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Second class particles in the rarefaction fan

P. A. Ferrari, C. Kipnis

Universidade de São Paulo and Université de Paris Dauphine

Summary. We consider the one dimensional totally asymmetric nearest neighbors simple exclusion process with drift to the right starting with the configuration "all one" to the left and "all zero" to the right of the origin. We prove that a second class particle initially added at the origin chooses randomly one of the characteristics with the uniform law on the directions and then moves at constant speed along the chosen one. The result extends to the case of a product initial distribution with densities $\rho > \lambda$ to the left and right of the origin respectively. Furthermore we show that, with a positive probability, two second class particles in the rarefaction fan never meet.

Résumé. On considère le processus d'exclusion simple totalement asymétrique unidimensionel avec dérive à droite. Le processus commence avec la configuration "tout un" à gauche de l'origine et "tout zéro" à droite. On prouve qu'une particule de deuxième classe placée à l'origine à l'instant zéro choisit une des charactéristiques avec une loi uniforme et ensuite suit cette charactéristique à vitesse constante. Le résultat s'etend au cas d'une mesure initiale produit avec densités ρ à gauche et λ à droite de l'origine, satisfaisant $\rho > \lambda$. Finalement, on prouve que deux particules de deuxième classe dans le front de raréfaction ont une probabilité positive de ne pas se rencontrer.

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1. Introduction.

The one dimensional asymmetric exclusion process is known to have the inviscid Burgers equation as hydrodynamic limit. Usually one takes advantage of the deep knowledge accumulated through the years on this non linear pde to prove results for the exclusion process. In this short note we intend to do exactly the contrary. We prove a result for the exclusion process and use it to guess a result for the pde.

For this purpose we study the trajectory of a second class particle which is known to give information on the characteristics of the pde. More precisely, it has been proved (Ferrari (1992), Rezakhanlou (1993)) except for the case of the rarefaction fan, that a second class particle added at a macroscopic site a has a position at macroscopic time determined by the characteristic emanating from a. Of course in these cases there is only one characteristic issued from a. When dealing with a rarefaction fan one has an infinite number of characteristics issued from a. We prove that the second class particle chooses instantaneously *at random* (uniformly) among the possible characteristics and then of course follows it.

This suggests the following result for the pde: if a small perturbation is added at a point of discontinuity that would give rise to a rarefaction fan, then the perturbation is smeared uniformly in the fan.

Informally the one dimensional asymmetric simple exclusion process we study here is described as follows. Only a particle is allowed per site and at rate one each particle independently of the others attemps to jump to its right nearest neighbor; the jump is realized only if the destination site is empty. A second class particle is a particle that jumps over empty sites to the right of it at rate 1 and interchanges positions with the other particles to the left of it at rate 1. Let $S(t)$ be the semigroup corresponding to the process without the second class particle.

Let $\nu_{\rho,\lambda}$ be the product distribution with marginals $\nu_{\rho,\lambda}(\eta(x)=1)=\rho 1\{x\leq 0\}+\lambda 1\{x>0\}$. Let $\nu_\rho=\nu_{\rho,\rho}$. The process with initial distribution $\nu_{\rho,\lambda}$ has hydrodynamic limit

$$\lim_{\varepsilon\to 0}\nu_{\rho,\lambda}S(\varepsilon^{-1}t)\tau_{[\varepsilon^{-1}r]}f=\nu_{u(r,t)}f \tag{1.1}$$

where f is a cylinder function, τ_x is the translation by x operator, $[.]$ is the integer part and for $t\geq 0$, $r\in$, $u(r,t)$ is the entropy solution of

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial(u(1-u))}{\partial r}=0\\ u(r,0)=u_0(r)\end{cases} \tag{1.2}$$

where $u_0(r)=\rho 1\{r\leq 0\}+\lambda 1\{r>0\}$. We consider $\rho>\lambda$. In this case the explicit form of $u(r,t)$ is the following:

$$u(r,t)=\begin{cases}\rho & \text{if } r\leq(1-2\rho)t\\ (t-r)/2t & \text{if } (1-2\rho)t<r\leq(1-2\lambda)t\\ \lambda & \text{if } r>(1-2\lambda)t\end{cases} \tag{1.3}$$

(See Rost (1982), Andjel and Vares (1987), Rezakhanlou (1990), Landim (1993) and references therein.) The characteristics emanating from a related to this equation are the solutions $r(t)$ of the ode

$$\begin{cases}\dfrac{dr}{dt}=1-2u(r,t)\\ r(0)=a\ .\end{cases} \tag{1.4}$$

Here $1-2u$ is the derivative with respect to u of the current $u(1-u)$ appearing in the non linear part of the Burgers equation (1.2). "Since u is not continuous, (1.4) is understood in the Filippov sense: an absolutely continuous function r is a solution if for almost all t, $\frac{dr}{dt}$ is between the essential infimum and the essential supremum of $1-2u(r,t)$ evaluated at the point $(r(t),t)$." (Rezakhanlou (1993); see Filippov (1960)). Under our initial condition there is only one characteristic for every $a \neq 0$ but there are infinitely many characteristics departing from the origin producing the fan in the region $[(1-2\rho)t,(1-2\lambda)t]$. Our first result says that the second class particle chooses one uniformly among those and follows it.

Theorem 1. *Consider the simple exclusion process starting with the product measure $\nu_{\rho,\lambda}$ with $1 \geq \rho > \lambda \geq 0$. At time zero put a second class particle in the origin regardless the configuration value at this point. Let X_t be the position of the second class particle at time t and let $X_t^\varepsilon = \varepsilon X_{\varepsilon^{-1}t}$. Then*

$$\lim_{\varepsilon\to 0} X_t^\varepsilon = U_t \quad \textit{in distribution,} \tag{1.5}$$

where U_t is a random variable uniformly distributed in the interval $[(1-2\rho)t,(1-2\lambda)t]$. Moreover for any $0<s<t$,

$$\lim_{\varepsilon\to 0}\left(s^{-1}X_s^\varepsilon - t^{-1}X_t^\varepsilon\right) = 0 \quad \textit{in probability.} \tag{1.6}$$

In other words, the initial perturbation to the right of the front smears in the rarefaction region instantaneously in the macroscopic scale. Once chosen a direction the second class particle follows this direction.

A perturbation of the initial condition of the Burgers equation in the rarefaction front has an analogous behavior. To describe it let $\delta > 0$ and $u_{0,\delta}(r)$ be a density that differs from $u_0(r)$ only in the interval $[0,\delta]$ and in this interval the density is equal to ρ (instead of λ). Then $u_\delta(r,t)$, the entropic solution of the Burgers equation (1.2) but with initial condition $u_{0,\delta}(r)$ is given by

$$u_\delta(r,t) = \begin{cases} \rho & \text{if } r \leq (1-2\rho)t+\delta \\ (t-r+\delta)/2t & \text{if } (1-2\rho)t+\delta < r \leq (1-2\lambda)t+\delta \\ \lambda & \text{if } r > (1-2\lambda)t+\delta \end{cases} \tag{1.7}$$

and the difference between this solution and the solution of the unperturbed system is given by $m_\delta(r,t) = u_\delta(r,t) - u(r,t)$. For $\delta < 2(\rho-\lambda)t$,

$$m_\delta(r,t) = \begin{cases} 0 & \text{if } r \leq (1-2\rho)t \\ (r-(1-2\rho)t)/2t & \text{if } (1-2\rho)t \leq r \leq (1-2\rho)t+\delta \\ \delta/2t & \text{if } (1-2\rho)t+\delta \leq r \leq (1-2\lambda)t \\ ((1-2\lambda)t-r+\delta)/2t & \text{if } (1-2\lambda)t \leq r \leq (1-2\lambda)t+\delta \\ 0 & \text{if } r > (1-2\lambda)t+\delta \end{cases} \tag{1.8}$$

In other words, the perturbation is smeared in the rarefaction front. The same result is presumably true for more general initial conditions and more general type of perturbations. To show (1.8) it suffices to compute $u(r,t)$ for the two different initial conditions and subtract. To do this computation observe that u_δ is just a translation of u by δ.

Theorem 1 is shown in the next section. Its proof is based in computing the same quantity using two different couplings. In Section 3 we consider the process starting with $\nu_{1,0}$, that is with the configuration that has 1's to the left of the origin (including it) and 0's to its right. We show that if two second class particles are added in sites 0 and 1 at time zero, then there is a positive probability that they never meet.

2. Couplings.

A coupling is a joint realization of two versions of the process with different initial configurations. To realize the "basic coupling" of Liggett (1985) one attaches a Poisson clock of parameter one to each site of . When the clock rings for site x, if there is a particle in x and there is no particle in $x+1$, then the particle jumps one unit to the right. Under this coupling the two configurations use the same realization of the clocks.

Under the "particle to particle" coupling we have to label the particles of the two configurations. We can also use the same realizations of the clocks attached to the sites, but only one of the configurations (say the first one) looks at the clocks. When a clock rings for the i-th particle of the first configuration, then the i-th particles of both configurations try to jump. On each marginal the jump is actually performed if the exclusion rules of the configuration of that marginal allow it.

Proof of (1.5). We want to show that for $r \in [(1-2\rho)t, (1-2\lambda)t]$,

$$\lim_{\varepsilon\to 0} (X_t^\varepsilon > r) = \frac{(1-2\lambda)t - r}{2(\rho-\lambda)t}. \tag{2.1}$$

For a given initial configuration η, let $J_{r,t}(\eta)$ be the number of particles of η to the left of the origin (including it) that end up at time t strictly to the right of r minus the number of particles of η strictly to the right of the origin that end up at time t to the left of r (including it). We call $J_{r,t}$ the current through r up to time t. Let $J^\varepsilon_{r,t} = J_{r\varepsilon^{-1},t\varepsilon^{-1}}$. Now we compute in two different ways

$$\int d\nu_{\rho,\lambda}(\eta) J^\varepsilon_{r,t}(\eta) - \int d(\tau_{-1}\nu_{\rho,\lambda})(\eta) J^\varepsilon_{r,t}(\eta).$$

where τ_x is the translation by x operator: $(\tau_x\eta)(z) = \eta(z-x)$. For *any* coupling $\bar\mu$ of $\nu_{\rho,\lambda}$ and $\tau_{-1}\nu_{\rho,\lambda}$ and *any* coupling $^-$ of the two processes the previous quantity is also equal to

$$\int d\bar{\mu}(\eta^0,\eta^1)(J^\varepsilon_{r,t}(\eta^0) - J^\varepsilon_{r,t}(\eta^1)). \tag{2.2}$$

In the sequel we write for the expectation with respect to the coupled process. We first couple $\nu_{\rho,\lambda}$ and $\tau_{-1}\nu_{\rho,\lambda}$ in such a way that if η^0 and η^1 are two configurations with those distributions respectively, then $\eta^0(x) = \eta^1(x)$ for all $x \neq 0$ and with probability $\rho - \lambda$ there is a particle in the origin for the first marginal and no particle for the second marginal: $\bar{\mu}(\eta^0(0) = 1 - \eta^1(0) = 1) = \rho - \lambda$. Now, in the event that there is a discrepancy in the origin, we use the basic coupling and observe that this discrepancy behaves like a second class particle. If we label the particles at the other sites and call them first class particles, then under this coupling the positions of these particles are exactly the same for both marginals. This implies that the current produced by the first class particles are identical for both marginals and that the only difference can arise from the second class particle. It is then easy to see that the currents through $\varepsilon^{-1}r$ at time $\varepsilon^{-1}t$ for the two marginals differ if and only if at time $\varepsilon^{-1}t$ the second class particle is beyond $\varepsilon^{-1}r$. Hence, taking expectations, from this coupling we see that (2.2) is equal to

$$(\rho - \lambda)(X^\varepsilon_t > r). \tag{2.3}$$

We now couple $\nu_{\rho,\lambda}$ and $\tau_{-1}\nu_{\rho,\lambda}$ in such a way that $\eta^1 = \tau_{-1}\eta^0$. Then we use the particle to particle coupling to obtain that the currents through $\varepsilon^{-1}r$ for the two marginals differ by one if and only if for the first marginal there is a particle at $[\varepsilon^{-1}r] + 1$ at time $\varepsilon^{-1}t$ and no particle in site 1 at time 0. Those currents differ by -1 if and only if for the first marginal there is a particle in site 1 at time 0 and there is no particle in site $[\varepsilon^{-1}r] + 1$ at time $\varepsilon^{-1}t$. Taking expectations and noting that the above events depend only on the first marginal, (2.2) is also equal to

$$\begin{aligned}(\eta_{\varepsilon^{-1}t}([\varepsilon^{-1}r]+1) = 1, \eta_0(1) = 0) - (\eta_{\varepsilon^{-1}t}([\varepsilon^{-1}r]+1) = 0, \eta_0(1) = 1)\\ = (\eta_{\varepsilon^{-1}t}([\varepsilon^{-1}r]+1) = 1) - (\eta_0(1) = 1).\end{aligned} \tag{2.4}$$

Since for the first marginal the initial distribution is $\nu_{\rho,\lambda}$, $(\eta_0(1) = 1) = \lambda$. Letting ε tending to zero we obtain by standard convergence to local equilibrium (1.1) that

$$\lim_{\varepsilon\to 0} (\eta_{\varepsilon^{-1}t}([\varepsilon^{-1}r]+1) = 1) = u(r,t), \tag{2.5}$$

where $u(r,t)$ is defined by (1.3). Putting (2.3), (2.4) and (2.5) together we get (2.1). $\square$

To show (1.6) we need the following lemma.

Lemma 2.6. *Consider the simple exclusion process starting with the product measure $\nu_{\rho,\lambda}$ with $1 \geq \rho > \lambda \geq 0$. Let $r \in (1-2\rho, 1-2\lambda)$. Assume that at time $\varepsilon^{-1}s$ we put a second class particle in position $[\varepsilon^{-1}rs]$ disregarding the occupation number in this position. For $t \geq s$ let R_t^ε be ε times the position of this particle at time $\varepsilon^{-1}t$. Then*

$$\lim_{\varepsilon\to 0} R_t^\varepsilon = rt \quad \textit{in probability.}$$

Proof. We first take $\delta > 0$ and use the basic coupling for the process starting with densities u_0 and u_δ, respectively, where u_δ is defined in (1.7). Define a family of initial distributions $\{\nu^\varepsilon\}_{\varepsilon>0}$, where ν^ε is a product distribution with marginals $\nu^\varepsilon(\eta(x)=1) = \rho 1\{x \leq \delta\varepsilon^{-1}\} + \lambda 1\{x > \delta\varepsilon^{-1}\}$. We couple the initial distributions $\nu_{\rho,\lambda}$ and ν^ε in such a way that if (η,σ) is a pair of configurations chosen from this coupling, then $\eta(x) \leq \sigma(x)$ for all x. We use the basic coupling to construct the process with initial configurations (η,σ). Calling η_t and σ_t the corresponding configurations at time t, we have $\eta_t(x) \leq \sigma_t(x)$ and calling $\xi_t(x) = \sigma_t(x) - \eta_t(x)$, we have that the ξ particles behave as second class particles interacting by exclusion among them. Define $J_{r,t}^{2,\varepsilon}$, the current of second class particles through the space–time line $(0,0)$–$(\varepsilon^{-1}r, \varepsilon^{-1}t)$ by

$$J_{r,t}^{2,\varepsilon} = \sum_{x \geq r\varepsilon^{-1}} \xi_{\varepsilon^{-1}t}(x) - \sum_{x\geq 0} \xi_0(x).$$

This is well defined because there is a finite number of second class particles at all times for all $\varepsilon > 0$. Rezakhanlou (1990) proved the following law of large numbers for the density fields. Let Φ be a bounded compact support continuous function, then (in our context),

$$\lim_{\varepsilon\to 0} \left| \varepsilon \sum_x \Phi(\varepsilon x)\sigma_{\varepsilon^{-1}t}(x) - \int \Phi(r) u_\delta(r,t)dr \right| = 0, \tag{2.7}$$

where u_δ is defined in (1.7) and denotes the expectation for the process with initial distribution ν^ε. The limit also holds if Φ is the indicator of a finite interval. The limit (2.7) implies a law of large numbers for the density fields of the second class particles:

$$\lim_{\varepsilon\to 0} \left| \varepsilon \sum_x \Phi(\varepsilon x)\xi_{\varepsilon^{-1}t}(x) - \int \Phi(r) m_\delta(r,t)dr \right| = 0, \tag{2.8}$$

where m_δ is the function given by (1.8) and denotes the expectation for the coupled process with coupled initial distribution with marginals $\nu_{\rho,\lambda}$ and ν^ε as described above. Call

$$M_\delta(r,t) = \int_r^\infty m_\delta(w,t)dw.$$

The limit (2.8) implies that

$$\lim_{\varepsilon\to 0}\left|\varepsilon J^{2,\varepsilon}_{r,t} - (M_\delta(r,t) - \delta(\rho-\lambda))\right| = 0 \tag{2.9}$$

We claim that for each $r \in ((1-2\rho),(1-2\lambda))$, $0 < s < t$, $\delta > 0$, there exists $r' = r'(r,s,t,\delta)$ with the following property:

$$\lim_{\varepsilon\to 0}\left|\varepsilon J^{2,\varepsilon}_{r't,t} - \varepsilon J^{2,\varepsilon}_{rs,s}\right| = 0. \tag{2.10}$$

In other words, the limit of the rescaled current of second class particles through the line determined by the space-time points (rs,s) and $(r't,t)$ is zero. To see that (2.10) is true one has to see how the extra particles evolve. Their (macroscopic) evolution is given by (1.8) hence r' is the point such that the area of the perturbation at time t to the right of the point $r't$ is the same as the area of the perturbation at time s to the right of rs. In other words, r' is the unique solution of

$$M_\delta(r't,t) = M_\delta(rs,s). \tag{2.11}$$

To see that there is a unique solution one checks that for each t and δ, $M_\delta(.,t)$ is strictly decreasing in the interval $[(1-2\rho)t,(1-2\lambda)t+\delta]$. This and (2.9) imply (2.10). From (2.11) it is easy to check that $r' = r - O(\delta)$, where $O(\delta)$ is some positive function that goes to zero as δ goes to zero. Let Z^ε_t be ε times the position at time $\varepsilon^{-1}t$ of the extra particle that at time $\varepsilon^{-1}s$ is located in site $\varepsilon^{-1}rs$ (if there is not we add one in this site disregarding the previous occupation number). Since by the exclusion interaction the current of second class particles through the space-time line $(\varepsilon^{-1}rs,\varepsilon^{-1}s)$-$(\varepsilon^{-1}Z^\varepsilon_t,\varepsilon^{-1}t)$ is zero, it is not hard to conclude that

$$\lim_{\varepsilon\to 0} Z^\varepsilon_t = r't \quad \text{in probability.}$$

It is simple to check that for $t \geq s$, $Z^\varepsilon_t \leq R^\varepsilon_t$ if the inequality holds for a precedent time. This holds indeed because $R^\varepsilon_s = Z^\varepsilon_s$. It is here where we use that the particles jump only to the right. Hence, for all $\delta > 0$ and $\gamma > 0$, since $r' = r - O(\delta)$,

$$\lim_{\varepsilon\to 0} P(t^{-1}R^\varepsilon_t - r < -O(\delta) - \gamma) = 0.$$

If $\delta < 0$ we perform again the basic coupling. In this case $r' = r + O(\delta)$ and the process with initial configuration η has extra particles. A similar argument shows that $Z^\varepsilon_t \geq R^\varepsilon_t$ and as before,

$$\lim_{\varepsilon\to 0} P(t^{-1}R^\varepsilon_t - r > O(\delta) + \gamma) = 0.$$

Putting the two limits together and taking δ to zero we get the result. $\square$

Proof of (1.6). By the first part of the Theorem we know that the rescaled position of the second class particle at macroscopic time s belongs to the interval $((1-2\rho)s,(1-2\lambda)s)$ with large probability. We fix $\gamma>0$ and partition this interval in N sub-intervals of length γs (without loss of generality we can take $N=2(\rho-\lambda)/\gamma$). Let

$$\ell_s^k = s(k\gamma+(1-2\rho)),\quad k=0,\dots,N.$$

Now

$$\begin{aligned}\left(\left|s^{-1}X_s^\varepsilon - t^{-1}X_t^\varepsilon\right|>2\gamma\right) \le \sum_{k=0}^{N-1} &\left(X_s^\varepsilon\in[\ell_s^k,\ell_s^{k+1}), \left|s^{-1}X_s^\varepsilon - t^{-1}X_t^\varepsilon\right|>2\gamma\right)\\ &+\left(s^{-1}X_s^\varepsilon\notin[1-2\rho,1-2\lambda)\right).\end{aligned}$$

The last term goes to zero as a consequence of the first part of the Theorem. Since the sum above has a finite number of terms, it suffices to show that each term goes to zero. We bound the k-th term by

$$\begin{aligned}&\left(X_s^\varepsilon\in[\ell_s^k,\ell_s^{k+1}), t^{-1}X_t^\varepsilon < s^{-1}\ell_s^k-\gamma\right)\\ &\quad+\left(X_s^\varepsilon\in[\ell_s^k,\ell_s^{k+1}), t^{-1}X_t^\varepsilon > s^{-1}\ell_s^{k+1}+\gamma\right).\end{aligned}\tag{2.12}$$

As in the proof of (1.5) the position of the second class particle is given by the position of a discrepancy initially at the origin. We consider two initial configurations η^0 picked from $\nu_{\rho,\lambda}$ and η^1 that differs from η^0 only in the origin, that is $\eta^1(0)=1-\eta^0(0)$. Performing the basic coupling for these configurations we get

$$X_t^\varepsilon = \varepsilon\sum_x x\mathbf{1}\{\eta^0_{\varepsilon^{-1}t}(x)=1-\eta^1_{\varepsilon^{-1}t}(x)\}.$$

Assume $\eta^0(0)=1$ (the other case is treated similarly) and suppose $\ell_s^k<X_s^\varepsilon$. For the process η_t^0, let $L_t^{\varepsilon,k}$ be ε times the position at time $\varepsilon^{-1}t$ of a second class particle that at time $\varepsilon^{-1}s$ is put in site $[\varepsilon^{-1}\ell_s^k]$. We describe the position of this particle by introducing a new family of processes $\eta^{2,\varepsilon}_{\varepsilon^{-1}u}$ defined, for each ε and $u\le s$, by

$$\eta^{2,\varepsilon}_{\varepsilon^{-1}u}(x) = \begin{cases}\eta^0_{\varepsilon^{-1}u}(x) & \text{if } u<s \text{ or } x\neq[\varepsilon^{-1}\ell_s^k]\\ 1-\eta^0_{\varepsilon^{-1}u}(x) & \text{if } u=s \text{ and } x=[\varepsilon^{-1}\ell_s^k].\end{cases}$$

After time $\varepsilon^{-1}s$, we perform the basic coupling for η_u^0, η_u^1 and $\eta_u^{2,\varepsilon}$. Hence, for $t>s$,

$$L_t^{\varepsilon,k} = \varepsilon\sum_x x\mathbf{1}\{\eta^{2,\varepsilon}_{\varepsilon^{-1}t}(x)=1-\eta^0_{\varepsilon^{-1}t}(x)\}.$$

There are two possibilities: either (a) $\eta^0_{\varepsilon^{-1}s}([\varepsilon^{-1}\ell^k_s]) = 0$, in this case it is easy to see that $L^{\varepsilon,k}_t$ and X^ε_t interact by exclusion and $L^{\varepsilon,k}_t < X^\varepsilon_t$ or (b) $\eta^0_{\varepsilon^{-1}s}([\varepsilon^{-1}\ell^k_s]) = 1$, and in this case $L^{\varepsilon,k}_t$ and X^ε_t may coalesce but can not interchange positions. Since $\ell^k_s = X^\varepsilon_s$ implies $L^{\varepsilon,k}_t = X^\varepsilon_t$, we have proved that if $\ell^k_s \le X^\varepsilon_s$ then $L^{\varepsilon,k}_t \le X^\varepsilon_t$. A similar argument shows that if $\ell^k_s \ge X^\varepsilon_s$ then $L^{\varepsilon,k}_t \ge X^\varepsilon_t$. This implies that the first term of (2.12) is bounded above by

$$\left(t^{-1}L^{\varepsilon,k}_t \le s^{-1}\ell^k_s - \gamma\right)$$

and using a similar argument, the second term of (2.12) is bounded above by

$$\left(t^{-1}L^{\varepsilon,k+1}_t \ge s^{-1}\ell^{k+1}_s + \gamma\right).$$

Both probabilities go to zero by Lemma 2.6. $\square$

3. Two second class particles.

In this section we show that two coalescing second class particles initially added in sites 0 and 1 do not meet with positive probability and that the expectation of the difference of positions at time t is of order t. We assume that when the second class particles are in sites x and $x+1$, at rate 1 both particles coalesce in site x.

Theorem 2. *Let η_t be the simple exclusion process with initial distribution $\nu_{1,0}$. Assume that at time* 0 *two coalescing second class particles are added in sites* 0 *and* 1*. Call X^0_t and X^1_t their positions at time t. Then,*

$$(X^0_t \ne X^1_t \textit{ for all } t \ge 0) \ge 1/4. \tag{3.1}$$

Furthermore,

$$\lim_{t\to\infty} t^{-1}(X^1_t - X^0_t) = 2/3. \tag{3.2}$$

Proof. Let $J_t(\eta)$ be the number of particles to the right of the origin by time t:

$$J_t(\eta) = \sum_{x\ge 1} \eta^\eta_t.$$

Let $\bar\eta$ the configuration $\dots 111000\dots$ with the rightmost particle in the origin. For the configuration $\bar\eta$ in a very small time interval only one jump can occur: the particle in the origin jumps to site 1. This is because the rightmost particle is the only particle that has an empty site to jump to. Hence, the Kolmogorov backwards equation applied to the expectation of $J_t(\bar\eta)$ gives

$$\frac{d}{dt}J_t(\bar\eta) = J_t(\bar\eta^{0,1}) - J_t(\bar\eta), \tag{3.3}$$

where the configuration $\bar\eta^{0,1}$ is defined by $\bar\eta^{0,1}(x) = \bar\eta(x)$ for $x \neq 0, 1$, $\bar\eta^{0,1}(0) = 0$ and $\bar\eta^{0,1}(1) = 1$. We perform the basic coupling for the processes starting with $\bar\eta$ and $\bar\eta^{0,1}$ to conclude that under this coupling

$$J_t(\bar\eta^{0,1}) - J_t(\bar\eta) = 1\{Y_t^0 \leq 0, Y_t^1 > 0\} \qquad \text{a.s.} \tag{3.4}$$

where Y_t^0 and Y_t^1 are the positions of the discrepancies that at time 0 were in the origin and in site 1 respectively. These discrepancies behave like annihilating second class particles. Hence, up to the meeting time, $X_t^i \equiv Y_t^i$. This implies that

$$\{X_s^0 \neq X_s^1 \text{ for all } 0 \leq s \leq t\} \supset \{Y_t^0 \leq 0, Y_t^1 > 0\}$$

and

$$(X_t^0 \neq X_t^1 \text{ for all } t \geq 0) \geq \lim_{t\to\infty} (Y_t^0 \leq 0, Y_t^1 > 0). \tag{3.5}$$

On the other hand,

$$\frac{d}{dt} J_t(\bar\eta) = [\eta_t(0)(1 - \eta_t(1))].$$

Hence, by standard convergence to local equilibrium (1.1),

$$\lim_{t\to\infty} \frac{d}{dt} J_t(\bar\eta) = u(0,1)(1 - u(0,1)) = \frac{1}{4}. \tag{3.6}$$

Putting (3.3), (3.4), (3.5) and (3.6) together we get (3.1). The same argument can be applied to show that for $r \in [-1, 1]$,

$$\lim_{t\to\infty} (Y_t^0 \leq rt, Y_t^1 > rt) = u(r,1)(1 - u(r,1)) - ru(r,1). \tag{3.7}$$

Write

$$(X_t^1 - X_t^0) = \sum_y P(X_t^0 \leq y < X_t^1) = \sum_y P(Y_t^0 \leq y < Y_t^1)$$

because up to the meeting time, $X_t^1 - X_t^0 = Y_t^1 - Y_t^0$ almost surely. Using (3.7) and dominated convergence,

$$\begin{aligned}\lim_{t\to\infty} t^{-1}(X_t^1 - X_t^0) &= \int_{-1}^{1} [u(r,1)(1 - u(r,1)) - ru(r,1)]\, dr \\ &= \int_{-1}^{1} \left[\frac{1-r}{2}\frac{1+r}{2} - r\frac{1-r}{2}\right] dr = \frac{2}{3}. \quad \square\end{aligned}$$

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References

- E. D. Andjel, M. E. Vares (1987). Hydrodynamic equations for attractive particle systems on . *J. Stat. Phys.* **47**
- P. A. Ferrari (1992) Shock fluctuations in asymmetric simple exclusion. *Probab. Theor. Related Fields.* **91**, 81–101.
- A. F. Filippov (1960) Differential equations with discontinuous right-hand side *Mat. Sbornik (N.S.)* **51(93)**, 99–128.
- C. Landim (1993). Conservation of local equilibrium for attractive particle systems on Z^d. *Ann. Probab.***21** 4 1782-1808
- T. M. Liggett (1985). *Interacting Particle Systems.* Springer, Berlin.
- H. Rezakhanlou (1990) Hydrodynamic limit for attractive particle systems on Z^d. *Comm. Math. Phys.* **140** 417-448.
- H. Rezakhanlou (1993) Microscopic structure of shocks in one conservation laws. Preprint submitted to *Ann. Inst. H. Poincaré, Analyse non lineaire.*
- H. Rost (1982) Nonequilibrium behaviour of a many particle process: density profile and local equilibrium. *Z. Wahrsch. verw. Gebiete,* **58** 41-53.

Instituto de Matemática e Estatística — Universidade de São Paulo
Cx. Postal 20570 — 01452-990 São Paulo SP — Brasil
`<pablo@ime.usp.br>`