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ON THE ONE-DIMENSIONAL MUCKENHOUPT CONDITION A_∞.

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Abstract

The exact value p for which the Muckenhoupt condition A_∞ implies A_p is obtained.

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Introduction. The necessary and sufficient condition for the boundedness of a series of classical operators in the weight spaces L_p is the A_p-condition of Muckenhoupt for the weight functions. The proof of the boundedness of operators is based precisely on the application of the following most important properties of A_p-weights [1], [2]:

$(i)J \quad (A_1)$ *implies the reverse Hölder inequality for some* $p > 1$;

$(ii) \quad$ *for* $1 < p < \infty$ *the condition* (A_p) *implies* $(A_{p-\varepsilon})$ *for some* $\varepsilon > 0$;

$(iii) \quad (A_\infty)$ *implies* (A_p) *for some* $p < \infty$.

The determination of extremal values of exponents is perhaps of interest for any of these properties. The upper bound for all p in the proposition (i) was obtained in [3]. The limiting value for ε in (ii) was determined in [4]. The present paper is devoted to the determination of the exact value of p in (iii). As in [3] and [4] we consider here the one-dimensional case.

1. Main results. Let f be a non negative measurable function on an interval $[\alpha, \beta]$. The nondecreasing rearrangement of the function f on a measurable set $E \subset [\alpha, \beta]$ is defined by the equality

$$f_*^{(r)}(t) \equiv f_{*,E}^{(r)}(t) = \sup\{\alpha > 0 : \ |\{x \in E : \ f(x) < \alpha\}| \leq t\}, \quad 0 < t < |E|,$$

where $|\cdot|$ denotes the Lebesgue measure. The so defined rearrangement $f_*^{(r)}(t)$ is the right-continuous function. If we replace the sign " $\leq$ " by " $<$ " in the definition of $f_*^{(r)}(t)$ then we obtain the left-continuous rearrangement $f_*^{(l)}(t)$. We define the function f_* by:

$$f_*(t) = \sqrt{f_*^{(r)}(t) \cdot f_*^{(l)}(t)}, \quad 0 < t < |E|, \ f_*(0) = f_*(0+), \ Jf_*(|E|) = f_*(|E| - 0). \tag{1.1}$$

The functions $f_*^{(r)}(t)$, $f_*^{(l)}(t)$ and $f_*(t)$ are nondecreasing on $(0, |E|)$ and equimeasurable with $(f \mid E)(x)$ where $f \mid E$ is the restriction of the function f to the set E.

Remark 1. The definition (1.1) of the rearrangement f_* was considered in [5]. It is convenient since, in the studied problems, the conditions are formulated in terms of values of the rearrangements at the specific points. Therefore, the application of f_* permits below in Theorem 1 does not exclude from the consideration the set of the discontinuity points. It is not essential to study the integral properties of f_* because the set of discontinuity points of the rearrangement is at most countable.

Let μ be a nonnegative absolutely continuous measure on an interval $[\alpha, \beta]$, e.g. $d\mu = w(x)dx$, where the weight function $w(x) \geq 0$, $x \in [\alpha, \beta]$, is nonequivalent to zero. In [5] the authors obtained different types of the necessary and sufficient conditions such that the measure μ satisfies the condition

$$\mu(I) \leq b \cdot m_I \cdot \mid I \mid, \quad I \subset [\alpha, \beta], \tag{1.2}$$

where $m_I \equiv m_{I,w} = (w \mid I)_*(\mid I \mid /2)$, and the constant b is independent of the interval I. In particular, one of the equivalent to (1.2) conditions is: *for some* p, $1 < p < \infty$, *the inequality*

$$\frac{1}{\mid I \mid}\int_I w(x)dx \left(\frac{1}{\mid I \mid}\int_I w^{-1/(p-1)}(x)dx\right)^{p-1} \leq B, \quad I \subset [\alpha, \beta], \tag{A_p}$$

holds, where the constant B *does not depend on interval* I.

The condition (1.2) is one of the several forms of the so-called A_∞-condition of Muckenhoupt. We consider this condition in the general case

$$\frac{1}{\mid I \mid}\int_I w(x)dx \leq c(w \mid I)_*\left(\frac{\mid I \mid}{a}\right), \quad I \subset [\alpha, \beta], \tag{A_∞}$$

where the constants $c > 0$ and $a \geq 1$ are independent of the interval I. We observe that from the trivial inequalities

$$\frac{1}{\mid I \mid}\int_I w(x)dx \geq \frac{1}{\mid I \mid}\int_{\{x \in I:\ w(x) \geq (w|I)_*(|I|/a)\}} w(x)dx \geq$$

$$\geq \frac{1}{\mid I \mid}\int_{|I|/a}^{|I|}(w \mid I)_*(t)dt \geq \left(1 - \frac{1}{a}\right)(w \mid I)_*\left(\frac{\mid I \mid}{a}\right)$$

it follows directly that in (A_∞) it is necessary that $c \geq 1 - 1/a$. Indeed we show below (cf. Proposition 1) that the condition (A_∞) is uninteresting for $c < 1$. If $c \geq 1$ and $a = 1$ then it is obvious that (A_∞) holds for every integrable function w. Therefore everywhere below, excepting the special stipulated cases, we assume that $a > 1$ and $c \geq 1$. We exclude also from the consideration the trivial cases when the studied functions are equivalent to zero.

In this article the important elements in the conditions (A_p) and (A_∞) are the constants B, a, c. Therefore below we denote by $A_p(B)$ and $A_\infty(a, c)$ the classes of all functions w satisfying conditions (A_p) and (A_∞) respectively. We denote also $A_p(B; E) \equiv A_p(B)$, $A_\infty(a, c; E) \equiv A_\infty(a, c)$, when the need arises to mark the set E on which the corresponding conditions are considered.

The following moment is important in determining the extremal exponents in the properties $(i), (ii)$ (see Introduction) in the papers [3] and [4]. ¿From the corresponding conditions for the function w the analogous condition for the rearrangement w_* with the same constant is deduced. This step permits to reduce the corresponding problem to the case when the function is monotone. In this case determining the extremal exponent is simpler. The analogous method has been applied before to prove the propositions $(i) - (iii)$ in some articles of the other authors, in particular in [1]. But the absence of the exact estimates of rarrangements was one of the basic obstacles in determining the exact values of parameters. One of the basic results of this paper is following: from the condition (A_∞) we deduce the analogous property of the rearrangement with the same constants. Namely, we have

Theorem 1. *Let w be a nonnegative function on an interval $[\alpha,\beta]$ belonging to the class $A_\infty(a,c)$ for some $a>1$ and $c\ge 1$. Then the inequality*

$$w_{**}(t)\equiv\frac{1}{t}\int_0^t w_*(u)du\le c\cdot w_*\left(\frac{t}{a}\right),\quad 0<t\le\beta-\alpha, \tag{1.3}$$

holds with the same constants a and c.

The proof of this Theorem is contained in Section 3. On its application is based the proof of the following below Theorem 4 where we obtain the limiting value p in property (iii).

The proof of (iii) for some $p=p(a,c)<\infty$ can be obtained by Theorem 1 and applying the well-known results. In fact, we have the following proposition.

Theorem 2. *Let $a_1,\ a_2,\ c_1$ be numbers such that $a_1,a_2>1$ and $c_1\ge 1$. Then there exists the number c_2, depending only on $a_1,\ a_2$ and c_1 such that from the belonging of an function w to the class $A_\infty(a_1,c_1;[\alpha,\beta])$ it follows that $w\in A_\infty(a_2,c_2;[\alpha,\beta])$.*

This Theorem is proved in Section 3. In particular, by Theorem 2 for $a_2=2$ we obtain that for every a and c the embedding $A_\infty(a,c)\subset A_\infty(2,c')$ holds for some $c'=c'(a,c)$. Now, it was shown in [5] that there exist $p=p(c')<\infty$ and $B=B(c')<\infty$ such that

$$A_\infty(2,c')\subset A_p(B). \tag{1.4}$$

Therefore we have the following

Theorem 3. *Let $a,\ c$ be numbers such that $a>1$ and $c\ge 1$. Then there exist $p=p(a,c)<\infty$ and $B=B(a,c)<\infty$ such that from the belonging of an function w to the class $A_\infty(a,c)$ it follows that $w\in A_p(B)$.*

But it is easy to show that the suggested in [5] method of the determining of value $p=p(c')$ such that the embedding (1.4) holds for some B, does not allow one to find the exact value p for this embedding. Furthermore, the above proposed proof of Theorem 3, which was based on the Theorem 2 and embedding (1.4), did not give of the optimal value of the parameter p.

We define the notion of the optimal value p in the proposition (iii) more precisely. By the Hölder inequality, the condition $w\in A_p(B)$ implies $w\in A_q(B)$ for every $q\ge p>1$. Then, from Theorem 3 it follows that there exists

$$p_0\equiv p_0(a,c)=\inf\{p>1:\exists\ B=B(a,c,p):\ A_\infty(a,c)\subset A_p(B)\}. \tag{1.5}$$

The number p_0 defined by (1.5), is called *the optimal value of the parameter p in proposition (iii).*

The key result of this paper is the determination of the value $p_0 = p_0(a, c)$.

Theorem 4. *Let $a,\ c$ be numbers such that $a > 1,\ c \geq 1$, and the number $p_0 \geq (\ln\ a)^{-1}$ be determined as the root of the equality*

$$a^{p-1} = p \cdot c. \tag{1.6}$$

Then:

a) for every $p > p_0$ there exists $B = B(a, c, p)$ such that if a function w belongs to the class $A_\infty(a, c; [\alpha, \beta])$ it follows that $w \in A_p(B; [\alpha, \beta])$;

b) if a and c such that $p_0 > 1$, there exists a function $w \in A_\infty(a, c; [\alpha, \beta])$ but it does not belong to $A_{p_0}(B; [\alpha, \beta])$ for every B;

c) if $p_0 = 1$ then the condition $w \in A_\infty(a, c; [\alpha, \beta])$ implies that the function $w(x)$ is equivalent to some constant on $[\alpha, \beta]$.

This Theorem is proved in Section 3.

Remark 2. The Theorem 4 shows that the value p_0 which is defined as the root of the equality (1.6) is equal to the value p_0 which is defined by (1.5). And, what is more, if we use the definition p_0 by (1.5) then for $p_0 > 1$ from the property of the A_p-weights (ii) we see that for every B the class $A_\infty(a, c)$ is not contained in $A_{p_0}(B)$. The part $b)$ of Theorem 4 asserts that in this case there exists a function $w \in A_\infty(a, c)$ but it does not belong to $\bigcup_B A_{p_0}(B)$.

Remark 3. It is obvious that the function $\varphi(p) = a^{p-1}/p,\ p > 0$, reach its smallest value when $p = (\ln\ a)^{-1}$. This value is equal to one for $a = e$ and smaller than one for the remaining values $a > 1$. Since $\varphi(1) = 1$ then for $c > 1$ the value $p_0 > (\ln\ a)^{-1}$, which is defined by (1.6) as the root of the equality $\varphi(p) = c$, is larger than one. If $c = 1$ then $p_0 > 1$ for $a < e$ and $p_0 = 1$ when $a \geq e$. Thus the special case $c = 1$ of Theorem 4 has the following form.

Theorem 5. $a)$ *If for some $a \geq e$ the integrable on $[\alpha, \beta]$ function $w(x) \geq 0$ satisfies the condition*

$$\frac{1}{|\ I\ |}\int_I w(x)dx \leq (w \mid I)_*(\frac{|\ I\ |}{a}), \quad I \subset [\alpha, \beta],$$

then $w(x)$ is equivalent to some constant on $[\alpha, \beta]$;

$b)$ *the assertion $a)$ ceases to be valid if $a < e$.*

In the other words, Theorem 5 asserts that, in the class $A_\infty(a, 1)$, nontrivial functions exist only if $a < e$.

Remark 4. In the limiting case $a = \infty$ the condition (A_∞) has the following form

$$\frac{1}{\mid I \mid}\int_I w(x)dx \leq c \cdot (w \mid I)_*(0), \quad I \subset [\alpha, \beta]. \tag{1.7}$$

Since $(w \mid I)_*(0) = \text{ess } \inf_{x \in I} \ w(x)$ then this inequality can be rewritten as

$$\frac{1}{\mid I \mid}\int_I w(x)dx \leq c \cdot \text{ess} \inf_{x \in I} \ w(x), \quad I \subset [\alpha, \beta]. \tag{A_1}$$

The condition (A_1) is well-known as the Muckenhoupt A_1-condition. Thus if we denote by $A_1(c) \equiv A_1(c; [\alpha, \beta])$ the class of all functions $w(x) \geq 0, \ x \in [\alpha, \beta]$, which satisfy (A_1), then we have the following limiting relation between the Muckenhoupt's classes A_1 and A_∞

$$A_1(c) \subset A_\infty(\infty, c).$$

It was noted above that the analogue of Theorem 1 for the functions from the class $A_1(c)$ has been proved in [3]. More precisely, the nonincreasing rearrangements of functions was considered in [3]. By the notations of this paper the equivalent formulation of the corresponding proposition from [3] has the following form.

Theorem 6.*Let the function $w \in A_1(c; [\alpha, \beta])$ and I a subinterval of $[\alpha, \beta]$. Then for each interval $J \subset [0, \mid I \mid]$ the inequality*

$$\frac{1}{\mid J \mid}\int_J (w \mid I)_*(u)du \leq c \cdot \text{ess} \inf_{t \in J} \ (w \mid I)_*(t) \tag{1.8}$$

holds.

The proof of Theorem 6 in [3] at first has been obtained for the continuous function w and the general case has also been obtained with the help of one special lemma [3]. In the fourth Section of this paper we give the more simple proof of Theorem 6. It is essentially the same modification of the proof of Theorem 1 for the case $a = \infty$.

2. Some auxilary results. For an integrable on an set $E, \ 0 <\mid E \mid< \infty$, function w we denote below

$$w_E = \frac{1}{\mid E \mid}\int_E w(x)dx.$$

It is clear that when $a < 1$ the condition (A_∞) has no sense for every $c > 0$. We show that (A_∞) is interesting only for $a > 1$ and $c \geq 1$.

Proposition 1. *Let the function $w(x) \geq 0, \ x \in [\alpha, \beta]$, satisfies the condition (A_∞) for some $c < 1$. Then:*

a) if $a > 1$ then $w(x)$ is equivalent to zero on $[\alpha, \beta]$;

b) if $a = 1$ then $w(x)$ is either equivelent to zero on $[\alpha, \beta]$ or $w(x)$ is essentially unbounded on every interval $I \subset [\alpha, \beta]$.

Proof. a). Let $x_0 \in [\alpha, \beta]$ be the Lebesgue point of w. Then for each interval $I \subset [\alpha, \beta]$ containing $x_0 \in [\alpha, \beta]$, by (A_∞), we obtain

$$\frac{2}{\mid I \mid} \int_I \mid w(x) - w(x_0) \mid dx \geq \frac{1}{\mid I \mid} \int_I \mid w(x) - w_I \mid dx = \frac{1}{\mid I \mid} \int_0^{|I|} \mid (w \mid I)_*(t) - w_I \mid dt \geq$$

$$\geq \frac{1}{\mid I \mid} \int_{|I|/a}^{|I|} ((w \mid I)_*(t) - w_I) dt \geq \left(1 - \frac{1}{a}\right) \left((w \mid I)_* \left(\frac{\mid I \mid}{a}\right) - w_I\right) \geq$$

$$\geq \left(1 - \frac{1}{a}\right) \left(\frac{1}{c} - 1\right) \frac{1}{\mid I \mid} \int_I w(x) dx.$$

If the intervals I subtend to x_0 then, by the Lebesgue Theorem [6], the left part of this inequality tends to zero, and the right part tends to $(1 - 1/a)(1/c - 1)w(x_0)$. Hence it follows that $w(x_0) = 0$. Since the almost every point x_0 is the Lebesgue point of function w then $w(x)$ is equivalent to zero on $[\alpha, \beta]$.

b). We assume that on some interval $I \subset [\alpha, \beta]$.

$$0 < M \equiv \text{ess} \sup_{x \in I} w(x) < \infty. \tag{2.1}$$

Then for every interval $J \subset I$, by (A_∞), it follows that

$$\frac{1}{\mid J \mid} \int_J w(x) dx \leq c \cdot (w \mid J)_*(\mid J \mid) \leq c \cdot (w \mid I)_*(\mid I \mid) = cM \equiv M_1 < M.$$

If the intervals J subtend to Lebesgue point of function w then we obtain from the last inequality that $w(x) \leq M_1 < M$ almost everywhere on I. This is contrary to (2.1). The proof is complete.

The two following Lemmas will be needed for the proof of Theorem 1. The first Lemma is one of the versions of the Riesz's Lemma on the "rising sun".

Lemma 1. *Let w be a function on an interval I and γ be a number not greater than w_I. Then there exists a family of pairwise nonintersecting intervals $I_\nu \subset I$, $\nu = 1, 2, \ldots$, such that $w_{I_\nu} = \gamma$, $\nu = 1, 2, \ldots$, and $w(x) \geq \gamma$ almost everywhere on the set $I \setminus E$ where $E \equiv \bigcup_\nu I_\nu$.*

Fof the proof of Lemma 1 see [7], [4].

For a function $w(x)$ measurable on an set E we define the distribution function

$$\lambda_{w,E}(y) =\mid \{x \in E: \ w(x) < y\} \mid, \quad 0 < y < \infty.$$

The function $\lambda_{w,E}(y)$ is nondecreasing left-continuous and in some sense the inverse function of the rearrangement $w_{*,E}(t)$.

Lemma 2. *Let the set $E = \bigcup_\nu E_\nu$ where E_ν, $\nu = 1, 2, ...,$ are measurable sets, $\mid E_\nu \bigcap E_k \mid= 0$, $\nu \neq k$, and $\mid E \mid< \infty$. Let $w(x)$ be a nonnegative measurable function on E and let a be a number not less than one. Then*

$$\inf_\nu (w \mid E_\nu)_* \left(\frac{\mid E_\nu}{a}\right) \leq w_{*,E}\left(\frac{\mid E \mid}{a}\right) \leq \sup_\nu (w \mid E_\nu)_* \left(\frac{\mid E_\nu}{a}\right). \tag{2.2}$$

Proof. If $a = 1$ then (2.2) is obvious. Let $a > 1$. If $w_{*,E}(t)$ is continuous at the point $t_0 =\mid E \mid /a$ and $w_{*,E}(t_0 - \delta) < w_{*,E}(t_0)$ for every $\delta > 0$ then (2.2) follows directly from the obvious equality

$$\lambda_{w,E}(y) = \sum_\nu \lambda_{w,E_\nu}(y), \quad 0 < y < \infty. \tag{2.3}$$

Let $w_{*,E}(t_0 - \delta) = w_{*,E}(t_0)$ for some $\delta > 0$ (this is equivalent that the function $\lambda_{w,E}(y)$ is discontinuous at the point $y_0 = w_{*,E}(t_0)$). Then, by (2.3), there exist ν_0 and $\delta_0 > 0$ such that $(w \mid E_{\nu_0})_*(\mid E_{\nu_0}/a - \delta_0) = (w \mid E_{\nu_0})_*(\mid E_{\nu_0}/a)$. Therefore (2.2) in this case is also true.

Finally, let the function $w_{*,E}(t)$ be discontinuous at the point t_0. In this case if at least for one ν the term on the right of (2.3) for $y = y_0 = w_{*,E}(t_0)$ is not equal to $\mid E_\nu \mid /a$ then, by (2.3), there exist numbers ν_1 and ν_2 such that

$$(w \mid E_{\nu_1})_* \left(\frac{\mid E_{\nu_1}}{a}\right) < w_{*,E}(t_0) < (w \mid E_{\nu_2})_* \left(\frac{\mid E_{\nu_2}}{a}\right),$$

and therefore (2.2) holds. If $\lambda_{w,E_\nu}(y_0) =\mid E_\nu \mid /a$ for every ν then (2.2) has the following form

$$\inf_\nu \sqrt{p_\nu \cdot q_\nu} \leq \sqrt{\sup_\nu p_\nu \cdot \inf_\nu q_\nu} \leq \sup_\nu \sqrt{p_\nu \cdot q_\nu},$$

where $p_\nu = (w \mid E_\nu)_*(\mid E_\nu/a - 0)$, $q_\nu = (w \mid E_\nu)_*(\mid E_\nu/a + 0)$, which is obviously true. The Lemma is proved.

The following Lemma is the Hardy inequality for negative exponent. Its proof is analogous to the proof of the Hardy inequality in [8] (cf. also [4]). We will apply it for the proof of Theorem 1 in the special case $r = q > 0$.

Lemma 3. *Let f be a nonnegative nondecreasing integrable function on an interval $[\alpha, \beta]$ and let r, q be numbers such that $r \geq q > 0$. Then the inequality*

$$\int_\alpha^\beta (t-\alpha)^{q/r-1} \left(\frac{1}{t-\alpha}\int_\alpha^t f(u)du\right)^{-q} dt \leq \left(\frac{r+1}{r}\right)^q \int_\alpha^\beta (t-\alpha)^{q/r-1} f^{-q}(t)dt \tag{2.4}$$

holds.

Proof. We will apply the Hölder inequality with negative exponent

$$\int_\alpha^\beta f(t)g(t)dt \ge \left(\int_\alpha^\beta f^{-q}(t)dt\right)^{-1/q} \left(\int_\alpha^\beta g^{q/(q+1)}(t)dt\right)^{(q+1)/q}, \tag{2.5}$$

where $q > 0$. The proof of (2.5) can be proved by the same argument that the proof of the usual Hölder inequality (see, for example [6]).

Without losing generality we will assume that in (2.4) $\alpha = 0,\ \beta = 1$. If the right part of (2.4) is finite then, by the monotonicity of f, it follows that $f(t) \ge f(\varepsilon) > 0$ for any $t \ge \varepsilon$, where ε be a small positive number, and

$$\int_\varepsilon^1 t^{q/r-1}\left(\frac{1}{t}\int_0^t f(u)du\right)^{-q} dt = \int_\varepsilon^1 t^{q/r-1+q}\left(\int_0^t f(u)du\right)^{-q} dt =$$

$$= \frac{1}{q/r+q}\int_\varepsilon^1 \frac{d}{dt}(t^{q/r+q})\left(\int_0^t f(u)du\right)^{-q} dt =$$

$$= \frac{1}{q/r+q}\left[t^{q/r+q}\left(\int_0^t f(u)du\right)^{-q}\right]_\varepsilon^1 + \frac{q}{q/r+q}\int_\varepsilon^1 t^{q/r+q}\left(\int_0^t f(u)du\right)^{-q-1} f(t)dt \ge$$

$$\ge -\frac{r}{q(r+1)}\varepsilon^{q/r}\left(\frac{1}{\varepsilon}\int_0^\varepsilon f(u)du\right)^{-q} +$$

$$+\frac{r}{r+1}\int_\varepsilon^1 t^{(q/r-1)(q+1)/q}\left(\frac{1}{t}\int_0^t f(u)du\right)^{-q-1} t^{(q/r-1)(-1/q)} f(t)dt.$$

Applying (2.5) to the second term on the right, we obtain

$$\int_\varepsilon^1 t^{q/r-1}\left(\frac{1}{t}\int_0^t f(u)du\right)^{-q} dt \ge -\frac{r}{q(r+1)}\varepsilon^{q/r}\left(\frac{1}{\varepsilon}\int_0^\varepsilon f(u)du\right)^{-q} +$$

$$+\frac{r}{r+1}\left(\int_\varepsilon^1 t^{q/r-1}\left(\frac{1}{t}\int_0^t f(u)du\right)^{-q} dt\right)^{(q+1)/q}\left(\int_\varepsilon^1 t^{q/r-1}f^{-q}(t)dt\right)^{-1/q}.$$

The first term on the right tends to zero when $\varepsilon \to 0+$. This follows from the convergence of the integral on the right in (2.4). Further, since $(\int_\varepsilon^1 t^{q/r-1}f^{-q}(t)dt)^{-1/q}$ does not tend to zero when $\varepsilon \to 0+$, then the left part in the last inequality does not tend to infinity. Letting ε tend to zero, from the last inequality, we obtain

$$\int_0^1 t^{q/r-1}\left(\frac{1}{t}\int_0^t f(u)du\right)^{-q} dt \geq$$

$$\geq \frac{r}{r+1}\left(\int_0^1 t^{q/r-1}\left(\frac{1}{t}\int_0^t f(u)du\right)^{-q} dt\right)^{(q+1)/q}\left(\int_0^1 t^{q/r-1} f^{-q}(t)dt\right)^{-1/q},$$

where the integrals are finite. From this (2.4) obviously follows. The Lemma is proved.

3. Proofs of Theorems 1, 2 and 4. *Proof of Theorem* 1. Fix a number $t \in (0, \beta-\alpha]$. By Lemma 1, in which $\gamma = w_{**}(t)$, we find the family of pairwise nonintersecting intervals I_ν, which satisfy the conditions of Lemma 1. Since

$$w_E = \frac{1}{\mid E \mid}\int_E w(x)dx = \frac{1}{\mid E \mid}\sum\int_{I_\nu} w(x)dx = w_{**}(t)$$

and

$$\frac{1}{\mid E \mid}\int_E w(x)dx \geq \frac{1}{\mid E \mid}\int_0^{|E} w_*(u)du,$$

where $E \equiv \bigcup_\nu I_\nu$ and $w_* \equiv w_{*,[\alpha,\beta]}$, then we have

$$\frac{1}{t}\int_0^t w_*(u)du \geq \frac{1}{\mid E \mid}\int_0^{|E} w_*(u)du.$$

Hence, by the monotonicity of w_*, it follows that

$$\mid E \mid \leq t. \tag{3.1}$$

In the case when $w_*(t/a) \geq w_{**}(t)$, the inequality (1.3) holds since $c \geq 1$. Let $w_*(t/a) < w_{**}(t)$. Since, by Lemma 1, $w(x) \geq w_{**}(t)$ almost everywhere on $I \setminus E$, then $(w \mid E)_*(u) = w_*(u)$ for every u such that $w_*(u) < w_{**}(t)$. Therefore, taking into account (3.1), we obtain

$$w_*\left(\frac{t}{a}\right) = (w \mid E)_*\left(\frac{t}{a}\right) \geq (w \mid E)_*\left(\frac{\mid E \mid}{a}\right). \tag{3.2}$$

Let ε be a small positive number. Applying the Lemma 2 in which $E_\nu = I_\nu$, we find the number ν_0 such that

$$\frac{(w \mid I_{\nu_0})_*(\mid I_{\nu_0} \mid /a)}{(w \mid E)_*(\mid E \mid /a)} < 1 + \varepsilon.$$

Then, by the condition $w \in A_\infty(a, c)$ and by (3.2), we obtain

$$\frac{w_{**}(t)}{w_*(t/a)} = \frac{w_{I_{\nu_0}}}{w_*(t/a)} \leq \frac{w_{I_{\nu_0}}}{(w \mid E)_*(\mid E \mid /a)} \leq c \cdot \frac{(w \mid I_{\nu_0})_*(\mid I_{\nu_0} \mid /a)}{(w \mid E)_*(\mid E \mid /a)} < c(1 + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, the proof of Theorem 1 is complete.

Proof of Theorem 2. The statement in Theorem 2 is trivial when $a_2 \leq a_1$. In this case we put $c_2 = c_1$.

Let $a_2 > a_1$. Fix any interval $I \subset [\alpha, \beta]$ and denote $f = w \mid I$. Then for every interval $J \subset I$, by the condition $w \in A_\infty(a_1, c_1; [\alpha, \beta])$ and by obvious embedding $A_\infty(a_1, c_1; [\alpha, \beta]) \subset A_\infty(a_1, c_1; I)$, we obtain

$$\frac{1}{\mid J \mid} \int_J f(x)dx \leq c_1 (f \mid J)_* \left(\frac{\mid J \mid}{a_1} \right). \tag{3.3}$$

¿From this we have, by application of Theorem 1 to the function f on I,

$$f_{**}(t) \leq c_1 f_* \left(\frac{t}{a_1} \right), \quad 0 < t \leq \mid I \mid . \tag{3.4}$$

Fix a number γ such that $1 < \gamma < a_1$. Then for every $s,\ 0 < s \leq \mid I \mid$, from (3.4) it follows that

$$f_* \left(\frac{s}{a_1} \right) \geq \frac{1}{c_1} f_{**}(s) \geq \frac{1}{c_1} \cdot \frac{1}{s} \int_0^s f_*(u)du \geq \frac{1}{c_1} \cdot \frac{1}{s} \int_{\gamma s / a_1}^s f_*(u)du \geq \frac{1}{c_1} \left(1 - \frac{\gamma}{a_1} \right) f_* \left(\frac{\gamma s}{a_1} \right).$$

Letting $\tau = \gamma s / a_1,\ \delta = (1 - \gamma / a_1)/c_1$, the obtained inequality we can rewrite in the form

$$f_*(\tau) \leq \frac{1}{\delta} f_* \left(\frac{\tau}{\gamma} \right), \quad 0 < \tau \leq \frac{\gamma}{a_1} \mid I \mid .$$

If we apply the preceeding inequality repeatedly then for $\nu = 1, 2, ...$ we obtain

$$f_* \left(\frac{\mid I \mid}{a_1} \right) \leq \frac{1}{\delta} f_* \left(\frac{\mid I \mid}{\gamma a_1} \right) \leq \frac{1}{\delta^2} f_* \left(\frac{\mid I \mid}{\gamma^2 a_1} \right) \leq ... \leq \frac{1}{\delta^\nu} f_* \left(\frac{\mid I \mid}{\gamma^\nu a_1} \right).$$

Select the number ν such that $\gamma^\nu a_1 \geq a_2$. Then

$$f_* \left(\frac{\mid I \mid}{a_1} \right) \leq \frac{1}{\delta^\nu} f_* \left(\frac{\mid I \mid}{\gamma^\nu a_1} \right) \leq \frac{1}{\delta^\nu} f_* \left(\frac{\mid I \mid}{a_2} \right).$$

Hence, by (3.3), it follows that

$$\frac{1}{\mid I \mid} \int_I w(x)dx = \frac{1}{\mid I \mid} \int_I f(x)dx \leq c_1 (f \mid I)_* \left(\frac{\mid I \mid}{a_1} \right) \leq \frac{c_1}{\delta^\nu} f_* \left(\frac{\mid I \mid}{a_2} \right) = \frac{c_1}{\delta^\nu} (w \mid I)_* \left(\frac{\mid I \mid}{a_2} \right).$$

Since the interval $I \subset [\alpha, \beta]$ is arbitrary then the Theorem 2 is proved where $c_2 = c_1 \delta^{-\nu}$.

Most important for the proof of Theorem 4 is the following

Proposition 2. *Let $a > 1,\ c \geq 1$ be numbers. Let f be a nonnegative nondecreasing function on interval $[0, 1]$ which satisfies the condition*

$$\frac{1}{t}\int_0^t f(u)du \leq c \cdot f\left(\frac{t}{a}\right), \quad 0 < t \leq 1. \tag{3.5}$$

Let the number $p \geq (\ln\ a)^{-1}$ is defined by equality

$$a^{p-1} = p \cdot c. \tag{3.6}$$

Then for every $r > p$ the integral

$$\int_0^1 f^{-1/(r-1)}(t)dt \tag{3.7}$$

is convergent.

Proof. Note that from the conditions of Proposition 2 it follows that $p \geq 1$ (cf. Remark 3). We start by proving that the integral

$$\int_0^1 t^{-p} f(t)dt \tag{3.8}$$

diverges. Multiply (3.5) by t^{-p} and integrate. Then for any $0 < \varepsilon < s \leq 1$ we obtain

$$\int_\varepsilon^s t^{-p-1}\int_0^t f(u)du\ dt \leq c\int_\varepsilon^s t^{-p} f\left(\frac{t}{a}\right)dt = \frac{c}{a^{p-1}}\int_{\varepsilon/a}^{s/a} t^{-p} f(t)dt.$$

A change of the order of integration, by (3.6), gives

$$(\varepsilon^{-p} - s^{-p})\int_0^\varepsilon f(u)du + \int_\varepsilon^s f(u)(u^{-p} - s^{-p})du \leq \int_{\varepsilon/a}^{s/a} t^{-p} f(t)dt.$$

¿From this follows

$$\varepsilon^{-p+1}\frac{1}{\varepsilon}\int_0^\varepsilon f(u)du + \int_\varepsilon^s u^{-p} f(u)du - \int_{\varepsilon/a}^{s/a} u^{-p} f(u)du \leq s^{-p}\int_0^s f(u)du. \tag{3.9}$$

Assume that the integral (3.8) is convergent. Then $u^{-p} f(u) = o(1/u)$, that is $f(u) = o(u^{p-1})$ when $u \to 0$. Therefore

$$\varepsilon^{-p+1}\frac{1}{\varepsilon}\int_0^\varepsilon f(u)du \leq \varepsilon^{-p+1} f(\varepsilon) = o(1), \quad \varepsilon \to 0.$$

Taking into account this and the convergence of the integral (3.8), going over to the limit by $\varepsilon \to 0+$ in (3.9), for $p > 1$ we find

$$s^{-p}\int_0^s f(u)du \geq \int_0^s u^{-p} f(u)du - \int_0^{s/a} u^{-p} f(u)du = \int_{s/a}^s u^{-p} f(u)du \geq$$

$$\geq f\left(\frac{s}{a}\right)\int_{s/a}^{s} u^{-p}du = f\left(\frac{s}{a}\right)\frac{s^{-p+1}}{p-1}(a^{p-1}-1).$$

¿From this, by (3.6), it follows that

$$\frac{1}{s}\int_0^s f(u)du \geq \frac{pc-1}{p-1} f\left(\frac{s}{a}\right). \tag{3.10}$$

If $c>1$ then, by Remark 3, $p>1$. Since in this case $(pc-1)/(p-1)>c$ then (3.10) contradicts (3.5).

Let $c=1$. If $p>1$ that is $1<a<e$ (cf. Remark 3), then (3.10) and (3.5) give

$$\frac{1}{t}\int_0^t f(u)du = f\left(\frac{t}{a}\right), \quad 0<t\leq 1.$$

¿From this for $0<s\leq 1$ it follows that

$$\int_0^{s/a} t^{-p}f(t)dt = a^{p-1}\int_0^s t^{-p} f\left(\frac{t}{a}\right)dt = a^{p-1}\int_0^s t^{-p-1}\int_0^t f(u)du\, dt =$$

$$= \frac{a^{p-1}}{p}\int_0^s f(u)(u^{-p}-s^{-p})du = \int_0^s u^{-p}f(u)du - s^{-p}\int_0^s f(u)du.$$

Therefore

$$\int_{s/a}^s t^{-p}f(t)dt = s^{-p}\int_0^s f(t)dt = s^{-p+1}\frac{1}{s}\int_0^s f(t)dt = s^{-p+1} f\left(\frac{s}{a}\right).$$

Applying this equallity and taking into account that $p\geq (\ln\ a)^{-1}$, we find

$$\int_0^{1/a} s^{-p}f(s)ds = a^{p-1}\int_0^1 s^{-p} f\left(\frac{s}{a}\right)ds = a^{p-1}\int_0^1 \frac{1}{s}\int_{s/a}^s t^{-p}f(t)dt\, ds =$$

$$= a^{p-1}\left(\int_0^{1/a} t^{-p}f(t)\int_t^{at}\frac{ds}{s}dt + \int_{1/a}^1 t^{-p}f(t)\int_t^1 \frac{ds}{s}dt\right) =$$

$$= a^{p-1} ln\ a\int_0^{1/a} t^{-p}f(t)dt + a^{p-1}\int_{1/a}^1 t^{-p}\ln\frac{1}{t}f(t)dt \geq$$

$$\geq \frac{a^{p-1}}{p}\int_0^{1/a} t^{-p}f(t)dt + a^{p-1}\int_{1/a}^1 t^{-p}\ln\frac{1}{t}f(t)dt =$$

$$= \int_0^{1/a} t^{-p}f(t)dt + a^{p-1}\int_{1/a}^1 t^{-p}\ln\frac{1}{t}f(t)dt.$$

¿From this it follows that the second term on the right is equal to zero, which is the only possible in trivial case $f(x) \equiv 0,\ 0 < x < 1$.

It remains consider the case $c = 1,\ p = 1$ that is $a \geq e$. From (3.9), tending $\varepsilon \to 0+$, we obtain

$$\frac{1}{s}\int_0^s f(u)du \geq f\left(\frac{s}{a}\right) \ln\ a, \quad 0 < s \leq 1.$$

If $a > e$ then this inequality contradicts (3.5). When $a = e$ then the contradiction we obtain by repeating the same argument that in the case $c = 1,\ p > 1$.

So, the proof of divergence of integral (3.8) is complete. Making use of this, we prove that the integral (3.7) is convergent.

Let $p < q < r$. Multiply (3.5) by t^{-q} and integrate. Then for any $0 < \varepsilon < 1$ we obtain

$$\int_\varepsilon^1 t^{-q-1} \int_0^t f(u)du\ dt \leq c \cdot a^{-q+1} \int_{\varepsilon/a}^{1/a} t^{-q} f(t)dt.$$

A change of the order of integration gives

$$\frac{\varepsilon^{-q} - 1}{q} \int_0^\varepsilon f(u)du + \frac{1}{q}\int_\varepsilon^1 f(u)(u^{-q} - 1)du \leq c\ a^{-q+1} \int_{\varepsilon/a}^{1/a} u^{-q} f(u)du,$$

that is

$$\varepsilon^{-q+1}\frac{1}{\varepsilon}\int_0^\varepsilon f(u)du - \int_0^1 f(u)du + \int_\varepsilon^1 u^{-q} f(u)du \leq c\ q\ a^{-q+1} \int_{\varepsilon/a}^{1/a} u^{-q} f(u)du.$$

¿From this it follows that

$$\int_\varepsilon^1 u^{-q} f(u)du \leq c\ q\ a^{-q+1} \left(\int_{\varepsilon/a}^{\varepsilon} u^{-q} f(u)du + \int_\varepsilon^1 u^{-q} f(u)du \right) + \int_0^1 f(u)du,$$

that is

$$\left(\frac{a^{q-1}}{c \cdot q} - 1\right) \int_\varepsilon^1 u^{-q} f(u)du \leq \int_{\varepsilon/a}^{\varepsilon} u^{-q} f(u)du + \frac{a^{q-1}}{c \cdot q}\int_0^1 f(u)du \leq$$

$$\leq f(\varepsilon)\frac{\varepsilon^{-q+1}}{q-1}(a^{q-1} - 1) + \frac{a^{q-1}}{c \cdot q}\int_0^1 f(u)du.$$

Since $q > p$ then we obtain

$$0 < \frac{a^{q-1}}{c \cdot q} - 1 \leq \frac{a^{q-1} - 1}{q - 1} \cdot \frac{\varepsilon^{-q+1} f(\varepsilon)}{\int_\varepsilon^1 u^{-q} f(u)du} + \frac{a^{q-1}}{c \cdot q} \cdot \frac{\int_0^1 f(u)du}{\int_\varepsilon^1 u^{-q} f(u)du}. \tag{3.11}$$

Let us now make use of the fact that the integral (3.8) is divergent. Then for $q > p$ the integral $\int_0^1 u^{-q} f(u)du$ diverges and therefore the second term on the right in (3.11) tends to zero when $\varepsilon \to 0+$. Therefore $f(\varepsilon)\varepsilon^{-q+1} \to \infty$ when $\varepsilon \to 0+$ and hence for $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(q)$ the inequality $f(\varepsilon) \geq \varepsilon^{q-1}$ holds. ¿From this, since $r > q$ we obtain

$$\int_0^1 f^{-1/(r-1)}(t)dt = \int_0^{\varepsilon_0} f^{-1/(r-1)}(t)dt + \int_{\varepsilon_0}^1 f^{-1/(r-1)}(t)dtJ \leq$$

$$\leq \int_0^{\varepsilon_0} t^{-(q-1)/(r-1)}dt + \int_{\varepsilon_0}^1 f^{-1/(r-1)}(t)dt < \infty.$$

By this the proof of Proposition 2 is complete.

Proof of Theorem 4. *a*) Fix any interval $I \subset [\alpha, \beta]$ and denote $g(t) = (w \mid I)_*(t),\ 0 < t \leq\mid I \mid$. Then the function g is nondecreasing on $[0, \mid I \mid]$ and, by Theorem 1, satisfies the condition

$$\frac{1}{t}\int_0^t g(u)du \leq c\ g\left(\frac{t}{a}\right), \quad 0 < t \leq\mid I \mid. \tag{3.12}$$

If we denote $f(t) = g(t \mid I \mid),\ 0 < t \leq 1$, then (3.12) will be have the following form

$$\frac{1}{t}\int_0^t f(u)du \leq c\ f\left(\frac{t}{a}\right), \quad 0 < t \leq 1. \tag{3.13}$$

Let the number $p_0 \geq (\ln\ a)^{-1}$ is defined as the root of the equation (1.6). Then by Proposition 2 it follows that for every $p > p_0 \geq 1$ the integral $\int_0^1 f^{-1/(p-1)}(t)dt$ is convergent. Fix $p > p_0$. Applying Lemma 3 with $r = q = 1/(p-1)$ and using (3.13) we find

$$p^{1/(p-1)}\int_0^1 f^{-1/(p-1)}(t)dt \geq \int_0^1 \left(\frac{1}{t}\int_0^t f(u)du\right)^{-1/(p-1)} dt \geq$$

$$\geq c^{-1/(p-1)}\int_0^1 f^{-1/(p-1)}\left(\frac{t}{a}\right) dt = a\ c^{-1/(p-1)}\int_0^{1/a} f^{-1/(p-1)}(t)dt =$$

$$= a\ c^{-1/(p-1)}\left(\int_0^1 f^{-1/(p-1)}(t)dt - \int_{1/a}^1 f^{-1/(p-1)}(t)dt\right). \tag{3.14}$$

Note that for $p > p_0$ the inequality $p^{1/(p-1)} < a\ c^{-1/(p-1)}$ holds. Therefore from (3.14) follows

$$(a\ c^{-1/(p-1)} - p^{1/(p-1)})\int_0^1 f^{-1/(p-1)}(t)dt \leq a\ c^{-1/(p-1)}\int_{1/a}^1 f^{-1/(p-1)}(t)dt. \tag{3.15}$$

¿From this and from (3.13), by the denotind

$$c_1 = \frac{a\ c^{-1/(p-1)}}{a\ c^{-1/(p-1)} - p^{1/(p-1)}},$$

we obtain

$$\int_0^1 f^{-1/(p-1)}(t)dt \le c_1 \int_{1/a}^1 f^{-1/(p-1)}(t)dt \le$$

$$\le c_1 \left(1-\frac{1}{a}\right) f^{-1/(p-1)}\left(\frac{1}{a}\right) \le c_1 \left(1-\frac{1}{a}\right) c^{1/(p-1)} \left(\int_0^1 f(t)dt\right)^{-1/(p-1)},$$

that is

$$\int_0^1 f(t)dt \cdot \left(\int_0^1 f^{-1/(p-1)}(t)dt\right)^{p-1} \le B \equiv c\left(c_1\left(1-\frac{1}{a}\right)\right)^{p-1}.$$

Reverting to the function g and taking into account the equimeasurability of functions g^γ on $[0, \mid I \mid]$ and w^γ on I, $\gamma \in \mathrm{R}$, the last inequality can be rewriten in the form

$$\frac{1}{\mid I \mid}\int_I w(t)dt \cdot \left(\frac{1}{\mid I \mid}\int_I w^{-1/(p-1)}(t)dt\right)^{p-1} \le B.$$

Since $I \subset [\alpha, \beta]$ is an arbitrary interval then the proof of part a) is complete.

b) It is obvious that the function $w(x) = x^{p_0-1}$ does not belong to the class $A_{p_0}(B; [0,1])$ for every B. For the proof that $w \in A_\infty(a, c; [0,1])$ we fix any interval $I \equiv [\gamma, \delta] \subset [0,1]$ and show that the inequality

$$\frac{\frac{1}{|I|}\int_I w(x)dx}{w(\gamma + \frac{\delta-\gamma}{a})} = \frac{1}{p_0}\cdot\frac{\delta^{p_0}-\gamma^{p_0}}{(\delta-\gamma)(\gamma+\frac{\delta-\gamma}{a})^{p_0-1}} \le \frac{a^{p_0-1}}{p_0} = \frac{\int_0^1 w(x)dx}{w(\frac{1}{a})} \tag{3.16}$$

holds. This inequality is equivalent to

$$\varphi_{\delta,\gamma}(a) \equiv (\delta-\gamma)((a-1)\gamma+\delta)^{p_0-1} - (\delta^{p_0}-\gamma^{p_0}) \ge 0, \quad a > 1,$$

which in turn follows by the obvious inequalities

$$\varphi_{\delta,\gamma}(1) \ge 0, \quad \frac{d\varphi_{\delta,\gamma}}{da}(a) \ge 0, \quad a > 1.$$

Thus from (3.16) we obtain that $w \in A_\infty(a, c; [0,1])$ where the constants a and c are connected with the value p_0 so as in the conditions of Theorem 4.

c). If $p_0 = 1$ then $c = 1$, $a \ge e$ (cf. Remark 3). Since $A_\infty(a,1) \subset A_\infty(e,1)$ then it is sufficient to concider a function $w \in A_\infty(e, 1; [, \beta])$.

Fix any interval $I \subset [\alpha, \beta]$. Then the function $f(t)$, $0 < t \le 1$, defined by the proof of part a), satisfies the condition

$$\frac{1}{t}\int_0^t f(u)du \le f\left(\frac{t}{e}\right), \quad 0 < t \le 1. \tag{3.17}$$

In this case the inequality (3.9), which was obtained in the proof of Proposition 2, has the following form

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} f(u)du + \int_{\varepsilon}^{s} f(u)\frac{du}{u} - \int_{\varepsilon/e}^{s/e} f(u)\frac{du}{u} \leq J\frac{1}{s}\int_0^s f(u)du, \quad 0 < \varepsilon < s \leq 1,$$

that is

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} f(u)du - \int_{\varepsilon/e}^{\varepsilon} f(u)\frac{du}{u} + \int_{s/e}^{s} f(u)\frac{du}{u} \leq J\frac{1}{s}\int_0^s f(u)du, \quad 0 < \varepsilon < s \leq 1, \qquad (3.18)$$

Since the two first terms on the left tended to $f(0)$ when $\varepsilon \to 0+$, their difference tends to zero. Therefore, by (3.17) we have

$$\int_{s/e}^{s} f(u)\frac{du}{u} \leq J\frac{1}{s}\int_0^s f(u)du, \quad 0 < \varepsilon < s \leq 1.$$

Considering the integral on the left and applying (3.17) we find

$$f(\frac{s}{e}) \leq \int_{s/e}^{s} f(u)\frac{du}{u} \leq J\frac{1}{s}\int_0^s f(u)du \leq f\left(\frac{s}{e}\right), \quad 0 < s \leq 1.$$

Therefore

$$f\left(\frac{s}{e}\right) = \int_{s/e}^{s} f(u)\frac{du}{u}, \quad 0 < s \leq 1,$$

and this, by the monotonicity f, implies $f(u) = f(s/e),\ s/e < u < s,\ 0 < s \leq 1$. Thus $f(t) = f(0),\ 0 < t < 1$. From this follows the assertion c).

The proof of Theorem 4 is complete.

Remark 5. The inequality (3.15) cannot be obtained from (3.13) by the simple passage to the limit used the function $f_{\varepsilon}(t) = \max(\varepsilon,\ f(t)),\ 0 < t \leq 1$, when $\varepsilon \to 0+$. In fact, the condition (3.13) does not imply the realisation of such inequality for $f_{\varepsilon}(t)$ with the same constants a and c. For the proof of part a) of Theorem 4 the Proposition 2 was needed for the ground of the convergence of integrals in (3.14).

4. Proof of Theorem 6. For the proof of Theorem 6 we apply the Lemma 1 in the following equivalent form.

Lemma 1′. *Let f be a integrable function on an interval I and γ a number not less than f_I. Then there exists a family of pairwise nonintersecting intervals $I_{\nu} \subset I$, $\nu = 1, 2, ...,$ such that $f_{I_{\nu}} = \gamma,\ \nu = 1, 2, ...,$ and $f(x) \leq \gamma$ almost everywhere on the set $I \setminus E$ where $E \equiv \bigcup_{\nu} I_{\nu}$.*

Proof of Theorem 6. In Theorem 6 it is convenient make use of the right-continuous rearrangement $f_*^{(r)}$.

Fix any interval $I \subset [\alpha, \beta]$ and denote $f(x) = (w \mid I)(x),\ x \in I$. Then from the condition $w \in A_1(c; [, \beta])$ it follows that for every interval $K \subset I$

$$\frac{1}{\mid K \mid} \int_K f(x)dx \leq c \text{ ess } \inf_{x \in K} f(x) = c\ (f \mid K)_*^{(r)}(0), \tag{4.1}$$

and inequality (1.8) has the following form

$$\frac{1}{\mid J \mid} \int_J f_*^{(r)}(u)du \leq c\ \min_{t \in J} f_*^{(r)}(t), J \quad J \subset [0, \mid I \mid]. \tag{4.2}$$

For the proof of (4.2) fix $J \subset [0, \mid I \mid]$ and denote $\gamma = \frac{1}{|J|} \int_J f_*^{(r)}(u)du$. If $\gamma \leq f_I$ then obviously there exists $s,\ 0 <\mid J \mid \leq s \leq \mid I \mid$, such that $\gamma = \frac{1}{s} \int_0^s f_*^{(r)}(u)du$. Then we obtain

$$\frac{1}{\mid J \mid} \int_J f_*^{(r)}(u)du = \frac{1}{s} \int_0^s f_*^{(r)}(u)du \leq \frac{1}{\mid I \mid} \int_0^{|IJ|} f_*^{(r)}(u)du = \frac{1}{\mid I \mid} \int_I w(x)dx \leq$$

$$\leq c\ (w \mid I)_*^{(r)}(0) = c\ f_*^{(r)}(0) \leq c\ \inf_{t \in J} f_*^{(r)}(t).$$

Therefore (4.2) holds in this case.

Let $\gamma > f_I$. Find $s,\ 0 <\mid J \mid \leq \mid I \mid -s$, such that $\gamma = \frac{1}{|I|-s} \int_s^{|IJ|} f_*^{(r)}(u)du$. Applying Lemma $1'$, we construct the family of pairwise nonintersecting intervals $I_\nu \subset I, \nu = 1, 2, ...,$ which satisfy the conditions of Lemma $1'$. From (4.1) it follows that $\gamma = f_{I_\nu} \leq c\ (f \mid I_\nu)_*^{(r)}(0),\ \nu = 1, 2, ...$ Therefore $\gamma \leq c\ \inf_\nu \text{ess } \inf_{x \in I_\nu} f(x) = c \text{ ess } \inf_{x \in E} f(x)$. If we will be show that

$$\text{ess } \inf_{x \in E} f(x) \leq f_*^{(r)}(s), \tag{4.3}$$

then we will obtain

$$\frac{1}{\mid J \mid} \int_J f_*^{(r)}(u)du = \gamma \leq c \text{ ess } \inf_{x \in E} f(x) \leq c\ f_*^{(r)}(s) \leq c\ f_*^{(r)}(t), \quad t \in J,$$

and the proof of Theorem 4 will be complete.

We prove (4.3). From the inequality

$$\frac{1}{\mid E \mid} \int_{|I|-|E|}^{|I|} f_*^{(r)}(u)du \geq \frac{1}{\mid E \mid} \int_E f(x)dx =$$

$$= \frac{1}{\mid E \mid} \sum_\nu \mid I_\nu \mid f_{I_\nu} = \gamma = \frac{1}{\mid I \mid -s} \int_s^{|I|} f_*^{(r)}(u)du$$

and from the monotonicity of $f_*^{(r)}$ it follows that $\mid E \mid \leq \mid I \mid -s$. Further, by Lemma $1'$ it follows that $\mid E \mid < \mid I \mid -s$ if and only if $\mid E \bigcap \{x \in I :\ f(x) < f_*^{(r)}(s)\} \mid > 0$. Therefore, if $\mid E \mid < \mid I \mid -s$ then the inequality (4.3) holds. If $\mid E \mid = \mid I \mid -s$ then

$f_*^{(r)}(u) = (f \mid E)_*^{(r)}(u), \ \mid I \mid - \mid E \mid < s \leq u \leq \mid I \mid$, and in this case (4.3) reduces to equality. Q.E.D.

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