Saddle-node cycles and prevalence of strange attractors

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Abstract

We consider parametrized families of diffeomorphisms bifurcating through the creation of critical saddle-node cycles and we show that they exhibit Hénon-like strange attractors for a set of parameter values which has positive Lebesgue density at the bifurcation value. This is the first example of a bifurcation mechanism displaying such prevalence of Hénon-like chaotic behaviour. Furthermore, for open classes of these families the bifurcation parameter is also a point of positive density of hyperbolic dynamics.

1 Introduction

In this paper we address the problem of determining which dynamical features are more frequently displayed by parametrized families of diffeomorphisms in the sequel of a bifurcation. *Frequency* is to be understood in the sense of Lebesgue measure in parameter space and the general question may be stated as follows, see [PT2, Chapter 7]. Let M be a smooth manifold of dimension $m \geq 2$ and $f\colon M \longrightarrow M$ be a diffeomorphism exhibiting a bifurcation of codimension $l \geq 1$. Let $k \geq l$ and $(f_\mu)_\mu$ be a generic k-parameter family of diffeomorphisms on M passing through f, say $f_0 = f$. Then one aims at describing the dynamical phenomena exhibited by f_μ for a set E of values of μ with positive Lebesgue measure (*persistent* phenomena) and at determining which of these are even *prevalent* near the bifurcation, meaning that $m(E \cap (-\varepsilon, \varepsilon)^k) \geq \text{const}\, \varepsilon^k$ for some const > 0 and every small $\varepsilon > 0$.

We deal with the unfolding of *critical saddle-node cycles* by smooth 1-parameter families $(f_\mu)_{\mu \in \mathbf{R}}$; *smooth* means that $\mathbf{R} \times M \ni (\mu, z) \mapsto f_\mu(z) \in M$ is a C^∞ map. Critical saddle-node cycles present a special interest from the point of view of the problem we have just stated. On the one hand, they exhibit a very complex bifurcation diagram, including nearly every known form of complicated and/or unstable dynamics. A converse is also true: the formation/destruction of such cycles is part of the unfolding of other main bifurcation mechanisms, e.g. homoclinic tangencies. On the other hand, for the unfolding of critical saddle-node cycles we are able to give here a fairly precise description of the dynamics related to the bifurcation, leading to stronger statements than have been obtained so far for such related mechanisms with a similar degree of complexity. We focus on analysing the occurrence of strange attractors *vs.* hyperbolicity of the dynamics, see below. In very brief terms, we show that Hénon-like strange attractors are always a prevalent phenomenon in the unfolding of such a cycle and the same holds for hyperbolicity in an open set of cases. Before we explain these ideas in more detail and state our results, let us give the precise definitions of the main notions involved.

By a *saddle-node* of a diffeomorphism $f\colon M \longrightarrow M$ we understand a periodic orbit $\mathcal{O}(p_0)$, of period $t_0 \geq 1$ say, such that $Df^{t_0}(p_0)$ has a unique eigenvalue in the unit circle and this is equal to 1. In what follows we always take saddle-nodes to be normally contracting: all the other eigenvalues are strictly less than 1 in norm. Thus the tangent space to M at p_0 admits a $Df^{t_0}(p_0)$-invariant splitting $T_{p_0}M = E^c \oplus E^{ss}$ with $\dim E^c = 1$, $Df^{t_0}(p_0)|E^c = \text{id}$ and $Df^{t_0}(p_0)|E^{ss}$ a contraction. According to invariant manifold theory, [HPS], there exist 1-dimensional *center manifolds* W^c_0, tangent to E^c at p_0 and locally invariant under f^{t_0} in the sense that $f^{t_0}(W^c_0)$ contains a neighbourhood of p_0 in W^c_0. We also assume all saddle-nodes to be nondegenerate – or *quadratic* – meaning that $(f^{t_0}|W^c_0)''(p_0) \neq 0$ for some choice of a (twice differentiable) center manifold W^c_0; actually, in this case one may take W^c_0 to be C^∞, see e.g. [Sh].

We say that the diffeomorphism f exhibits a *saddle-node ℓ-cycle*, $\ell \geq 1$, if it has periodic orbits $\mathcal{O}(p_0)$, $\mathcal{O}(p_1)$, …, $\mathcal{O}(p_{\ell-1})$, where $\mathcal{O}(p_0)$ is a saddle-node and $\mathcal{O}(p_i)$, $1 \leq i \leq \ell - 1$, are hyperbolic saddles, satisfying

- $W^u(\mathcal{O}(p_{i-1}))$ has some transverse intersection with $W^s(\mathcal{O}(p_i))$, for every $1 \leq i \leq \ell$;

- $W^u(\mathcal{O}(p_{\ell-1}))$ intersects the interior of $W^s(\mathcal{O}(p_0))$.

For $\ell = 1$ we require $W^u(\mathcal{O}(p_0))$ to be contained in the interior of $W^s(\mathcal{O}(p_0))$. Note that $W^u(\mathcal{O}(p_0)) = \cup_{i=0}^{t_0-1} f^i(W^u(p_0))$ and $W^u(p_0)$ is (strictly) contained in W^c. On the other hand, $W^s(\mathcal{O}(p_0))$ is the union of the $f^i(W^s(p_0))$, $0 \leq i \leq t_0 - 1$, and $W^s(p_0) \subset M$ is a (codimension zero) submanifold with boundary: $\partial W^s(p_0) = W^{ss}$ is the *strong-stable manifold* of p_0, i.e. the unique f^{t_0}-invariant submanifold satisfying $T_{p_0} W^{ss} = E^{ss}$, see [HPS].

Following [NPT], we say that a saddle-node cycle is *critical* if $W^u(\mathcal{O}(p_{\ell-1}))$ has some point χ of nontransverse intersection with a leaf F of $\mathcal{F}^{ss}$. Here $\mathcal{F}^{ss}$ denotes the *strong-stable foliation* of p_0, i.e. the unique $f_0^{t_0}$-invariant foliation of $W^s(p_0)$ by codimension-1 submanifolds, having W^{ss} as a leaf, see [NPT, Theorem II.3.4]. We always assume the criticality to be generic: $W^u(p_1)$ and F have a *quadratic* contact at χ. Two examples of critical cycles – a 1-cycle and a 2-cycle – are depicted in the figure. Note that this 2-cycle may be may be thought of as the result of collapsing a periodic attractor with a saddle point contained in a horseshoe; clearly, such an example also exhibits ℓ-cycles for every $\ell \geq 2$.

Now let $(f_\mu)_{\mu \in \mathbf{R}}$ be a smooth family of diffeomorphisms with $f_0 = f$. Then, [HPS], there exist two-dimensional submanifolds W^c of $\mathbf{R} \times M$, tangent to $\mathbf{R} \times E^c$ at $(0, p_0)$ and locally invariant under $\mathbf{R} \times M \ni (\mu, z) \mapsto (\mu, f_\mu^{t_0}(z)) \in \mathbf{R} \times M$. For each such W^c we have $W_0^c = W^c \cap (\{0\} \times M)$ a center manifold for f_0. We say that $(f_\mu)_\mu$ *unfolds* the saddle-node – and the saddle-node cycle – *generically* if one may take W^c so that $\partial_\mu(\varphi | W^c)(0, p_0) \neq 0$, where $\varphi(\mu, z) = f_\mu^{t_0}(z)$. In more precise terms: there are local coordinates (μ, x) on W^c, with respect to which $\varphi | W^c$ is given by

$$\varphi(\mu, x) = x + v\mu + \alpha x^2 + \beta \mu x + \gamma \mu^2 + O(|\mu|^3 + |x|^3),$$

with $v \neq 0$ (and $\alpha \neq 0$, recall above). Such coordinates (and parametrized center manifold W^c) may be taken with an arbitrarily high degree of smoothness, up to restricting the parameter range to a sufficiently small neighbourhood of $\mu = 0$. Here, for the sake of technical simplicity, we take them to be C^∞ but a finite amount of smoothness is sufficient for all our arguments.

Families of diffeomorphisms bifurcating *via* saddle-node cycles, both critical and non-critical, were studied in [NPT] from the point of view of structural stability. Their analysis shows that the presence of a criticality leads to a much richer and unstable unfolding. Indeed, [NPT, Theorem III.4.1], any family of diffeomorphisms going through a critical saddle-node cycle exhibits homoclinic tangencies, at parameter values arbitrarily close to zero. As a consequence, it must display a wide variety of phenomena which are known to be involved in the unfolding of such tangencies: cascades of period-doubling bifurcations [YA], [Mr]; frequency of hyperbolicity *vs.* bifurcations determined by fractal dimensions [PT1], [PY]; persistence of tangencies, coexistence of infinitely many sinks [N2], [Rb], [PV]; Hénon-like strange attractors [BC], [MV], [V]. In particular, the results in [MV], [V], in view of the previous conclusion from [NPT], imply that strange attractors are persistent in the unfolding of any critical saddle-node cycle. Our first result here asserts a much stronger fact: they even constitute a prevalent phenomenon.

Theorem A *Let $(f_\mu)_\mu$ be a generic smooth family of diffeomorphisms on a manifold M, unfolding a critical saddle-node cycle. Then the set $\mathcal{S}$ of values of μ for which f_μ exhibits Hénon-like strange attractors satisfies*

$$\liminf_{\varepsilon\to 0} \frac{m\left(\mathcal{S}\cap(-\varepsilon,\varepsilon)\right)}{2\varepsilon} > 0.$$

By a *strange attractor* of a transformation $g\colon M \longrightarrow M$ we mean a compact g-invariant set $A \subset M$ such that

- the stable set $W^s(A) = \{z \in M\colon \lim_{n\to+\infty} \operatorname{dist}(g^n(z), A) = 0\}$ has nonempty interior;
- there exists $\bar z \in A$ such that the orbit $\{g^n(\bar z)\colon n \geq 0\}$ is dense in A and there exist $c > 0$, $\sigma > 1$ and $v \in T_{\bar z}M$ such that $\|Dg^n(\bar z)v\| \geq c\sigma^n$ for every $n \geq 0$.

The strange attractors we encounter here have some additional properties, namely

- A coincides with the closure of $W^u(\mathcal{O}(q))$ for some hyperbolic saddle point q;
- A is not (uniformly) hyperbolic: it contains *critical points*, admitting tangent directions which are contracted by both positive and negative iteration under Dg.

We call them *Hénon-like* strange attractors, cf. [BC], [MV]. See also [BY] for important ergodic features of this kind of attractors.

Let us point out that the converse to [NPT, Theorem III.4.1] is also true, as observed by L. Mora: critical saddle-node cycles are formed whenever a homoclinic tangency is generically unfolded. On the other hand, it remains an interesting open question whether strange attractors can be a prevalent phenomenon in the unfolding of a homoclinic tangency. It follows from [PT1] that this can not always be the case: in a large (open) class of examples *hyperbolicity* is fully prevalent, in the sense that the limit set of f_μ, see below, is hyperbolic for a set of values of μ with Lebesgue density 1 at $\mu = 0$. On the other hand, it may be that prevalence of strange attractors does hold in some relevant situations in this setting, e.g. for homoclinic tangencies occurring in the Hénon family $(x, y) \mapsto (1 - ax^2 + y, bx)$ at parameter values (a, b) close to $(2, 0)$.

In a somewhat opposite direction, we stress that hyperbolicity of the dynamics related to the cycle may also be a prevalent feature, at least for an open set of parametrized families $(f_\mu)_\mu$ unfolding a critical saddle-node cycle. In order to illustrate this fact, we consider two different situations, corresponding to the two types of cycles described in the figure above. In both cases we fix a compact set $V \subset M$ such that all the periodic orbits involved in the cycle are contained in $\operatorname{int}(V)$ and, morever, $f_0(V) \subset \operatorname{int}(V)$. This last condition remains true for f_μ, any μ close to zero, and then $L(f_\mu|V) = L(f_\mu) \cap V$. Recall that the *limit set* of f_μ is defined by

$$L(f_\mu) = \overline{\bigcup_{x \in M} \alpha(x)} \cup \overline{\bigcup_{x \in M} \omega(x)},$$

where the α-limit $\alpha(x)$, resp. the ω-limit $\omega(x)$, is the set of accumulation points of $f_\mu^n(x)$ as $n \to -\infty$, resp. $n \to +\infty$. Moreover, $L(f_\mu|V)$ is defined in a similar way, restricting to those orbits which are contained in V. Then we denote by $\mathcal{H}$ the set of values of μ for which $L(f_\mu|V)$ is a hyperbolic set of f_μ (and so the dynamics of f_μ restricted to V is stable [N1]). Note that $\mathcal{H}$ is disjoint from $\mathcal{S}$ since Hénon-like strange attractors are, by definition, nonhyperbolic.

Theorem B *There exist open sets of smooth families of diffeomorphisms $(f_\mu)_{\mu \in \mathbf{R}}$ unfolding a critical saddle-node cycle, for which $\mathcal{H}$ satisfies*

$$\liminf_{\varepsilon \to 0} \frac{m\left(\mathcal{H} \cap (-\varepsilon, \varepsilon)\right)}{2\varepsilon} > 0.$$

For the proof of Theorem A we combine the results in [MV], [V] with a careful analysis of the distribution of parameter values corresponding to homoclinic tangencies. A main ingredient is the notion of transition maps, which we borrow from [NPT] and further develop in Section 2. Our construction yields attractors which are related to the dynamics in the vicinity of homoclinic orbits and thus have just a semi-global character. Under additional hypotheses, we can further push our arguments to obtain, for the first time, a Hénon-like attractor of a truly global nature. For that we consider the setting of 1-cycles. Given any diffeomorphism $f_0 \colon M \longrightarrow M$ exhibiting a saddle-node 1-cycle, one may construct a compact subset $V \subset M$ as above, such that $f_0(V) \subset \operatorname{int}(V)$ and the closure of $W^u(\mathcal{O}(p_0))$ is contained in $\operatorname{int}(V)$; see Section 4.1. Then we also have $f_\mu(V) \subset \operatorname{int}(V)$ for every small μ and it is a most natural question to ask what is the dynamical nature of $A_\mu = \cap_{n \geq 0} f_\mu^n(V)$. It follows from Theorem A that A_μ contains strange attractors for a large set of parameter values. Actually, a much stronger statement holds, at least in an open set of cases: A_μ itself *is* a Hénon-like strange attractor. Observe also that, by construction, the basin $W^s(A_\mu)$ of A_μ contains V, which depends only on f_0. The construction of such *global strange attractors* will appear in [DRV].

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2 Saddle-node local dynamics: transition maps

2.1 The one-dimensional case

Throughout this section $\varphi = (\varphi_\mu)_\mu$ denotes a *saddle-node arc* of 1-dimensional maps. By this we mean a C^∞ map $\varphi: (-\delta_0, \delta_0) \times (-\delta_0, \delta_0) \longrightarrow \mathbf{R}$, $(\mu, x) \mapsto \varphi(\mu, x) = \varphi_\mu(x)$, satisfying

$$\varphi(0,0) = 0,\ \partial_x \varphi(0,0) = 1,\ \partial_x^2 \varphi(0,0) \neq 0 \text{ and } \partial_\mu \varphi(0,0) \neq 0.$$

Up to performing simple (affine) changes of the coordinate x and of the parameter μ we may, and do from now on, assume that $2\alpha = \partial_x^2 \varphi(0,0) > 0$ and $\partial_\mu \varphi(0,0) = 1$. Then

$$\varphi_\mu(x) = x + \mu + \alpha x^2 + \beta x \mu + \gamma \mu^2 + O(|\mu|^3 + |x|^3). \tag{1}$$

Saddle-node arcs of 1-dimensional *vector fields* $X = (X(\mu, \cdot))_\mu$ are defined in a similar way, satisfying

$$X(0,0) = 0,\ \partial_x X(0,0) = 1,\ \partial_x^2 X(0,0) \neq 0 \text{ and } \partial_\mu X(0,0) \neq 0.$$

As before, we may write

$$X(\mu, x) = \mu + \alpha x^2 + \beta x \mu + \gamma \mu^2 + O(|\mu|^3 + |x|^3) \text{ with } \alpha > 0. \tag{2}$$

We say that X is *tangent* to φ if its time-1 arc $X_1 = (X_1(\mu, \cdot))_\mu$ satisfies

$$(\partial_\mu^k \partial_x^l X_1)(0, x) = (\partial_\mu^k \partial_x^l \varphi)(0, x) \tag{3}$$

for every $k, l \geq 0$ and x in a neighbourhood of zero. Our study of the local dynamics of saddle-node arcs makes use of the following theorem, which is an improvement of a result in [NPT], see also [L], [T1], [T2].

Theorem 2.1 *For any saddle-node arc there exist tangent saddle-node arcs of vector fields.*

For the sake of completeness we include in the Appendix a proof of Theorem 2.1. We also note that a somewhat stronger statement was obtained in [IY] using different methods.

Now let φ be a saddle-node arc and fix X a saddle-node arc of vector fields tangent to φ. Let moreover $a < 0 < b$ be fixed, close enough to zero so that $X(0, x) > 0$ for all $x \in [\varphi_0^{-2}(a), \varphi_0^2(b)] \setminus \{0\}$. For $k \in \mathbf{N}$ and small $\mu > 0$ we define $\sigma_k(\mu)$ by the relation

$$X_{k+\sigma_k(\mu)}(\mu, a) = b,$$

where $X_s(\mu, \cdot)$ denotes the time-s map of $X(\mu, \cdot)$.

Proposition 2.2 *For sufficiently large $k \geq 1$ there is a unique $\mu_k^* > 0$ satisfying $\sigma_k(\mu_k^*) = 0$. Moreover, $\sigma_k\colon [\mu_{k+1}^*, \mu_k^*] \longrightarrow [0,1]$ is a C^∞ (decreasing) diffeomorphism onto $[0,1]$.*

We also get that the *reparametrization maps* $\mu_k = (\sigma_k | \, [\mu_{k+1}^*, \mu_k^*])^{-1} : [0,1] \longrightarrow [\mu_{k+1}^*, \mu_k^*]$ have uniformly bounded metric distorsion:

Proposition 2.3 *Given $\varepsilon > 0$ there is $k_0 \in \mathbf{N}$ such that for every $k \geq k_0$ and every Borel subset A of $[0,1]$ we have*

$$(1-\varepsilon)\, m(A) \leq \frac{m(\mu_k(A))}{m([\mu_{k+1}^*, \mu_k^*])} \leq (1+\varepsilon)\, m(A) \quad (m = \text{ Lebesgue measure }).$$

For $\varphi_0^{-1}(a) \leq x \leq \varphi_0(a)$ and $|\mu|$ small we define $t(\mu, x)$ by $x = X_{t(\mu,x)}(\mu, a)$ and then the main result of this section is

Theorem 2.4 *The sequence $T_k\colon [0,1] \times [\varphi_0^{-1}(a), \varphi_0(a)] \longrightarrow \mathbf{R}$, $T_k(\sigma, x) = \varphi^k_{\mu_k(\sigma)}(x)$, converges in the C^∞ topology to the* transition map $T_\infty\colon [0,1] \times [\varphi_0^{-1}(a), \varphi_0(a)] \longrightarrow \mathbf{R}$ *defined by $T_\infty(\sigma, x) = X_{t(0,x)-\sigma}(0, b)$.*

Starting the proof of these statements, we define for $\mu > 0$ small

$$f(\mu) = \int_a^b \frac{dx}{X(\mu, x)}.$$

Then $b = X_{f(\mu)}(\mu, a)$, that is $f(\mu) = k + \sigma_k(\mu)$ for any $k \in \mathbf{N}$.

Lemma 2.5 *For every $i \geq 0$,*

$$\lim_{\mu \to 0^+} \mu^{i+1/2} f^{(i)}(\mu) = \tilde{C}_i = \frac{(-1)^i i!}{\sqrt{\alpha}} \int_{-\pi/2}^{\pi/2} \cos^{2i} x \, dx.$$

Proof: A direct calculation gives

$$f^{(i)}(\mu) = (-1)^i i! \int_a^b \frac{(\partial_\mu X)^i(\mu,x)}{X(\mu,x)^{i+1}}\,dx + \sum_{j=1}^{i}\int_a^b \frac{F_j(\mu,x)}{X(\mu,x)^j}\,dx, \tag{4}$$

where each F_j is a polynomial function of the partial derivatives of X, $(\partial_\mu^r\partial_x^s X)$, $r+s\le i$. We claim that

$$\lim_{\mu\to 0^+} \mu^{i+1/2}\int_a^b \frac{dx}{X(\mu,x)^{i+1}} = \frac{1}{\sqrt{\alpha}}\int_{-\pi/2}^{\pi/2}\cos^{2i} x\,dx \quad \text{for } i\ge 0 \tag{5}$$

$$\lim_{\mu\to 0^+} \mu^{i+1/2}\int_a^b \frac{|x|\,dx}{X(\mu,x)^{i+1}} = 0 \quad \text{for } i\ge 1. \tag{6}$$

First we explain how the lemma follows from these claims. Observe that case $i=0$ is a direct consequence of claim (5). From now on we consider $i\ge 1$. Returning to (4), we get for every $1\le j\le i$

$$|\mu^{i+1/2}\int_a^b \frac{F_j}{X^j}\,dx| \le M\mu^{i-j+1}|\mu^{j-1/2}\int_a^b\frac{dx}{X^j}| \le M\mu|\mu^{(j-1)+1/2}\int_a^b\frac{dx}{X^j}|,$$

where $M=\sup\{|F_j(\mu,x)|\colon 1\le j\le i,\ a\le x\le b,\ |\mu|\le\delta_0/2\}$. Then claim (5) implies

$$\lim_{\mu\to 0^+}\mu^{i+1/2}\int_a^b\frac{F_j}{X^j}\,dx = 0, \text{ for every } 1\le j\le i. \tag{7}$$

Now, in order to prove that

$$\lim_{\mu\to 0^+}\mu^{i+1/2}\int_a^b\left(\frac{(\partial_\mu X)^i}{X^{i+1}} - \frac{1}{X^{i+1}}\right)dx = 0 \text{ for each } i\ge 1, \tag{8}$$

we use (2) and then

$$(\partial_\mu X)^i - 1 = (1+\beta x+2\gamma\mu+O(\mu^2+x^2))^i - 1 = A_i(\mu,x)(\beta x+2\gamma\mu+O(\mu^2+x^2)),$$

where $A_i(\mu,x)=\sum_{j=0}^{i-1}(1+\beta x+2\gamma\mu+O(\mu^2+x^2))^j$. Note also that for some $c>0$ we have $X(\mu,x)\ge c(\mu^2+x^2)$ for all small x and $\mu>0$. Hence,

$$\int_a^b\frac{|(\partial_\mu X)^i-1|}{X^{i+1}}\,dx \le K\left(|\beta|\int_a^b\frac{|x|\,dx}{X^{i+1}} + 2|\gamma|\mu\int_a^b\frac{dx}{X^{i+1}} + K\int_a^b\frac{dx}{X^i}\right)$$

for some $K=K_i>0$. Now, it is clear that the claims imply (8) and, on their turn, (7), (8) and claim (5) imply the conclusion of the lemma. Now we come to the proof of the claims. First we note that for the particular arc $\tilde X(\mu,x)=\mu+\alpha x^2$, (5) and (6) follow, easily, from an explicit calculation of the integrals. On the other hand, for a general saddle-node arc of vector fields $X(\mu,x)$ as in (2),

$$1-O(|\mu|+|x|)\le\frac{X(\mu,x)}{\tilde X(\mu,x)}\le 1+O(|\mu|+|x|).$$

Hence, given any $\varepsilon > 0$ we may fix $\delta > 0$ and $\nu > 0$ such that

$$1 - \varepsilon \le \frac{X(\mu, x)}{\tilde{X}(\mu, x)} \le 1 + \varepsilon \qquad \text{for every } (\mu, x) \in (0, \nu) \times [-\delta, \delta].$$

Observing that

$$\limsup_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{Y^{i+1}} = \limsup_{\mu \to 0^+} \mu^{i+1/2} \int_{-\delta}^{\delta} \frac{dx}{Y^{i+1}}$$

for both $Y = X$ and $Y = \tilde{X}$, we obtain

$$\limsup_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X^{i+1}} \le \frac{1}{(1-\varepsilon)^{i+1}} \lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{\tilde{X}^{i+1}}$$

and, analogously,

$$\liminf_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X^{i+1}} \ge \frac{1}{(1+\varepsilon)^{i+1}} \lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{\tilde{X}^{i+1}}.$$

Since $\varepsilon > 0$ is arbitrary this proves that

$$\lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{\tilde{X}^{i+1}} = \lim_{\mu \to 0^+} \mu^{i+1/2} \int_a^b \frac{dx}{X^{i+1}}$$

implying that claim (5) holds for X. Claim (6) is proved in the same way. $\square$

In particular, $\lim_{\mu \to 0^+} f(\mu) = +\infty$ and $\lim_{\mu \to 0^+} f'(\mu) = -\infty$. Hence, for each sufficiently large $k \in \mathbf{N}$ there is a unique $\mu = \mu_k^*$ such that $f(\mu) = k$, i.e. $\sigma_k(\mu_k^*) = 0$. Note that the sequence $(\mu_k^*)_k \to 0$: in fact

$$\lim_{k \to \infty} k \sqrt{\mu_k^*} = \tilde{C}_0 = \frac{\pi}{\sqrt{\alpha}} \tag{9}$$

The relation $\sigma_k(\mu_{k+1}^*) = 1$ is obvious and so the proof of Proposition 2.2 is complete.

Lemma 2.6 *For each $i \in \mathbf{N}$ there exists $C_i \in \mathbf{R}$ such that*

$$\lim_{k \to \infty} \frac{\mu_k^{(i)}(\sigma)}{\mu_k(\sigma)^{1+i/2}} = C_i, \text{ uniformly on } \sigma \in [0, 1].$$

Proof: ¿From $f(\mu_k(\sigma)) = k + \sigma$ we get $f'(\mu_k)\mu_k' = 1$. Then, by Lemma 2.5,

$$\lim_{k \to \infty} \frac{\mu_k'}{\mu_k^{1+1/2}} = \lim_{k \to \infty} \frac{1}{\mu_k^{1+1/2} f'(\mu_k)} = \frac{1}{\tilde{C}_1}.$$

Now we proceed by induction on i. A simple inductive argument shows that for every $i \ge 2$

$$f'(\mu_k)\mu_k^{(i)} + \sum_{j=1}^{n(i)} G_j = 0 \tag{10}$$

with $n(i) \geq 1$ and each G_j, $1 \leq j \leq n(i)$, having the form $f^{(l)}(\mu_k) \cdot (\mu_k')^{m_1} \cdots (\mu_k^{(i-1)})^{m_{i-1}}$ with $m_1, \ldots, m_{i-1} \geq 0$ satisfying $1 \leq l = \sum_{s=1}^{i-1} m_s \leq \sum_{s=1}^{i-1} s m_s = i$. Then, using this last property, we find

$$\lim_{k\to\infty} \frac{G_j}{f'(\mu_k)\mu_k^{1+i/2}} = \lim_{k\to\infty} \frac{f^{(l)}(\mu_k)\mu_k^{l+1/2}}{f'(\mu_k)\mu_k^{1+1/2}} \prod_{s=1}^{i-1} \left(\frac{\mu_k^{(s)}}{\mu_k^{1+s/2}}\right)^{m_s} = \frac{\tilde{C}_l}{\tilde{C}_1} \prod_{s=1}^{i-1} C_s^{m_s}$$

for every $1 \leq j \leq n(i)$. In view of (10), this proves that $\mu_k^{(i)}/\mu_k^{1+i/2}$ converges as $k \to \infty$ and it is also clear from the argument that this convergence is uniform. $\square$

Proposition 2.3 is a direct consequence of (9) and Lemma 2.6. Indeed, let $\varepsilon > 0$; if k is large enough then for every $\sigma_1, \sigma_2 \in [0,1]$

$$\frac{\mu_k'(\sigma_1)}{\mu_k'(\sigma_2)} \leq (1 + \frac{\varepsilon}{3}) \left(\frac{\mu_k(\sigma_1)}{\mu_k(\sigma_2)}\right)^{3/2} \leq (1 + \frac{2\varepsilon}{3})(\frac{k+1}{k})^3 \leq 1 + \varepsilon.$$

In particular,

$$1 - \varepsilon \leq \frac{\mu_k'(\sigma)}{m([\mu_{k+1}^*, \mu_k^*])} \leq 1 + \varepsilon \qquad \text{for every } \sigma \in [0,1]$$

and the proposition follows immediately.

Next, we define $\beta_k\colon [\mu_{k+1}^*, \mu_k^*] \times [\varphi_0^{-1}(a), \varphi_0(a)] \longrightarrow \mathbf{R}$ by $X_{k+\beta_k(\mu,x)}(\mu, x) = \varphi_\mu^k(x)$. Then we may write

$$T_k(\sigma, x) = \varphi_{\mu_k(\sigma)}^k(x) = X_{\beta_k(\mu_k(\sigma),x)+t(\mu_k(\sigma),x)-\sigma}(\mu_k(\sigma), b).$$

Clearly, $\beta_k(\mu, x)$, $t(\mu, x)$ and $X_s(\mu, x)$ are C^∞ functions. On the other hand, (9) and Lemma 2.6 imply that $(\mu_k)_k$ converges in the C^∞ topology to the null function. Notice that if φ_μ coincides (for every μ) with the time-1 map of $X(\mu, \cdot)$ then β_k is identically zero and so Theorem 2.4 is already proved in that particular case. Moreover, the general case of the theorem is now a direct consequence of

Lemma 2.7 *For every r, $s \geq 0$*

$$\lim_{k\to\infty} \left(\sup\{|\partial_\mu^r \partial_x^s \beta_k(\mu, x)|\colon (\mu, x) \in [\mu_{k+1}^*, \mu_k^*] \times [\varphi_0^{-1}(a), \varphi_0(a)]\}\right) = 0$$

Proof: Let $\alpha(\mu, x)$ be defined by

$$\alpha(\mu, x) = \int_{X_1(\mu,x)}^{\varphi(\mu,x)} \frac{d\xi}{X(\mu, \xi)} \qquad \text{i.e. } X_{1+\alpha(\mu,x)}(\mu, x) = \varphi_\mu(x).$$

Then $\beta_k(\mu, x) = \sum_{j=0}^{k-1} \alpha(\mu, \varphi_\mu^j(x))$ and it follows that, for every r, $s \geq 0$

$$\partial_\mu^r \partial_x^s \beta_k(\mu, x) = \sum_{j=0}^{k-1} \sum_{l=1}^{n(r,s)} H_{j,l} \tag{11}$$

where $n(r,s) \geq 1$ and each $H_{j,l}$ has the form

$$\partial_\mu^p \partial_x^q \alpha(\mu, \varphi_\mu^j(x)) \prod_{i=1}^{s} \partial_\mu^{u_i} \partial_x^{v_i} \varphi_\mu^j(x)$$

with p, q, s, u_i, $v_i \geq 0$ and $(p+q)$, $(u_i+v_i) \leq (r+s)$ for every i. We claim that

(a) given u, $v \geq 0$ there are $E_1 > 0$ and $m_1 \geq 0$ such that

$$|\partial_\mu^u \partial_x^v \varphi_\mu^j(x)| \leq E_1 k^{m_1} \text{ for all } x \in [\varphi_0^{-1}(a), \varphi_0(a)], 0 < \mu \leq \mu_k^* \text{ and } 0 \leq j \leq k;$$

(b) there is $\delta > 0$ and given p, q, $m \geq 0$ there is $E_2 > 0$ such that

$$|\partial_\mu^p \partial_x^q \alpha(\mu, z)| \leq E_2 \mu^m \text{ for all } (\mu, z) \in (0, \delta] \times [\varphi_0^{-1}(a), \varphi_0(b)].$$

In order to deduce the lemma from these claims we observe that the number of terms in (11) is bounded by const k. Replacing (a) and (b) there we conclude that there is $m_0 = m_0(r,s) \geq 1$ and given any $m \geq 1$ there is $E_0 = E_0(r,s,m) > 0$ such that

$$|\partial_\mu^r \partial_x^s \beta_k(\mu, x)| \leq E_0 \mu^m k^{m_0} \quad \text{for every } (\mu, x) \in (0, \mu_k^*] \times [\varphi_0^{-1}(a), \varphi_0(a)].$$

Recall also (from (9)) that $\mu_k^* = O(k^{-2})$. We fix $m > m_0/2$ and the lemma follows.

For the proof of claim (a) we note first that given any smooth function $\psi_\mu(x) = \psi(\mu, x)$, we have $\partial_\mu^u \partial_x^v (\varphi_\mu(\psi_\mu(x))) = \partial_x \varphi_\mu(\psi_\mu(x)) \cdot \partial_\mu^u \partial_x^v \psi_\mu(x) + \sum_{i=1}^{m(u,v)} I_i$, where $m(u,v) \geq 0$ and each I_i has the form

$$\partial_\mu^a \partial_x^b \varphi_\mu(\psi_\mu(x)) \prod_{t=1}^{s} \partial_\mu^{c_t} \partial_x^{d_t} \psi_\mu(x)$$

with a, b, s, c_t, $d_t \geq 0$, $(a+b) \leq (u+v)$, $s \leq (u+v)$ and $(c_t+d_t) \leq (u+v-1)$. Then, by induction on j

$$\partial_\mu^u \partial_x^v \varphi_\mu^j(x) = \sum_{i=1}^{n(u,v,j)} J_i \tag{12}$$

where $n(u,v,j) \leq j m(u,v) + 1$ and each J_i can be written

$$\partial_x \varphi_\mu^{j-e}(\varphi_\mu^e(x)) \cdot \partial_\mu^a \partial_x^b \varphi_\mu(\varphi_\mu^e(x)) \cdot \prod_{t=1}^{s} \partial_\mu^{c_t} \partial_x^{d_t} \varphi_\mu^e(x) \tag{13}$$

with a, b, s, c_t, d_t, $e \geq 0$, $(a+b) \leq (u+v)$, $s \leq (u+v)$, $(c_t+d_t) \leq (u+v-1)$ and $e \leq j$. We also use the fact that, cf. [NPT, Lemma 3.7], there is $E_3 > 0$ such that for k large enough

$$\partial_x \varphi_\mu^i(\varphi_\mu^e(x)) \leq \frac{E_6}{\mu} \; \left(\leq \frac{2\alpha E_3}{\pi^2} k^2 \right) \tag{14}$$

for every $(\mu, x) \in (0, \mu_k^*] \times [\varphi_0^{-1}(a), \varphi_0(a)]$ and e, $i \geq 0$ with $e + i \leq k$. Case $u, v = 0$ of the claim is trivial. For a general (u, v), (12), (13) and (14) give, by induction on $(u + v)$,

$$|\partial_\mu^u \partial_x^v \varphi_\mu^j(x)| \leq \sum_{i=1}^{n(u,v,j)} E_4 k^2 \prod_{t=1}^{s} E_2(c_t, d_t) k^{m_2(c_t, d_t)}$$

with $E_4 = (2\alpha E_3/\pi^2) \sup\{|\partial_\mu^a \partial_x^b \varphi_\mu(z)| : (a + b) \leq (u + v), (\mu, z) \in (0, \delta] \times [\varphi_0^{-1}(a), \varphi_0(a)]\}$. Therefore, it is sufficient to take $m_1(u, v) = (u + v) \sup\{m_1(c, d) : (c + d) \leq (u + v - 1)\} + 3$ and $E_1(u, v) = E_4(m(u, v) + 1)(\sup\{E_1(c, d) : (c + d) \leq (u + v - 1)\})^{u+v}$.

In order to prove claim (b) we remark that every $\partial_\mu^p \partial_x^q \alpha$ may be written

$$\partial_\mu^p \partial_x^q \alpha(\mu, z) = \int_{X_1(\mu,z)}^{\varphi(\mu,z)} \frac{A_{p,q}(\mu, \xi)}{X(\mu, \xi)^{p+q+1}} \, d\xi + B_{p,q}(\mu, z) \tag{15}$$

where $A_{p,q}$ is a C^∞ function defined on $[-\delta, \delta] \times [\varphi_0^{-1}(a), \varphi_0(b)]$ for some (fixed) $\delta > 0$ and $B_{p,q}$ is flat at $\mu = 0$ in the sense that given any i, j, $m \geq 0$ there is $E_5 > 0$ such that

$$|\partial_\mu^i \partial_x^j B_{p,q}(\mu, z)| \leq E_5 \mu^m \quad \text{for all } (\mu, z) \in (0, \delta] \times [\varphi_0^{-1}(a), \varphi_0(b)].$$

We prove (15) by induction on (p, q). Note that it is obvious for $p, q = 0$. Suppose now that it holds for some p, $q \geq 0$ and let $*$ denote either μ or x. Then $\partial_* \partial_\mu^p \partial_x^q \alpha$ may be written as

$$\int_{X_1}^{\varphi} \frac{X \partial_* A_{p,q} - (p + q + 1) A_{p,q} \partial_* X}{X^{p+q+2}} \, d\xi + \partial_* B_{p,q} + \left(\frac{A_{p,q}(\cdot, \varphi) \partial_* \varphi}{X(\cdot, \varphi)^{p+q+1}} - \frac{A_{p,q}(\cdot, X_1) \partial_* X_1}{X(\cdot, X_1)^{p+q+1}} \right). \tag{16}$$

Clearly, $\partial_* B_{p,q}$ is flat at $\mu = 0$ and so the inductive step of our argument will be complete if we show that the same holds for the last term in (16). This, on its turn, is a straightforward consequence of the fact that the functions $(\partial_* \varphi - \partial_* X_1)$, $(A_{p,q}(\cdot, \varphi) - A_{p,q}(\cdot, X_1))$ and $(X(\cdot, \varphi)^{p+q+1} - X(\cdot, X_1)^{p+q+1})$ are flat at $\mu = 0$ and, moreover, $X(\mu, z) > E_6 \mu$ for some $E_6 > 0$ and every $(\mu, z) \in (0, \delta] \times [\varphi_0^{-1}(a), \varphi_0(b)]$. Finally, claim (b) follows from (15) just by observing that

$$\left| \int_{X_1}^{\varphi} \frac{A_{p,q}}{X^{p+q+1}} \, d\xi \right| \leq \frac{\sup\{|A_{p,q}|\}}{(E_6 \mu)^{p+q+1}} |\varphi - X_1|$$

and so, $(\varphi - X_1)$ being flat at $\mu = 0$, given any $m \geq 0$ there is $E_7 > 0$ such that

$$\left| \int_{X_1(\mu,x)}^{\varphi(\mu,x)} \frac{A_{p,q}(\mu, \xi)}{X(\mu, \xi)^{p+q+1}} \, d\xi \right| \leq E_7 \mu^m$$

for $(\mu, x) \in (0, \delta] \times [\varphi_0^{-1}(a), \varphi_0(b)]$. $\square$

2.2 The general setting

Now we prove a higher-dimensional version of Theorem 2.4. Let M be a smooth manifold, $m = \dim M \geq 2$, and let $(f_\mu)_{\mu\in\mathbf{R}}$ be a smooth arc of diffeomorphisms unfolding a saddle-node bifurcation on M, recall the Introduction. Let the point $(0, p_0) \in \mathbf{R}\times M$ to correspond to the saddle-node. Then there are C^∞ coordinates $(\mu, x, Y) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^{m-1}$ on a neighbourhood U of $(0, p_0)$, such that $f_\mu(x, Y) = (\varphi_\mu(x), \Psi(\mu, x, Y))$ with

(a) φ a 1-dimensional saddle-node arc and

(b) $\Psi(\mu, x, 0) = 0$ and $\|\partial_Y\Psi(\mu, x, Y)\| \leq \lambda < 1$ for every $(\mu, x, Y) \in U$.

As in the previous section, we denote by μ_k the reparametrization maps of φ and by T_∞ its transition map. We fix V a small neighbourhood of $0 \in \mathbf{R}^{m-1}$, and using the coordinates above we define $\hat{T}_k, \hat{T}_\infty\colon [0,1]\times[\varphi_0^{-1}(a), \varphi_0(a)]\times V \longrightarrow \mathbf{R}\times\mathbf{R}^{m-1}$ by

$$\hat{T}_k(\sigma, x, Y) = f^k_{\mu_k(\sigma)}(x, Y) \quad\text{and}\quad \hat{T}_\infty(\sigma, x, Y) = (T_\infty(\sigma, x), 0).$$

Theorem 2.8 *The sequence $(\hat{T}_k)_k$ converges to the map $\hat{T}_\infty$ in the C^∞ topology.*

Proof: We have $\hat{T}_k(\sigma, x, Y) = (\varphi^k_{\mu_k(\sigma)}(x), Y_k(\mu_k(\sigma), x, Y)$, where the Y_k are defined inductively by $Y_0(\mu, x, Y) = Y$ and $Y_{j+1}(\mu, x, Y) = \Psi(\mu, x_j, Y_j)$, $x_j = \varphi^j_\mu(x)$. We claim that given any $p, q, r \geq 0$ there are $C_0 > 0$ and $m_0 \geq 0$ such that for every $1 \leq j \leq k$

(17) $\sup\{\|\partial^p_\mu\partial^q_x\partial^r_Y Y_j(\mu, x, Y)\|\colon (\mu, x, Y) \in [\mu^*_{k+1}, \mu^*_k]\times[\varphi_0^{-1}(a), \varphi_0(a)]\times V\} \leq C_0 j\lambda^j k^{m_0}.$

In particular, $\sup\{\|\partial^p_\mu\partial^q_x\partial^r_Y Y_k\|\} \leq C_0\lambda^j k^{m_0+1} \to 0$ which, together with Lemma 2.6, implies that $(Y_k)_k$ converges to zero in the C^∞ topology. In view of Theorem 2.4, this proves our statement. Hence, we are left to prove the claim and we do this by induction on $p+q+r$. First, we note that (b) implies $\|Y_j(\mu, x, Y)\| \leq \lambda^j\|Y\|$ and so in case $p, q, r = 0$ it is sufficient to take $C_0 = 1$ (say) and $m_0 = 0$. Now, given any $p, q, r \geq 0$ we assume that (17) holds for every derivative of order strictly less than $p+q+r$ and we conclude that it also holds for $\partial^p_\mu\partial^q_x\partial^r_Y Y_j$, any $1 \leq j \leq k$. Case $j = 1$ is easy: clearly, $\|\partial^p_\mu\partial^q_x\partial^r_Y Y_1\| \leq C_1$, where $C_1 = \sup\|\partial^p_\mu\partial^q_x\partial^r_Y\Psi\|$, and so we only have to take $C_0 \geq C_1/\lambda$. Observe now that successive derivation of $Y_{j+1}(\mu, x, Y) = \Psi(\mu, x_j, Y_j)$ leads to

$$\partial^p_\mu\partial^q_x\partial^r_Y Y_{j+1}(\mu, x, Y) = \partial_Y\Psi(\mu, x_j, Y_j)\partial^p_\mu\partial^q_x\partial^r_Y Y_j(\mu, x, Y) + \sum_{l=1}^{l_0} H_l \tag{18}$$

where $l_0 = l_0(p, q, r) \geq 0$ and each H_l has the form

$$\partial^a_\mu\partial^b_x\partial^c_Y\Psi(\mu, x_j, Y_j)\prod_{s=1}^{s_0}\partial^{\alpha_s}_\mu\partial^{\beta_s}_x x_j\prod_{t=1}^{t_0}\partial^{u_t}_\mu\partial^{v_t}_x\partial^{w_t}_Y Y_j$$

with $a, b, c, s_0, \alpha_s, \beta_s, t_0, u_t, v_t, w_t \geq 0$ depending only on p, q, r, l and satisfying $a+b+c \leq p+q+r$, $\sum_s(\alpha_s+\beta_s) + \sum_t(u_t+v_t+w_t) \leq p+q+r$ and $c \leq \sum_t(u_t+v_t+w_t) \leq p+q+r-1$.

Recall that, according to claim (a) in the proof of Lemma 2.7, $|\partial_\mu^{\alpha_s}\partial_x^{\beta_s}x_j| \le C_{2,s}k^{m_{2,s}}$ for some $C_{2,s} > 0$ and $m_{2,s} \ge 0$ depending only on α_s, β_s. Observe, moreover, that (b) above implies $\|\partial_\mu^a\partial_x^b\Psi(\mu,x,Y)\| \le C_3\|Y\|$ for some $C_3 = C_3(a,b)$. Finally, we denote $C_4 = \sup\|\partial_\mu^a\partial_x^b\partial_Y^c\Psi\|$, where the supremum is taken over all $a+b+c \le p+q+r$. In order to estimate $|H_l|$ we distinguish two cases. Suppose first that $c \ge 1$. Then it must be $t_0 \ge 1$ (because $c \le \sum_1^{t_0}(u_t+v_t+w_t)$) and, by induction, we get

$$|H_l| \le C_4 \prod_{s=1}^{s_0}(C_{2,s}k^{m_{2,s}}) \prod_{t=1}^{t_0}(C_{0,t}j\lambda^j k^{m_{0,t}}) \le C_5\lambda^j k^{m_5}$$

where $C_{0,t} = C_0(u_t,v_t,w_t)$, $m_{0,t} = m_0(u_t,v_t,w_t)$ and $C_5 > 0$, $m_5 \ge 1$, depend only on p,q,r,l. On the other hand, if $c = 0$ we have

$$|H_l| \le C_3\lambda^j \prod_{s=1}^{s_0}(C_{2,s}k^{m_{2,s}}) \prod_{t=1}^{t_0}(C_{0,t}j\lambda^j k^{m_{0,t}}) \le C_5\lambda^j k^{m_5}$$

(for possibly larger C_5, m_5). Replacing these estimates in (18) we get

$$\|\partial_\mu^p\partial_x^q\partial_Y^r Y_{j+1}\| \le \lambda\|\partial_\mu^p\partial_x^q\partial_Y^r Y_j\| + C_6\lambda^j k^{m_6},$$

where C_6, m_6 are determined by p,q,r. Thus, by induction on j,

$$\|\partial_\mu^p\partial_x^q\partial_Y^r Y_{j+1}\| \le C_0 j\lambda^{j+1}k^m + C_6\lambda^j k^{m_6} \le C_0(j+1)\lambda^{j+1}k^m,$$

as long as we fix $C_0 \ge C_6/\lambda$ and $m_0 \ge m_6$. This finishes the proof of the claim (17) and so our argument is complete. $\square$

3 Proof of Theorem A

As we said before, the strange attractors we exhibit for the proof of Theorem A are associated to homoclinic tangencies occurring in the unfolding of any critical saddle-node cycle. The existence of such tangencies is a particularly subtle fact in the setting of 1-cycles, see [NPT], and so we treat first the relatively simpler case of ℓ-cycles, $\ell \ge 2$.

3.1 Saddle-node ℓ-cycles, $\ell \ge 2$

Let $f_\mu: M \longrightarrow M$, $\mu \in \mathbf{R}$, be a smooth arc of diffeomorphisms unfolding a critical saddle-node ℓ-cycle, $\ell \ge 2$. For the sake of clearness we treat first the case when $\ell = 2$ and the periodic orbits involved in the cycle $\mathcal{O}(p_0), \mathcal{O}(p_1)$, both consist of fixed points. On the other hand, our arguments extend easily to the general case, as we will explain afterwards.

Let (x,Y) be smooth coordinates on a small neighbourhood U of p_0 as in Section 2.2: $f_\mu(x,Y) = (\varphi_\mu(x), \Psi_\mu(x,Y))$ with φ a saddle-node arc on the real line, $\Psi(\mu,x,0) = 0$ and

$\|\partial_Y \Psi(\mu, x, Y)\| \leq \lambda < 1$ for every (μ, x, Y). For simplicity we assume the diffeomorphism f_0 to be C^N-linearizable near p_1: there exist a neighbourhood $\tilde{U}$ of p_1 and C^N coordinates $(\tilde{x}, \tilde{Y})$ on $\tilde{U}$ such that $f_0(\tilde{x}, \tilde{Y}) = (\rho\tilde{x}, \Lambda\tilde{Y})$, where $\rho \in \mathbf{R}$ and $\Lambda \in \mathcal{GL}(\mathbf{R}^{m-1})$ satisfy $\|\Lambda\| < 1 < |\rho|$. Here N is a fixed integer; for all our purposes it is sufficient to take $N \geq 3$. Let $\chi = (a, Y_0) \in U$, $a < 0$, be a critical point, i.e. a point of tangency of $W^u(p_1)$ with some strong-stable leaf of p_0. We fix $l \geq 1$ such that $\tilde{\chi} = f_0^{-l}(\chi) \in \tilde{U}$; up to rescaling the coordinate $\tilde{x}$, we may suppose $\tilde{\chi} = (-1, 0)$. We assume the criticality to be quadratic and then for $(\tilde{x}, \tilde{Y})$ near $(-1, 0)$

$$f_0^l(\tilde{x}, \tilde{Y}) = (a + \beta_1(\tilde{x}+1)^2 + h_1(\tilde{x}+1, \tilde{Y}), Y_0 + H_1(\tilde{x}+1, \tilde{Y}))$$

with $\beta_1 \neq 0$ and $h_1 = \partial_{\tilde{x}} h_1 = \partial_{\tilde{x}}^2 h_1 = 0 = H_1$ at $(0,0)$. We also fix $\kappa = (q, 0) \in U$, a point of transverse intersection of $W^u(p_0)$ and $W^s(p_1)$ close to p_0. Extending $\tilde{U}$ by negative iteration if necessary, we may suppose that $\kappa \in \tilde{U}$. Then we let $\tilde{Y} = \Gamma(\tilde{x})$ be the expression of $W^u(p_0)$ in $(\tilde{x}, \tilde{Y})$ coordinates, near κ. On the other hand, we take X to be an arc of vector fields tangent to φ and, using a above and $b = X_{1/2}(0, q)$, we define the transition maps $T_\infty(\sigma, x)$ of φ and $\hat{T}_\infty(\sigma, x, Y) = (T_\infty(\sigma, x), 0)$ of f, recall Section 2. Then $\hat{T}_\infty(1/2, \chi) = \kappa$ and so there is a neighbourhood V of $(1/2, \chi)$ such that $\hat{T}_\infty(V) \subset \tilde{U}$. It is clear from the definition $T_\infty(\sigma, x) = X_{t(0,x)-\sigma}(0, b)$ that $\partial_\sigma T_\infty$ and $\partial_x T_\infty$ are nonzero. Hence we may write $\hat{T}_\infty : V \longrightarrow \tilde{U}$ as

$$\hat{T}(\sigma, x, Y) = (\tilde{x}, \Gamma(\tilde{x})), \qquad \tilde{x} = \beta_2(\sigma - 1/2) + \beta_3(x - a) + h_2(\sigma - 1/2, x - a),$$

with $\beta_2\beta_3 \neq 0$ and $h_2 = \partial_\sigma h_2 = \partial_x h_2 = 0 = H_2$ at the origin.

Now we introduce n-dependent parameter $\theta = \tau_n(\sigma)$ and coordinates $(\xi, \Theta) = \phi_n(\tilde{x}, \tilde{Y})$ as follows. Let $x_n, \sigma_n \in \mathbf{R}$, close to zero, be given implicitly by

(a) $2\beta_1 x_n + \partial_{\tilde{x}} h_1(w_n) = 0, \quad w_n = (x_n, \Lambda^n \Gamma(\rho^{-n}(-1 + x_n)))$

(b) $\beta_2\sigma_n + \rho^{-n} + \beta_1\beta_3 x_n^2 + \beta_3 h_1(w_n) + h_2(z_n) = 0, \quad z_n = (\sigma_n, \beta_1 x_n^2 + h_1(w_n))$.

Note that $\partial_{\tilde{x}} h_1(0,0) = \partial_\sigma h_2(0,0) = 0 \neq \beta_1\beta_2$, together with $\|\Lambda\| < 1$, assures the existence of x_n and σ_n for every large n. Then we define

$$\theta = \rho^{2n}(\sigma - 1/2 - \sigma_n) \qquad \xi = \rho^n(\tilde{x} + 1 - x_n) \qquad \Theta = \rho^{3n}(\tilde{Y} - \Lambda^n\Gamma(\rho^{-n}\tilde{x}))$$

and we let ψ_n be the expression of $f_0^n \circ \hat{T}_\infty \circ f_0^l$ with respect to (θ, ξ, Θ), that is

$$\psi_n(\theta, \xi, \Theta) = \phi_n \circ f_0^n \circ T_\infty(\tau_n^{-1}(\theta), \cdot) \circ f_0^l \circ \phi_n^{-1}(\xi, \Theta).$$

Proposition 3.1 *Given any compact $K \subset \mathbf{R}^{m+1}$, ψ_n is defined on K for every sufficiently large n. Moreover, $(\psi_n|K)_n$ converges in the C^N topology to the map $\psi\colon K \longrightarrow \mathbf{R}^m$ given by $(\theta, \xi, \Theta) \mapsto (\beta_2\theta + \beta_1\beta_3\xi^2, 0)$.*

Proof: The first part is an easy consequence of the fact that $(\tau_n^{-1}, \phi_n^{-1})(K) \to (1/2, \tilde{\chi})$ as $n \to \infty$. This, on its turn, follows immediately from the definitions above: note, in particular, that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sigma_n = 0$. As to the second part, a direct calculation gives

$$\psi_n(\theta, \xi, \Theta) = (\beta_2\theta + \beta_2\rho^{2n}\sigma_n + \rho^n + \beta_1\beta_3\rho^{2n}(x_n + \rho^{-n}\xi)^2 + \beta_3\rho^{2n}h_1(w) + \rho^{2n}h_2(z),\ 0),$$

where

$$w = (x_n + \rho^{-n}\xi, \Lambda^n\Gamma(\rho^{-n}(-1 + x_n + \rho^{-n}\xi)) + \rho^{-3n}\Theta),\ z = (\sigma_n + \rho^{-2n}\theta, \beta_1(x_n + \rho^{-n}\xi)^2 + h_1(w)).$$

We set

$$F_n(\xi, \Theta) = \beta_1\rho^{2n}(x_n + \rho^{-n}\xi)^2 + \rho^{2n}h_1(w) \quad \text{and} \quad G_n(\theta, \xi, \Theta) = \rho^{2n}h_2(z)$$

and then, in view of our definition of σ_n,

$$\beta_2\rho^{2n}\sigma_n + \rho^n + \beta_3 F_n(0,0) + G_n(0,0) = 0.$$

Let us estimate the partial derivatives of F_n. Let $p, q \geq 0$ be such that $1 \leq p + q \leq N$. It is easy to see (e.g. by induction on $p + q$) that $\partial_\xi^p \partial_\Theta^q (h_1(w))$ may be written as $\sum_{i=1}^{u(p,q)} H_i$, each H_i having the form

$$\partial_{\tilde{x}}^k \partial_{\tilde{Y}}^{q+l} h_1(w) \rho^{-kn-3qn} \prod_{j=1}^{l} \left(\Lambda^n \Gamma^{(s_j)}(\rho^{-n}(-1 + x_n + \rho^{-n}\xi))\rho^{-2s_j n}\right),$$

with $0 \leq k, l \leq p$ and $1 \leq s_1, \ldots, s_l \leq p$ depending only on p, q, i and satisfying $k + \sum_1^l s_j = p$. As a direct consequence,

(1) $$\|\partial_\xi^p \partial_\Theta^q F_n(\xi, \Theta)\| \leq \text{const}\, \rho^{(2-(p+3q))n}$$

and so $\partial_\xi^p \partial_\Theta^q F_n$ converges to zero, for all $p+3q>2$. Moreover, due to our definition of x_n,

$$\partial_\xi F_n(0,0) = \partial_{\tilde{Y}} h_1(w_n) \Lambda^n \Gamma'(y_n), \quad y_n = \rho^{-n}(-1+x_n),$$

and so $\partial_\xi F_n(0,0) \to 0$. On the other hand,

$$\partial_\xi^2 F_n(0,0) = 2\beta_1 + \partial_{\tilde{x}}^2 h_1(w_n) + \partial_{\tilde{x}} \partial_{\tilde{Y}} h_1(w_n) \Lambda^n \Gamma'(y_n) \rho^{-n} +$$
$$+ \partial_{\tilde{Y}}^2 h_1(w_n) \left(\Lambda^n \Gamma'(y_n)\right)^2 \rho^{-2n} + \partial_{\tilde{Y}} h_1(w_n) \Lambda^n \Gamma''(y_n) \rho^{-2n},$$

which, together with $\partial_{\tilde{x}}^2 h_1(0,0) = 0$, proves that $\partial_\xi^2 F_n(0,0) \to 2\beta_1$. Now we show that all partial derivatives of G_n converge to zero. Observe that $z = (\sigma_n + \rho^{-2n}\theta, \rho^{-2n} F_n(\xi, \Theta))$ and so, for every $1 \le p+q+r \le N$, we may write $\partial_\xi^p \partial_\Theta^q \partial_\theta^r (h_2(z)) = \sum_{i=1}^{v(p,q,r)} K_i$, each K_i having the form

$$\rho^{-2rn} \partial_\sigma^r \partial_x^k h_2(z) \prod_{j=1}^{k} \partial_\xi^{s_j} \partial_\Theta^{t_j} \left(\rho^{-2n} F_n(\xi, \Theta)\right)$$

with $0 \le k \le (p+q)$, $1 \le (s_j + t_j) \le (p+q)$, and $\sum_1^k s_j = p$, $\sum_1^k t_j = q$. Using (1) to bound the $\partial_\xi^{s_j} \partial_\Theta^{t_j} (\rho^{-2n} F_n)$ we conclude that

$$\|\partial_\xi^p \partial_\Theta^q \partial_\theta^r G_n(\theta, \xi, \Theta)\| \le \rho^{(2-(2r+p+3q))n}$$

and so $\partial_\xi^p \partial_\Theta^q \partial_\theta^r G_n \to 0$ whenever $2r+p+3q>2$. This leaves out three cases, which we treat separately. For $(p,q,r) = (0,0,1)$ we have

$$\partial_\theta G_n(0,0,0) = \partial_\sigma h_2(z_n) \to 0,$$

because $\partial_\sigma h_2(0,0) = 0$. For $(p,q,r) = (1,0,0)$ we recall that $\partial_x h_2(0,0) = 0$ and then

$$\partial_\xi G_n(0,0,0) = \partial_x h_2(z_n) \partial_\xi F_n(0,0) \to 0,$$

since both factors converge to zero. Finally, for $(p,q,r) = (2,0,0)$

$$\partial_\xi^2 G_n(0,0,0) = \partial_x h_2(z_n) \partial_\xi^2 F_n(0,0) + \rho^{-2n} \partial_x^2 h_2(z_n) \left(\partial_\xi F_n(0,0)\right)^2 \to 0$$

where we use once more the fact that $\partial_x h_2(0,0) = 0$. Altogether this shows that

$$\beta_2\theta + \beta_2\rho^{2n}\sigma_n + \rho^n + \beta_3 F_n(\xi,\Theta) + G_n(\theta,\xi,\Theta) \to \beta_2\theta + \beta_1\beta_3\xi^2$$

in the C^N topology as $n \to \infty$ and so the proof of the proposition is complete. □

We set $(\hat{\theta}, \hat{\xi}, \hat{\Theta}) = (\hat{\tau}_n(\sigma), \hat{\phi}_n(\tilde{x}, \tilde{Y})) = (-\beta_1\beta_2\beta_3\theta, (\beta_2\theta)^{-1}\xi, \Theta)$ and then the expression $\hat{\psi}_n$ of $f_0^n \circ \hat{T}_\infty \circ f_0^l$ with respect to $(\hat{\theta}, \hat{\xi}, \hat{\Theta})$ converges to the map $\hat{\psi}(\hat{\theta}, \hat{\xi}, \hat{\Theta}) = (1 - \hat{\theta}\hat{\xi}^2, 0)$ as $n \to \infty$. We take $n \gg 1$ to be fixed from now on. On the other hand, by Theorem 2.8,

$$\hat{\psi}_{n,k}(\hat{\theta}, \hat{\xi}, \hat{\Theta}) = \hat{\phi}_n \circ f_\mu^{n+k+l} \circ \hat{\phi}_n^{-1}(\xi, \Theta), \quad \mu = \mu_k(\hat{\tau}_n^{-1}(\hat{\theta})),$$

converges to $\hat{\psi}_n$ as $k \to \infty$. Hence, for k large $\hat{\psi}_{n,k}$ is a Hénon- (or quadratic-) like family, in the sense that it is C^N-close to the family of quadratic maps $\hat{\psi}$, some $N \geq 3$. According to [MV], [V], any such family has Hénon-like strange attractors for a positive Lebesgue measure set of parameter values. Actually, the proof of this fact even provides a uniform lower bound for the measure of these sets on a neighbourhood of the family $\hat{\psi}$. We conclude that there are $c_1 > 0$ and $k_1 \geq 1$ such that for every $k \geq k_1$ the set Σ_k of values of $\hat{\theta}$ for which the diffeomorphism $\hat{\psi}_{n,k}(\hat{\theta}, \cdot)$ exhibits Hénon-like attractors has Lebesgue measure $m(\Sigma_k) \geq c_1$. Now we denote $S_k = \mu_k(\hat{\tau}_n^{-1}(\Sigma_k)) \subset [\mu^*_{k+1}, \mu^*_k]$. Then, by construction, the dynamics of f_μ contains Hénon-like strange attractors for every $\mu \in S_k$ and $k \geq k_1$. Moreover, using also Proposition 2.3, $m(S_k) \geq c_2(\mu^*_k - \mu^*_{k+1})$ for every $k \geq k_1$ and some $c_2 > 0$ independent of k. Recalling, in addition, that $\mu_{k+1}/\mu_k \to 1$ we conclude that

$$\frac{m((\cup_{k \geq k_1} S_k) \cap [-\varepsilon, \varepsilon])}{2\varepsilon} \geq \frac{c_2}{3} \quad \text{for every small } \varepsilon > 0$$

and the statement of the theorem follows immediately.

Finally, we discuss the general case of saddle-node ℓ-cycles, $\ell \geq 2$. Let $\mathcal{O}(p_0)$, $\mathcal{O}(p_1)$, $\ldots$, $\mathcal{O}(p_{\ell-1})$ be the periodic orbits involved in the cycle and let $t_0, t_1, \ldots, t_{\ell-1}$ be the corresponding periods. We fix a point p_0 in the saddle-node orbit. On the other hand, we assume $f_0^{t_0 t_i}$ to be C^N-linearizable on a neighbourhood of $\mathcal{O}(p_i)$ for some $1 \leq i \leq \ell - 1$ and we fix such an i from now on. Note also that, by the inclination lemma, we may take the point p_i and some $0 \leq t < t_0$ satisfying

- $W^u(p_i)$ has a critical intersection χ with $W^s(p_0)$;
- $W^u(f_0^t(p_0))$ has a transverse intersection $\tilde{\kappa}$ with $W^s(p_i)$.

We choose neighbourhoods U of p_0 and $\tilde{U}$ of p_i as before and we may suppose $\chi = (a, Y_0) \in U$ and $\tilde{\kappa} \in \tilde{U}$. We fix $l \geq 1$ such that $\tilde{\chi} = f_0^{l t_0 t_i} = (-1, 0) \in \tilde{U}$ and we also denote $\kappa = f_0^{-t}(\tilde{\kappa}) = (q, 0) \in U$. Then we define the transition map $\hat{T}_\infty$ of $f_0^{t_0 t_i}$ in U. From this point on the argument proceeds in just the same way as before, with $f_0^{n t_0 t_i} \circ (f_0^t \circ \hat{T}_\infty) \circ f_0^{l t_0 t_i}$, resp. $f_\mu^{(n+k+l) t_0 t_i + t}$, in the place of $f_0^n \circ \hat{T}_\infty \circ f_0^l$, resp f_μ^{n+k+l}.

3.2 Saddle-node 1-cycles

Now we prove Theorem A for critical cycles involving a unique (saddle-node) periodic orbit $\mathcal{O}(p_0)$. For simplicity we take p_0 to be a fixed point, the general case following from this in just the same way as in the previous section. Once more, let U be a small open neighbourhood of p_0 and (x, Y) be coordinates on U such that $f_\mu(x, Y) = (\varphi_\mu(x), \Psi_\mu(x, Y))$, where φ is a 1-dimensional saddle-node arc, $\Psi(\mu, x, 0) = 0$ and $\|\partial_Y \Psi(\mu, x, Y)\| \leq \lambda < 1$. We also continue to denote by X some saddle-node arc of vector fields tangent to φ.

Let $\approx_\mu$ be the equivalence relation on U spanned by

$$z \approx_\mu f_\mu(z) \text{ for every } z \in U \cap f_\mu^{-1}(U)$$

and denote by $\pi_\mu: U \longrightarrow (U/\approx_\mu)$ the canonical projection associated to it. For $a^- < 0 < a^+$ close to zero we let $D_\mu^\pm = \{(x,0) \in U: a^\pm \le x \le \varphi_\mu(a^\pm)\}$ and $\tilde{D}_\mu^\pm = \pi_\mu(D_\mu^\pm)$. Then, as long as $|\mu| \ll |a^\pm|$, each $\tilde{D}_\mu^\pm$ is a smooth circle and diffeomorphisms $\phi_\mu^\pm: \tilde{D}_\mu^\pm \longrightarrow S^1$ may be taken depending smoothly on μ. In addition, up to replacing $(\phi_\mu^\pm)_\mu$ by $(\tau^\pm \circ (\phi_0^\pm)^{-1} \circ \phi_\mu^\pm)_\mu$ with $\tau^\pm(\pi_0(x,0)) = \int_{a^\pm}^x X(0,s)^{-1} ds \bmod 1$, we may suppose that $\theta_\mu^\pm = \phi_\mu^\pm \circ \pi_\mu$ satisfies

$$\theta_0^\pm(X_t(0,a^\pm)) = t \bmod 1, \quad \text{for every } t. \tag{2}$$

Next, we let $C_\mu^\pm = \{(x,Y) \in U: a^\pm \le x \le \varphi_\mu(a^\pm)\}$ and endow $\tilde{C}_\mu^\pm = \pi_\mu(C_\mu^\pm)$ with the quotient manifold structure. Clearly, every $\tilde{C}_\mu^\pm$ is diffeomorphic to the solid m-torus $S^1 \times B$, B =unit ball in $\mathbf{R}^{m-1}$, and we may choose C^∞ diffeomorphisms

$$\Phi_\mu^\pm: \tilde{C}_\mu^\pm \longrightarrow S^1 \times B, \quad \Phi_\mu^\pm(\pi_\mu(x,Y)) = (\theta_\mu^\pm(x), \Theta_\mu^\pm(x,Y))$$

depending smoothly on μ, with $\theta_\mu^\pm$ as above and $\Theta_\mu^\pm(x,0) = 0$ for every μ and x.

Now we construct smooth maps $\psi, \psi_k: [0,1] \times S^1 \times B \longrightarrow S^1 \times B$ in the following way. Up to taking $|a^\pm|$ small enough, we may fix a (large) integer l such that $f_0^l(D_0^+)$ is contained in $\{(x,Y) \in U: x < \varphi_0(a^-)\}$. Then f_0^l induces a smooth map $h_l: \tilde{D}_0^+ \longrightarrow \tilde{C}_0^-$: $h_l \circ \pi_0 = \pi_0 \circ f_0^l$. Let $\hat{T}_\infty = (T_\infty, 0)$ be the transition map of f, with $a = a^-$, $b = a^+$, recall Section 2. Given $0 \le \sigma \le 1$, we denote by $\tau_\sigma: \tilde{C}_0^- \longrightarrow \tilde{D}_0^+ \subset \tilde{C}_0^+$ the map induced by $\hat{T}_\infty(\sigma, \cdot)$. Then we define $\psi(\sigma, \cdot) = \Phi_0^- \circ h_l \circ \tau_\sigma \circ (\Phi_0^-)^{-1}$. Note that $\Phi_0^+ \circ \tau_\sigma \circ (\Phi_0^-)^{-1}(\theta, \Theta) = (\theta - \sigma \bmod 1, 0)$, in view of (2) and the definition of $T_\infty, \hat{T}_\infty$. We write $\Phi_0^- \circ h \circ (\Phi_0^+)^{-1}(\theta, 0) = (g(\theta), G(\theta))$ and then

$$\psi(\sigma, \theta, \Theta) = (g \circ R_{-\sigma}(\theta), G \circ R_{-\sigma}(\theta)), \quad R_{-\sigma}(\theta) = \theta - \sigma \bmod 1.$$

On the other hand, we set $\psi_k(\sigma, \cdot) = \Phi_0^- \circ h_{k,l,\sigma} \circ (\Phi_0^-)^{-1}$, where $h_{k,l,\sigma}: \tilde{C}_0^- \longrightarrow \tilde{C}_0^-$ denotes the map induced by $f_{\mu_k(\sigma)}^{l+k}$. By Theorem 2.8, this is well defined for every large k and $(\psi_k)_k$ converges to ψ in the C^∞ topology.

By [NPT, Theorem 3.7], the family of circle maps $(g_\sigma = g \circ R_{-\sigma})_{\sigma \in [0,1]}$ unfolds generically homoclinic tangencies associated to some hyperbolic periodic point. In other words, there are $I \subset [0,1]$ an open interval, $\tilde{\sigma} \in I$ and smooth maps $I \ni \sigma \mapsto p_\sigma, c_\sigma$ with p_σ a repelling periodic point and c_σ a critical point of g_σ, such that $c_{\tilde{\sigma}} \in W^u(p_{\tilde{\sigma}})$, $g_{\tilde{\sigma}}^j(c_{\tilde{\sigma}}) = p_{\tilde{\sigma}}$ and $\partial_\sigma \left(g_\sigma^j(c_\sigma) - p_\sigma\right)(\tilde{\sigma}) \neq 0$ for some $j \ge 1$. We assume g to be a Morse function and then such tangencies are quadratic. As a consequence, for every large k the family $\psi_k = (\psi_k(\sigma, \cdot))_{\sigma \in [0,1]}$ of diffeomorphisms of $S^1 \times B$ unfolds generically quadratic homoclinic tangencies associated to some hyperbolic saddle point. Hence, we may conclude from [MV], [V], that $\psi_k(\sigma, \cdot)$ has Hénon-like strange attractors for a set Σ_k of values of σ with positive Lebesgue measure (smooth linearizability near the saddle is not necessary for this conclusion, see [Rm].) Moreover, the fact that the limit family ψ itself undergoes generic quadratic tangencies permits to apply the renormalization scheme in [MV], [V], uniformly to the sequence $(\psi_k)_k$, to conclude that $m(\Sigma_k)$ is uniformly bounded away from zero. Let us explain this in more detail. First one constructs for ψ a sequence $(\phi_n)_n$ of local coordinates such that, up to an appropriate n-dependent reparametrization $\sigma = \tau_n(\nu)$, the expression of ψ^n in these coordinates converges to $\hat{\psi}: (\nu, x, Y) \mapsto (1 - \nu x^2, 0)$ as $n \to \infty$. More precisely:

$(\phi_n \circ \psi(\tau_n(\nu),\cdot) \circ \phi_n^{-1})(x,Y) \to \hat{\psi}(\nu,x,Y)$ in the C^N topology, for some fixed $N \geq 3$. Thus, for every n and k sufficiently large $(\phi_n \circ \psi_k(\tau_n(\nu),\cdot) \circ \phi_n^{-1})_\nu$ is a Hénon-like family and so it exhibits strange attractors for a set of values of the parameter ν whose Lebesgue measure is positive and even uniformly bounded away from zero. We fix n large and conclude in this way that there are $c_1 > 0$ and $k_1 \geq 1$ such that $m(\Sigma_k) \geq c_1$ for every $k \geq k_1$. Now we let $S_k = \mu_k(\Sigma_k) \subset [\mu^*_{k+1}, \mu^*_k]$ and then precisely the same argument as in the previous case shows that $(\cup_{k \geq k_1} S_k)$ has positive density at $\mu = 0$. Finally, it is immediate to check that f_μ has Hénon-like strange attractors for every $\mu \in (\cup_{k \geq k_1} S_k)$. The proof of Theorem A is complete.

4 Proof of Theorem B

Here we construct open classes of families of diffeomorphisms bifurcating through critical saddle-node cycles, for which hyperbolicity of the dynamics near the cycle is also a prevalent feature. We exhibit two such classes $\mathcal{A}_1$, $\mathcal{A}_2$, corresponding, respectively, to 1- and 2-cycles, cf. the Introduction. The conditions in the definition of $\mathcal{A}_1$, $\mathcal{A}_2$, are just meant as simple sets of assumptions assuring hyperbolic behaviour and no effort was made to optimize them.

4.1 Hyperbolicity for 1-cycles

In the present section $(f_\mu)_\mu$ refers always to a smooth family of diffeomorphisms unfolding a critical saddle-node 1-cycle of f_0. We keep the notations of Section 3.2.

First we introduce the map $h\colon [-1,1] \longrightarrow [-1,1]$ given by $h(x) = x(-a + (1+a)x^2)$, where $a \in (0,1)$ is fixed. Note that h has three fixed points $-1, 0, 1$. Since $h'(-1) = h'(1)$, we may identify h with a C^1 endomorphism of $S^1 = [-1,1]/(-1 \sim 1)$ and we do so in the sequel. On the other hand, the origin is an attractor, $|h'(0)| = a < 1$, and its basin $W^s(0)$ contains both critical points of h. This last statement follows, for instance, from the fact that h has negative schwarzian derivative, [Si], using also that it is an odd function. As a consequence, [Ma], $K = S^1 \backslash W^s(0)$ is a hyperbolic set for h: there are $N \geq 1$ and $\rho > 1$ such that $|(h^N)'(\theta)| \geq \rho$ for all $\theta \in K$. Moreover, these dynamical features are robust under small perturbations of the map: there exists a neighbourhood $\mathcal{N}$ of h in $C^1(S^1, S^1)$ such that, given any $\tilde{h} \in \mathcal{N}$, $\tilde{h}$ has a fixed attractor $x(\tilde{h})$ close to zero and $K(\tilde{h}) = S^1 \backslash W^s(x(\tilde{h}))$ is a hyperbolic set for $\tilde{h}$.

Now we define $\mathcal{A}_1$, simply, by the condition

(C) $\quad (g \circ R_{-\frac{1}{2}}) \in \mathcal{N}$.

Let us show that the conclusion of Theorem B is indeed satisfied by every family $(f_\mu)_\mu$ as above. This requires a few preliminary considerations. We introduce

$$U_0 = \{(x,Y) \in U \colon c \leq x \leq \varphi_0(a^+), \|Y\| \leq \xi(x)\} \text{ and } \hat{C} = U_0 \cap C_0^+.$$

Here $c < a^-$ and ξ are chosen in such a way that $f_0^l(\hat{C}) \subset \operatorname{int}(U_0)$ and $f_0(U_0 \backslash \hat{C}) \subset \operatorname{int}(U_0)$. Then there are compact sets $U_{l-1}, \ldots, U_1$ such that

- $f_0(\hat{C}) \subset \text{int}\,(U_1)$, $f_0(U_j) \subset \text{int}\,(U_{j+1})$ for $1 \leq j \leq l-2$ and $f_0(U_{l-1}) \subset \text{int}\,(U_0)$.

As a consequence, $V = U_0 \cup U_1 \cup \cdots \cup U_{l-1}$ satisfies $f_0(V) \subset \text{int}\,(V)$. Now our goal is to show that $L(f_\mu|V)$ is a hyperbolic set of f_μ, for a set of values of μ with positive Lebesgue density at $\mu = 0$.

In order to do this we consider the maps $\psi, \psi_k: [0,1] \times S^1 \times B \longrightarrow S^1 \times B$ introduced in Section 3.2. Recall that $\psi(\sigma, \theta, \Theta) = (g \circ R_{-\sigma}(\theta), G \circ R_{-\sigma}(\theta))$ and $(\psi_k)_k \to \psi$ as $k \to \infty$. Condition (C) implies that $g_\sigma = g \circ R_{-\sigma} \in \mathcal{N}$ for σ in some interval $I = [1/2 - \delta, 1/2 + \delta]$; from now on we always consider $\sigma \in I$. Then $\psi(\sigma, \cdot): S^1 \times B \longrightarrow S^1 \times B$ has a fixed attractor z_σ close to $\{\theta = 0\}$ and $W^s(z_\sigma) = (K(g_\sigma) \times B)^c$. We fix $N \geq 1$, $\rho_1 > 1$ and W an open neighbourhood of $K(g_\sigma)$ such that $|(g_\sigma^N)'(\theta)| \geq \rho_1$ for every $\theta \in W$. Then, for k sufficiently large, $\psi_k(\sigma, \cdot)$ has a fixed attractor $z_{k,\sigma}$ satisfying $(z_{k,\sigma})_k \to z_\sigma$ as $k \to \infty$ and $W^s(z_{k,\sigma}) \supset (W \times B)^c$. Moreover,

$$\Lambda_{k,\sigma} = (S^1 \times B) \backslash W^s(z_{k,\sigma}) = \{z \in S^1 \times B: \psi_k^n(\sigma, z) \in W \times B \text{ for all } n \in \mathbf{Z}\}$$

is a hyperbolic set of the diffeomorphism $\psi_k(\sigma, \cdot)$. Indeed, invariant stable and unstable cone fields for $\psi_k(\sigma, \cdot)$ on $W \times B$ may be constructed in a fairly straightforward way, see for instance [PT2, Chapter 6.3], where a similar situation is treated.

Now we go back to analysing the limit set of f_μ in V. We consider $\mu = \mu_k(\sigma)$, with $\sigma \in I$ and k large. Let w be any point in $L(f_\mu|V)$. In particular, the orbit of w is contained in V and it follows from our construction that there are $n_\pm = n_\pm(\mu, w)$ with $n_- < 0 \leq n_+$ such that both points $w_\pm = f_\mu^{n_\pm}(w)$ belong to C_μ^-. We take $n_\pm$ minima (in absolute value) with that property and then they satisfy

$$|n_\pm(\mu, w)| \leq k + \text{const} \tag{1}$$

where const is independent of μ, w, k (basically, it is determined by the choice of $a^\pm$). Moreover, $\psi_k(\sigma, \tilde{w}_-) = \tilde{w}_+$, where $\tilde{w}_\pm = \pi_\mu(w_\pm)$. At this point we distinguish two cases. Suppose first that $\tilde{w}_\pm \in W^s(z_{k,\sigma})$. Since $\tilde{w}_\pm$ belong to the limit set of $\psi_k(\sigma, \cdot)$, we must have $\tilde{w}_+ = \tilde{w}_- = z_{k,\sigma}$ and so $w \in \mathcal{O}(\hat{z}_\mu)$, where $\hat{z}_\mu$ denotes any point in $\pi_\mu^{-1}(z_{k,\sigma})$. Clearly, $\mathcal{O}(\hat{z}_\mu)$ is a periodic attractor for f_μ. Now assume that $\tilde{w}_\pm \in \Lambda_{k,\sigma}$ and let $\tilde{E}^s_\pm \oplus \tilde{E}^u_\pm$ be the corresponding splitting of the tangent space of $(S^1 \times B)$ at $\tilde{w}_\pm$. We take $T_w M = E^s \oplus E^u$ to be given by $D\pi_\mu \cdot Df_\mu^{n_+}(E^*_w) = E^*_+$ for $* = s, u$. We set this for every $w \in \hat{\Lambda}_\mu = L(f_\mu|V) \backslash W^s(\hat{z}_\mu)$ and then the splitting $E^s \oplus E^u$ of the tangent bundle over $\hat{\Lambda}_\mu$ defined in this way is invariant under Df_μ. Moreover, the uniform bound (1) assures that this is a hyperbolic splitting: there are $\lambda < 1$ and $L \geq 1$ (depending on k) such that $\|Df_\mu^L|E^s\|, \|Df_\mu^{-L}|E^u\| \leq \lambda$.

Altogether, we have proved that $L(f_\mu|V)$ is hyperbolic for every $\mu \in \cup_k \sigma_k(I)$: more precisely, it may be written as $\mathcal{O}(\hat{z}_\mu) \cup \hat{\Lambda}_\mu$, where $\mathcal{O}(\hat{z}_\mu)$ is a hyperbolic periodic attractor and $\hat{\Lambda}_\mu$ is a hyperbolic set of saddle-type. On the other hand, $\cup_k \sigma_k(I)$ has positive Lebesgue density at $\mu = 0$, as a consequence of Proposition 2.3. Our argument is complete.

4.2 Hyperbolicity for 2-cycles

For the construction of $\mathcal{A}_2$ we depart from a (linear) horseshoe map $\psi\colon R \longrightarrow R$ as described in the figure. More precisely, we let R be a compact rectangle in $\mathbf{R}^m = \mathbf{R} \times \mathbf{R}^{m-1}$, $m \geq 2$, and we write $R = R_1 \cup \cdots \cup R_5$, satisfying $\psi(R_1 \cup R_5) \subset \operatorname{int}(R_1)$, $\psi(R_3) \subset \operatorname{int}(R_5)$ and

$$D\psi|R_1 = \begin{pmatrix} \xi & 0 \\ 0 & \Lambda \end{pmatrix}, \quad D\psi|R_2 = \begin{pmatrix} \rho_1 & 0 \\ 0 & \Lambda \end{pmatrix}, \quad D\psi|(R_4 \cup R_5) = \begin{pmatrix} -\rho_2 & 0 \\ 0 & -\Lambda \end{pmatrix},$$

where $\rho_1, \rho_2 > 1$, Λ is a contraction on $\mathbf{R}^{m-1}$ and ξ is some smooth function defined on $\mathbf{R}$. For the sake of notational simplicity we take the fixed saddle point p_0 of ψ in R_2 to coincide with the origin. Moreover, ψ has a fixed sink s_0 in R_1, with $s_0 = (0, S)$ for some $S \in \mathbf{R}^{m-1}$.

Now we modify ψ in order to collapse p_0 and s_0 to a saddle-node point, in the following way. We fix compact intervals W_0, W in $\mathbf{R} \times \{0\} \subset \mathbf{R} \times \mathbf{R}^{m-1}$ such that

$$\{p_0, s_0\} \subset \operatorname{int}(W_0), \quad W_0 \subset \operatorname{int}(W) \text{ and } \psi(W) \subset \operatorname{int}(R_1 \cup \cdots \cup R_4).$$

Then we let $\varphi_\mu\colon W \longrightarrow \mathbf{R}$, $\mu \in [-1, 1]$, be a C^∞ saddle-node arc, in the sense of Section 2.1, such that

- $\varphi_{-1} = \pi_1 \circ \psi|W$ and $\varphi_\mu|(W \backslash W_0) = \pi_1 \circ \psi|(W \backslash W_0)$ for every μ;
- the map φ_0 has a fixed saddle-node point at the origin;
- the derivative φ_μ' is a monotone function, for every small μ.

We also consider $\phi\colon \mathbf{R}^{m-1} \to \mathbf{R}$ to be a C^∞ bump function with support contained in $\pi_2(R)$, satisfying $\phi|\pi_2(\psi(R_2)) = 1$ and $\phi|\pi_2(\psi(R_4)) = 0$. Then we set

$$f_\mu(x, Y) = \psi(x, Y) + \phi(Y)(\varphi_\mu(x) - \pi_1 \circ \psi(x, Y), 0)$$

if $x \in W$ and $f_\mu(x, Y) = \psi(x, Y)$ otherwise. This defines a family of smooth transformations of R into itself, unfolding a critical saddle-node cycle and, clearly, given any m-dimensional manifold M one may extend $(f_\mu)_\mu$ to a family of diffeomorphisms on M. Now we show that ψ and $(\varphi_\mu)_\mu$ may be chosen in such a way that this family satisfies the conclusion of Theorem B, with $V = R$.

In order to do this we fix $a < 0 < b$ such that $(a, 0) \in \operatorname{int}(R_1)$ and $(b, 0) \in \operatorname{int}(R_4)$ and we let $\mu_k(\sigma)$, $T_k(\sigma, x) = \varphi^k_{\mu_k(\sigma)}(x)$, $\hat{T}_k(\sigma, x, Y) = f^k_{\mu_k(\sigma)}(x, Y)$ be as in Section 2. It follows from Theorem 2.4 that there is $\tau > 0$ such that $|(\varphi^k_{\mu_k(\sigma)})'(x)| > \tau$, for every $x, y \in [\varphi^{-1}_{\mu_k(\sigma)}(a), a]$, $\sigma \in [0, 1]$, and k large enough. From now on we assume that $\rho_2\tau > 1$ and $\psi(R_5) \subset \operatorname{int}(C_\mu)$ for all small μ, where $C_\mu = \{(x, Y) \in R : x \in [\varphi^{-1}_\mu(a), a]\}$. These properties may be obtained just by supposing that $\pi_1(R_4)$ and $\pi_1(R_5)$ have sufficiently small diameters. Actually, by further reducing the diameter of $\pi_1(R_5)$ if necessary, we may assume that $\hat{T}_\infty(\sigma, \psi(R_5)) \subset \operatorname{int}(R_3)$, for some interval J of values of σ. As a consequence, $f^{k+2}_{\mu_k(\sigma)}(R_5) \subset R_5$ for every $\sigma \in J$ and k sufficiently large; moreover, it is easy to see that this construction may be performed in such a way that $\|Df^{k+2}_{\mu_k(\sigma)}|R_5\| < 1$. Hence, every f_μ, $\mu \in \mu_k(J)$, has an attracting $(k+2)$-periodic orbit $\mathcal{O}(s_\mu)$ whose basin contains R_5 and R_3. From now on we restrict to values of $\mu \in \cup_{k>k_0}\mu_k(J)$, for some large k_0, and we prove that $L(f_\mu|R)$ is a hyperbolic set for f_μ.

First we note that $L(f_\mu|R) \subset H_\mu \cup \mathcal{O}(s_\mu)$, where H_μ is the maximal invariant set of f_μ in $R_1 \cup R_2 \cup R_4$, and so we only have to show that H_μ is a hyperbolic set. We introduce the (constant) cone fields on $U = \operatorname{int}((R_1 \cup R_2 \cup R_4) \cap \psi(R))$ defined by

$$C^u(z) = \{w = (u, V) \in T_zR : |u| \geq \|V\|\} \text{ and } C^s(z) = \{w = (u, V) \in T_zR : |u| \leq \|V\|\}.$$

In the sequel we always take $\|w\| = \sup\{|u|, \|V\|\}$. Note that $f_\mu(R) = \psi(R)$ and so $H_\mu \subset U$; moreover, $Df_\mu|U$ has the form

$$\begin{pmatrix} \eta_\mu & 0 \\ 0 & \Lambda \end{pmatrix}.$$

Clearly, we may assume, right from the start, that $\|\Lambda\|$ is small enough so as to assure that $\|\Lambda\eta_\mu^{-1}\| < 1$ on U, for every small μ. This implies the invariance of the two cone fields above

$$Df_\mu(C^u(z)) \subset C^u(f_\mu(z)) \quad \text{and} \quad Df_\mu^{-1}(C^s(z)) \subset C^s(f_\mu^{-1}(z))$$

and it also follows that vectors in $C^s(z)$ are geometrically expanded by Df_μ^{-1}. Now we claim that there is $\theta > 1$ such that, given any $z \in H_\mu$, there exists $j = j(z) \leq k+2$ for which

$$\|Df^j_\mu(z)w\| \geq \theta\|w\| \text{ for all } w \in C^u(z)$$

(k is determined by the condition $\mu \in \mu_k(J)$). Note that hyperbolicity of H_μ is an immediate consequence of this claim together with the previous arguments. On the other hand, the claim may be justified as follows. Given $z \in H_\mu$, let $l \geq 0$ be minimum such that $f^l_\mu(z) \in R_4$; then, by construction, $l \leq k+1$ (at most k iterates are spent in $R_1 \cup R_2$, near the origin). We take $j = l+1$ and then

$$\|Df^j_\mu(z)w\| = \|Df_\mu(f^l_\mu(z))) \cdot Df^l_\mu(z)w\| \geq \rho_2\tau\|w\|,$$

(here we make use of the monotonicity of φ'_μ). This proves the claim with $\theta = \rho_2 \tau$.

Observe now that $\cup_{k>k_0} \mu_k(J)$ has positive Lebesgue density at zero, as a consequence of Proposition 2.3. Therefore, we have proved that the conclusion of the theorem holds for the family $(f_\mu)_\mu$. Finally, it is clear that the previous arguments and conclusions carry on to every one-parameter family close enough to $(f_\mu)_\mu$ and so we may take $\mathcal{A}_2$ to be any sufficiently small neighbourhood of $(f_\mu)_\mu$.

A Appendix: Tangent arcs of vector fields

Here we prove Theorem 2.1.

For $\psi: V_1 \times V_2 \longrightarrow \mathbf{R}$ an arbitrary C^∞ map, V_1, V_2 neighbourhoods of zero, we denote by $[\psi]_\infty$ the ∞-jet of ψ at $(0,0)$. First we state a formal version of the theorem.

Proposition A.1 *There exists a unique ∞-jet of vector field arcs $J_\infty = [Z]_\infty$ such that for every representative Z of J_∞*

(a) $[Z_1]_\infty = [\varphi]_\infty$ and

(b) $[\varphi_* Z]_\infty = [Z]_\infty$, where $\varphi_* Z(\mu, x) = \partial_x \varphi(\mu, \varphi_\mu^{-1}(x)) \cdot Z(\mu, \varphi_\mu^{-1}(x))$.

The existence of a unique J_∞ satisfying (a) is proved in [L], [T2]; on the other hand (b) is a direct consequence of (a).

Our goal now is to find a representative Z of J_∞ such that Z_1 has an ∞-order contact with φ along the x-axis. First we describe the asymptotic behaviour of the $\partial_\mu^k(\varphi_*^n Z)|_{\mu=0}$ as $|n| \to \infty$.

Proposition A.2 *There exist V_0 a neighbourhood of zero and C^∞ maps $H_k: V_0 \longrightarrow \mathbf{R}$, $k \geq 0$, such that for all $k \geq 0$ and $Z \in J_\infty$ the sequence $\partial_\mu^k(\varphi_*^n Z)(0, \cdot) \to H_k$ in the C^∞ topology, on $V_0 \cap [0, +\infty)$ when $n \to +\infty$ and on $V_0 \cap (-\infty, 0]$ when $n \to -\infty$.*

By a natural extension of Borel lemma, there is a C^∞ map $X: V \times V \to \mathbf{R}$, $V \subset V_0$ a neighbourhood of zero, such that $\partial_\mu^k X(0, x) = H_k(x)$ for every $k \geq 0$ and $x \in V$. We claim that the arc of vector fields X is tangent to φ. In order to see this note first that $X(0, \cdot) = H_0$ implies that the corresponding flows coincide, i. e. $X_t(0, \cdot) = H_{0,t}$ for all t. On the other hand, by construction, $(\varphi_0)_* H_0 = H_0$ and so $\varphi_0 \circ H_{0,t} = H_{0,t} \circ \varphi_0$ for all t. In this 1-dimensional setting it follows that $\varphi_0 = H_{0,s}$ for some s and, since $H''_{0,s}(0) = sH''_0(0) = 2s\alpha$, it must be $s = 1$. Altogether, this proves that $X_1(0, \cdot) = \varphi_0$, i.e. case $k = 0$ of (3), Section 2. Observe now that each map $\big(t \mapsto (\partial_\mu^k X_t)(0, x)\big)$, $k \geq 1$, $x \in V$, is the solution of the Cauchy problem

(E_k) $\frac{d}{dt} \partial_\mu^k X_t(0, x) = X^{[k]}(0, X_t(0, x), \ldots, \partial_\mu^k X_t(0, x))$, $\partial_\mu^k X_0(0, x) = 0$;

where the $X^{[k]}$ are defined by $X^{[0]}(\mu, x_0) = X(\mu, x_0)$ and

$$(1) \qquad X^{[k+1]}(\mu, x_0, \ldots, x_{k+1}) = \partial_\mu X^{[k]}(\mu, x_0, \ldots, x_k) + \sum_{j=0}^{k} \partial_{x_j} X^{[k]}(\mu, x_0, \ldots, x_k) x_{j+1}.$$

On the other hand, each $(t \mapsto \partial_\mu^k(\varphi_*^n Z)_t(\mu,x))$, $k \geq 1$, $n \in \mathbf{Z}$, $(\mu,x) \in V \times V$, $Z \in J_\infty$, is the solution of

$$(E_{k,n}) \quad \frac{d}{dt}\partial_\mu^k(\varphi_*^n Z)_t(\mu,x) = F^{[k,n]}(\mu,(\varphi_*^n Z)_t(\mu,x),\ldots,\partial_\mu^k(\varphi_*^n Z)_t(\mu,x)), \quad \partial_\mu^k(\varphi_*^n Z)_0(\mu,x) = 0;$$

where $F^{[0,n]}(\mu,x) = (\varphi_*^n Z)(\mu,x)$ and the equations $(E_{k,n})$, $k \geq 1$, are obtained by successive μ-derivation of $(E_{0,n})$; in particular the $F^{[k,n]}$ satisfy recursive relations analogous to (1). Using these relations we conclude, by induction on k, that $F^{[k,n]}(0,x_0,\ldots,x_k)$ converges to $X^{[k]}(0,x_0,\ldots,x_k)$ uniformly on compacts of $(V \cap [0,+\infty)) \times \mathbf{R}^k$ (resp. $(V \cap (-\infty,0]) \times \mathbf{R}^k$) when $n \to +\infty$ (resp. $n \to -\infty$). It follows that

$$(2) \qquad \partial_\mu^k(\varphi_*^n Z)_t(0,x) \to \partial_\mu^k X_t(0,x) \quad \text{for every } t \text{ and } k \geq 1$$

uniformly on $V \cap [0,+\infty)$ (resp. $V \cap (-\infty,0]$) when $n \to +\infty$ (resp. $n \to -\infty$). On the other hand,

Proposition A.3 *For all $k \geq 1$ and $Z \in J_\infty$ the sequence $\partial_\mu^k(\varphi_*^n Z)_1(0,\cdot) \to \partial_\mu^k \varphi_0$, uniformly on $V \cap [0,+\infty)$ (resp. $V \cap (-\infty,0]$) when $n \to +\infty$ (resp. $n \to -\infty$).*

Thus, we have reduced the proof of the theorem to proving Propositions A.2 and A.3. Moreover, when doing this we only have to prove convergence on $\{x \geq 0\}$ as $n \to +\infty$: convergence on $\{x \leq 0\}$ as $n \to -\infty$ follows just by replacing φ_μ by its inverse φ_μ^{-1}; note also that if $H_{k,\pm} = \lim_{n\to\pm\infty} \partial_\mu^k(\varphi_*^n Z)|\{\mu = 0, \pm x \geq 0\}$ then, recall Proposition A.1, $H_{k,+}^{(l)}(0) = H_{k,-}^{(l)}(0)$ for every $l \geq 0$ and so $H_k|\{x \geq 0\} = H_{k,+}$, $H_k|\{x \leq 0\} = H_{k,-}$ defines a C^∞ function H_k on V_0.

Lemma A.4 *Let $\psi(x) = x - \alpha x^2 + O(|x|^3)$ with $\alpha > 0$. Then there is $\delta > 0$ such that*

(a) $\lim_{n\to+\infty} n\psi^n(x) = 1/\alpha$, uniformly on every $[\delta',\delta]$, $0 < \delta' < \delta$;

(b) Given l, $m \in \mathbf{N}$ with $l - m \geq 2$ then $\sum_{n=1}^{+\infty} n^m(\psi^n(x))^l$ is uniformly convergent on $[0,\delta]$;

(c) $\lim_{n\to+\infty} n^2(\psi^n)'(x)$ exists and is nonzero for all $x \in (0,\delta]$; moreover, the convergence is uniform on every $[\delta',\delta]$, $0 < \delta' < \delta$.

Proof: Let us denote $x_n = \psi^n(x)$ and $a_n = n\alpha x_n$. Then $a_{n+1} = a_n(1+1/n)(1-\alpha x_n + O(x_n^2))$ and so

$$a_{n+1} - 1 = (a_n - 1)(1 - \frac{a_n}{n}) + a_n(-\frac{\alpha x_n}{n} + (1+\frac{1}{n})O(x_n^2)).$$

Now, ([DV, Lemma 3.2]) given $0 < \delta' < \delta$ there are C_1, $C_2 > 0$ such that if $x \in [\delta',\delta]$ then $a_n \in [C_1,C_2]$ for every $n \geq 0$. It follows that

$$|a_{n+1} - 1| \leq |a_n - 1||1 - \frac{C_1}{n}| + \frac{C_3}{n^2} \quad \text{for some } C_3 > 0$$

Let us denote $\xi_n = |a_n - 1|$, $\alpha_n = |1 - C_1/n|$ and $\beta_n = C_3/n^2$. Then $\xi_{n+1} \leq A_p + B_p$ for every $p \in \{1, \ldots, n\}$, where $A_p = \beta_n + \alpha_n\beta_{n-1} + \cdots + \alpha_n\alpha_{n-1}\cdots\alpha_{p+2}\beta_{p+1}$ and $B_p = \alpha_n\alpha_{n-1}\cdots\alpha_{p+1}\beta_p + \cdots + \alpha_n\alpha_{n-1}\cdots\alpha_2\beta_1 + \alpha_n\alpha_{n-1}\cdots\alpha_1\xi_1$. It is easy to check that

$$A_p \leq \sum_{j=p+1}^{n} \frac{C_3}{j^2} \leq \frac{C_4}{p} \text{ and } B_p \leq \prod_{j=p+1}^{n} \alpha_j \left(\xi_1 + \sum_{i=1}^{p} \frac{C_3}{n^2}\right) \leq C_4 \left(\frac{p}{n}\right)^{\bar{\gamma}},$$

for some C_4, $\bar{\gamma} > 0$. Then, fixing $p \in [\sqrt{n}, \sqrt{n}+1)$ and replacing above we get that $\xi_{n+1} \leq C_5/n^{\gamma}$ for some C_5, $\gamma > 0$. Thus $\lim a_n = 1$, which proves (a). Moreover, we also get

$$|n^m(\psi^n(x))^l| \leq n^m(\psi^n(\delta))^l \leq C_6 n^{m-l}$$

for some $C_6 > 0$ and every $x \in [0, \delta]$, thus proving (b). In order to prove (c) we put $b_n = n^2(\psi^n)'(x)$ and then from $b_{i+1} = b_i(1+1/i)^2(1 - 2\alpha x_i + O(x_i^2))$ we find

$$b_n = b_1 \prod_{i=1}^{n-1} \left(1 + \frac{2}{i}(1 - a_i) + \frac{1 - 2a_i - 2\alpha x_i}{i^2} + (1+\frac{1}{i})^2 O(x_i^2)\right)$$

where, as before, $a_i = \alpha i x_i$. Now the fact that $(b_n)_n$ converges to a nonzero real number is an immediate consequence of

$$\sum_i |\frac{2}{i}(1 - a_i) + \frac{1 - 2a_i - 2\alpha x_i}{i^2} + (1+\frac{1}{i})^2 O(x_i^2)| \leq \sum_i (\frac{2}{i}\frac{C_5}{i^{\gamma}} + \frac{4}{i^2} + 4\frac{C_7}{i^2}) < +\infty$$

(some $C_7 > 0$). $\square$

Lemma A.5 *Let $\varphi = (\varphi_\mu)_\mu$ be a saddle-node arc. If $\delta > 0$ is small enough then for each $(k, l) \in \mathbf{N}_0^2$ there are constants $A(k, l)$, $B(k, l) > 0$ and nonnegative integers $i(k, l)$, $j(k, l)$ such that for all $n \geq 1$ and $x \in [0, \delta]$*

(a) $|\partial_\mu^k \partial_x^l (\varphi_\mu^{-n}(x))|_{\mu=0}| \leq A(k, l) n^{i(k,l)}$;

(b) $|\partial_\mu^k \partial_x^l (\partial_x \varphi_\mu^n(\varphi_\mu^{-n}(x)))|_{\mu=0}| \leq B(k, l) n^{j(k,l)}$.

Proof: For the sake of simplicity we denote $\psi_\mu = \varphi_\mu^{-1}$. First we prove (a), by induction on $k + l$. For $(k, l) = (0, 0)$ we take $A(0, 0) = \delta$ and $i(0, 0) = 0$ and then the statement is obvious. For $(k, l) = (0, 1)$ we observe that for $z > 0$ close to zero

$$\partial_x \psi_0^n(z) = \prod_{i=0}^{n-1} \partial_x \psi_0(\psi_0^i(z)) < 1 \quad \text{for all } n \geq 1$$

so that we may take $A(0, 1) = 1$ and $i(0, 1) = 0$. For $(k, l) = (1, 0)$ we let $A(1, 0) = C_1 = \sup\{|\partial_\mu \psi_\mu(z)|_{\mu=0}| : 0 \leq z \leq \delta\}$ and $i(1, 0) = 1$ and then we proceed by induction on n:

$$\begin{aligned} |\partial_\mu \psi_\mu^{n+1}(x)|_{\mu=0}| &\leq |\partial_\mu \psi_\mu(\psi_0^n(x))|_{\mu=0}| + |\partial_x \psi_0(\psi_0^n(x))|_{\mu=0}||\partial_\mu \psi_\mu^n(x)|_{\mu=0}| \\ &\leq C_1 + A(1, 0) n^{i(1,0)} \leq A(1, 0)(n+1)^{i(1,0)}. \end{aligned}$$

Let us assume now that (a) is true for every $(p,q)\in\mathbf{N}_0^2$ such that $p+q<k+l$. It is not difficult to see that

$$\partial_\mu^k\partial_x^l\psi_\mu^{n+1}(x)=\partial_x\psi_\mu(\psi_\mu^n(x))\partial_\mu^k\partial_x^l\psi_\mu^n(x)+\sum_{j=1}^{m(k,l)}F_j \tag{3}$$

where $m(k,l)\geq 1$ and each F_j has the form $\partial_\mu^\alpha\partial_x^\beta\psi(\mu,\psi_\mu^n(x))\prod_{i=1}^s\partial_\mu^{p_i}\partial_x^{q_i}\psi_\mu^n(x)$ with $s\geq 0$, $\alpha+\beta\leq k+l$ and $1\leq p_i+q_i\leq k+l-1$ for all i. Thus

$$|\partial_\mu^k\partial_x^l\psi_\mu^{n+1}|\leq|\partial_\mu^k\partial_x^l\psi_\mu^n|+C_2n^{j_2}$$

for some $C_2>0$ and $j_2\geq 0$ depending only on: (i) $A(p,q)$, $i(p,q)$ with $p+q\leq k+l-1$; (ii) $\sup_{z\in[0,\delta]}|\partial_\mu^\alpha\partial_x^\beta\psi_\mu(0,z)|$ with $\alpha+\beta\leq k+l$; (iii) the form of (3), which is determined by (k,l). Hence, (a) follows by induction on n, as long as

$$A(k,l)n^{i(k,l)}+C_2n^{j_2}\leq A(k,l)(n+1)^{i(k,l)}$$

and this is satisfied if we fix $A(k,l)=C_2$ and $i(k,l)=j_2+1$.

Now we prove (b). Remark that Lemma A.4 applied to $\psi=\psi_0$ gives $\partial_x{\varphi_0^{-}}^n(x)\geq(An^2)^{-1}$ for some $A>0$ and all $x\in[0,\delta]$. Hence, we may take $B(0,0)=A$ and $j(0,0)=2$. For general (k,l) we have

$$\partial_\mu^k\partial_x^l(\partial_x\varphi_\mu^n({\varphi_\mu^{-}}^n(x)))=\partial_\mu^k\partial_x^l((\partial_x{\varphi_\mu^{-}}^n(x))^{-1})=(\partial_x{\varphi_\mu^{-}}^n(x))^{-k-l-1}\sum_{j=1}^{n(k,l)}G_j \tag{4}$$

where $n(k,l)\geq 1$ and each G_j has the form $\prod_{i=1}^s\partial_\mu^{p_i}\partial_x^{q_i}{\varphi_\mu^{-}}^n(x)$ with $s\geq 0$ and $p_i+q_i\leq k+l+1$ for all i. From (a) and the previous remark we get

$$|\partial_\mu^k\partial_x^l(\partial_x\varphi_\mu^n({\varphi_\mu^{-}}^n(x)))|\leq(An^2)^{k+l+1}C_3n^{j_3}$$

where C_3 and j_3 depend only on: (i) $A(p,q)$ and $i(p,q)$ with $p+q\leq k+l+1$; (ii) the form of (4), which is determined by (k,l). Therefore, (b) follows just by taking $B(k,l)=C_3A^{k+l+1}$ and $j(k,l)=2(k+l+1)+j_3$. $\square$

Proof of Proposition A.2: By derivation of $\varphi_*^nZ(\mu,x)=\partial_x\varphi_\mu^n({\varphi_\mu^{-}}^n(x))\cdot Z(\mu,{\varphi_\mu^{-}}^n(x))$ we get that $\partial_\mu^k\partial_x^l(\varphi_*^nZ)(\mu,x)$ can be written as a sum

$$\sum_{j=1}^{m(k,l)}\left(\partial_\mu^{\alpha_j}\partial_x^{\beta_j}(\partial_x\varphi_\mu^n({\varphi_\mu^{-}}^n(x)))\cdot\partial_\mu^{p_j}\partial_x^{q_j}Z(\mu,{\varphi_\mu^{-}}^n(x))\cdot\prod_{i=1}^{s_j}\partial_\mu^{\gamma_{j,i}}\partial_x^{\delta_{j,i}}{\varphi_\mu^{-}}^n(x)\right) \tag{5}$$

where $m(k,l)\geq 1$, $s_j\geq 0$ and $(\alpha_j+\beta_j)$, (p_j+q_j), $(\gamma_{j,i}+\delta_{j,i})\leq(k+l)$ for all i and j. From Lemma A.5 we conclude that there exist $C_1>0$ and $m\geq 0$ (depending only on $A(p,q)$, $B(p,q)$, $i(p,q)$, $j(p,q)$ with $p+q\leq k+l$ and on the form of (5)) such that for every pair Z, Y, of saddle-node arcs of vector fields

$$|\partial_\mu^k\partial_x^l(\varphi_*^nZ-\varphi_*^nY)(0,x)|\leq C_1n^m\sum_{p+q\leq k+l}|\partial_\mu^k\partial_x^l(Z-Y)(0,{\varphi_0^{-}}^n(x))| \tag{6}$$

for all $n \geq 1$ and $x \in [0, \delta]$. On the other hand, given any $Z \in J_\infty$ then Z and $Y = \varphi_* Z$ have ∞-order contact at $(0,0)$ and so there is $C_2 > 0$ such that

$$(7) \qquad |\partial_\mu^p \partial_x^q (Z - \varphi_* Z)(0, z)| \leq C_2 z^{m+2}$$

for all $p + q \leq k + l$ and $z \in [0, \delta]$. From (6), (7) and Lemma A.4 we get

$$|\partial_\mu^k \partial_x^l (\varphi_*^i Z - \varphi_*^j Z)(0, x)| \leq C_3 \sum_{n=i}^{j-1} n^m (\varphi_0^{-n}(x))^{m+2} \leq C_4 \sum_{n \geq i} \frac{1}{n^2}$$

for some C_3, $C_4 > 0$ and all $1 \leq i \leq j$, $x \in [0, \delta]$. This implies that, for each k, $l \geq 0$, $\left(\partial_\mu^k \partial_x^l (\varphi_*^n Z)(0, \cdot)\right)_n$ is a Cauchy sequence and so each $\left(\partial_\mu^k (\varphi_*^n Z)(0, \cdot)\right)_n$, $k \geq 0$, converges in the C^∞ topology to some C^∞ function H_k. $\square$

Proof of Proposition A.3: It follows from (2) and the previous proposition that $\lim_{n \to \infty} \partial_\mu^k (\varphi_*^n Z)(0, \cdot)$ is independent of $Z \in J_\infty$. Therefore, it is no restriction to assume here

$$(8) \qquad Z_1(0, \cdot) = \varphi_0;$$

observe that, given any $\tilde{Z} \in J_\infty$, $Z(\mu, x) = H_0(x) + (\tilde{Z}(\mu, x) - \tilde{Z}(0, x))$ is also in J_∞ and satisfies (8). By derivation of $(\varphi_*^n Z)_1 = \varphi_\mu^n \circ Z_1(\mu, \cdot) \circ \varphi_\mu^{-n}$ we get that $\partial_\mu^k (\varphi_*^n Z)_1(\mu, x)$ may be written

$$(9) \quad \sum_{j=1}^{m(k,l)} \left(\partial_\mu^{p_j} \partial_x^{q_j} (\varphi_\mu^n (Z_1(\mu, \varphi_\mu^{-n}(x)))) \cdot \prod_{i=1}^{r_j} \partial_\mu^{\alpha_{j,i}} \partial_x^{\beta_{j,i}} (Z_1(\mu, \varphi_\mu^{-n}(x))) \cdot \prod_{i=1}^{s_j} \partial_\mu^{\gamma_{j,i}} \partial_x^{\delta_{j,i}} (\varphi_\mu^{-n}(x)) \right)$$

where $m(k, l) \geq 1$, r_j, $s_j \geq 0$ and $(p_j + q_j)$, $(\alpha_{j,i} + \beta_{j,i})$, $(\gamma_{j,i} + \delta_{j,i}) \leq (k + l)$ for all i and j. We also write $\varphi_\mu = \varphi_\mu^n \circ \varphi_\mu \circ \varphi_\mu^{-n}$ and then Lemma A.5 and (8) imply

$$(10) \; |\partial_\mu^k (\varphi^n \circ Z_1 \circ \varphi^{-n} - \varphi)(0, x)| \leq C_1 n^m \sum_j |(\prod_1^{r_j} \partial_\mu^{\alpha_{j,i}} \partial_x^{\beta_{j,i}} Z_1 - \prod_1^{r_j} \partial_\mu^{\alpha_{j,i}} \partial_x^{\beta_{j,i}} \varphi)(0, \varphi_0^{-n}(x))|$$

for some $C_1 > 0$ and $m \geq 1$ depending only on $A(p, q)$, $B(p, q)$, $i(p, q)$, $j(p, q)$ with $p + q \leq k + l$ and on the form of (9). On the other hand, as $[Z_1]_\infty = [\varphi]_\infty$, we have $[\prod_1^r \partial_\mu^{\alpha_i} \partial_x^{\beta_i} Z_1]_\infty = [\prod_1^r \partial_\mu^{\alpha_i} \partial_x^{\beta_i} \varphi]_\infty$ for all r, $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_r \geq 0$. Hence, there exists $C_2 > 0$ such that

$$(11) \qquad \sum_j |(\prod_1^{r_j} \partial_\mu^{\alpha_{j,i}} \partial_x^{\beta_{j,i}} Z_1 - \prod_1^{r_j} \partial_\mu^{\alpha_{j,i}} \partial_x^{\beta_{j,i}} \varphi)(0, \varphi_0^{-n}(x))| \leq C_2 (\varphi_0^{-n}(x))^{m+1}$$

for all $x \in [0, \delta]$. from (10) and (11) we deduce that there is $C_3 > 0$ such that

$$|\partial_\mu^k (\varphi^n \circ Z_1 \circ \varphi^{-n} - \varphi)(0, x)| \leq C_3 n^m (\varphi_0^{-n}(x))^{m+1}$$

for all $x \in [0, \delta]$. In view of Lemma A.4, this implies that $\partial_\mu^k (\varphi_*^n Z)_1(0, \cdot)$ converges to $\partial_\mu^k \varphi(0, \cdot)$, uniformly on $[0, \delta]$. $\square$

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