$
\begin{aligned}
\int_{\mathbb{R}_+} dk\; g(k)\,\psi_1(k^2-i\sigma;x) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} d\theta \int_{\mathbb{R}_+} dk \sum_{\varepsilon=\pm} e^{ik(x_1\sin\theta + \varepsilon\,x_2\cos\theta)}\\
&\times\; k\,\cos\theta\; g(k)\,\tilde f_\varepsilon(k^2-i\sigma;\theta)\,. && (5.7)
\end{aligned}
$$

The functions $\tilde f_\varepsilon(k^2 - i\sigma;\theta)$ can be easily extracted from (5.4). As $\psi_2 \in L^2(\mathbb{R}^2)$ and the RHS of (5.7) is a Fourier–Plancherel integral rewritten in the polar coordinates (k,θ), one finds that F_σ is L^2–integrable. Moreover, F_σ converges to F_0, as $\sigma \downarrow 0$, in the Hilbert space $L^2(\mathbb{R}^2)$. Because

$$
\hat H(U(z)\varphi + U(\bar z)\,\tilde V(z)\varphi) = z\,U(z)\varphi + \bar z\,U(\bar z)\,\tilde V(z)\varphi
$$

and $\hat H$ is closed one obtains in the limit

$$
\hat H\,F_0(x) = \int_{\mathbb{R}_+} dk\; k^2\,g(k)\,\psi_+(k^2 - i0;x)\,. \tag{5.8}
$$

Notice that, because of the definition of $\Lambda(z)$, $f(k^2-i0;\kappa,u)$ is supported in $\{|\kappa|\le k\}$. Set

$$\psi_0(k;x):=\frac{1}{\sqrt{2\pi}}\int_{|\kappa|<k}d\kappa\ \exp(i\kappa\,x_1-i\sqrt{k^2-\kappa^2}\,x_2)\,h(\kappa)\,. \tag{5.9}$$

Dropping off the terms decaying exponentially in x_2 and using the substitution $\kappa=k\cos\theta$, with $\theta\in(0,\pi)$ for $x_2>0$ and $\theta\in(\pi,2\pi)$ for $x_2<0$, one obtains the asymptotics as $|x_2|\to+\infty$,

$$\begin{aligned}\psi_+(k^2-i0;x)\ &\simeq\ \psi_0(k;x)\,\vartheta(x_2)+\int_0^\pi d\theta\ \exp[ik(x_1\cos\theta+x_2\sin\theta)]\,S(\theta)\,\vartheta(x_2)\\ &+\ \int_\pi^{2\pi}d\theta\ \exp[ik(x_1\cos\theta+x_2\sin\theta)]\,S(\theta)\,\vartheta(-x_2)\,.\end{aligned} \tag{5.10}$$

$S(\theta)$ is some function belonging to $L^2((0,2\pi))$.

Theorem 5.6. *The functions $\psi_+(k^2-i0;x)$ given in (5.5), with $k>k_0$ large enough, are generalized eigen–functions of the Hamiltonian $\hat{H}$ corresponding to the value k^2. Moreover, it holds $W_-\psi_0(k;x)=\psi_+(k^2-i0;x)$ in the usual sense, i.e., for an arbitrary density $g\in C_0^\infty((k_0,+\infty))$ it is true*

$$W_-\left(\int_{I\!R_+}dk\ g(k)\,\psi_0(k;x)\right)=\int_{I\!R_+}dk\ g(k)\,\psi_+(k^2-i0;x)\,. \tag{5.11}$$

Proof. It remains to show that the limit $t\to-\infty$ of the expression

$$\int_{I\!R^2}d^2x\,\left|\int_{I\!R_+}dk\ e^{-it\,k^2}\,g(k)\,(\psi_+(k^2-i0;x)-\psi_0(k;x))\right|^2$$

is zero. Compare (5.4), (5.5) with (5.9) to find that one can approximate this integral with an error, arbitrarily small and independent of t, by replacing $\psi_+(k^2-i0;x)-\psi_0(k;x)$ by $\xi(z;x)=\mathcal{F}^{-1}_{\kappa\to x_1}\rho(z;\kappa,x_2)$, where

$$\begin{aligned}\rho(z;\kappa,u)\ &=\ -\exp(-\sqrt{\kappa^2-\bar{z}}\,u)\,h(\kappa)+2\,\exp(-\sqrt{\kappa^2-\bar{z}}\,u)\,\Xi(\bar{z})^{-1}\Lambda(\bar{z})\,h(\kappa)\,,\\ &\qquad\text{for } u>0\,,\\ &=\ -\exp(\sqrt{\kappa^2-\bar{z}}\,u)\,h(\kappa)+2\,\exp(\sqrt{\kappa^2-\bar{z}}\,u)\,\Omega\,\Xi(\bar{z})^{-1}\Lambda(\bar{z})\,h(\kappa)\,,\\ &\qquad\text{for } u<0\,,\end{aligned}$$

$z=k^2-i\sigma$ and $\sigma>0$ is sufficiently small. Consequently, it is enough to verify that the limit is zero after this replacement. By the Lebesgue criterion, one can interchange the limit and the integration in the variable x_1. Concerning the variable x_2, the stationary phase method is applicable since $\xi(k^2-i\sigma;x)$ depends on k smoothly (even analytically). ∎

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DEPARTMENT OF MATHEMATICS AND DOPPLER INSTITUTE,
FACULTY OF NUCLEAR SCIENCE, CTU,
TROJANOVA 13, 120 00 PRAGUE, CZECH REPUBLIC